ASYMPTOTIC REPRESENTATIONS AND DRINFELD RATIONAL FRACTIONS

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Abstract. We introduce and study a category of representations of the Borel algebra, associated with a quantum loop algebra of non-twisted type. We construct fundamental representations for this category as a limit of the Kirillov-Reshetikhin modules over the quantum loop algebra and we establish explicit formulas for their characters. We prove that general simple modules in this category are classified by n-tuples of rational functions in one variable, which are regular and non-zero at the origin but may have a zero or a pole at infinity.

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1. Introduction

The theory of finite dimensional representations of quantum loop algebras is a rich subject, developed already by many authors. It has many applications and connections to various branches of mathematics, including representation theory, algebraic geometry, combinatorics and classical/quantum integrable systems. For a recent review the reader is referred to [CH, L].

Let us briefly recall the classification of irreducible finite dimensional representations. Let $U_q(\mathfrak{g})$ be a quantum loop algebra of non-twisted type, and let $x_i^\pm(z)$, $\phi_i^\pm(z)$ $(1 \leq i \leq n)$ be the Drinfeld currents (for the notation, see Section 2 below). Then any irreducible (type 1) finite dimensional $U_q(\mathfrak{g})$-module V is presented as $V = U_q(\mathfrak{g})\, v$, with v a non-zero vector satisfying

$$x_i^+(z)\, v = 0, \quad \phi_i^\pm(z)\, v = \Psi_i(z)\, v \quad (i = 1, \cdots, n)\,.$$

The eigenvalues of $\phi_i^\pm(z)$ are the power series expansions in $z^{\pm 1}$ of a rational function $\Psi_i(z)$ of a specific type,

$$\Psi_i(z) = q_i^{\deg P_i} \frac{P_i(q_i^{-1} z)}{P_i(q_i z)}\,,$$

where $P_i(z) \in \mathbb{C}[z]$ is a polynomial such that $P_i(0) = 1$. These are called Drinfeld polynomials. The n-tuple $\mathbf{\Psi} = \big(\Psi_i\big)_{i=1,\cdots,n}$ is called the highest ℓ-weight of V. Note that each $\Psi_i(z)$ is regular and non-zero at $z = 0, \infty$. The correspondence between V and $\big(P_i(z)\big)_{1\leq i\leq n}$ is bijective [CP1, CP2].

A particularly well-studied case is the Kirillov-Reshetikhin (KR) module defined by

$$P_i(z) = \prod_{l=1}^{k}(1 - aq_i^{k-2l+1}z)\,, \quad P_j(z) = 1 \ (j \neq i)$$

for some $i = 1, \cdots, n$, $k \in \mathbb{Z}_{\geq 0}$ and $a \in \mathbb{C}^*$. In this article we denote it by $L(M^{(i)}_{k,a})$. The KR modules have various nice properties. Among others, it is known [N2, H2] that as $k \to \infty$ the normalized q-character $\tilde{\chi}_q\big(L(M^{(i)}_{k,q_i^{-2k+1}})\big)$ [FR] has a well-defined limit as a formal power series.

In view of these results, one is naturally led to ask the following questions:

(i) In the description of highest ℓ-weight modules, what will happen if $\Psi_i(z)$'s are more general rational fractions?

(ii) What is the representation theoretical content of $\lim_{k\to\infty} \tilde{\chi}_q\big(L(M^{(i)}_{k,q_i^{-2k+1}})\big)$?

Motivated by these questions, we introduce and study in this article a certain category $\mathcal{O}$ of $U_q(\mathfrak{b})$-modules, where $U_q(\mathfrak{b})$ is the Borel subalgebra of $U_q(\mathfrak{g})$. Basic notions for $U_q(\mathfrak{g})$ such as highest ℓ-weight carry over in a straightforward manner to $U_q(\mathfrak{b})$ as well. We prove that a simple highest ℓ-weight $U_q(\mathfrak{b})$-module belongs to category $\mathcal{O}$ if and only if its highest ℓ-weight $\mathbf{\Psi} = (\Psi_i(z))_{1\leq i\leq n}$ consists of rational functions which are regular and non-zero at $z = 0$. We call them the Drinfeld rational fractions. Since no condition at $z = \infty$ is required, this gives an answer to question (i). These simple modules are infinite dimensional in general, but we prove that their weight spaces are finite dimensional.

Among simple highest ℓ-weight modules, of particular significance are the two kinds of fundamental modules $L^{\pm}_{i,a}$ $(1 \leq i \leq n,\ a \in \mathbb{C}^*)$ defined by the highest ℓ-weight

$$\Psi_i(z) = (1 - az)^{\pm 1}\,, \quad \Psi_j(z) = 1 \ (j \neq i)\,.$$

General simple highest ℓ-weight modules are subquotients of tensor products of $L^{\pm}_{i,a}$'s. For the proof of the classification mentioned above, the key point lies in showing that $L^{\pm}_{i,a}$'s belong to category $\mathcal{O}$. We do this by constructing the module $L^{-}_{i,a}$ as the limit $k \to \infty$ of KR modules $M^{(i)}_{i,q_i^{-2k+1}a}$ viewed as $U_q(\mathfrak{b})$-modules (with an appropriate shift of grading). The module $L^{+}_{i,a}$ are then constructed either as a similar limit of $M^{(i)}_{i,q_i^{2k-1}a}$ or using a duality argument. In particular the limit of the normalized q-characters of $M^{(i)}_{i,q_i^{-2k+1}a}$ is the q-character of the $U_q(\mathfrak{b})$-module $L^{-}_{i,1}$. This gives an answer to question (ii). These are the main results of the present paper.

Actually we prove that the module $L^{-}_{i,a}$ admits an action of a 'larger' algebra $\widetilde{U}_q(\mathfrak{g})$ which we call asymptotic algebra. It is defined in the same way as the quantum loop algebra $U_q(\mathfrak{g})$ wherein invertibility of the $\phi_i^{-}(0)$'s is not assumed. Asymptotic algebra does not contain $U_q(\mathfrak{b})$ as a subalgebra, but it is 'larger' in the sense that Q-graded $\widetilde{U}_q(\mathfrak{g})$-modules ($Q$ denoting the root lattice of the classical part of $\mathfrak{g}$) can be regarded uniquely as $U_q(\mathfrak{b})$-modules. See Proposition 2.4 for the precise statement. In contrast, the action of $U_q(\mathfrak{b})$ on $L^{+}_{i,a}$ cannot be extended to that of the full asymptotic algebra $\widetilde{U}_q(\mathfrak{g})$.

Since explicit character formulas for KR modules are known [N2, H2], our asymptotic construction of the fundamental representations $L^{\pm}_{i,a}$ implies explicit formulas for their characters $\chi(L^{\pm}_{i,a})$. Moreover, our results imply that $\chi(L^{+}_{i,a}) = \chi(L^{-}_{i,b})$ for any $i = 1, \cdots, n$ and $a, b \in \mathbb{C}^*$.

Some particular cases of fundamental representations $L^{\pm}_{i,a}$ of the Borel algebra have been discussed in the literature under the name of 'q-oscillator representations', see e.g. [BLZ], Appendix D for $\widehat{\mathfrak{sl}}_2$, [BHK], Appendix B for $\widehat{\mathfrak{sl}}_3$, [BTs], Section 5 for $\widehat{\mathfrak{sl}}(2|1)$, and [Ko] for $L^{\pm}_{1,a}$ of $\widehat{\mathfrak{sl}}_n$. The corresponding q-characters are intimately connected with Baxter's Q-operators important in quantum integrable systems. Giving a systematic account to this construction has been another motivation for our study.

As mentioned above, the theory of finite dimensional representations of quantum affine algebras has been intensively studied. We can expect similar developments for the category $\mathcal{O}$ of the Borel algebra. For example, it would be very interesting to find the defining relations for its Grothendieck ring.

The standard asymptotic representation theory [V] involves limits with respect to the rank of a Lie algebra or a group. In the present paper, the limits are taken with respect to the level of representations. Here, by 'level' we mean that of representations of the quantum group associated to the underlying finite-dimensional Lie algebra. The relation between these two kinds of asymptotic representation theory should be understood in the spirit of the level/rank duality of [F].

Another direction is to understand the connection of our work to results about Demazure modules for classical current algebras, such as in [FL].

We hope to return to these issues in the near future.

The text is organized as follows.

In section 2 we set up the notation concerning $U_q(\mathfrak{g})$, introduce the asymptotic algebra $\widetilde{U}_q(\mathfrak{g})$ and discuss the connection between representations of $\widetilde{U}_q(\mathfrak{g})$ and that of the Borel algebra $U_q(\mathfrak{b})$. In section 3 we discuss the basics of highest ℓ-weight modules of $U_q(\mathfrak{b})$. We introduce the category $\mathcal{O}$, and state the classification of simple modules in Theorem 3.11, whose proof will be given in the following sections. We also discuss the q-characters and summarize some facts about finite dimensional representations of $U_q(\mathfrak{g})$, including the results from [H4] which will be used in subsequent sections. We analyze analogous properties for a category $\mathcal{O}^*$ dual to $\mathcal{O}$. In section 4 we show that the family of KR modules $\{M^{(i)}_{k,q_i^{-2k+1}}\}_{k\geq 0}$ has a well defined limit V_∞, and that it has the structure of a module over the asymptotic algebra $\widetilde{U}_q(\mathfrak{g})$. As a consequence, the simple module $L^{-}_{i,1}$ is shown to belong to category $\mathcal{O}$.

The next section concerns the simple module $L^{+}_{i,1}$. In section 5 we consider a dual module $(V^{\sigma}_{\infty})^*$ of the module V_∞ constructed in the previous section, over the algebra $\widetilde{U}_{q^{-1}}(\mathfrak{g})$. From this we conclude that $L^{+}_{i,1}$ is in category $\mathcal{O}$. By using this result, we prove in section 6 that V_∞ is irreducible, so that it coincides with the simple module $L^{-}_{i,1}$. By using duality, we also prove that $(V^{\sigma}_{\infty})^*$ is irreducible isomorphic to $L^{+}_{i,1}$. In particular, we get an explicit character formula for $L^{\pm}_{i,a}$. In the last section 7 we discuss the asymptotic construction of $L^{+}_{i,1}$.

2. Quantum loop algebra and related algebras

In this section, we introduce our notation concerning the quantum loop algebra and related algebras.

2.1. **Quantum loop algebra.** Let $C = (C_{i,j})_{0\leq i,j\leq n}$ be an indecomposable Cartan matrix of non-twisted affine type. We denote by $\mathfrak{g}$ the Kac-Moody Lie algebra associated with C. Set $I = \{1, \ldots, n\}$, and denote by $\dot{\mathfrak{g}}$ the finite-dimensional simple Lie algebra associated with the Cartan matrix $(C_{i,j})_{i,j\in I}$. Let $\{\alpha_i\}_{i\in I}$, $\{\alpha_i^\vee\}_{i\in I}$, $\{\omega_i\}_{i\in I}$ be the simple roots, the simple coroots and the fundamental weights of $\dot{\mathfrak{g}}$, respectively. We set $Q = \oplus_{i\in I}\mathbb{Z}\alpha_i$, $Q^+ = \oplus_{i\in I}\mathbb{Z}_{\geq 0}\alpha_i$. Let $D = \mathrm{diag}(d_0 \ldots, d_n)$ be the unique diagonal matrix such that $B = DC$ is symmetric and d_i's are relatively prime positive integers. We denote by $(\ , \ ) : Q \times Q \to \mathbb{Z}$ the invariant symmetric bilinear form such that $(\alpha_i, \alpha_i) = 2d_i$. Let $a_0, \cdots, a_n$ stand for the Kac label ([Kac], pp.55-56).

Throughout this paper, we fix a non-zero complex number q which is not a root of unity. We set $q_i = q^{d_i}$.

The quantum loop algebra $U_q(\mathfrak{g})$ is the $\mathbb{C}$-algebra defined by generators e_i, f_i, $k_i^{\pm 1}$ $(0 \leq i \leq n)$ and the following relations for $0 \leq i, j \leq n$.

$$k_ik_j = k_jk_i, \quad k_0^{a_0}k_1^{a_1}\cdots k_n^{a_n} = 1,$$
$$k_ie_jk_i^{-1} = q_i^{C_{i,j}}e_j, \quad k_if_jk_i^{-1} = q_i^{-C_{i,j}}f_j,$$
$$[e_i, f_j] = \delta_{i,j}\frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},$$
$$\sum_{r=0}^{1-C_{i.j}}(-1)^re_i^{(1-C_{i,j}-r)}e_je_i^{(r)} = 0 \quad (i \neq j), \qquad \sum_{r=0}^{1-C_{i.j}}(-1)^rf_i^{(1-C_{i,j}-r)}f_jf_i^{(r)} = 0 \quad (i \neq j)\,.$$

Here we have set $x_i^{(r)} = x_i^r/[r]_{q_i}!$ $(x_i = e_i, f_i)$, and used the standard symbols for q-integers

$$[m]_z = \frac{z^m - z^{-m}}{z - z^{-1}}, \quad [m]_z! = \prod_{j=1}^m[j]_z, \quad \begin{bmatrix} s \\ r \end{bmatrix}_z = \frac{[s]_z!}{[r]_z![s-r]_z!}.$$

The algebra $U_q(\mathfrak{g})$ has a Hopf algebra structure. We choose the coproduct and the antipode given by

$$\Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i, \quad \Delta(k_i) = k_i \otimes k_i\,,$$
$$S(e_i) = -k_i^{-1}e_i, \quad S(f_i) = -f_ik_i, \quad S(k_i) = k_i^{-1}\,,$$

where $i = 0, \cdots, n$. In particular,

$$S^{-1}(e_i) = -e_ik_i^{-1}, \quad S^{-1}(f_i) = -k_if_i, \quad S^{-1}(k_i) = k_i^{-1}\,.$$

2.2. **Drinfeld generators and asymptotic algebras.** The algebra $U_q(\mathfrak{g})$ has another presentation in terms of the Drinfeld generators. For our purposes it is convenient to introduce them in the following manner.

Let $\widetilde{U}_q(\mathfrak{g})$ be the $\mathbb{C}$-algebra defined by generators

$$\tilde{x}_{i,r}^{\pm} \ (i \in I, r \in \mathbb{Z}), \quad \tilde{\phi}_{i,\pm m}^{\pm} \ (i \in I, m \geq 0), \quad \kappa_i \ (i \in I)$$

and the following defining relations for all $i,j\in I$, $r,r'\in\mathbb{Z}$, $m,m'\geq 0$:

$$\begin{aligned}
&\tilde{\phi}^+_{i,0}=1\,,\tilde{\phi}^-_{i,0}=\kappa_i^2\,,\quad [\tilde{\phi}^\pm_{i,\pm m},\tilde{\phi}^\pm_{j,\pm m'}]=0,\ [\tilde{\phi}^\pm_{i,\pm m},\tilde{\phi}^\mp_{j,\mp m'}]=0\,,\\
&\kappa_i\tilde{x}^\pm_{j,r}=q_i^{\mp C_{i,j}}\tilde{x}^\pm_{j,r}\kappa_i,\\
&\tilde{\phi}^+_{i,m}\tilde{x}^\pm_{j,r}=\sum_{0\leq l\leq m}q_i^{\pm lC_{i,j}}\tilde{x}^\pm_{j,r+l}\tilde{\phi}^+_{i,m-l}-\sum_{0\leq l\leq m-1}q_i^{\pm(l-1)C_{i,j}}\tilde{x}^\pm_{j,r+l+1}\tilde{\phi}^+_{i,m-l-1}\,,\\
&\tilde{\phi}^-_{i,-m}\tilde{x}^\pm_{j,r}=-\sum_{0\leq l\leq m-1}q_i^{\mp(l+1)C_{i,j}}\tilde{x}^\pm_{j,r-l-1}\tilde{\phi}^-_{i,-m+l+1}+\sum_{0\leq l\leq m}q_i^{\mp(l+2)C_{i,j}}\tilde{x}^\pm_{j,r-l}\tilde{\phi}^-_{i,-m+l}\,,\\
&q_i^{-C_{i,j}}\tilde{x}^+_{i,r}\tilde{x}^-_{j,r'}-\tilde{x}^-_{j,r'}\tilde{x}^+_{i,r}=\delta_{i,j}\frac{\tilde{\phi}^+_{i,r+r'}-\tilde{\phi}^-_{i,r+r'}}{q_i-q_i^{-1}}\,,\\
&\tilde{x}^\pm_{i,r+1}\tilde{x}^\pm_{j,r'}-q_i^{\pm C_{i,j}}\tilde{x}^\pm_{j,r'}\tilde{x}^\pm_{i,r+1}=q_i^{\pm C_{i,j}}\tilde{x}^\pm_{i,r}\tilde{x}^\pm_{j,r'+1}-\tilde{x}^\pm_{j,r'+1}\tilde{x}^\pm_{i,r}\,,\\
&\sum_{\pi\in\Sigma_s}\sum_{k=0}^{s}(-1)^k\begin{bmatrix}s\\k\end{bmatrix}_{q_i}\tilde{x}^\pm_{i,r_{\pi(1)}}\cdots\tilde{x}^\pm_{i,r_{\pi(k)}}\tilde{x}^\pm_{j,r'}\tilde{x}^\pm_{i,r_{\pi(k+1)}}\cdots\tilde{x}^\pm_{i,r_{\pi(s)}}=0\,.
\end{aligned}$$

In the last relation, $i\neq j$, $s=1-C_{i,j}$, $r_1,\cdots,r_s$ run over all integers, and Σ_s stands for the symmetric group on s letters. We have set also $\tilde{\phi}^\pm_{i,\pm m}=0$ for $m<0$. We call $\widetilde{U}_q(\mathfrak{g})$ the asymptotic algebra. Notice that the elements $\kappa_i\in\widetilde{U}_q(\mathfrak{g})$ are not assumed to be invertible. Indeed, we shall consider later representations of $\widetilde{U}_q(\mathfrak{g})$ on which κ_i act as 0.

We have then

Proposition 2.1. [Dr2, Be] *There is an isomorphism of $\mathbb{C}$-algebras*

$$U_q(\mathfrak{g})\simeq\widetilde{U}_q(\mathfrak{g})\otimes_{\mathbb{C}[\kappa_i]_{i\in I}}\mathbb{C}[\kappa_i,\kappa_i^{-1}]_{i\in I}\,. \tag{2.1}$$

The standard Drinfeld generators $x^\pm_{i,r}$, $\phi^\pm_{i,\pm m}$, $k_i^{\pm1}$ are related to those in the right hand side by

$$x^+_{i,r}=\tilde{x}^+_{i,r},\quad x^-_{i,r}=\kappa_i^{-1}\tilde{x}^-_{i,r},\quad \phi^\pm_{i,\pm m}=\kappa_i^{-1}\tilde{\phi}^\pm_{i,\pm m},\quad k_i=\kappa_i^{-1}\,. \tag{2.2}$$

With this identification, we shall regard $\widetilde{U}_q(\mathfrak{g})$ as a subalgebra of $U_q(\mathfrak{g})$.

For $\mathfrak{g}=\widehat{\mathfrak{sl}}_2$, the isomorphism (2.1) is given explicitly by

$$\begin{aligned}
&e_1\mapsto\tilde{x}^+_{1,0},\quad f_1\mapsto\kappa_1^{-1}\tilde{x}^-_{1,0},\quad e_0\mapsto\tilde{x}^-_{1,1},\quad f_0\mapsto\tilde{x}^+_{1,-1}\kappa_1^{-1},\\
&k_0\mapsto\kappa_1,\quad k_1\mapsto\kappa_1^{-1},\quad [e_1,k_1e_0]\mapsto\frac{1}{q-q^{-1}}\kappa_1^{-1}\tilde{\phi}^+_{1,1},
\end{aligned}$$

that is $k_1e_0\mapsto x^-_{1,1}$ and $f_0k_1^{-1}\mapsto x^+_{1,-1}$.

In the general case, the isomorphism (2.1) takes e_i to $x^+_{i,0}$, f_i to $x^-_{i,0}$, and k_i to κ_i^{-1} for $i\in I$. The image y of e_0 is described as follows (see e.g. [CP2], p.393). In the Lie algebra $\dot{\mathfrak{g}}$, choose simple root vectors $\dot{x}^+_i$ and suppose that the maximal root vector is written as a commutator $\lambda[\dot{x}^+_{i_1},[\dot{x}^+_{i_2},\cdots,[\dot{x}^+_{i_k},\dot{x}^+_{j_0}]\cdots]]$ with some $\lambda\in\mathbb{C}^\times$. Quite generally we say that an element x of $U_q(\mathfrak{g})$ has weight $\beta\in Q$ if $k_ix=q^{(\beta,\alpha_i)}xk_i$ for all $i\in I$. For an element A

(resp., B) of weight α (resp., β), introduce the notation $[A,B]_q = AB - q^{(\alpha,\beta)}BA$. Then we have

$$y = [\tilde{x}^-_{i_1,0}, [\tilde{x}^-_{i_2,0}, \cdots, [\tilde{x}^-_{i_k,0}, \tilde{x}^-_{j_0,1}]_q \cdots]_q]_q \,. \tag{2.3}$$

In what follows we shall use the generating series

$$\phi_i^\pm(z) = \sum_{m\geq 0} \phi^\pm_{i,\pm m} z^{\pm m}, \quad \tilde{\phi}_i^\pm(z) = \sum_{m\geq 0} \tilde{\phi}^\pm_{i,\pm m} z^{\pm m}. \tag{2.4}$$

Let us define the following isomorphism.

Proposition 2.2. *There is a unique isomorphism of $\mathbb{C}$-algebras*

$$\sigma : U_q(\mathfrak{g}) \to U_{q^{-1}}(\mathfrak{g})$$

satisfying

$$\tilde{x}^+_{i,m} \mapsto q_i^2 \tilde{x}^-_{i,m}, \quad \tilde{x}^-_{i,m} \mapsto \tilde{x}^+_{i,m}, \quad \phi^\pm_{i,\pm r} \mapsto \phi^\pm_{i,\pm r}$$

for $i \in I$, $m \in \mathbb{Z}$, $r \geq 0$.

In particular we have $\sigma(k_i) = k_i$ and $\sigma(\tilde{\phi}^\pm_{i,\pm r}) = \tilde{\phi}^\pm_{i,\pm r}$ for $i \in I$ and $r \geq 0$. This implies that σ restricts to an isomorphism of $\mathbb{C}$-algebras $\tilde{U}_q(\mathfrak{g}) \to \tilde{U}_{q^{-1}}(\mathfrak{g})$.

Proof. As $\sigma(k_i) = k_i$, it suffices to check that the defining relations of $\tilde{U}_q(\mathfrak{g})$ are mapped to the defining relations of $\tilde{U}_{q^{-1}}(\mathfrak{g})$. This is immediate for all relations except

$$q_i^{-C_{i,j}} \tilde{x}^+_{i,r} \tilde{x}^-_{j,r'} - \tilde{x}^-_{j,r'} \tilde{x}^+_{i,r} = \delta_{i,j} \frac{\tilde{\phi}^+_{i,r+r'} - \tilde{\phi}^-_{i,r+r'}}{q_i - q_i^{-1}}. \tag{2.5}$$

The left hand side of (2.5) is mapped to

$$q_i^{-C_{i,j}} q_i^2 \tilde{x}^-_{i,r} \tilde{x}^+_{j,r'} - q_i^2 \tilde{x}^+_{j,r'} \tilde{x}^-_{i,r} = -q_i^{-C_{i,j}+2}(-\tilde{x}^-_{i,r}\tilde{x}^+_{j,r'} + q_j^{C_{j,i}} \tilde{x}^+_{j,r'} \tilde{x}^-_{i,r}).$$

If $i \neq j$, this is zero in $\tilde{U}_{q^{-1}}(\mathfrak{g})$. If $i = j$, this is

$$-(q_i^2 \tilde{x}^+_{i,r'} \tilde{x}^-_{i,r} - \tilde{x}^-_{i,r} \tilde{x}^+_{i,r'}).$$

In this case, the right hand side of (2.5) is mapped to

$$\frac{\tilde{\phi}^+_{i,r+r'} - \tilde{\phi}^-_{i,r+r'}}{q_i - q_i^{-1}} = -\frac{\tilde{\phi}^+_{i,r+r'} - \tilde{\phi}^-_{i,r+r'}}{q_i^{-1} - q_i},$$

and so we get the correct relation in $\tilde{U}_{q^{-1}}(\mathfrak{g})$. □

For example, in the case of $\mathfrak{sl}_2$, we get $\sigma(e_1) = q^2 \tilde{x}^-_{1,0}$, $\sigma(f_1) = k_1 x^+_{1,0}$, $\sigma(e_0) = \sigma(\tilde{x}^-_{1,1}) = x^+_{1,1}$ and $\sigma(f_0) = \sigma(x^+_{1,-1}k_1) = q^2 \tilde{x}^-_{1,-1} k_1$.

2.3. **Borel algebra.**

Definition 2.3. *The Borel algebra $U_q(\mathfrak{b})$ is the subalgebra of $U_q(\mathfrak{g})$ generated by e_i and $k_i^{\pm 1}$ with $0 \leq i \leq n$.*

This is a Hopf subalgebra of $U_q(\mathfrak{g})$. It is well known (see e.g.[Ja, Section 4.21], [H1, Lemma 4]) that $U_q(\mathfrak{b})$ is isomorphic to the algebra defined by the generators e_i, $k_i^{\pm 1}$ $(0 \leq i \leq n)$ and the relations

$$k_i k_j = k_j k_i, \quad k_i e_j = q_i^{C_{i,j}} e_j k_i,$$
$$\sum_{r=0}^{s} (-1)^r (e_i)^{(s-r)} e_j (e_i)^{(r)} = 0 \quad (s = 1 - C_{i,j},\ i \neq j)\,.$$

For $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$, the Borel algebra $U_q(\mathfrak{b})$ can be easily described in terms of the Drinfeld generators : it is the subalgebra of $U_q(\widehat{\mathfrak{sl}}_2)$ generated by the $x_{1,m}^+$, $x_{1,r}^-$, $k_1^{\pm 1}$, $\phi_{1,m}^+$ where $m \geq 0$ and $r > 0$. Such a simple description does not hold in the general case. Nevertheless it is known [Be] that the algebra $U_q(\mathfrak{b})$ contains the Drinfeld generators $x_{i,m}^+$, $x_{i,r}^-$, $k_i^{\pm 1}$, $\phi_{i,r}^+$ where $i \in I$, $m \geq 0$ and $r > 0$.

Let $U_q(\mathfrak{g})^{\pm}$ (resp. $U_q(\mathfrak{g})^0$) be the subalgebra of $U_q(\mathfrak{g})$ generated by the $x_{i,r}^{\pm}$ where $i \in I, r \in \mathbb{Z}$ (resp. by the $k_i^{\pm 1}$, $\phi_{i,\pm r}^{\pm}$ where $i \in I, r > 0$).

Then by [Be], [BCP, Proposition 1.3] we have a triangular decomposition (isomorphism of vector spaces) :

$$U_q(\mathfrak{g}) \simeq U_q(\mathfrak{g})^- \otimes U_q(\mathfrak{g})^0 \otimes U_q(\mathfrak{g})^+.$$

Let further $U_q(\mathfrak{b})^{\pm} = U_q(\mathfrak{g})^{\pm} \cap U_q(\mathfrak{b})$ and $U_q(\mathfrak{b})^0 = U_q(\mathfrak{g})^0 \cap U_q(\mathfrak{b})$. Then we have

$$U_q(\mathfrak{b})^+ = \langle x_{i,m}^+ \rangle_{i \in I, m \geq 0}, \quad U_q(\mathfrak{b})^0 = \langle \phi_{i,r}^+, k_i^{\pm 1} \rangle_{i \in I, r > 0}.$$

In general, $U_q(\mathfrak{b})^-$ does not have such a nice description in terms of Drinfeld generators, except when $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$ for which $U_q(\mathfrak{b})^- = \langle x_{1,m}^- \rangle_{m \geq 1}$.

By using the PBW basis of [Be], we have a triangular decomposition

$$U_q(\mathfrak{b}) \simeq U_q(\mathfrak{b})^- \otimes U_q(\mathfrak{b})^0 \otimes U_q(\mathfrak{b})^+. \tag{2.6}$$

2.4. **From $\widetilde{U}_q(\mathfrak{g})$ to $U_q(\mathfrak{b})$.** A representation V of the asymptotic algebra $\widetilde{U}_q(\mathfrak{g})$ is said to be Q-graded if there is a decomposition into a direct sum of linear subspaces $V = \bigoplus_{\alpha \in Q} V^{(\alpha)}$ such that

$$\tilde{x}_{i,r}^{\pm} V^{(\alpha)} \subset V^{(\alpha \pm \alpha_i)}, \quad \tilde{\phi}_{i,\pm m}^{\pm} V^{(\alpha)} \subset V^{(\alpha)}, \quad \kappa_i V^{(\alpha)} \subset V^{(\alpha)}$$

hold for any $\alpha \in Q$, $i \in I$, $r \in \mathbb{Z}, m \geq 0$.

Proposition 2.4. *For a Q-graded $\widetilde{U}_q(\mathfrak{g})$-module V, there is a unique structure of $U_q(\mathfrak{b})$-module on V such that*

$$e_i\, v = \tilde{x}_{i,0}^+ v, \quad e_0\, v = y\, v, \quad k_i v = q_i^{\alpha(\alpha_i^\vee)} v \quad (i \in I,\ v \in V^{(\alpha)})\,, \tag{2.7}$$

where the element $y \in \widetilde{U}_q(\mathfrak{g})$ is given in (2.3).

Similarly, for a Q-graded $\tilde{U}_{q^{-1}}(\mathfrak{g})$-module V, there is a unique structure of $U_q(\mathfrak{b})$-module on V such that

$$e_i\, v = \sigma(\tilde{x}^+_{i,0})v, \quad e_0\, v = \sigma(y)v, \quad k_i v = q_i^{-\alpha(\alpha_i^\vee)} v \quad (i \in I,\ v \in V^{(\alpha)})\,. \tag{2.8}$$

Proof. Let us check that the relations of $U_q(\mathfrak{b})$ are satisfied. Since each $V^{(\alpha)}$ is a joint eigenspace of k_i, it is clear that $k_i k_j = k_j k_i$. From $\tilde{x}^\pm_{i,r} V^{(\alpha)} \subset V^{(\alpha\pm\alpha_i)}$, we get the relations $k_i e_j = q_i^{C_{i,j}} e_j k_i$. Finally the elements $y, \tilde{x}^+_{1,0}, \cdots, \tilde{x}^+_{n,0}$ satisfy Serre relations in the algebra $\widetilde{U}_q(\mathfrak{g})$ (resp. $\tilde{U}_{q^{-1}}(\mathfrak{g})$). □

In general, the action of $U_q(\mathfrak{b})$ can not be extended to an action of the full quantum affine algebra $U_q(\mathfrak{g})$, because the κ_i's are allowed to have non-trivial kernels on V.

As $\sigma(U_{q^{-1}}(\mathfrak{b})) \neq U_q(\mathfrak{b})$, it is not sufficient to have a $U_{q^{-1}}(\mathfrak{b})$-module structure on V to get a $U_q(\mathfrak{b})$-module structure via σ. That is why we will construct $\tilde{U}_{q^{-1}}(\mathfrak{g})$-modules.

Remark 2.5. For a Q-graded $U_q(\mathfrak{b})$-module $V = \oplus_{\alpha\in Q} V^{(\alpha)}$, one can freely shift the action of the k_i's. Namely, for any $\beta_i \in \mathbb{C}^\times$ $(i \in I)$, a new $U_q(\mathfrak{b})$-module structure on V is obtained by setting

$$k_i\ v = \beta_i\, q^{\alpha(\alpha_i)} v \qquad (v \in V^{(\alpha)})$$

and retaining the same action of e_i's.

2.5. **Coproduct.** The algebra $U_q(\mathfrak{g})$ has a natural Q-grading defined by

$$\deg\bigl(x^\pm_{i,m}\bigr) = \pm\alpha_i, \quad \deg\bigl(h_{i,r}\bigr) = \deg\bigl(k_i^\pm\bigr) = 0\,.$$

Let $\mathcal{U}_q^+(\mathfrak{g})$ (resp. $\mathcal{U}_q^-(\mathfrak{g})$) be the subalgebra of $U_q(\mathfrak{g})$ consisting of elements of positive (resp. negative) Q-degree. These subalgebras should not be confused with the subalgebras $U_q(\mathfrak{g})^\pm$ previously defined in terms of Drinfeld generators. Let

$$X^+ = \sum_{j\in I, m\in\mathbb{Z}} \mathbb{C}x^+_{j,m} \subset \mathcal{U}_q^+(\mathfrak{g}).$$

Theorem 2.6. [Da] *Let $i \in I$. For $m \in \mathbb{Z}$, we have*

$$\Delta\left(x^+_{i,m}\right) \in x^+_{i,m} \otimes 1 + U_q(\mathfrak{g}) \otimes \bigl(U_q(\mathfrak{g})X^+\bigr)\,. \tag{2.9}$$

For $r > 0$, we have

$$\Delta\left(\phi^\pm_{i,\pm r}\right) \in \sum_{0\le l\le r} \phi^\pm_{i,\pm l} \otimes \phi^\pm_{i,\pm(r-l)} + \mathcal{U}_q^-(\mathfrak{g}) \otimes \mathcal{U}_q^+(\mathfrak{g}), \tag{2.10}$$

$$\Delta\left(x^-_{i,r}\right) \in x^-_{i,r} \otimes k_i + 1 \otimes x^-_{i,r} + \sum_{1\le j<r} x^-_{i,r-j} \otimes \phi^+_{i,j} + U_q(\mathfrak{g}) \otimes \bigl(U_q(\mathfrak{g})X^+\bigr)\,. \tag{2.11}$$

For $r \le 0$, we have

$$\Delta\left(x^-_{i,r}\right) \in x^-_{i,r} \otimes k_i^{-1} + 1 \otimes x^-_{i,r} + \sum_{1\le j\le -r} x^-_{i,r+j} \otimes \phi^-_{i,-j} + U_q(\mathfrak{g}) \otimes \bigl(U_q(\mathfrak{g})X^+\bigr)\,. \tag{2.12}$$

3. Category $\mathcal{O}$ for $U_q(\mathfrak{b})$

3.1. **Highest ℓ-weight modules.** Denote by $\mathfrak{t}$ the subalgebra of $U_q(\mathfrak{b})$ generated by $\{k_j^{\pm 1}\}_{j\in I}$. Set $\mathfrak{t}^* = \left(\mathbb{C}^\times\right)^I$, and endow it with a group structure by pointwise multiplication. We define a group morphism $^- : Q \longrightarrow \mathfrak{t}^*$ by setting $\overline{\alpha}_i(j) = q_i^{C_{i,j}}$ for a simple root α_i. We shall use the standard partial ordering on $\mathfrak{t}^*$:

$$\omega \le \omega' \quad \text{if } \omega\omega'^{-1} \text{ is a product of } \{\overline{\alpha}_i^{-1}\}_{i\in I}. \tag{3.13}$$

For a $U_q(\mathfrak{b})$-module V and $\omega \in \mathfrak{t}^*$, we set

$$V_\omega = \{v \in V \mid k_i\, v = \omega(i) v \ (\forall i \in I)\}\,, \tag{3.14}$$

and call it the weight space of weight ω. For any $i \in I$, $r \in \mathbb{Z}$ we have $\phi^\pm_{i,r}(V_\omega) \subset V_\omega$ and $x^\pm_{i,r}(V_\omega) \subset V_{\omega\overline{\alpha}_i^{\pm 1}}$. We say that V is $\mathfrak{t}$-diagonalizable if $V = \bigoplus_{\omega\in\mathfrak{t}^*} V_\omega$.

Definition 3.1. *A series $\mathbf{\Psi} = (\Psi_{i,m})_{i\in I, m\ge 0}$ of complex numbers such that $\Psi_{i,0} \neq 0$ for all $i \in I$ is called an ℓ-weight.*

We denote by $\mathfrak{t}^*_\ell$ the set of ℓ-weights. Identifying $(\Psi_{i,m})_{m\ge 0}$ with its generating series we shall write

$$\mathbf{\Psi} = (\Psi_i(z))_{i\in I}, \quad \Psi_i(z) = \sum_{m\ge 0} \Psi_{i,m} z^m.$$

Since each $\Psi_i(z)$ is an invertible formal power series, $\mathfrak{t}^*_\ell$ has a natural group structure. We have a surjective morphism of groups $\varpi : \mathfrak{t}^*_\ell \to \mathfrak{t}^*$ given by $\varpi(\mathbf{\Psi})(i) = \Psi_{i,0}$. For a $U_q(\mathfrak{b})$-module V and $\mathbf{\Psi} \in \mathfrak{t}^*_\ell$, the linear subspace

$$V_{\mathbf{\Psi}} = \{v \in V \mid \exists p \ge 0, \forall i \in I, \forall m \ge 0, (\phi^+_{i,m} - \Psi_{i,m})^p v = 0\} \tag{3.15}$$

is called the ℓ-weight space of V of ℓ-weight $\mathbf{\Psi}$.

Definition 3.2. *A $U_q(\mathfrak{b})$-module V is said to be of highest ℓ-weight $\mathbf{\Psi} \in \mathfrak{t}^*_\ell$ if there is $v \in V$ such that $V = U_q(\mathfrak{b})v$ and the following hold:*

$$e_i\, v = 0 \quad (i \in I)\,, \qquad \phi^+_{i,m} v = \Psi_{i,m} v \quad (i \in I,\ m \ge 0)\,.$$

The ℓ-weight $\mathbf{\Psi} \in \mathfrak{t}^*_\ell$ is uniquely determined by V. It is called the highest ℓ-weight of V. The vector v is said to be a highest ℓ-weight vector of V. See [N1, H1] for an analogous definition in the context of representations of quantum affinizations.

Lemma 3.3. *Let V be a highest ℓ-weight $U_q(\mathfrak{b})$-module with highest ℓ-weight vector v. Then $x^+_{i,m} v = 0$ holds for all $i \in I, m \ge 0$.*

Consequently $V = U_q(\mathfrak{b})^- v$. Moreover V is $\mathfrak{t}$-diagonalizable and $V = \bigoplus_{\lambda \le \varpi(\mathbf{\Psi})} V_\lambda$.

Proof. The first statement can be verified by induction using the formula

$$[\tilde{\phi}^+_{i,1}, x^+_{i,m}] = (q_i^2 - q_i^{-2}) x^+_{i,m+1}\,.$$

The rest of the assertions are clear from (2.6). □

Example. For any $\mathbf{\Psi} \in \mathfrak{t}_\ell^*$, define the Verma module $M(\mathbf{\Psi})$ to be the quotient of $U_q(\mathfrak{b})$ by the left ideal generated by e_i ($i \in I$) and $\phi_{i,m}^+ - \Psi_{i,m}$ ($i \in I, m \geq 0$). From (2.6), $M(\mathbf{\Psi})$ is a free $U_q(\mathfrak{b})^-$-module of rank 1. In particular it is non trivial and it is a highest ℓ-weight module of highest ℓ-weight $\mathbf{\Psi}$.

3.2. **Simple highest ℓ-weight modules.** Let $\mathbf{\Psi} \in \mathfrak{t}_\ell^*$. By a standard argument, the Verma module $M(\mathbf{\Psi})$ has a unique proper submodule, and so we get the following.

Proposition 3.4. *For any $\mathbf{\Psi} \in \mathfrak{t}_\ell^*$, there exists a simple highest ℓ-weight module $L(\mathbf{\Psi})$ of highest ℓ-weight $\mathbf{\Psi}$. This module is unique up to isomorphism.*

Let us give two fundamental examples.

Example. Let $\omega \in \mathfrak{t}^*$. We define $\mathbf{\Psi}_\omega \in \mathfrak{t}_\ell^*$ by $(\mathbf{\Psi}_\omega)_{i,0} = \omega(i)$, $(\mathbf{\Psi}_\omega)_{i,m} = 0$ for $i \in I$ and $m \geq 1$. Then $L(\mathbf{\Psi}_\omega) = \mathbb{C}v$ is 1-dimensional.

Example. For $i \in I$, let $P_i(z) \in \mathbb{C}[z]$ be a polynomial with constant term 1. Set

$$\mathbf{\Psi} = (\Psi_i(z))_{i \in I}, \quad \Psi_i(z) = q_i^{\deg(P_i)} \frac{P_i(zq_i^{-1})}{P_i(zq_i)}.$$

Then $L(\mathbf{\Psi})$ is finite-dimensional. Moreover the action of $U_q(\mathfrak{b})$ can be uniquely extended to an action of the full quantum affine algebra $U_q(\mathfrak{g})$, and all (type 1) irreducible finite-dimensional $U_q(\mathfrak{g})$-modules are of this form. This follows from the classification of simple finite dimensional modules of quantum affine algebras [CP2] by Drinfeld polynomials along with the following result.

Proposition 3.5. *Let V be a simple finite dimensional $U_q(\mathfrak{g})$-module. Then V is simple as a $U_q(\mathfrak{b})$-module.*

This result was proved in [BT] for $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$, and in [Bo], [CG, Proposition 2.7] in the general case. For completeness, let us give a short elementary proof of this statement, independent from the proof of [Bo].

Proof. Let $\pi : U_q(\mathfrak{g}) \to \mathrm{End}(V)$ be the representation morphism. Let $i \in I$. We prove for any $r \in \mathbb{Z}$ that $\pi(x_{i,r}^\pm)$ is in the space linearly spanned by $(\pi(x_{i,m}^\pm))_{m>0}$. Since $\mathrm{End}(V)$ is finite dimensional, there is a non-trivial linear relation $\sum_{r=a}^b \lambda_r \pi(x_{i,r}^\pm) = 0$ with $a \geq 1$. On the other hand, for any $r \in \mathbb{Z}$ we have

$$\phi_{i,-1}^- x_{i,r}^\pm - q_i^{\mp 2} x_{i,r}^\pm \phi_{i,-1}^- = (q_i^{\mp 4} - 1) x_{i,r-1}^\pm \phi_{i,0}^- \, .$$

It follows that for any $R \geq 0$ we have $\sum_{r=a}^b \lambda_r \pi(x_{i,r-R}^\pm) = 0$. Now it is clear that V is cyclic as a $U_q(\mathfrak{b})$-module generated by a highest ℓ-weight vector, and that all primitive vectors are of highest ℓ-weight. $\square$

Note that in the proof, we proved a weak version of the quasi-polynomiality property (see [BK, Proposition 6.2], [H3, Proposition 3.8]).

Remark 3.6. From the relation (2.10), the submodule of $L(\mathbf{\Psi}) \otimes L(\mathbf{\Psi}')$ generated by the tensor product of the highest ℓ-weight vectors is of highest ℓ-weight $\mathbf{\Psi}\mathbf{\Psi}'$. In particular, $L(\mathbf{\Psi}\mathbf{\Psi}')$ is a subquotient of $L(\mathbf{\Psi}) \otimes L(\mathbf{\Psi}')$.

Definition 3.7. *For $i \in I$ and $a \in \mathbb{C}^\times$, let*

$$L_{i,a}^{\pm} = L(\mathbf{\Psi}) \quad \text{where} \quad \Psi_j(z) = \begin{cases} (1-za)^{\pm 1} & (j=i)\,, \\ 1 & (j \neq i)\,. \end{cases} \tag{3.16}$$

The representations $L_{i,a}^{\pm}$ $(i \in I, a \in \mathbb{C}^\times)$ are called fundamental representations.

For $a \in \mathbb{C}^\times$, we have an algebra automorphism $\tau_a : U_q(\mathfrak{g}) \longrightarrow U_q(\mathfrak{g})$ such that

$$\tau_a(x_{i,m}^{\pm}) = a^m x_{i,m}^{\pm}\,, \quad \tau_a(\phi_i^{\pm}(z)) = \phi_i^{\pm}(az)\,. \tag{3.17}$$

The subalgebra $U_q(\mathfrak{b})$ is stable by τ_a. Denote its restriction to $U_q(\mathfrak{b})$ by the same letter. Then the pullbacks of the $U_q(\mathfrak{b})$-modules $L_{i,b}^{\pm}$ by τ_a is $L_{i,ab}^{\pm}$.

We remark that for $a = 0$, we get an algebra morphism $\tau_0 : U_q(\mathfrak{b}) \longrightarrow U_q(\mathfrak{b})$ which is not invertible. Its image is the subalgebra $U_q(\dot{\mathfrak{b}})$ of $U_q(\mathfrak{b})$ generated by the $x_{i,0}^-$, $k_i^{\pm 1}$ $(i \in I)$, that is the Borel algebra of the quantum algebra of finite type associated to $U_q(\dot{\mathfrak{g}})$. The pullback of the 1-dimensional simple representations of $U_q(\dot{\mathfrak{b}})$ are the representations $L(\mathbf{\Psi}_\omega)$.

3.3. **Category $\mathcal{O}$.**

For $\lambda \in \mathfrak{t}^*$, we set $D(\lambda) = \{\omega \in \mathfrak{t}^* \mid \omega \leq \lambda\}$.

Definition 3.8. *A $U_q(\mathfrak{b})$-module V is said to be in category $\mathcal{O}$ if:*

i) V is $\mathfrak{t}$-diagonalizable,

ii) for all $\omega \in \mathfrak{t}^*$ we have $\dim(V_\omega) < \infty$,

iii) there exist a finite number of elements $\lambda_1, \cdots, \lambda_s \in \mathfrak{t}^*$ such that the weights of V are in $\bigcup_{j=1,\cdots,s} D(\lambda_j)$.

The category $\mathcal{O}$ is a tensor category. In general a simple highest ℓ-weight module is not necessarily in category $\mathcal{O}$.

Lemma 3.9. *Let V be a $U_q(\mathfrak{b})$-module, $\omega \in \mathfrak{t}^*$ and $i \in I$. We suppose that V_ω, $V_{\omega\overline{\alpha}_i}$ and $V_{\omega\overline{\alpha}_i^{-1}}$ are finite-dimensional. Then for $\mathbf{\Psi} \in \mathfrak{t}_\ell^*$ such that $\varpi(\mathbf{\Psi}) = \omega$ and $V_{\mathbf{\Psi}} \neq 0$, $\Psi_i(z)$ is rational.*

Proof. The proof is a generalization of the proof of [H1, Lemma 14].

There is a non zero $v \in V_{\mathbf{\Psi}}$ such that $\phi_{i,m}^+ v = \Psi_{i,m} v$ for any $i \in I$, $m \geq 0$. Choose a basis $(u_1, \cdots, u_R)$ of $V_{\omega\overline{\alpha}_i^{-1}}$, a basis $(v_1, \cdots, v_P)$ of $V_{\omega\overline{\alpha}_i}$, and a linear map $\pi : V_\omega \to \mathbb{C}v$ such that $\pi(v) = v$.

For $m \geq 0$ and $p \geq 1$, we can find $\lambda_{p,j}, \lambda'_{p,j}, \mu_{m,j}, \mu'_{m,j} \in \mathbb{C}$ so that

$$x_{i,p}^- v = \sum_{k=1}^{R} \lambda_{p,k} u_k, \quad (q_i - q_i^{-1})\pi\big(x_{i,p}^- v_j\big) = \lambda'_{p,j} v \quad (1 \leq j \leq P)\,,$$

$$x_{i,m}^+ v = \sum_{j=1}^{P} \mu_{m,j} v_j, \quad (q_i - q_i^{-1})\pi\big(x_{i,m}^+ u_k\big) = \mu'_{m,k} v \quad (1 \leq k \leq R).$$

Using $(q_i - q_i^{-1})[x_{i,m}^+, x_{i,p}^-]v = \Psi_{i,m+p}v$ and applying π on both sides we obtain

$$\Psi_{i,m+p} = \sum_{k=1}^{R} \lambda_{p,k}\mu'_{m,k} - \sum_{j=1}^{P} \mu_{m,j}\lambda'_{p,j}\,.$$

We set $\lambda_k(z) = \sum_{p\geq 1} \lambda_{p,k} z^p$, $\lambda'_j(z) = \sum_{p\geq 1} \lambda'_{p,j} z^p$ and

$$\Psi_i^{>m}(z) = z^{-m}\Big(\Psi_i(z) - \sum_{p=0}^{m} \Psi_{i,p} z^p\Big) \quad (m \geq 0)\,.$$

Then for $m \geq 0$ we have

$$\Psi_i^{>m}(z) = \sum_{k=1}^{R} \lambda_k(z)\mu'_{m,k} - \sum_{j=1}^{P} \lambda'_j(z)\mu_{m,j}.$$

Hence $(\Psi_i^{>m}(z))_{m\geq 0}$ is not linearly independent, and we have a relation of the form

$$\sum_{m=0}^{N} a_m z^{N-m}\Big(\Psi_i(z) - \sum_{p=0}^{m} \Psi_{i,p} z^p\Big) = 0.$$

This shows that $\Psi_i(z)$ is rational. $\square$

As a direct consequence, we have the following.

Proposition 3.10. *Let V be in category $\mathcal{O}$. Then for $\mathbf{\Psi} \in \mathfrak{t}_\ell^*$ such that $V_{\mathbf{\Psi}} \neq 0$, $\Psi_i(z)$ is rational for any $i \in I$.*

The following is one of the main results of this paper. It is a complete classification of simple objects in category $\mathcal{O}$.

Theorem 3.11. *For $\mathbf{\Psi} \in \mathfrak{t}_\ell^*$, the simple module $L(\mathbf{\Psi})$ is in category $\mathcal{O}$ if and only if $\Psi_i(z)$ is rational for any $i \in I$.*

Proof. The "only if" part follows directly from Proposition 3.10.

From Remark 3.6, an arbitrary simple representation in category $\mathcal{O}$ is a subquotient of a tensor product of fundamental representations. So, to prove the "if" part, it suffices to show that all fundamental representations are in category $\mathcal{O}$. Furthermore, with the aid of the twist automorphism (3.17), the proof is reduced to the case of $L_{i,1}^{\pm}$.

In the next sections we shall show that $L_{i,1}^{\pm}$ are indeed in category $\mathcal{O}$ (see Corollary 4.8 and Corollary 5.1). $\square$

3.4. **q-characters in category $\mathcal{O}$.** Let $\mathfrak{r}$ be the subgroup of $\mathfrak{t}_\ell^*$ consisting of $\mathbf{\Psi}$ such that $\Psi_i(z)$ is rational for any $i \in I$. (The letter $\mathfrak{r}$ stands for 'rational'.)

Let $\mathcal{E}_\ell \subset \mathbb{Z}^{\mathfrak{r}}$ be the ring of maps $c : \mathfrak{r} \to \mathbb{Z}$ satisfying $c(\mathbf{\Psi}) = 0$ for all $\mathbf{\Psi}$ such that $\varpi(\mathbf{\Psi})$ is outside a finite union of sets of the form $D(\mu)$ and such that for each $\omega \in \mathfrak{t}^*$, there are finitely many $\mathbf{\Psi}$ such that $\varpi(\mathbf{\Psi}) = \omega$ and $c(\mathbf{\Psi}) \neq 0$. Similarly, let $\mathcal{E} \subset \mathbb{Z}^{\mathfrak{t}^*}$ be the ring of maps $c : \mathfrak{t}^* \to \mathbb{Z}$ satisfying $c(\omega) = 0$ for all ω outside a finite union of sets of the form $D(\mu)$. The map ϖ is naturally extended to a surjective ring morphism $\varpi : \mathcal{E}_\ell \to \mathcal{E}$.

For $\mathbf{\Psi} \in \mathfrak{r}$ (resp. $\omega \in \mathfrak{t}^*$), we define $[\mathbf{\Psi}] = \delta_{\mathbf{\Psi},\cdot} \in \mathcal{E}_\ell$ (resp. $[\omega] = \delta_{\omega,\cdot} \in \mathcal{E}$).

Let V be a $U_q(\mathfrak{b})$-module in category $\mathcal{O}$. We define the q-character of V to be the element of $\mathcal{E}_\ell$

$$\chi_q(V) = \sum_{\mathbf{\Psi} \in \mathfrak{r}} \dim(V_{\mathbf{\Psi}})[\mathbf{\Psi}] \,. \tag{3.18}$$

Similarly we define the ordinary character of V to be an element of $\mathcal{E}$

$$\chi(V) = \varpi(\chi_q(V)) = \sum_{\omega \in \mathfrak{t}^*} \dim(V_\omega)[\omega] \,. \tag{3.19}$$

For V in category $\mathcal{O}$ which has a unique ℓ-weight $\mathbf{\Psi}$ whose weight is maximal (for example a highest ℓ-weight module), we also consider its normalized q-character $\tilde{\chi}_q(V)$ and normalized character $\tilde{\chi}(V)$ by

$$\tilde{\chi}_q(V) = [\mathbf{\Psi}^{-1}] \cdot \chi_q(V) \,, \quad \tilde{\chi}(V) = \varpi(\tilde{\chi}_q(V)) \,.$$

Let $\mathrm{Rep}(U_q(\mathfrak{b}))$ be the Grothendieck ring of the category $\mathcal{O}$. We define the q-character morphism as the group morphism

$$\chi_q : \mathrm{Rep}(U_q(\mathfrak{b})) \to \mathcal{E}_\ell$$

which sends a class of a representation V to $\chi_q(V)$. The map is well-defined as χ_q is clearly compatible with exact sequences.

By using Theorem 2.6, we can prove the following as in [FR, Lemma 3] and [FR, Theorem 3].

Proposition 3.12. *The q-character morphism is an injective ring morphism.*

Remark 3.13. As $\mathcal{E}_\ell$ is clearly a commutative ring, this implies that $\mathrm{Rep}(U_q(\mathfrak{b}))$ is commutative. The argument is the same as for the commutativity of the Grothendieck ring $\mathrm{Rep}(U_q(\mathfrak{g}))$ of finite-dimensional modules of $U_q(\mathfrak{g})$ given in [FR]. Note that $\mathrm{Rep}(U_q(\mathfrak{g}))$ is naturally a subring of $\mathrm{Rep}(U_q(\mathfrak{b}))$. The category $\mathcal{O}$ is not braided (it is easy to construct a counter-example by using the category of finite dimensional representations of $U_q(\mathfrak{g})$ which is known to be not braided). Moreover, in contrast to the case of quantum affine algebras, no meromorphic R-matrix (in the sense of [KS]) is known for $U_q(\mathfrak{b})$. That is why the commutativity is a bit more surprising in this context.

3.5. **Finite dimensional representations of $U_q(\mathfrak{g})$.** In this subsection we quote some results for finite-dimensional $U_q(\mathfrak{g})$-modules which will be used in subsequent sections. For more details, the reader may refer to the book [CP2] and to the recent review papers [CH, L]. We consider only representations of type 1, namely such that the eigenvalues of the k_i ($i \in I$) are in $q^{\mathbb{Z}}$.

We have reminded above the parametrization of simple irreducible $U_q(\mathfrak{g})$-modules by n-tuples $(P_i(z))_{i\in I}$ of polynomials of constant term 1.

Following [FR], consider the ring of Laurent polynomials $\mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^*}$ in the indeterminates $\{Y_{i,a}\}_{i\in I, a\in\mathbb{C}^*}$. Let $\mathcal{M}$ be the group of monomials of $\mathcal{Y}$. For a monomial $m = \prod_{i\in I, a\in\mathbb{C}^*} Y_{i,a}^{u_{i,a}}$, we consider its 'evaluation on $\phi^+(z)$'. By definition it is an element $m(\phi(z)) \in \mathfrak{r}$ given by

$$m\big(\phi(z)\big) = \prod_{i\in I, a\in\mathbb{C}^*} \big(Y_{i,a}(\phi(z))\big)^{u_{i,a}} \,,$$

where

$$\Big(Y_{i,a}\big(\phi(z)\big)\Big)_j = \begin{cases} q_i \dfrac{1-aq_i^{-1}z}{1-aq_iz} & (j=i),\\ 1 & (j\neq i).\end{cases}$$

This defines an injective group morphism $\mathcal{M}\to\mathfrak{r}$. We identify a monomial $m\in\mathcal{M}$ with its image in $\mathfrak{r}$. Note that $\varpi(Y_{i,a})=\overline{\omega_i}$.

It is proved in [FR] that a finite-dimensional $U_q(\mathfrak{g})$-module V satisfies $V=\bigoplus_{m\in\mathcal{M}} V_{m(\phi(z))}$. In particular, $\chi_q(V)$ can be viewed as an element of $\mathcal{Y}$.

Note that for V a finite dimensional $U_q(\mathfrak{g})$-module and $\mathbf{\Psi}\in\mathfrak{r}$, the ℓ-weight spaces (3.15) can be characterized alternatively as [FR]

$$V_{\mathbf{\Psi}}=\{v\in V\mid \exists p\geq 0, \forall i\in I, \forall m\geq 0, (\phi^-_{i,-m}-\Psi^-_{i,-m})^p v=0\}\,.$$

Here $\Psi_i^-(z^{-1})=\sum_{m\geq 0}\Psi^-_{i,-m}z^{-m}$ is the expansion of $\Psi_i(z)\in\mathbb{C}(z)$ in z^{-1}.

A monomial $M\in\mathcal{M}$ is said to be dominant if $M\in\mathbb{Z}[Y_{i,a}]_{i\in I,a\in\mathbb{C}^*}$. For $L(\mathbf{\Psi})$ a finite-dimensional simple $U_q(\mathfrak{g})$ module, $\mathbf{\Psi}=M\big(\phi(z)\big)$ holds for some dominant monomial $M\in\mathcal{M}$. The representation will be denoted by $L(M)$.

For example, for $i\in I$, $a\in\mathbb{C}^*$ and $k\geq 0$, let

$$M^{(i)}_{k,a}=Y_{i,a}Y_{i,aq_i^2}\cdots Y_{i,aq_i^{2(k-1)}}\,. \tag{3.20}$$

Then $W^{(i)}_{k,a}=L(M^{(i)}_{k,a})$ is called a Kirillov-Reshetikhin module (KR module). An explicit formula for $\chi(W^{(i)}_{k,a})$ is known [N2, H2].

For $i\in I, a\in\mathbb{C}^*$, define $A_{i,a}\in\mathcal{M}$ to be

$$Y_{i,aq_i^{-1}}Y_{i,aq_i}\Big(\prod_{\{j\in I|C_{j,i}=-1\}}Y_{j,a}\prod_{\{j\in I|C_{j,i}=-2\}}Y_{j,aq^{-1}}Y_{j,aq}\prod_{\{j\in I|C_{j,i}=-3\}}Y_{j,aq^{-2}}Y_{j,a}Y_{j,aq^2}\Big)^{-1}.$$

Note that $\varpi(A_{i,a})=\overline{\alpha_i}$.

Theorem 3.14. [FR, FM] *Le V a simple finite-dimensional $U_q(\mathfrak{g})$-module. Then*

$$\tilde{\chi}_q(V)\in\mathbb{Z}[A^{-1}_{i,a}]_{i\in I,a\in\mathbb{C}^*}.$$

Let $M=\prod_{\substack{i\in I,\\ r\in\mathbb{Z}}}Y_{i,q^r}^{u_{i,r}}$ be a dominant monomial, and let $l\in\mathbb{Z}$. We set

$$M^{\geq l}=\prod_{i\in I,r\geq l}Y_{i,q^r}^{u_{i,r}},\quad M^{<l}=\prod_{i\in I,r<l}Y_{i,q^r}^{u_{i,r}},$$

so that $M=M^{\geq l}M^{<l}$. It is well-known that the results in [C2, Kas, VV] imply the existence of a surjective morphism of $U_q(\mathfrak{g})$-modules (see references in [H4, Corollary 5.5]) :

$$L(M^{\geq l})\otimes L(M^{<l})\longrightarrow L(M)\,. \tag{3.21}$$

Consider the linear subspace

$$L(M)^{\geq l}=\bigoplus_{m\in M\mathbb{Z}[A^{-1}_{i,q^{r+d_i}}]_{i\in I,r\geq l}}(L(M))_m\,.$$

The following result will play a key role in subsequent sections.

Theorem 3.15. [H4] *(i) The map* (3.21) *restricts to an isomorphism of vector spaces*

$$L(M^{\geq l}) \otimes v^{<l} \longrightarrow L(M)^{\geq l}, \tag{3.22}$$

where $v^{<l}$ is a highest weight vector of $L(M^{<l})$.

(ii) Let $F : L(M^{\geq l}) \to L(M)^{\geq l}$ be the composition of the map (3.22) *with the natural map $L(M^{\geq l}) \to L(M^{\geq l}) \otimes v^{<l}$. Then for any $i, j \in I$ and $r \in \mathbb{Z}$ we have*

$$x_{i,r}^+ F = F\, x_{i,r}^+, \tag{3.23}$$

$$\phi_j^\pm(z)\, F = F\, M^{<l}\big(\phi_j^\pm(z)\big) \times \phi_j^\pm(z)\,. \tag{3.24}$$

Statement (i) appears as Proposition 5.6 in [H4], and statement (ii) follows from Remark 5.7, *loc. cit.*. A particular case of this result had been proved in [HL].

3.6. **The dual category $\mathcal{O}^*$.** For V a $\mathfrak{t}$-diagonalizable $U_q(\mathfrak{b})$-module, we define a structure of $U_q(\mathfrak{b})$-module on its graded dual $V^* = \oplus_{\beta \in \mathfrak{t}^*} V_\beta^*$ by

$$(x\, u)(v) = u\big(S^{-1}(x) v\big) \quad (u \in V^*,\ v \in V,\ x \in U_q(\mathfrak{b})).$$

The reason why we use S^{-1} and not the antipode in the definition of V^* is discussed in Remark 3.19 below.

Definition 3.16. *Let $\mathcal{O}^*$ be the category of $\mathfrak{t}$-diagonalizable $U_q(\mathfrak{b})$-modules V such that V^* is in category $\mathcal{O}$.*

Lemma 3.17. *A $\mathfrak{t}$-diagonalizable $U_q(\mathfrak{b})$-module V is in category $\mathcal{O}$ if and only if V^* is in category $\mathcal{O}^*$.*

Proof. As $S^2(k_i) = k_i$ for any $i \in I$, the weight spaces of $(V^*)^*$ can be identified with the weight spaces of V. The result follows. □

A $U_q(\mathfrak{b})$-module V is said to be of lowest ℓ-weight $\mathbf{\Psi} \in \mathfrak{t}_\ell^*$ if there is $v \in V$ such that $V = U_q(\mathfrak{b})v$ and the following hold:

$$U_q(\mathfrak{b})^- v = \mathbb{C}v\,, \qquad \phi_{i,m}^+ v = \Psi_{i,m} v \quad (i \in I,\ m \geq 0)\,.$$

For $\mathbf{\Psi} \in \mathfrak{t}_\ell^*$, we have the simple $U_q(\mathfrak{b})$-module $L'(\mathbf{\Psi})$ of lowest ℓ-weight Ψ, which is not in category $\mathcal{O}^*$ in general. We define the fundamental representation $L'^{\pm}_{i,a}$ whose lowest ℓ-weight is the highest ℓ-weight of $L^\pm_{i,a}$.

The analog of Proposition 3.10 holds and we have the notion of characters and q-characters for category $\mathcal{O}^*$ as in Section 3.4.

For V a $\mathfrak{t}$-diagonalizable $U_q(\mathfrak{b})$-module with finite-dimensional weight spaces, we have

$$\chi(V^*) = \chi^{-1}(V)$$

where $\chi^{-1}(V)$ is obtained from $\chi(V)$ by replacing each $[\omega]$ by $[\omega^{-1}]$ for $\omega \in \mathfrak{t}_\ell^*$.

Proposition 3.18. *Let V be a $U_q(\mathfrak{b})$-module of lowest ℓ-weight $\mathbf{\Psi}$ with finite-dimensional weight spaces. Then V^* contains a highest ℓ-weight vector of ℓ-weight $\mathbf{\Psi}^{-1}$. In the case V irreducible, we get $(L'(\mathbf{\Psi}))^* \simeq L(\mathbf{\Psi}^{-1})$.*

Proof. Let v be a lowest ℓ-weight vector in V. Let $v^* \in V^*$ such that $v^*(v) = 1$ and v^* is zero on higher weight spaces of V. As $\chi(V^*) = \chi^{-1}(V)$ and the weight of v^* is the opposite of the weight of v, v^* has maximal weight in V^*, and so $e_j\, v^* = 0$ for any $j \in I$. Let us compute the ℓ-weight $\mathbf{\Psi}^* = (\Psi_j^*(z))_{j\in I}$ of v^* in V^* in terms of $\mathbf{\Psi} = (\Psi_j(z))_{j\in I}$.

As $U_q(\mathfrak{b})$ is a Hopf algebra, the linear morphism

$$\mathcal{D} : V \otimes V^* \to \mathbb{C}\,, \quad \mathcal{D}(u \otimes v) = v(u)$$

is a morphism of $U_q(\mathfrak{b})$-module.

Let $j \in I$. From Theorem 2.6, we have in $V \otimes V^*$

$$\phi_j^+(z)(v \otimes v^*) = \phi_j^+(z)v \otimes \phi_j^+(z)v^* = \Psi_j(z)\Psi_j^*(z)(v \otimes v^*).$$

As $\mathcal{D}(v \otimes v^*) = 1$ and the action of $\phi_j^+(z)$ on the trivial module is 1, by applying $\mathcal{D}$ we get $\Psi_j^*(z) = \Psi_j^{-1}(z)$.

For the last point, by construction, V is irreducible if and only if V^* is irreducible. □

Remark 3.19. If we had used S instead of S^{-1} to define V^*, we should have used a map $V^* \otimes V \to \mathbb{C}$ instead of $\mathcal{D}$ (as for example in the proof of [FM, Lemma 6.8]). But then by using Theorem 2.6 as above, we could get additional terms because v^* has maximal weight and v has minimal weight.

Let $L(m)$ be an irreducible finite-dimensional representation of $U_q(\mathfrak{g})$. As a $U_q(\mathfrak{b})$-module, $L(m)$ is in category $\mathcal{O}$ and in category $\mathcal{O}^*$, and is irreducible by Proposition 3.5. By [FM, Corollary 6.9], the lowest weight monomial of $L(m)$ is

$$m' = \prod_{i\in I, a\in\mathbb{C}^*} Y_{\bar{i}, aq^{r^\vee h^\vee}}^{-u_{i,a}}$$

if we denote $u_{i,a}$ the multiplicity of $Y_{i,a}$ in m. Here $\bar{i}$ is defined so that $w_0(\alpha_i) = -\alpha_{\bar{i}}$ where w_0 is the longest element of the Weyl group, $h^\vee$ is the dual Coxeter number, and $r^\vee$ is the maximal number of edges connecting two vertices of the Dynkin diagram of $\dot{\mathfrak{g}}$. So we have

$$L(m) = L'(m') \text{ and } (L(m))^* \simeq L((m')^{-1}). \tag{3.25}$$

This last isomorphism is analogous to [CP2, Proposition 5.1(b)] where the standard duality is used.

4. Asymptotic representations and $L_{i,1}^-$

In this section, we shall construct the irreducible $U_q(\mathfrak{b})$-module $L_{i,1}^-$ as a limit of KR modules of $U_q(\mathfrak{g})$. Throughout this section we assume that $|q| > 1$ (except when it is mentioned explicitly).

4.1. **Example.** As an illustration, let us first consider the simplest example $U_q(\widehat{\mathfrak{sl}}_2)$. Consider the KR modules $L(M_k)$ with $M_k = Y_{1,q^{-1}}Y_{1,q^{-3}}\cdots Y_{1,q^{-2k+1}}$. Its normalized q-character is

$$1 + A_{1,1}^{-1} + A_{1,1}^{-1}A_{1,q^{-2}}^{-1} + \cdots + A_{1,1}^{-1}\cdots A_{1,q^{-2(k-1)}}^{-1}\,.$$

When $k \to \infty$, it converges as a formal power series in the $A_{1,b}^{-1}$ to

$$\sum_{j\geq 0} A_{1,1}^{-1}A_{1,q^{-2}}^{-1}\cdots A_{1,q^{-2(j-1)}}^{-1}.$$

Let us explain this in terms of representations.

The module $L(M_k)$ carries a basis $(v_0,\cdots,v_k)$ with the action of $U_q(\widehat{\mathfrak{sl}}_2)$ given by

$$x_{1,r}^+ v_j = q^{2r(-j+1)}v_{j-1}\,,\quad x_{1,r}^- v_j = q^{-2rj}[j+1]_q[k-j]_q v_{j+1}\,,$$
$$\phi_1^{\pm}(z)v_j = q^{k-2j}\frac{(1-q^{-2k}z)(1-q^2z)}{(1-q^{-2j+2}z)(1-q^{-2j}z)}v_j\,.$$

Observe that the action of the $x_{1,r}^+$ on this basis does not depend on k. In contrast, that of the $x_{1,r}^-$ depends on k and diverges as $k\to\infty$. Nevertheless the actions of $\tilde{x}_{1,r}^-$ and $\tilde{\phi}_1^{\pm}(z)$ converge. Altogether these limiting operators give rise to the following 'asymptotic' representation of $\widetilde{U}_q(\widehat{\mathfrak{sl}}_2)$ on the space $V_\infty = \oplus_{j=0}^{\infty}\mathbb{C}v_j$:

$$\tilde{x}_{1,r}^+ v_j = q^{2r(-j+1)}v_{j-1}\,,\quad \tilde{x}_{1,r}^- v_j = q^{-2rj+j+2}\frac{[j+1]_q}{q-q^{-1}}v_{j+1}\,,$$
$$\tilde{\phi}_1^{+}(z)v_j = \frac{1-q^2z}{(1-q^{-2j+2}z)(1-q^{-2j}z)}v_j\,,$$
$$\tilde{\phi}_1^{-}(z)v_j = -q^{4j}\frac{z^{-1}(1-q^{-2}z^{-1})}{(1-q^{2j-2}z^{-1})(1-q^{2j}z^{-1})}v_j\,.$$

Introducing a Q-grading by $\deg(v_j)=-j\alpha_1$, and defining $k_1v_j=q^{-2j}v_j$, we obtain by Proposition 2.4 a structure of a $U_q(\mathfrak{b})$-module on V_∞. This representation is clearly irreducible, and $\phi_1^+(z)v_0=(1-z)^{-1}v_0$. So we get the following.

Lemma 4.1. *V_∞ is isomorphic to $L_{1,1}^-$ as a $U_q(\mathfrak{b})$-module.*

It is easy to show that this action cannot be extended to an action of the full quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$.

As another example, let us study the KR modules for $U_q(\widehat{\mathfrak{sl}}_3)$

$$V_k = L(M_k),\quad M_k = Y_{1,q^{-1}}Y_{1,q^{-3}}\cdots Y_{1,q^{-2k+1}}\,.$$

Then V_k has a basis $\{v_{n,n'}\}_{0\leq n'\leq n\leq k}$ consisting of ℓ-weight vectors $v_{n,n'}$ of ℓ-weight $M_k\cdot A_{1,1}^{-1}\cdots A_{1,q^{-2n+2}}^{-1}\cdot A_{2,q}^{-1}\cdots A_{2,q^{-2n'+3}}^{-1}$. The action of the generators is given explicitly as follows.

$$x_{1,r}^+v_{n,n'} = q^{r(-2n+2)}[n-n']_q v_{n-1,n'}\,,\quad x_{2,r}^+v_{n,n'} = q^{r(-2n'+3)}v_{n,n'-1},$$
$$x_{1,r}^-v_{n,n'} = q^{-2nr}[k-n]_q v_{n+1,n'},\quad x_{2,r}^-v_{n,n'} = q^{r(-2n'+1)}[n'+1]_q[n-n']_q v_{n,n'+1},$$
$$\phi_1^{\pm}(z)v_{n,n'} = q^{k-2n+n'}\frac{(1-q^{-2k}z)(1-q^{-2n'+2}z)}{(1-q^{-2n}z)(1-q^{-2n+2}z)}v_{n,n'},$$
$$\phi_2^{\pm}(z)v_{n,n'} = q^{n-2n'}\frac{(1-q^3z)(1-q^{-2n+1}z)}{(1-q^{-2n'+1}z)(1-q^{-2n'+3}z)}v_{n,n'}\,.$$

Here we set $v_{n,n'} = 0$ unless $0 \le n' \le n \le k$. Note that the action of $\tilde{x}^-_{2,m}$ and $\tilde{\phi}^\pm_2(z)$ do not depend on k. Letting $k \to \infty$, we get a representation of $\widetilde{U}_q(\widehat{\mathfrak{sl}}_3)$ on $V_\infty = \oplus_{0 \le n' \le n} \mathbb{C} v_{n,n'}$:

$$
\begin{aligned}
\tilde{x}^+_{1,r} v_{n,n'} &= q^{r(-2n+2)}[n-n']_q \, v_{n-1,n'} \,, \quad \tilde{x}^+_{2,r} v_{n,n'} = q^{r(-2n'+3)} \, v_{n,n'-1} \,, \\
\tilde{x}^-_{1,r} v_{n,n'} &= \frac{q^{n-n'+2-2nr}}{q-q^{-1}} \, v_{n+1,n'} \,, \\
\tilde{x}^-_{2,r} v_{n,n'} &= q^{2n'-n+2+r(-2n'+1)}[n'+1]_q[n-n']_q \, v_{n,n'+1} \,, \\
\tilde{\phi}^+_1(z) v_{n,n'} &= \frac{1-q^{-2n'+2}z}{(1-q^{-2n}z)(1-q^{-2n+2}z)} \, v_{n,n'} \,, \\
\tilde{\phi}^-_1(z) v_{n,n'} &= -q^{4n-2n'} \frac{z^{-1}(1-q^{2n'-2}z^{-1})}{(1-q^{2n}z^{-1})(1-q^{2n-2}z^{-1})} \, v_{n,n'} \,, \\
\tilde{\phi}^\pm_2(z) v_{n,n'} &= \frac{(1-q^3 z)(1-q^{-2n+1}z)}{(1-q^{-2n'+1}z)(1-q^{-2n'+3}z)} \, v_{n,n'} \,.
\end{aligned}
$$

These examples have appeared in [BLZ, BHK] as representations of the Borel algebra, but the action of the full asymptotic algebra was not discussed.

4.2. **General case.** We now proceed to the construction of $L^-_{i,1}$ in the general case. Since a direct computational method is hardly applicable in general, we take an alternative approach.

Fix $i \in I$, and consider a family of KR modules labeled by $k \ge 0$:

$$
V_k = L(M_k) \,, \quad M_k = Y_{i,q_i^{-1}} Y_{i,q_i^{-3}} \cdots Y_{i,q_i^{-2k+1}} \,,
$$

where we set $M_0 = 1$.

For each k, l satisfying $k \ge l \ge 0$, we decompose M_k as $M_k^{\ge(-2l+1)d_i} M_k^{<(-2l+1)d_i}$, so that $M_k^{\ge(-2l+1)d_i} = M_l$. Let

$$
F_{k,l} \ : \ V_l \longrightarrow V_k^{\ge(-2l+1)d_i} \tag{4.26}
$$

be the corresponding isomorphism of $U_q(\mathfrak{g})^+$-modules given in Theorem 3.15. Fixing a highest weight vector $v_k \in V_k$ we normalize (4.26) by $F_{k,l} v_l = v_k$. We have then for all $k \ge l \ge m$

$$
F_{k,l} \circ F_{l,m} = F_{k,m} \,, \quad F_{k,k} = \mathrm{id} \,.
$$

Thus $(\{V_k\}, \{F_{k,l}\})$ constitutes an inductive system of $U_q(\mathfrak{g})^+$-modules. Set

$$
V_\infty = \varinjlim V_k \,, \quad v_\infty = F_{\infty,k} \, v_k \,,
$$

where $F_{\infty,k} : V_k \to V_\infty$ denotes the injective morphism satisfying the condition $F_{\infty,k} \circ F_{k,l} = F_{\infty,l}$.

It is known [N2, H2] that, in the limit $k \to \infty$, the normalized q-character $\tilde{\chi}_q(V_k)$ converges to a formal power series in $\mathbb{Z}[[A^{-1}_{i,q^a}]]_{i \in I, a \in \mathbb{Z}}$. In particular, the dimension of each weight subspace of V_k stabilizes as $k \to \infty$.

Proposition 4.2. *For $k \geq l \geq 0$ we have*

$$\phi_i^{\pm}(z)\, F_{k,l} = q_i^{k-l} \frac{1-q_i^{-2k}z}{1-q_i^{-2l}z} \times F_{k,l}\, \phi_i^{\pm}(z)\,, \tag{4.27}$$

$$\phi_j^{\pm}(z)\, F_{k,l} = F_{k,l}\, \phi_j^{\pm}(z) \quad (j \neq i)\,. \tag{4.28}$$

For each l, the limits $\lim_{k\to\infty} F_{k,l}^{-1}\tilde{\phi}_i^{\pm}(z)F_{k,l}$ exist in $\mathrm{End}(V_l)$ and gives rise to an endomorphism of V_∞.

Remark 4.3. It follows from Proposition 4.2 that the operator $F_{k,l}^{-1}\tilde{\phi}_j^{\pm}(z)F_{k,l}$ is of the form $A(z) + q_i^{-2k}B(z)$ where the operators $A(z)$, $B(z)$ do not depend on k and are well-defined without any assumption on $|q|$. So the limiting operator $A(z)$ on V_∞ makes sense for such a q.

Proof. Formulas (4.27), (4.28) follow from (3.24) and

$$M_k\big(\phi_j^{\pm}(z)\big) = \begin{cases} q_i^k \dfrac{1-q_i^{-2k}z}{1-z} & (j=i)\,, \\ 1 & (j \neq i)\,. \end{cases}$$

Clearly $\lim_{k\to\infty} F_{k,l}^{-1}\tilde{\phi}_j^{\pm}(z)F_{k,l}$ exists, and

$$\big(\lim_{k\to\infty} F_{k,l}^{-1}\tilde{\phi}_j^{\pm}(z)F_{k,l}\big) \circ F_{l,m} = F_{l,m} \circ \big(\lim_{k\to\infty} F_{k,m}^{-1}\tilde{\phi}_j^{\pm}(z)F_{k,m}\big)$$

holds for all $l \geq m$. Hence they give rise to a well defined operator on V_∞. ☐

Next let us study the convergence property for $\tilde{x}_{j,r}^{-}$.

Lemma 4.4. *Suppose $k \geq l \geq 0$. Then we have*

$$\tilde{x}_{j,r}^{-} V_k^{\geq(-2l+1)d_i} \subset V_k^{\geq(-2l+1-2\delta_{i,j})d_i} \quad (j \in I,\ r \in \mathbb{Z}).$$

Proof. Let v be a highest ℓ-weight vector of $V_k^{<(-2l+1)d_i}$, and let $w \in V_l$. From the formula for coproduct in Theorem 2.6 we have

$$x_{j,r}^{-}(w \otimes v) = \sum_{p=0}^{r-1} x_{j,r-p}^{-} w \otimes \phi_{j,p}^{+} v + w \otimes x_{j,r}^{-} v \quad (r > 0)\,,$$

$$x_{j,-r}^{-}(w \otimes v) = \sum_{p=0}^{r-1} x_{j,-r-p}^{-} w \otimes \phi_{j,-p}^{-} v + w \otimes x_{j,-r}^{-} v \quad (r \geq 0)\,.$$

It follows that

$$F_{k,l}\big(x_{j,r}^{-}(w \otimes v)\big) \in V_k^{\geq(-2l+1)d_i} + F_{k,l}\big(w \otimes x_{j,r}^{-} v\big) \quad (r \in \mathbb{Z}).$$

If $j \neq i$, the second term is absent. If $j = i$, then by [H2, Lemma 4.4], $x_{i,r}^{-}v$ belongs to the generalized eigenspace corresponding to the monomial $M_k^{<(-2l+1)d_i} A_{i,q_i^{-2l-1}}^{-1}$. The assertion of the Lemma follows from these. ☐

Proposition 4.5. *For $j \in I$, $r \in \mathbb{Z}$ and $k > l \geq 1$, the operator $F_{k,l+1}^{-1}\tilde{x}_{j,r}^{-}F_{k,l} \in \mathrm{Hom}(V_l, V_{l+1})$ is of the form*

$$F_{k,l+1}^{-1}\tilde{x}_{j,r}^{-}F_{k,l} = C + q_i^{-2k}D\,, \tag{4.29}$$

where $C, D \in \mathrm{Hom}(V_l, V_{l+1})$ do not depend on k. In particular, the limit

$$\lim_{k\to\infty} F_{k,l+1}^{-1}\tilde{x}_{j,r}^{-}F_{k,l} \in \mathrm{Hom}(V_l, V_{l+1}) \tag{4.30}$$

exists and gives rise to an endomorphism of V_∞.

Proof. Let $\omega_l \in \mathfrak{t}^*$ be the highest weight of V_l. We prove the convergence on each weight subspace $(V_l)_{\omega_l \overline{\beta}^{-1}}$ by induction on $\beta \in Q$ with respect to the ordering (3.13). If $\beta \notin Q^+ = \oplus_{i\in I}\mathbb{Z}_{\geq 0}\alpha_i$, then there is nothing to show. Suppose the assertion is true for elements of Q smaller than β, and let $v \in (V_l)_{\omega_l \overline{\beta}^{-1}}$. Since $x_{j',r'}^{+}$ commutes with $F_{k,l}$ for any $j' \in I$ and $r' \in \mathbb{Z}$, we have

$$\begin{aligned} q_{j'}^{-C_{j',j}} x_{j',r'}^{+}\left(F_{k,l+1}^{-1}\tilde{x}_{j,r}^{-}F_{k,l}\right)v &= \left(F_{k,l+1}^{-1}\tilde{x}_{j,r}^{-}F_{k,l}\right)x_{j',r'}^{+}v \\ &+ \delta_{j',j}\frac{1}{q_j - q_j^{-1}}F_{k,l+1}^{-1}\left(\tilde{\phi}_{j,r+r'}^{+} - \tilde{\phi}_{j,r+r'}^{-}\right)F_{k,l}v\,. \end{aligned}$$

By the induction hypothesis, the first term in the right hand side is of the form (4.29). The operator in second term which is equal to

$$\delta_{j',j}\frac{1}{q_j - q_j^{-1}}F_{l+1,l}\left(F_{k,l}^{-1}\left(\tilde{\phi}_{j,r+r'}^{+} - \tilde{\phi}_{j,r+r'}^{-}\right)F_{k,l}\right)$$

is also of the form (4.29) due to Remark 4.3. Hence we have a sequence of vectors $w_k = F_{k,l+1}^{-1}\tilde{x}_{j,r}^{-}F_{k,l}v$ in a finite dimensional vector space $(V_{l+1})_{\omega_{l+1}\beta\overline{\alpha}_j^{-1}}$ such that for any j', r', the vector $x_{j',r'}^{+}w_k$ is of the form

$$x_{j',r'}^{+}w_k = C_{j',r'} + q_i^{-2k}D_{j',r'} \tag{4.31}$$

where the vectors $C_{j',r'}$, $D_{j',r'}$ do not depend on k. On the other hand, since V_{l+1} is simple, the joint kernel of the $x_{j',r'}^{+}$ on weight subspaces of weight lower than ω_{l+1} is zero. It follows that the sequence $\{w_k\}$ is also of the form (4.31). So the operator $F_{k,l+1}^{-1}\tilde{x}_{j,r}^{-}F_{k,l}$ is of the form (4.29). The existence of the limit follows immediately. The well-defineness on V_∞ holds by the same reason as in Proposition 4.2. □

Remark 4.6. By construction, the limiting operator C in Proposition 4.5 is well-defined for all q which is not a root of unity.

Taking the limit $k \to \infty$ we get a structure of $\widetilde{U}_q(\mathfrak{g})$-module on V_∞, since the relations of the algebra are preserved. For example, on V_l we have

$$\begin{aligned} &q_i^{-C_{i,j}}\left(F_{k,l+1}^{-1}\tilde{x}_{i,r}^{+}F_{k,l+1}\right)\left(F_{k,l+1}^{-1}\tilde{x}_{j,r'}^{-}F_{k,l}\right) - \left(F_{k,l+1}^{-1}\tilde{x}_{j,r'}^{-}F_{k,l}\right)\left(F_{k,l}^{-1}\tilde{x}_{i,r}^{+}F_{k,l}\right) \\ &= \delta_{i,j}F_{l+1,l}\left(F_{l,k}^{-1}\frac{\tilde{\phi}_{i,r+r'}^{+} - \tilde{\phi}_{i,r+r'}^{-}}{q_i - q_i^{-1}}F_{k,l}\right), \end{aligned}$$

and so the relation is satisfied by the asymptotic operators on V_∞. In particular, since $F_{k,l}^{-1}k_i^{-1}F_{k,l} = q_i^{-k+l}k_i^{-1}$, κ_i acts as 0 on V_∞.

This structure of $\widetilde{U}_q(\mathfrak{g})$-module on V_∞ makes sense without any assumption on $|q|$. Indeed, the action of the Borel algebra on KR modules is well-defined for such a q : there is a basis of V_k such that the coefficients of the action of the generators e_i, f_i, $k_i^{\pm 1}$ $(0 \leq i \leq n)$ on V_k are Laurent polynomials in q, see [CP3, Section 4]. Although the process of taking the limit is not well-defined if $|q| \leq 1$, it suffices to check that the limiting operators on V_∞ make sense at q. The operators $F_{k,l}^{-1}\tilde{x}_{j,m}^+F_{k,l}$ are constant when $k \to \infty$, so the corresponding limiting operators on V_∞ make sense for q. We have seen in Remarks 4.3 and 4.6 that the limiting operators of $F_{k,l}^{-1}\tilde{\phi}^\pm(z)F_{k,l}$ and of $F_{k,l}^{-1}\tilde{x}_{j,m}^-F_{k,l}$ make sense for q. So, for the next results, we do not assume necessarily that $|q| > 1$.

The $\widetilde{U}_q(\mathfrak{g})$-module V_∞ has a unique Q-grading such that v_∞ has degree 0.

Applying Proposition 2.4 we obtain the following.

Theorem 4.7. *The space V_∞ has a structure of a $U_q(\mathfrak{b})$-module which is in category $\mathcal{O}$. The vector v_∞ satisfies $e_j\, v_\infty = 0$ for any $j \in I$ and has ℓ-weight*

$$\mathbf{\Psi}(z) = (1, \cdots, \overset{i-th}{(1-z)^{-1}}, \cdots, 1)\,.$$

Corollary 4.8. *The module $L_{i,1}^-$ is in category $\mathcal{O}$.*

Proof. Let V'_∞ be the submodule of V_∞ generated by v_∞. Then V'_∞ is in category $\mathcal{O}$ of highest ℓ-weight $\mathbf{\Psi}$. As $L_{i,1}^-$ is a quotient of V'_∞, we get the result. □

5. The representations $L_{i,1}^+$

Let $i \in I$, and let V_∞ be the $\tilde{U}_{q^{-1}}(\mathfrak{g})$-module constructed as in the last section, with quantum parameter q^{-1} in place of q (we have seen at the end of the last section that this action is defined without any assumption on $|q|$). We use the Q-grading $V_\infty = \oplus_{\beta \in Q}(V_\infty)_\beta$ such that its highest weight vector has degree 0. Then from Proposition 2.4, (2.8), there is a natural structure of $U_q(\mathfrak{b})$-module on V_∞. This representation is denoted by V_∞^σ. Note that the highest weight vector of V_∞ becomes the lowest weight vector in V_∞^σ. The representation V_∞^σ is in category $\mathcal{O}^*$, and so we have a structure of $U_q(\mathfrak{b})$-module on the graded dual $(V_\infty^\sigma)^*$ as in Section 3.6. By Lemma 3.17, the $U_q(\mathfrak{b})$-module $(V_\infty^\sigma)^*$ is in category $\mathcal{O}$.

Let $v_\infty^* \in (V_\infty^\sigma)^*$ be defined by $v_\infty^*(v_\infty) = 1$ and $v_\infty^* = 0$ on $\bigoplus_{\omega \neq 0}(V_\infty^\sigma)_\omega$. The vector v_∞^* satisfies $e_j\, v_\infty^* = 0$ for any $j \in I$ and has ℓ-weight

$$\mathbf{\Psi}(z) = (1, \cdots, \overset{i-th}{1-z}, \cdots, 1)\,.$$

We get the following consequence.

Corollary 5.1. *The module $L_{i,1}^+$ is in category $\mathcal{O}$.*

Let us look at the example of $\widehat{\mathfrak{sl}}_2$ in more details. We have $V_\infty = \oplus_{j=0}^\infty \mathbb{C}v_j$ with the action of $\widetilde{U}_{q^{-1}}(\widehat{\mathfrak{sl}}_2)$

$$\tilde{x}_{1,r}^+ v_j = q^{2r(j-1)}v_{j-1}\,, \quad \tilde{x}_{1,r}^- v_j = -q^{2rj-j-2}\frac{[j+1]_q}{q-q^{-1}}v_{j+1}\,, \quad k_1 v_j = q^{2j}v_j.$$

Hence the action of $U_q(\mathfrak{b})$ on V_∞^σ is given by

$$e_1\, v_j = -q^{-j}\frac{[j+1]_q}{q-q^{-1}}v_{j+1}\,, \quad e_0\, v_j = q^{2(j-1)}v_{j-1}\,, \quad k_1 v_j = q^{2j}v_j\,.$$

We have $(V_\infty^\sigma)^* = \oplus_{j=0}^\infty \mathbb{C}v_j^*$, where $(v_j^*)_{j\geq 0}$ is a basis dual to $(v_j)_{j\geq 0}$. We have

$$e_1\, v_j^* = q^{-3j+3}\frac{[j]_q}{q-q^{-1}}v_{j-1}^*\,, \quad e_0\, v_j^* = -q^{4j+2}v_{j+1}^*\,, \quad k_1 v_j^* = q^{-2j}v_j^*\,.$$

There is a unique basis $(w_j^*)_{j\geq 0}$ of $(V_\infty^\sigma)^*$ such that $w_0^* = v_0^*$, $w_j^* \in \mathbb{C}^* v_j^*$ and $e_1\, w_j^* = w_{j-1}^*$. We obtain

$$e_0\, w_j^* = -q^{j+2}\frac{[j+1]_q}{q-q^{-1}}w_{j+1}^*\,, \quad k_1 w_j^* = q^{-2j}w_j^*\,.$$

Let us write the example of $\widehat{\mathfrak{sl}}_3$ in details. We have $V_\infty^\sigma = \oplus_{0\leq n'\leq n}\mathbb{C}v_{n,n'}$ with the action of $U_q(\mathfrak{b})$ given by

$$\begin{aligned}
&e_1 v_{n,n'} = \frac{q^{-n+n'}}{q^{-1}-q}\,, \quad e_2 v_{n,n'} = q^{-2n'+n}[n'+1]_q[n-n']_q\, v_{n,n'+1}\,,\\
&e_0 v_{n,n'} = -q^{n+n'-2}v_{n-1,n'-1}\,,\\
&k_1 v_{n,n'} = q^{2n-n'}v_{n,n'}\,, \quad k_2 v_{n,n'} = q^{2n'-n}v_{n,n'}\,, \quad k_0 v_{n,n'} = q^{-n-n'}v_{n,n'}\,.
\end{aligned}$$

We get the action of $U_q(\mathfrak{b})$ on $(V_\infty^\sigma)^* = \bigoplus_{n,n'\geq 0}\mathbb{C}v_{n,n'}^*$

$$\begin{aligned}
&e_1 v_{n,n'}^* = \frac{q^{-3n+2n'+3}}{q-q^{-1}}\, v_{n-1,n'}^*\,, \quad e_2 v_{n,n'}^* = -q^{2(n-2n'+2)}[n']_q[n-n'+1]_q\, v_{n,n'-1}^*\,,\\
&e_0 v_{n,n'}^* = q^{2(n+n'+1)}v_{n+1,n'+1}^*\,,\\
&k_1 v_{n,n'}^* = q^{-2n+n'}v_{n,n'}^*\,, \quad k_2 v_{n,n'}^* = q^{-2n'+n}v_{n,n'}^*\,, \quad k_0 v_{n,n'}^* = q^{n+n'}v_{n,n'}^*\,.
\end{aligned}$$

There is a unique basis $(w_{n,n'}^*)_{n,n'\geq 0}$ of $(V_\infty^\sigma)^*$ such that $e_1 w_{n,n'}^* = [n-n']_q w_{n-1,n'}^*$, $e_2 w_{n,n'}^* = w_{n,n'-1}^*$, $w_{0,0}^* = v_{0,0}^*$ and $w_{n,n'}^* = \lambda_{n,n'}v_{n,n'}^*$ where $\lambda_{n,n'} \in \mathbb{C}^*$. We get

$$\lambda_{n,n'} = \lambda_{n-1,n'}(q^{n-n'} - q^{n'-n})q^{3n-2n'-3}\,,\ \lambda_{n,n'}[n']_q[n-n'+1]_q = -\lambda_{n,n'-1}q^{4n'-2n-4}.$$

This implies $e_0 w_{n,n'}^* = -[n'+1]_q\frac{q^{n+4}}{q-q^{-1}}w_{n+1,n'+1}^*$.

6. Irreducibility of asymptotic representations and character formulas

6.1. **Irreducibility of V_∞ and the character of $L_{i,1}^-$.** Let $i \in I$. We recall the $U_q(\mathfrak{b})$-module V_∞ constructed in Section 4.2. We have proved that $L_{i,1}^-$ is a subquotient of V_∞.

Theorem 6.1. *The module V_∞ is irreducible and is isomorphic to $L_{i,1}^-$. In particular*

$$\tilde{\chi}_q(L_{i,1}^-) = \lim_{k\to\infty}\tilde{\chi}_q\big(L(M_k)\big)$$

as a formal power series in $\mathbb{Z}[[A_{j,q^r}^{-1}]]_{j\in I, r\in\mathbb{Z}}$.

Proof. Consider the KR module $L(M_k)$ with $M_k = Y_{i,q_i^{-1}} Y_{i,q_i^{-3}} \cdots Y_{i,q_i^{-2k+1}}$, viewed as a $U_q(\mathfrak{b})$-module. In view of Remark 2.5, we modify its Q-grading so that $k_j v = v$ $(j \in I)$ holds on the highest weight vector v. Denote the resulting $U_q(\mathfrak{b})$-module by $\tilde{L}(M_k)$.

We set

$$\tilde{\chi}_q(\tilde{L}(M_k)) = \sum_{\mathbf{\Psi} \in \mathfrak{r}} n_{\mathbf{\Psi}}^{(k)} [\mathbf{\Psi}] , \quad \tilde{\chi}_q(V_\infty) = \sum_{\mathbf{\Psi} \in \mathfrak{r}} n_{\mathbf{\Psi}}^{(\infty)} [\mathbf{\Psi}] , \quad \tilde{\chi}_q(L_{i,1}^-) = \sum_{\mathbf{\Psi} \in \mathfrak{r}} n_{\mathbf{\Psi}}^- [\mathbf{\Psi}] .$$

Both $\tilde{\chi}_q(\tilde{L}(M_k))$, $\tilde{\chi}_q(V_\infty)$ belong to $\mathbb{Z}[[A_{j,b}^{-1}]]_{j \in I, b \in \mathbb{C}^*}$, and $\tilde{\chi}_q(\tilde{L}(M_k))$ converges to $\tilde{\chi}_q(V_\infty)$ as $k \to \infty$. Since $L_{i,1}^-$ is a subquotient of V_∞, we have

$$n_{\mathbf{\Psi}}^- \le n_{\mathbf{\Psi}}^{(\infty)} \quad (\mathbf{\Psi} \in \mathfrak{r}).$$

To prove the theorem, it suffices to show the reverse inequality. Fix $\mathbf{\Psi} \in \mathfrak{r}$. We may assume $n_{\mathbf{\Psi}}^{(\infty)} \neq 0$. Comparing the highest ℓ-weights we see that $\tilde{L}(M_k)$ is a subquotient of $L_{i,q_i^{-2k}}^+ \otimes L_{i,1}^-$. Hence we must have

$$\mathbf{\Psi}(z) = \mathbf{\Psi}_k^+(q_i^{-2k} z) \mathbf{\Psi}_k^-(z) \tag{6.32}$$

for some $\mathbf{\Psi}_k^\pm(z)$ which occur in $\chi_q(L_{i,1}^\pm)$. We show that we must necessarily have $\mathbf{\Psi}_k^+(z) = 1$ for sufficiently large k. Indeed, if we write $\varpi\big(\mathbf{\Psi}(z)\big) = (\overline{\alpha}_{i_1} \cdots \overline{\alpha}_{i_l})^{-1}$, $\varpi\big(\mathbf{\Psi}_k^-(z)\big) = (\overline{\alpha}_{j_1} \cdots \overline{\alpha}_{j_m})^{-1}$, then from

$$\varpi\big(\mathbf{\Psi}(z)\big) = \varpi\big(\mathbf{\Psi}_k^+(q_i^{-2k} z)\big) \varpi\big(\mathbf{\Psi}_k^-(z)\big)$$

we have $m \le l$. Given $\mathbf{\Psi}$, there are only finitely many possibilities for such $\mathbf{\Psi}_k^-(z)$'s. Since (6.32) holds for any k, $\mathbf{\Psi}_k^+(z)$ must be independent of z for k large enough. It follows that $(\varpi(\mathbf{\Psi}))^{-1} \mathbf{\Psi} = (\varpi(\mathbf{\Psi}_k^-))^{-1} \mathbf{\Psi}_k^-$. On the other hand, $\mathbf{\Psi}, \mathbf{\Psi}_k^-$ are both monomials in $A_{j,b}^{-1}$'s. Therefore we must have $\mathbf{\Psi} = \mathbf{\Psi}_k^-$ and $\mathbf{\Psi}_k^+ = 1$.

The multiplicity of the term $[1]$ in $\tilde{\chi}_q(L_{i,q_i^{-2k}}^+)$ is 1. We conclude that $n_{\mathbf{\Psi}}^{(k)} \le n_{\mathbf{\Psi}}^-$, which shows the opposite inequality

$$n_{\mathbf{\Psi}}^- \ge n_{\mathbf{\Psi}}^{(\infty)} \quad (\mathbf{\Psi} \in \mathfrak{r}).$$

□

6.2. Irreducibility of $(V_\infty^\sigma)^*$ and the character of $L_{i,1}^+$.

Proposition 6.2. *For $\mathbf{\Psi} \in \mathfrak{t}_\ell^*$, the simple module $L'(\mathbf{\Psi})$ is in category $\mathcal{O}^*$ if and only if $\Psi_i(z)$ is rational for any $i \in I$.*

Proof. As for Theorem 3.11, it suffices to prove that for $i \in I$, the fundamental representations $L_{i,1}'^\pm$ are in category $\mathcal{O}^*$. Theorem 3.11 is already proved, so the representations $L_{i,1}^\pm$ of $U_{q^{-1}}(\mathfrak{b})$ are in category $\mathcal{O}$. Since $(L_{i,1}'^\pm)^* \simeq L_{i,1}^\mp$ by Proposition 3.18, we get the result. □

We define a partial ordering $\preceq$ on characters : $\chi \preceq \chi'$ if the multiplicities of weights are lower in χ than in χ'.

Theorem 6.3. *$(V_\infty^\sigma)^*$ is irreducible isomorphic to $L_{i,1}^+$ and we have $\chi(L_{i,1}^+) = \chi(L_{i,1}^-)$.*

Proof. We have proved that $L^+_{i,1}$ is a subquotient of $(V^\sigma_\infty)^*$ and $\chi((V^\sigma_\infty)^*) = \chi(V_\infty) = \chi(L^-_{i,1})$. So $\chi(L^+_{i,1}) \preceq \chi(L^-_{i,1})$.

To prove the reverse inequality, let $k \geq 1$ and consider the KR module $L'(P_k^{-1})$ with $P_k = Y_{i,q_i} Y_{i,q_i^3} \cdots Y_{i,q_i^{2k+1}}$ (this is a KR module by (3.25)). This is an irreducible $U_q(\mathfrak{b})$-module by Proposition 3.5. We modify its Q-grading so that $k_j v_k = v_k$ ($j \in I$) holds on the lowest weight vector v_k. In the resulting module $\tilde{L}'(P_k^{-1})$, we have

$$\phi_i^+(z)\, v_k = \frac{1 - z q_i^{2(k+1)}}{1-z}\, v_k\,, \quad \phi_j^+(z)\, v_k = v_k \quad \text{for } j \neq i\,.$$

So $\tilde{L}'(P_k^{-1})$ is a subquotient of $L'^-_{i,1} \otimes L'^+_{i,q_i^{2(k+1)}}$.

By twisting by σ, the q-character of $L'(P_k^{-1})$ is equal to the q-character of the $U_{q^{-1}}(\mathfrak{g})$-KR module $L(Y_{i,(q_i^{-1})^{-2k-1}} Y_{i,(q_i^{-1})^{-2k+1}} \cdots Y_{i,(q_i^{-1})^{-1}})$. Hence, both $P_k \chi_q(\tilde{L}'(P_k^{-1}))$ and $\mathbf{\Psi}^{-1}\chi_q(V^\sigma_\infty)$ belong to $\mathbb{Z}[[A_{j,b}]]_{j\in I, b\in\mathbb{C}^*}$ ($\mathbf{\Psi}$ is the ℓ-weight of V^σ_∞ of minimal weight), and

$$\mathbf{\Psi}^{-1}\chi_q(V^\sigma_\infty) = \lim_{k\to\infty} P_k \chi_q(\tilde{L}'(P_k^{-1}))$$

as a formal power series in $\mathbb{Z}[[A_{j,b}]]_{j\in I, b\in\mathbb{C}^*}$.

Now we can conclude as for Theorem 6.1 that $\chi(L'^-_{i,1}) \succeq \chi^{-1}(L^-_{i,1})$.

So we have $\chi(L^+_{i,1}) = \chi^{-1}(L'^-_{i,1}) \succeq \chi(L^-_{i,1})$. This implies the result. $\square$

6.3. **Explicit character formulas for fundamental representations.** While the convergence of the normalized q-characters has been proven [N2, H2], no explicit formula for the limit is known in general. For the ordinary characters $\chi\big(L(M_k)\big)$, explicit formulas are known [N2, H2], from which one can extract the following formula for the limit.

Theorem 6.4. *For any $i \in I$, $a, b \in \mathbb{C}^*$, we have*

$$\chi(L^+_{i,a}) = \chi(L^-_{i,b}) = \frac{\displaystyle\sum_{N=(N_k^{(j)})} \prod_{j\in I, k>0} \binom{N_k^{(j)} + \delta_{i,j}k - \sum_{h\in I, l>0} N_l^{(h)} r_j C_{j,h} min(\frac{k}{r_h}, \frac{l}{r_j})}{N_k^{(j)}} [\overline{\alpha_j}]^{-kN_k^{(i)}}}{\displaystyle\prod_{\alpha\in\Delta_+} (1 - [\overline{\alpha}]^{-1})}$$

where Δ_+ is the set of positive roots of $\dot{\mathfrak{g}}$, $\binom{a}{b} = \Gamma(a+1)/\big(\Gamma(a-b+1)\Gamma(b+1)\big)$, and the sum is taken over all non-negative integers $N_k^{(j)}$ with $j \in I$, $k \in \mathbb{Z}_{>0}$.

We have proved an explicit character formula for all fundamental representations in category $\mathcal{O}$.

7. Asymptotic representations and $L^+_{i,1}$

In this section we study another limit of the KR modules and discuss its relation to the module $L^+_{i,1}$. We assume $|q| < 1$ throughout.

7.1. **First examples.** We begin with the simplest example of $U_q(\widehat{\mathfrak{sl}}_2)$. Consider the KR module $W_k = L(N_k)$ with highest monomial $N_k = Y_{1,q}Y_{1,q^3}\cdots Y_{1,q^{2k-1}}$. It has a basis $(v_0,\cdots,v_k)$ with the action

$$\begin{aligned}
&x^+_{1,r}v_j = q^{2r(k-j+1)}v_{j-1}\,, \quad x^-_{1,r}v_j = q^{2r(k-j)}[k-j]_q[j+1]_q v_{j+1}\,,\\
&\phi^\pm_1(z)v_j = q^{k-2j}\frac{(1-z)(1-zq^{2(k+1)})}{(1-zq^{2(k-j+1)})(1-zq^{2(k-j)})}v_j\,.
\end{aligned}$$

Unlike the case of $L^-_{1,1}$ discussed in the previous section, only a 'half' of these operators converge as $k\to\infty$.

$$\begin{aligned}
&\lim_{k\to\infty}\tilde{x}^+_{1,r}v_j = \delta_{r,0}v_{j-1} \quad (r\ge 0)\,,\\
&\lim_{k\to\infty}\tilde{x}^-_{1,p}v_j = \frac{-q^{j+2}\delta_{p,1}}{q-q^{-1}}[j+1]_q v_{j+1} \quad (p\ge 1)\,,\\
&\lim_{k\to\infty}\tilde{\phi}^+_1(z)v_j = (1-z)v_j\,.
\end{aligned}$$

By setting $k_1 v_j = q^{-2j}v_j$, we get an action of $U_q(\mathfrak{b})$ on $W_\infty = \oplus_{j\ge 0}\mathbb{C}v_j$:

$$\begin{aligned}
&x^+_{1,r}v_j = \delta_{r,0}v_{j-1}\,, \quad x^-_{1,p}v_j = \frac{-q^{-j}\delta_{p,1}}{q-q^{-1}}[j+1]_q v_{j+1}\,,\\
&\phi^+_1(z)v_j = q^{-2j}(1-z)v_j.
\end{aligned}$$

This representation is simple and isomorphic to $L^+_{1,1}$. We recover the action of the example of the last section

$$e_1\, v_j = v_{j-1}\,, \quad e_0\, v_j = -q^{j+2}\frac{[j+1]_q}{q-q^{-1}}v_{j+1}\,, \quad k_1 v_j = q^{-2j}v_j\,.$$

It is easy to check that this action cannot be extended to that of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$.

We note that, in contrast to the case $L^-_{1,1}$, the normalized q-character

$$\tilde{\chi}_q\big(L^+_{1,1}\big) = \sum_{j=0}^{\infty}[q^{-2j}]$$

is independent of z and is *not* a formal power series in $A^{-1}_{1,a}$'s.

Let us study another example for $U_q\big(\widehat{\mathfrak{sl}}_3\big)$. Consider the KR modules

$$W_k = L(N_k), \quad N_k = Y_{1,q}Y_{1,q^3}\cdots Y_{1,q^{2k-1}}.$$

It has a basis $\{v_{n,n'}\}_{0\le n'\le n\le k}$ with the action

$$\begin{aligned}
\tilde{x}^+_{1,m} v_{n,n'} &= q^{2m(1+k-n)}[n-n']_q\, v_{n-1,n'}\,, \quad \tilde{x}^+_{2,m} v_{n,n'} = q^{m(3+2k-2n')}\, v_{n,n'-1}\,,\\
\tilde{x}^-_{1,p} v_{n,n'} &= q^{-k+2n-n'+2+2p(k-n)}[k-n]_q\, v_{n+1,n'},\\
\tilde{x}^-_{2,p} v_{n,n'} &= q^{-n+2n'+2+p(1+2k-2n')}[n'+1]_q[n-n']_q\, v_{n,n'+1},\\
\tilde{\phi}^\pm_1(z) v_{n,n'} &= \frac{(1-z)(1-zq^{2(1+k-n')})}{(1-zq^{2(k-n)})(1-zq^{2(1+k-n)})}\, v_{n,n'},\\
\tilde{\phi}^\pm_2(z) v_{n,n'} &= \frac{(1-zq^{1+2k-2n})(1-zq^{3+2k})}{(1-zq^{1+2k-2n'})(1-zq^{3+2k-2n'})}\, v_{n,n'}\,.
\end{aligned}$$

For $m \ge 0$ and $p > 0$, these operators converge when $k \to \infty$,

$$\begin{aligned}
&\tilde{x}^+_{1,m} v_{n,n'} = \delta_{m,0}[n-n']_q\, v_{n-1,n'}\,, \quad \tilde{x}^-_{1,p} v_{n,n'} = \delta_{p,1}\frac{-q^{n-n'+2}}{q-q^{-1}}\, v_{n+1,n'},\\
&\tilde{x}^+_{2,m} v_{n,n'} = \delta_{m,0}\, v_{n,n'-1}\,, \quad \tilde{x}^-_{2,p} v_{n,n'} = 0\,,\\
&\tilde{\phi}^+_1(z) v_{n,n'} = (1-z) v_{n,n'}\,, \quad \tilde{\phi}^+_2(z) v_{n,n'} = v_{n,n'}\,.
\end{aligned}$$

In addition, the operator $e_0 = \tilde{x}^-_{2,1}\tilde{x}^-_{1,0} - q\tilde{x}^-_{1,0}\tilde{x}^-_{2,1}$ also converges since

$$e_0\, v_{n,n'} = [n'+1]_q[k-n]_q q^{k+4}\, v_{n+1,n'+1} \quad \longrightarrow \quad [n'+1]_q\frac{-q^{n+4}}{q-q^{-1}}\, v_{n+1,n'+1}.$$

In particular we get an asymptotic action of the Borel algebra on $W_\infty = \oplus_{0\le n'\le n}\mathbb{C}v_{n,n'}$. The action of $\tilde{x}^-_{1,0}$ does not converge but the action of $\tilde{x}^-_{2,0}$ is constant. This example appeared in [BHK].

7.2. **First approach.** From now on, we shall be concerned with the family of KR modules of $U_q(\mathfrak{g})$

$$W_k = L(N_k), \quad N_k = Y_{i,q_i^{2k-1}} \cdots Y_{i,q_i^3} Y_{i,q_i}\,,$$

with the convention $N_0 = 1$. More generally we consider for $k \ge l \ge 0$ the modules $W_{k,l} = L(N_{k,l})$ with $N_{k,l} = Y_{i,q_i^{2k-1}} \cdots Y_{i,q_i^{2k-2l+1}}$. We fix a highest weight vector $w_{k,l} \in W_{k,l}$ and write $w_k = w_{k,k}$.

We have a unique isomorphism of vector spaces

$$H_{k,l} \;:\; W_l \longrightarrow W_{k,l},$$

such that $H_{k,l} w_l = w_{k,l}$ and

$$x\, H_{k,l} = H_{k,l}\, \tau_{q_i^{2(k-l)}}(x) \quad (x \in U_q(\mathfrak{g}))\,, \tag{7.33}$$

where τ_a denotes the automorphism of $U_q(\mathfrak{g})$ given in (3.17).

Decomposing the monomial as $N_k = N_{k,l}N_{k-l}$, we consider the corresponding morphism of $U_q(\mathfrak{g})^+$-modules in Theorem 3.15

$$G_{k,l} \;:\; W_{k,l} \longrightarrow W_k^{\ge(2k-2l+1)d_i}\,,$$

normalized as $G_{k,l}w_{k,l} = w_k$. Set further

$$I_{k,l} = G_{k,l} \circ H_{k,l} \ : \ W_l \longrightarrow W_k^{\geq (2k-2l+1)d_i}.$$

Clearly

$$I_{k,l} \circ I_{l,m} = I_{k,m} \quad (k \geq l \geq m), \quad I_{k,k} = \mathrm{id},$$

so that $(\{W_k\}, \{I_{k,l}\})$ constitutes an inductive system of linear spaces. Let

$$W_\infty = \varinjlim W_k, \quad w_\infty = I_{\infty,k} w_k,$$

where $I_{\infty,k} : W_k \to W_\infty$ denotes the injective linear map satisfying the condition $I_{\infty,k} \circ I_{k,l} = I_{\infty,l}$.

Combining (7.33) with (3.23), (3.24) we find that

$$x^+_{j,r} I_{k,l} = q_i^{2(k-l)r} I_{k,l} x^+_{j,r}\,, \tag{7.34}$$

$$\phi^\pm_j(z) I_{k,l} = I_{k,l} \phi^\pm_j(z q_i^{2(k-l)}) \times \begin{cases} q_i^{k-l} \frac{1-z}{1-q_i^{2(k-l)}z} & (j=i), \\ 1 & (j \neq i). \end{cases} \tag{7.35}$$

In particular, k_j is constant if $j \neq i$ and we have $k_i I_{k,l} = q_i^{k-l} I_{k,l} k_i$.

Proposition 7.1. *Consider the limit $k \to \infty$.*

(i) For $r \geq 1$, the operator $I_{k,l}^{-1} x^+_{j,r} I_{k,l}$ converges to 0. For $r = 0$ it stays constant.

(ii) The operator $I_{k,l}^{-1} \tilde{\phi}^+_j(z) I_{k,l}$ converges to $1 - \delta_{i,j} z$.

Proof. This follows from (7.34) and (7.35). □

Proposition 7.2. *Let $j \in I \setminus \{i\}$ and $r \in \mathbb{Z}$. The action of the operator*

$$I_{k,l+1}^{-1} \big(q_i^{-2kr} \tilde{x}^-_{j,r}\big) I_{k,l} \ \in \mathrm{Hom}(W_l, W_{l+1})$$

stays constant. In particular, $I_{k,l+1}^{-1} \tilde{x}^-_{j,r} I_{k,l}$ converges to 0 if $r \geq 1$ and stays constant if $r = 0$.

Proof. We adapt the argument in the proof of Proposition 4.2. Let $\omega_l \in \mathfrak{t}^*$ be the highest weight of W_l. We prove by induction on $\beta \in Q$ that when $k \to \infty$ the operator $I_{k,l+1}^{-1}\big(q_i^{-2kr}\tilde{x}^-_{j,r}\big) I_{k,l}$ stays constant on $(W_l)_{\omega_l \overline{\beta}^{-1}}$. When $\beta \notin Q^+$ this is clear as $(W_l)_{\omega_l \overline{\beta}^{-1}} = 0$. For an arbitrary β, it suffices to prove that $\tilde{x}^+_{j',r'} I_{k,l+1}^{-1}\big(q_i^{-2kr}\tilde{x}^-_{j,r}\big) I_{k,l}$ is constant on $(W_L)_{\omega_l \overline{\beta}^{-1}}$ for all $j' \in I$ and $r' \geq 0$. Furthermore, by the argument of the proof of Proposition 3.5, we may assume without loss of generality that $r' > r$.

We have

$$\begin{aligned} \tilde{x}^+_{j',r'} \left(I_{k,l+1}^{-1} q_i^{-2kr} \tilde{x}^-_{j,r} I_{k,l} \right) &= q_j^2 \left(I_{k,l+1}^{-1} q_i^{-2kr} \tilde{x}^-_{j,r} I_{k,l} \right) \tilde{x}^+_{j',r'} \\ &\quad + \delta_{j,j'} \frac{q_j^2 q_i^{-2kr+2r'(l-k)} \left(I_{k,l+1}^{-1} \tilde{\phi}^+_{j,r+r'} I_{k,l} \right)}{q_j - q_j^{-1}}. \end{aligned}$$

In the right hand side, the first term is constant by the induction hypothesis. The second term equals up to a constant multiple

$$\delta_{j,j'} q_i^{2r'(l-k)-2kr} \left(I_{k,l+1}^{-1} \tilde{\phi}^+_{j,r+r'} I_{k,l} \right) = \delta_{j,j'} q_i^{-2lr} \tilde{\phi}^+_{j,r+r'} \,,$$

which is constant. □

Proposition 7.3. *Let $r \in \mathbb{Z}$. We have*

$$I_{k,l+1}^{-1}\big(q_i^{2k(1-r)} \tilde{x}^-_{i,r}\big) I_{k,l} \ = A_r + q_i^{2k} B_r \in \mathrm{Hom}(W_l, W_{l+1})$$

where $A_r, B_r \in \mathrm{Hom}(W_l, W_{l+1})$ are constant operators.

In particular, $I_{k,l+1}^{-1} \tilde{x}^-_{i,r} I_{k,l}$ converges to 0 if $r \geq 2$ and converges if $r = 1$.

Proof. The proof is analogous to the proof of the previous proposition and we retain the notation there. We prove by induction on $\beta \in Q$ that the operator $I_{k,l+1}^{-1} \tilde{x}^-_{i,r} I_{k,l}$ is of the form $A_r + q_i^{2k} B_r$ on $(W_l)_{\omega_l \overline{\beta}^{-1}}$. When $\beta \notin Q^+$ this is clear. For an arbitrary β, it suffices to prove that $\tilde{x}^+_{j',r'} I_{k,l+1}^{-1}\big(q_i^{2k(1-r)} \tilde{x}^-_{i,r}\big) I_{k,l}$ is of this form on $(W_L)_{\omega_l \overline{\beta}^{-1}}$ for all $j' \in I$, $r' \geq 0$. We may assume that $r' > r$ and we have

$$\begin{aligned} \tilde{x}^+_{j',r'} \left(I_{k,l+1}^{-1} q_i^{2k(1-r)} \tilde{x}^-_{i,r} I_{k,l} \right) &= q_i^2 \left(I_{k,l+1}^{-1} q_i^{2k(1-r)} \tilde{x}^-_{i,r} I_{k,l} \right) \tilde{x}^+_{j',r'} \\ + \delta_{i,j'} q_i^2 & \frac{q_i^{2k(1-r)} q_i^{2r'(l-k)} \left(I_{k,l+1}^{-1} \tilde{\phi}^+_{i,r+r'} I_{k,l} \right)}{q_i - q_i^{-1}} . \end{aligned}$$

The first term in the right side is of the correct form from the induction hypothesis. The second term for $i = j'$ is equal, up to a constant multiple, to

$$\begin{aligned} q_i^{2k(1-r)+2r'(l-k)} I_{k,l+1}^{-1} \tilde{\phi}^+_{i,r+r'} I_{k,l} &= \left[\frac{q_i^{-2lr} (q_i^{2k} - z q_i^{2l}) \tilde{\phi}^+_i(z)}{1-z} \right]_{r+r'} \\ &= \left[\frac{-z q_i^{2l(1-r)} \tilde{\phi}^+_i(z)}{1-z} \right]_{r+r'} + q_i^{2k-2lr} \left[\frac{\tilde{\phi}^+_i(z)}{1-z} \right]_{r+r'} , \end{aligned}$$

where we write $c_n = [f(z)]_n$ for a formal power series $f(z) = \sum_{n=0}^{\infty} c_n z^n$. This is also in the correct form. □

Let $L \geq 1$ be the length of the maximal root of $\dot{\mathfrak{g}}$ and $\mathcal{N}_i \geq 1$ be the multiplicity of α_i in the maximal root of $\dot{\mathfrak{g}}$.

Proposition 7.4. *We have*

$$I_{k,l+L}^{-1} e_0 \, I_{k,l} \ = q_i^{2k(1-\mathcal{N}_i)} \sum_{0 \leq p \leq N} q_i^{2kp} A_p \in \mathrm{Hom}(W_l, W_{l+L})$$

where for $0 \leq p \leq N$, $A_p \in \mathrm{Hom}(W_l, W_{l+L})$ is a constant operator.

In particular, $I_{k,l+L}^{-1} e_0 \, I_{k,l}$ is convergent if $\mathcal{N}_i = 1$.

Proof. Let us write e_0 as in (2.3). Then it is clear that for any $k \geq l + L$, the operator $I_{k,l+L}^{-1} x_0^+ I_k$ makes sense in $\mathrm{Hom}(W_l, W_{l+L})$. Then the result follows immediately from the last propositions. □

From here until the end of this section 7.2, we assume that $\mathcal{N}_i = 1$. We get a structure of $U_q(\mathfrak{b})$-module on W_∞ in category $\mathcal{O}$ (with the natural Q-grading such that the highest weight vector has degree 0).

Theorem 7.5. *W_∞ is irreducible isomorphic to $L^+_{i,1}$ and we have $\tilde{\chi}_q(L^+_{i,1}) = \chi(L^+_{i,1})$.*

In this case, we get another proof that $L^+_{i,1}$ is in category $\mathcal{O}$. We also get an explicit q-character formula for $L^+_{i,1}$.

Proof. The representation $L^+_{i,1}$ is a subquotient of W_∞. For any $k \geq 0$, we have $\tilde{\chi}(W_k) = \tilde{\chi}(V_k)$. By construction, $\chi(V_\infty)$ (resp. $\chi(W_\infty)$) is the limit of the $\tilde{\chi}(V_k)$ (resp. of the $\tilde{\chi}(W_k)$). Hence $\chi(W_\infty) = \chi(V_\infty)$ which is equal to $\chi(L^+_{i,1})$ by Theorem 6.3. The result follows. For the second point, it follows from Proposition 7.1 that $\tilde{\chi}_q(W_\infty) = \chi(W_\infty)$. $\square$

Let $W'_k \subset W_k$ be the sum of the $U_q(\dot{\mathfrak{g}})$-submodules of W_k which do not contain the highest weight vector. For any $j \in I$, $k \geq l$, we have $x^+_{j,0} I_{k,l} = I_{k,l} x^+_{j,0}$. In particular we have $I_{k,l}(W'_l) \subset W'_k$. Let us consider $W'_\infty = \bigcup_{k\geq 0} I_{\infty,k}(W'_k)$.

Lemma 7.6. *We have $W'_\infty = \{0\}$.*

Proof. First let us prove that $U_q(\mathfrak{b})W'_\infty$ is a submodule of W_∞. The subspace W'_∞ is stable under the action of the $x^+_{j,0}, k_j^{\pm 1}$ for $j \in I$. Since $\phi^+_{j,1}$ acts as a scalar, it follows that W'_∞ is stable also by $x^+_{j,r}$ with $r > 0$. Hence $U_q(\mathfrak{b})^+ W'_\infty \subset W'_\infty$ and we have

$$U_q(\mathfrak{b})W'_\infty = U_q(\mathfrak{b})^- U_q(\mathfrak{b})^0 W'_\infty.$$

Consequently, for reasons of weights, $w_\infty \notin U_q(\mathfrak{b})W'_\infty$. Since W_∞ is irreducible, we obtain the assertion. $\square$

As a consequence, we get a second explicit formula for $\chi(L^+_{i,1})$ in this case : it is the limit of characters given by the Weyl character formula.

Remark 7.7. We also get that for any $k \geq 0, a \in \mathbb{C}^*$, the KR module $L(Y_{i,a}Y_{i,aq_i^2}\cdots Y_{i,aq_i^{2(k-1)}})$ with $\mathcal{N}_i = 1$ is irreducible as a representation of $U_q(\dot{\mathfrak{g}})$. We recover the result of [C1].

7.3. **Another example.** Let us look at the $\widehat{\mathfrak{sl}}_2$-case from a different angle. We can choose a basis $(v'_0, v'_1, \cdots, v'_k)$ of W_k such that

$$x^+_{1,r} v'_j = q^{2r(k-j+1)+2k-4(j-1)}[j]_q[k-j+1]_q v'_{j-1}\,, \quad x^-_{1,r} v'_j = q^{2r(k-j)+4j-2k} v'_{j+1}\,,$$

$$\phi^\pm_1(z) v'_j = q^{k-2j} \frac{(1-z)(1-zq^{2(k+1)})}{(1-zq^{2(k-j+1)})(1-zq^{2(k-j)})} v'_j\,.$$

In this basis we see that $(f_1 k_1^2)v'_j = v'_{j+1}$ and

$$\begin{aligned}\lim_{k\to\infty}(e_1 k_1^{-1})v'_j &= \lim_{k\to\infty} q^{k-2j+4}[j]_q[k-j+1]_q v'_{j-1} \\ &= -[j]_q \frac{q^{-j+3}}{q-q^{-1}}[k-j+1]_q v'_{j-1}.\end{aligned}$$

Then $(e_0 k_0^{-1})v'_j = q^{2(j+1)} v'_{j+1}$ is constant and $\tilde{\phi}^+_1(z)v'_j$ converges to $(1-z)v'_j$.

Let us generalize this calculation in the next subsection.

7.4. **Second approach.** For $k \geq 0$, let V_k be as in Section 4.2 but for the parameter q^{-1}, that is, V_k is the $U_{q^{-1}}(\mathfrak{g})$-module $L(Y_{i,(q_i^{-1})^{-1}}Y_{i,(q_i^{-1})^{-3}}\cdots Y_{i,(q_i^{-1})^{-(2k-1)}})$. Let v_k be the highest ℓ-weight vector of V_k. Pulling back by σ, we get the $U_q(\mathfrak{g})$-module V_k^σ. We have in V_k^σ

$$\phi_i^+(z)\, v_k = q_i^{-k}\frac{1-zq_i^{2k}}{1-z}\, v_k\,,\quad \phi_j^+(z)\, v_k = 1 \text{ for } j\neq i\,.$$

So the ℓ-weight of v_k in V_k^σ is $Y_{i,q_i}^{-1}Y_{i,q_i^3}^{-1}\cdots Y_{i,q_i^{2k-1}}^{-1} = N_k^{-1}$. We define its dual module $(V_k^\sigma)^*$ as in Section 3.6. By Proposition 3.18, we have a highest ℓ-weight vector v_k^* in $(V_k^\sigma)^*$ of ℓ-weight N_k. In particular we can identify $(V_k^\sigma)^*$ with W_k and v_k with w_k.

The injective morphism of $U_q(\mathfrak{g})^+$-module $F_{k,l} : V_l \to V_k$ $(1 \leq l \leq k)$ gives rise to an injective linear morphism $F_{k,l}^\sigma : V_l^\sigma \to V_k^\sigma$ and to its surjective dual $\Pi_{l,k} = (F_{k,l}^\sigma)^* : W_k \to W_l$. Then $(\{W_k\},\{\Pi_{l,k}\})$ constitutes a projective system of vector spaces. Set

$$W_\infty^\Pi = \varprojlim W_k,$$

and let $w_\infty \in W_\infty^\Pi$ be the unique vector satisfying $w_k = \Pi_{k,\infty}\, w_\infty^\Pi$, where $\Pi_{k,\infty} : W_\infty^\Pi \to W_k^\Pi$ denotes the surjective linear morphism satisfying $\Pi_{l,k}\circ\Pi_{k,\infty} = \Pi_{l,\infty}$ for $l \leq k$. Note that W_∞^Π can be identified with $(V_\infty^\sigma)^*$.

Now for $j \in I$ and $l \leq k$, we have $F_{k,l}e_j = e_jF_{k,l}$ and $F_{k,l}k_j = q_i^{2\delta_{i,j}(k-l)}k_jF_{k,l}$. So $F_{k,l}^\sigma(k_j^{-1}f_j) = (k_j^{-1}f_j)F_{k,l}^\sigma$ and $F_{k,l}^\sigma k_j = q_i^{2\delta_{i,j}(k-l)}k_jF_{k,l}^\sigma$. This implies

$$\Pi_{l,k}(f_jk_j^2) = (f_jk_j^2)\Pi_{l,k} \quad\text{ and }\quad \Pi_{l,k}k_j = q_i^{2\delta_{i,j}(k-l)}k_j\Pi_{l,k}\,.$$

For $0 \leq j \leq n$, we set $\tilde{e}_j = e_jk_j^{-1}$.

Proposition 7.8. *Let $v \in W_\infty^\Pi$, $0 \leq j \leq n$ and $l \geq 1$. When $k \to \infty$, $(\Pi_{l,k}\tilde{e}_j\Pi_{k,\infty})(v)$ converges to a vector $v_{j,l} \in W_l$. Moreover we have $\Pi_{l,l'}(v_{j,l'}) = v_{j,l}$ for $l \leq l'$.*

Proof. First assume that $j \in I$, and let $v \in W_\infty^\Pi$. Using $S^{-1}(\tilde{e}_j) = -q_j^{-2}e_j$ in $U_{q^{-1}}(\mathfrak{g})$ and $\sigma(e_j) = q_j^2\tilde{x}_{j,0}^-$, we have for $k > l$

$$(\Pi_{l,k}\tilde{e}_j\Pi_{k,\infty})(v) = (\tilde{e}_j\Pi_{k,\infty}(v))F_{k,l}^\sigma = -q_j^{-2}\Pi_{k,\infty}(v)\, e_j\, F_{k,l}^\sigma = -vF_{\infty,k}\,\tilde{x}_{j,0}^-\,F_{k,l}\,.$$

The right hand side can be written as $-vF_{\infty,l+1}((F_{l+1,k})^{-1}\tilde{x}_{j,0}^-\,F_{k,l})$. This converges as $k \to \infty$ since $F_{k,l+1}^{-1}\tilde{x}_{j,0}^-F_{k,l}$ does.

For $k \geq l' \geq l$, we have

$$(\Pi_{l,k}\tilde{e}_j\Pi_{k,\infty})(v) = \Pi_{l,l'}((\Pi_{l',k}\tilde{e}_j\Pi_{k,\infty})(v)).$$

This implies the relation $\Pi_{l,l'}(v_{j,l'}) = v_{j,l}$.

We have $\sigma(S^{-1}(\tilde{e}_0)) = -q_0^{-2}\sigma(e_0) \in \mathbb{C}\langle\tilde{x}_{j,m}^+\rangle_{j\in I,m\in\mathbb{Z}}$. As each $F_{k,l}^{-1}\tilde{x}_{j,m}^+F_{k,l}$ converges when $k \to +\infty$ in $\mathrm{End}(V_l)$, we can conclude as above. □

Let $\tilde{e}_j(v) \in W_\infty^\Pi$ be the projective limit of the $(v_{j,l})_{l\geq 1}$. We get a linear operator $\tilde{e}_j \in \mathrm{End}(W_\infty^\Pi)$. W_∞^Π has a natural Q-grading. We define the action of $k_j^{\pm 1}$ on W_∞^Π so that w_∞^Π has weight 1. Then we get a structure of $U_q(\mathfrak{b})$-module on W_∞^Π.

Theorem 7.9. *W_∞^Π is irreducible isomorphic to $L_{i,1}^+$.*

Proof. We have seen that each v_k has ℓ-weight N_k. So the ℓ-weight of W_∞^Π is the highest ℓ-weight of $L_{i,1}^+$. In particular $L_{i,1}^+$ is a subquotient of W_∞^Π. By construction, we have $\chi(W_\infty^\Pi) = \chi(V_\infty)$. Now the result follows from Theorem 6.3. $\square$

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