Symmetries of Grothendieck rings in representation theory

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Grothendieck group

- General construction of the Grothendieck group of an additive category.
- $\mathcal{C}$: additive category with distinguished short exact sequences

$$A \to B \to C$$

(for example, an abelian category with all ordinary exact sequences).

Definition

Grothendieck group $K(\mathcal{C})$: group generated by the elements $[M]$ for each isomorphism class of objects M in $\mathcal{C}$ with the relations

$$[A] + [C] = [B]$$

for each exact sequence as above.

- $K(\mathcal{C})$: important algebraic structure attached to the category $\mathcal{C}$.

Grothendieck group - Examples

- Original motivation from algebraic geometry : $\mathcal{C}$ category of coherent sheaves on an algebraic variety X. The Grothendieck group

$$K(X) = K(\mathcal{C})$$

is the starting point of K-theory in algebraic geometry.

- $\mathcal{C} = \mathcal{V}$ category of finite-dimensional k-vector spaces. Isomorphism classes parametrized by the dimension. $\mathcal{V}$ is semi-simple and the direct sum descends to the sum in the Grothendieck group :

$$[V \oplus W] = [V] + [W]$$

$$K(\mathcal{V}) \simeq \mathbb{Z} = \mathbb{Z}[k]$$

generated by the class $[k]$ of the one-dimensional vector space as

$$[V] = \dim(V).[k].$$

Grothendieck group - Representation theory

- $\mathcal{A}$: an algebra.
- $\mathcal{F}$: the category of finite-dimensional representations of $\mathcal{A}$.

Theorem (Jordan-Hölder series)

Each object M in $\mathcal{F}$ admits a series of subobjects

$$M = M_0 \supset M_1 \supset \cdots \supset M_N = \{0\}$$

with M_i/M_{i+1} simple object.
The number $n_V(M)$ of occurrence of a simple object V (as a quotient M_i/M_{i+1}) depends only on M.

Grothendieck group - Representation theory

- Hence
$$K(\mathcal{F}) = \bigoplus_{V \text{ simple class}} \mathbb{Z}[V]$$
has a canonical basis parametrized by simple classes.
- For M object in $\mathcal{F}$, one has
$$[M] = \sum_{V \text{ simple class}} n_V(M)[V].$$
- An element
$$\sum_{V \text{ simple class}} n_V[V]$$
with all $n_V \geq 0$ is called the class of an *actual* representation.
- If one of the $n_V < 0$, this the class of a *virtual* representation.

Monoidal categories

- We now assume that an abelian category $\mathcal{C}$ has a monoidal structure : in particular
  - tensor products $M \otimes N$ for M, N objects in $\mathcal{C}$.
  - a unit object $\mathbf{1}$ in $\mathcal{C}$ so that for M object in $\mathcal{C}$

$$M \otimes \mathbf{1} \simeq \mathbf{1} \otimes M \simeq M.$$

- Example : Category $\mathcal{V}$ of finite-dimensional k-vector spaces.
- Example : G a group. Category of finite-dimensional representations of G (that is of its group-algebra $k[G]$).
- Example : a Lie algebra $\mathfrak{g}$. Category of finite-dimensional representations of $\mathfrak{g}$ (that is of its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$).

Monoidal categories

- Example : more generally : for a Hopf algebra H, that is an algebra with a coproduct and a counit

$$\Delta : H \to H \otimes H \text{ and } \epsilon : H \to k.$$

This includes all examples above.

- And new examples, in particular Drinfeld-Jimbo **quantum groups**

$$\mathcal{U}_q(\mathfrak{g})$$

associated to $q \in \mathbb{C}^*$ quantum parameter and $\mathfrak{g}$ simple Lie algebra. It is a q-deformation of $\mathcal{U}(\mathfrak{g})$, originated from mathematical physics.

Grothendieck ring

- $\mathcal{C}$ monoidal category.

Proposition

The Grothendieck group $K(\mathcal{C})$ of $\mathcal{C}$ inherits a ring structure such that

$$[M].[N] = [M \otimes N]$$

for M, N objects in $\mathcal{C}$ and the class $[\mathbf{1}] \in K(\mathcal{C})$ is the neutral element.

- $K(\mathcal{C})$ is called the Grothendieck ring of $\mathcal{C}$.
- Example : $\mathcal{C} = \mathcal{V}$, then $K(\mathcal{C}) \simeq \mathbb{Z}$ as a ring.
- If the category is braided, that is

$$M \otimes N \simeq N \otimes M$$

  for all objects M, N in $\mathcal{C}$, then the Grothendieck ring is commutative. The converse statement is false.

Grothendieck rings in representation theory

- $\mathcal{C}$ category of finite-dimensional representations of a Hopf algebra H. The Grothendieck ring

$$K(\mathcal{C}) = \bigoplus_{V \text{ simple}} \mathbb{Z}[V]$$

has a canonical basis $([V])_{V \text{ simple}}$ with positive constant structures.

- Example : $\mathcal{C}$ category of finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$.

$$K(\mathcal{C}) \simeq \mathbb{Z}[X]$$

polynomial ring in one variable

$$X = [V] \text{ where } V = \mathbb{C}^2 \text{ natural representation.}$$

The basis $(Q_n)_{n\geq 0})$ of simples satisfies

$$Q_n^2 = Q_{n+1} + 1$$

$$(Q_0, Q_1, Q_2, \cdots) = (1, X, X^2 - 1, X^3 - 2X, \cdots).$$

Grothendieck rings in representation theory : applications

- *Representation Theory* : description of the structure of simple representations from the structure of more elementary representations. Algorithm "à la Kazhdan-Lusztig".
- *Categorification* : Realization of an algebra $\mathcal{A}$ as a Grothendieck ring:

$$\mathcal{A} \simeq K(\mathcal{C}).$$

  The categorified structure $\mathcal{A}$ inherits a canonical basis.
- *Quantum integrable models* : certain Grothendieck rings are identified with the ring of commuting operators of a quantum integrable model (transfer-matrix construction).

  Algebra of symmetries of a quantum integrable model

  The relations in $K(\mathcal{C})$ are satisfied by eigenvalues of quantum operators on any related quantum integrable model.
  The relations in $K(\mathcal{C})$ are universal relations.

Symmetries of Grothendieck rings

- Aim of this talk : certain symmetries of Grothendieck rings (and their consequences).
- Main examples :
  - Cluster symmetries.
  - Weyl group symmetries.

Cluster algebra symmetries

- Theory of cluster algebras : Fomin-Zelevinsky (2000).
- Commutative algebra (the cluster algebra) with a distinguished set of generators (the cluster variables) defined by a combinatorial process.
- Many incarnations in Poisson geometry, discrete dynamical systems, algebraic geometry, representation theory...
- Certain Grothendieck rings admit a cluster algebra structure :

A Cluster symmetry.

Cluster algebras (quick review)

- Cluster algebra $\mathcal{A}_Q$ attached to a quiver Q (with set of vertices Q_0).
- $\mathcal{A}_Q$: subalgebra of $\mathcal{F} = \mathbb{Q}(X_i)_{i \in Q_0}$.
- $\mathcal{A}_Q$ is the subalgebra of $\mathcal{F}$ generated by cluster variables.
- Initial cluster variables : X_i ($i \in Q_0$).
- New cluster variables : obtained from the initial variables by mutations. For example, the first step mutated variables X_i^* are :

$$X_i X_i^* = \prod_{j \to i} X_j + \prod_{j \leftarrow i} X_j.$$

- The cluster variables are grouped into overlapping subsets : the clusters.
- Cluster monomials : monomials in the cluster variables from the same cluster.
- There might be additional non-mutable cluster variable : frozen variables.

Cluster algebras (example)

- Example : Q quiver of type A_2 :

$$\bullet \longrightarrow \bullet$$

- Initial variables (X_1, X_2). Five cluster variables :

$$X_1, X_2, \frac{1+X_2}{X_1}, \frac{1+X_1}{X_2}, \frac{1+X_1+X_2}{X_1X_2}.$$

- Five clusters :

$$(X_1, X_2), \left(\frac{1+X_2}{X_1}, X_2\right), \left(\frac{1+X_2}{X_1}, \frac{1+X_1+X_2}{X_1X_2}\right),$$

$$\left(\frac{1+X_1}{X_2}, \frac{1+X_1+X_2}{X_1X_2}\right), \left(\frac{1+X_1}{X_2}, X_1\right).$$

- Five families of cluster monomials.

Monoidal categorification of cluster algebras

Definition (H.-Leclerc 2010)

A monoidal category $\mathcal{M}$ is a *monoidal categorification* of a cluster algebra $\mathcal{A}$ is there exists a ring isomorphism

$$\phi : \mathcal{A} \to K(\mathcal{M})$$

so that the cluster monomials are sent to simple classes.

- This can be seen as a cluster symmetry of the Grothendieck ring, or as a categorical realization of the cluster algebra.
- It gives useful information on the monoidal category :
- It points out remarkable simple representations (corresponding to cluster variables).
- It gives a factorization of simple representations corresponding to cluster monomials.

Monoidal categorification - examples

- Original examples (H.-Leclerc, 2010) : from the finite-dimensional representations of quantum affine algebras.
- Many developments about monomial categorifications in this context : Nakajima, Qin, Kashiwara-Kim-Oh-Park, Bittmann, Brito-Chari...
- Other monoidal categorifications of cluster algebras : perverse sheaves on quiver varieties, representations of quiver Hecke algebras, coherent sheaves on affine Grassmanians, representations of shifted quantum groups...
- In all known examples, the cluster symmetry is related to the defect of symmetry of the category which produces relations in the Grothendieck ring :

$$V \otimes W \text{ not isomorphic to } W \otimes V$$

but $[V \otimes W] = [W \otimes V]$ in the Grothendieck ring.

Quantum affine algebra

- $\mathfrak{g}$ simple Lie algebra (for example, $\mathfrak{g} = \mathfrak{sl}_2$)of rank n ($n = 1$ for $\mathfrak{g} = \mathfrak{sl}_2$).
- $\hat{\mathfrak{g}}$: affine Kac-Moody algebra.
- (Central extension of the loop algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$).
- $q \in \mathbb{C}^*$: quantum parameter (not root of unity).
- Quantum affine algebra : $\mathcal{U}_q(\hat{\mathfrak{g}})$.
- $\mathcal{U}_q(\hat{\mathfrak{g}})$: Hopf algebra, q-deformation of $\mathcal{U}(\hat{\mathfrak{g}})$.

Quantum affine algebra

- $\mathcal{C}$: Category of finite-dimensional representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$: very interesting (and intricated) category.
- $\mathcal{C}$: monoidal category, but not semi-simple and not braided.
- The simple objects of $\mathcal{C}$ have been parametrized in terms of Drinfeld polynomials (Chari-Pressley).
- In particular, for $1 \leq i \leq n$ and $a \in \mathbb{C}^*$: the fundamental representation $V_i(a)$.

Theorem (Frenkel-Reshetikhin)

The Grothendieck ring $K(\mathcal{C})$ is commutative and polynomial

$$K(\mathcal{C}) \simeq \mathbb{Z}[X_{i,a}]_{1\leq i\leq n, a\in\mathbb{C}^*},$$

with $X_{i,a} = [V_{i,a}]$ class of a fundamental representation.

Monoidal categorification - example

- Example : for $\mathfrak{g} = \mathfrak{sl}_3$.
- $\mathcal{M}$: monoidal and Serre subcategory of the category $\mathcal{C}$ of finite-dimensional representations of $\mathcal{U}_q(\hat{\mathfrak{sl}}_3)$-representations generated by four fundamental representations

$$V_1(1), V_1(q^2), V_2(q), V_2(q^3).$$

- Then we obtain a monoidal categorification

$$K(\mathcal{M}) \simeq \mathcal{A}_Q$$

with Q of type A_2 (as above) with two additional frozen variables. 7 cluster variables in total.

- Application : every simple representations in $\mathcal{M}$ factorized into only 7 representations (prime objects).

Monoidal categorifications - quantum affine algebras

- A cluster algebra structure has been obtained on $K(\mathcal{C}^-)$ for $\mathcal{C}^-$ a large subcategory of $\mathcal{C}$ (H.-Leclerc 2016).
- Kashiwara-Kim-Oh-Park established (2022) the following conjectured in [H-Leclerc].

Theorem

The category $\mathcal{C}$ of finite-dimensional representations of a quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ is a monoidal categorification of a cluster algebra.

- Application (proof of a Conjecture for non simply-laced type, H. 2002) :

Theorem (Fujita-H.-Oh-Oya, 2022)

The dimension (and character) of simple representations corresponding to cluster monomials can be obtained from an algorithm (à la Kazhdan-Lusztig-Nakajima).

Shifted quantum affine algebras

- Next step : Shifted quantum groups.
- **Shifted** quantum affine algebras. : new classes of quantum groups related to Coulomb branches.
- Algebras introduced by Finkelberg-Tsymbaliuk in the study of K-theoretical Coulomb branches (in the sense of Braverman-Finkelberg-Nakajima).
- $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$: variations of quantum affine algebras depending on a shift parameter : a coweight μ of $\mathfrak{g}$.
- $\mu = 0$: $\mathcal{U}_q^0(\hat{\mathfrak{g}})$ is (a central extension) of $\mathcal{U}_q(\hat{\mathfrak{g}})$.

Representations of shifted quantum affine algebras

Theorem (H. 2020)

$\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ has non-zero finite-dimensional representation if and only if μ is codominant.

- In general, $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ has an abelian category $\mathcal{O}^\mu$ of representations.

Theorem (H. 2020)

The simple objects in $\mathcal{O}^\mu$ are parametrized by n-tuples of rational fractions $(\psi_i(z))_{1\leq i\leq n}$ regular at 0 (with a condition on the degree).

- For $1 \leq i \leq n$, $a \in \mathbb{C}^*$, $\psi_j(z) = (1 - za)^{\delta_{i,j}}$: positive prefundamental representation $L_{i,a}^+$. It is of dimension 1 !
- For $1 \leq i \leq n$, $a \in \mathbb{C}^*$, $\psi_j(z) = (1 - za)^{-\delta_{i,j}}$: negative prefundamental representation $L_{i,a}^-$. It is infinite dimensional.

Grothendieck ring

- One obtains an abelian category

$$\mathcal{O} = \bigoplus_{\mu} \mathcal{O}^{\mu}$$

- The sum of Grothendieck groups

$$K_0(\mathcal{O}) = \bigoplus_{\mu} K_0(\mathcal{O}^{\mu})$$

  has a ring structure (analog to a Grothendieck ring structure).
- Commutative ring.

Cluster structure - finite-dimensional representations

- $\mathcal{C}^{sh} \subset \mathcal{O}$ category of finite-dimensional representations of shifted quantum affine algebras.

Theorem (H.-Leclerc 2016, Kashiwara-Kim-Oh-Park 2020, H. 2020)

$K_0(\mathcal{C}^{sh})$ is isomorphic to a cluster algebra $\mathcal{A}_{\Gamma_\infty}$ (explicit quiver Γ_∞).
Initial cluster variables : classes of positive prefundamental representations.
The cluster monomials correspond to certain classes of simple modules.

Cluster structure - finite-dimensional representation

- Example : $\mathfrak{g} = \mathfrak{sl}_2$. Infinite linear quiver :

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

- Initial seed of 1-dimensional representations

$$\cdots \longrightarrow L^+_{1,q^{-2}} \longrightarrow L^+_{1,1} \longrightarrow L^+_{1,q^2} \longrightarrow \cdots$$

- First step mutations are given by Baxter TQ-relations

$$[L^+_{1,a}][V_1(a)] = [L^+_{1,aq^2}] + [L^+_{1,aq^{-2}}]$$

where V_1 : 2-dimensional fundamental representation.

- Relevant from the point of view of quantum integrable models.

Cluster structure - category $\mathcal{O}$

- New families of quivers Γ'_∞ attached to each $\mathfrak{g}$.

Theorem (Geiss-H.-Leclerc 2023)

The Grothendieck ring $K_0(\mathcal{O})$ is isomorphic to (a completion of) $\mathcal{A}_{\Gamma'_\infty}$.

- The proof is partly based on additional symmetries of the Grothendieck ring.

Cluster structure - category $\mathcal{O}$

- Example : $\mathfrak{g} = \mathfrak{sl}_2$

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

- Initial seed :

$$\cdots \longrightarrow L^+_{1,q^2} \longrightarrow L^+_{1,1} \longleftarrow L^-_{1,q^{-2}} \longrightarrow L^-_{1,q^{-4}} \longrightarrow \cdots$$

- Mutation at $L^+_{1,1}$:

$$\cdots \longrightarrow L^+_{1,q^2} \longleftarrow L^-_{1,1} \longrightarrow L^-_{1,q^{-2}} \longrightarrow L^-_{1,q^{-4}} \longrightarrow \cdots$$

- Mutation at $L^-_{1,q^{-2}}$:

$$\cdots \longrightarrow L^+_{1,q^2} \longrightarrow L^+_{1,1} \longrightarrow L^+_{1,q^{-2}} \longleftarrow L^-_{1,q^{-4}} \longrightarrow \cdots$$

Conjecture

- We conjecture : all cluster monomials in $\mathcal{A}_{\Gamma'_\infty}$ correspond to classes of simple objects in $\mathcal{O}$ through our isomorphism.

Theorem (Geiss-H.-Leclerc 2023)

The conjecture is true for $\mathfrak{g} = sl_2$.

Classical Theory

- $\mathfrak{g}$ complex finite-dimensional simple Lie algebra of rank n.
- Character morphism :

$$\chi : \mathrm{Rep}(\mathfrak{g}) \to \mathbb{Z}[y_i^{\pm 1}]_{1\leq i\leq n}$$

- Image of the character morphism :

$$\mathrm{Im}(\chi) = (\mathbb{Z}[y_i^{\pm 1}]_{1\leq i\leq n})^W$$

W : Weyl group W generated by the simple reflexions s_i

$$s_i(y_j) = y_j a_i^{-\delta_{ij}} \text{ where } a_i = \prod_{k\in I} y_k^{C_{ji}}.$$

a_i corresponds to a simple root (C is the Cartan matrix of $\mathfrak{g}$).

Classical Theory

- Example, for $\mathfrak{g} = sl_2$, $C = (2)$:

$$a_1 = y_1^2$$

$$s_1(y_1) = y_1 a_1^{-1} = y_1^{-1},$$

$$s_1^2(y_1) = y_1$$

$$s_1(y_1 + y_1^{-1}) = y_1 + y_1^{-1}.$$

$$\mathrm{Im}(\chi) = (\mathbb{Z}[y_1^{\pm 1}])^W = \mathbb{Z}[y_1 + y_1^{-1}].$$

q-characters

- Analogue of character for finite-dimensional representations of quantum affine algebras : q-character (Frenkel-Reshetikhin) :

$$\chi_q : K(\mathcal{C}) \to \mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{1 \leq i \leq n, a \in \mathbb{C}^*}.$$

- Injective ring morphism on the Grothendieck ring of $\mathcal{C}$.
- Example : fundamental representations of $\mathcal{U}_q(\hat{\mathfrak{sl}}_2)$:

$$\chi_q(V_1(a)) = Y_{1,a} + Y_{1,aq^2}^{-1}.$$

- Weyl-group symmetry ?
- Monomial analogs of simple roots (formula for simply-laced types) :

$$A_{i,a} = Y_{i,aq^{-1}} Y_{i,aq} \prod_{j | C_{j,i} = -1} Y_{j,a}^{-1}.$$

Example for $\mathfrak{g} = \mathfrak{sl}_2$: $A_{1,a} = Y_{1,aq^{-1}} Y_{1,aq}$.

Symmetry of q-characters

- Operators (Frenkel-H. 2022) :

$$\Theta_i(Y_{j,a}) = Y_{j,a} A_{i,aq^{-1}}^{-\delta_{i,j}} \frac{\Sigma_{i,aq^{-3}}^{\delta_{i,j}}}{\Sigma_{i,aq^{-1}}^{\delta_{i,j}}}$$

- Here $\Sigma_{i,a}$ is the solution of the q-difference equation

$$\Sigma_{i,a} = 1 + A_{i,a}^{-1} \Sigma_{i,aq^{-2}}$$

in a sum

$$\Pi = \bigoplus_{w \in W} \tilde{\mathcal{Y}}^w$$

of completions $\tilde{\mathcal{Y}}^w$ of $\mathcal{Y}$.

Symmetry of q-character

- Example : $\mathfrak{g} = sl_2$,

$$\Theta_1(Y_{1,a}) = Y_{1,aq^{-2}}^{-1}\frac{\Sigma_{1,aq^{-3}}}{\Sigma_{1,aq^{-1}}}.$$

- Here $\Sigma_{1,a}$ is the couple

$$(1 + A_{1,a}^{-1}(1 + A_{1,aq^{-2}}^{-1}(1 + \cdots),$$

$$-A_{1,aq^2}(1 + A_{1,aq^4}(1 + \cdots))).$$

It belongs to

$$\Pi = \mathcal{Y}^e \oplus \mathcal{Y}^{s_1}.$$

Symmetry of q-characters

- $\mathcal{Y}$ embeds in Π diagonally.

Theorem (Frenkel-H. 2022)

The Θ_i define a Weyl group action and

$$\mathcal{Y}^W = Im(\chi_q).$$

Interpretation of the Weyl group action

Theorem (Frenkel-H. 2015)

The replacement of the variables :

$$Y_{i,a} = \frac{[L^+_{i,aq^{-1}}]}{[L^+_{i,aq}]}$$

in the q-character of finite-dimensional representation gives a well-defined relation in (the fraction field) of $K(\mathcal{O})$.

- Interpretation of the operator Θ_i : replace the class $[L^+_{i,a}]$ by the class of another simple representation $\tilde{L}^+_{i,a}$.

QQ-system

- For a general $w \in W$, $i \in I$, we introduce
$$Q_a^{w(\omega_i^\vee)} \in K(\mathcal{O}).$$

Theorem (Frenkel-H. 2023)

The series $Q_a^{w(\omega_i^\vee)}$ satisfy the QQ-system

$$Q_{aq}^{(ws_i)(\omega_i^\vee)} Q_{aq^{-1}}^{w(\omega_i^\vee)} - Q_{aq^{-1}}^{(ws_i)(\omega_i^\vee)} Q_{aq}^{w(\omega_i^\vee)} = \prod_{j | C_{i,j} = -1} Q_a^{w(\omega_j^\vee)}.$$

- Motivations from quantum integrable models.
- Example ($\mathfrak{g} = sl_2$) : Quantum Wronskian relation :
$$\tilde{Q}_{aq} Q_{aq^{-1}} - \tilde{Q}_{aq^{-1}} Q_{aq} = 1,$$
for $Q_a = Q_a^{\omega_1^\vee}$ and $\tilde{Q}_a = Q_a^{-\omega_1^\vee}$.
- Crucial point of the construction of the cluster algebra structure.

Quantization of Grothendieck rings

- Many interesting additional symmetries of Grothendieck rings.
- In particular : in relation to their quantizations.
- Fujita-H. (2023) : theory of Monoidal Jantzen filtrations for generically braided monoidal categories.
- Distinguished filtrations by submodules :

$$M = M_0 \supset M_1 \supset \cdots \supset M_N = \{0\}$$

which lead to deformed classes :

$$[M]_t = \sum_{r \geq 0} t^r [M_r / M_{r-1}],$$

and a deformation of the product of the Grothendieck ring.