Representations of quantum affine algebras at the crossroads between cluster categorification and quantum integrable models.

David Hernandez
U. Paris Cité
IMJ-PRG

ICM - Philadelphia - July 2026

Introduction

Drinfeld 1986 ICM contribution :

Introduction

Drinfeld 1986 ICM contribution :

- "Quantum groups"

Introduction

Drinfeld 1986 ICM contribution :

- "Quantum groups"
- *This is a report on recent works on Hopf algebras (or quantum groups, which is more or less the same) motivated by (...) a method for constructing and studying integrable quantum systems.*

Introduction

- This talk : recent developments of the theory in two (related) directions :

Introduction

- This talk : recent developments of the theory in two (related) directions :
- **Quantum integrable models** QIM (original motivation).

Introduction

- This talk : recent developments of the theory in two (related) directions :
- **Quantum integrable models** QIM (original motivation).
- Categorification of **cluster algebras** (new subject).

Introduction

$$
\begin{tikzcd}
 & \text{Cluster Algebras} & \\
\text{Quantum Affine Algebras} \arrow[ur, dashed, "\text{Grothendieck ring}"] \arrow[dr, dashed, "\text{Transfer-matrices}"'] & & \text{T/TQ/QQ-relations} \arrow[ul, dashed, leftrightarrow, "\text{Mutation relations}"'] \\
 & \text{QIM} \arrow[ur, dashed, "\text{Bethe Ansatz}"'] &
\end{tikzcd}
$$

Introduction

$$\begin{tikzcd}
 & \text{Cluster Algebras} \arrow[dr, dashed, leftrightarrow, "\text{Mutation relations}"] & \\
\textcolor{red}{\text{Quantum Affine Algebras}} \arrow[ur, dashed, "\text{Grothendieck ring}"] \arrow[dr, dashed, "\text{Transfer-matrices}"'] & & \text{T/TQ/QQ-relations} \\
 & \text{QIM} \arrow[ur, dashed, "\text{Bethe Ansatz}"'] &
\end{tikzcd}$$

Quantum groups

- Main examples of quantum groups : quantized enveloping algebras (Drinfeld, Jimbo).

Quantum groups

- Main examples of quantum groups : quantized enveloping algebras (Drinfeld, Jimbo).
- $\mathfrak{g} = Lie(G)$ Lie algebra of G complex simple Lie group.

Quantum groups

- Main examples of quantum groups : quantized enveloping algebras (Drinfeld, Jimbo).
- $\mathfrak{g} = Lie(G)$ Lie algebra of G complex simple Lie group.
- For instance
$$\mathfrak{sl}_2(\mathbb{C}) = Lie(SL_2(\mathbb{C})).$$

Quantum groups

- Main examples of quantum groups : quantized enveloping algebras (Drinfeld, Jimbo).
- $\mathfrak{g} = Lie(G)$ Lie algebra of G complex simple Lie group.
- For instance
$$\mathfrak{sl}_2(\mathbb{C}) = Lie(SL_2(\mathbb{C})).$$
- For simplicity of notations : A, D, E types (results hold for general types). I : set of simple roots of $\mathfrak{g}$.

Quantum groups

- $q \in \mathbb{C}^*$: quantum parameter (not root of unity).

Quantum groups

- $q \in \mathbb{C}^*$: quantum parameter (not root of unity).
- $\mathcal{U}_q(\mathfrak{g})$: q-deformation of $\mathcal{U}(\mathfrak{g})$.

Quantum groups

- $q \in \mathbb{C}^*$: quantum parameter (not root of unity).
- $\mathcal{U}_q(\mathfrak{g})$: q-deformation of $\mathcal{U}(\mathfrak{g})$.
- For instance $\mathcal{U}_q(\mathfrak{sl}_2) =< E, F, K^{\pm 1} >$ with relations

$$KE = q^2 EK \ , \ FK = q^{-2} KF \ , \ EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

Quantum groups

- $q \in \mathbb{C}^*$: quantum parameter (not root of unity).
- $\mathcal{U}_q(\mathfrak{g})$: q-deformation of $\mathcal{U}(\mathfrak{g})$.
- For instance $\mathcal{U}_q(\mathfrak{sl}_2) =< E, F, K^{\pm 1} >$ with relations

$$KE = q^2 EK \ , \ FK = q^{-2} KF \ , \ EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

- Hopf algebra with many remarkable properties.

Quantum groups

- $q \in \mathbb{C}^*$: quantum parameter (not root of unity).
- $\mathcal{U}_q(\mathfrak{g})$: q-deformation of $\mathcal{U}(\mathfrak{g})$.
- For instance $\mathcal{U}_q(\mathfrak{sl}_2) =< E, F, K^{\pm 1} >$ with relations

$$KE = q^2 EK \ , \ FK = q^{-2} KF \ , \ EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

- Hopf algebra with many remarkable properties.
- Originally motivated in Physics, found applications and connections in various fields of Mathematics.

Quantum affine algebra

- Even more interesting : quantum groups from *infinite dimensional* Lie groups.

Quantum affine algebra

- Even more interesting : quantum groups from *infinite dimensional* Lie groups.
- Loop group LG of regular loops
$$\text{Unit circle} \rightarrow G.$$

Quantum affine algebra

- Even more interesting : quantum groups from *infinite dimensional* Lie groups.
- Loop group LG of regular loops

$$\text{Unit circle} \rightarrow G.$$

- Its Lie algebra : the loop algebra

$$\mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}].$$

Quantum affine algebra

- Even more interesting : quantum groups from *infinite dimensional* Lie groups.
- Loop group LG of regular loops

$$\text{Unit circle} \rightarrow G.$$

- Its Lie algebra : the loop algebra

$$\mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}].$$

- Associated Drinfeld-Jimbo quantum group : quantum loop algebra (or quantum affine algebra) :

$$\mathcal{U}_q(\mathcal{L}\mathfrak{g})$$

Quantum affine algebra

- $\mathcal{C}$: Category of finite-dimensional representations of $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$.

Quantum affine algebra

- $\mathcal{C}$: Category of finite-dimensional representations of $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$.

- Very interesting (and intricated) category.

Quantum affine algebra

- $\mathcal{C}$: Category of finite-dimensional representations of $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$.
- Very interesting (and intricated) category.
- Studied intensively by various methods in connection to various fields.

Quantum affine algebra

- $\mathcal{C}$: Category of finite-dimensional representations of $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$.
- Very interesting (and intricated) category.
- Studied intensively by various methods in connection to various fields.
- Tensor category, but not semi-simple and not braided.

Quantum affine algebra

- $\mathcal{C}$: Category of finite-dimensional representations of $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$.
- Very interesting (and intricated) category.
- Studied intensively by various methods in connection to various fields.
- Tensor category, but not semi-simple and not braided.
- There are examples of V, W simples in $\mathcal{C}$ with

$$V \otimes W \not\cong W \otimes V.$$

Quantum affine algebra

Theorem (Chari-Pressley, 92)

Simple finite-dimensional representations of $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ *(of type* 1*) are parameterized by n-tuple of complex polynomials*

$$(P_i(z))_{i\in I}$$

with $P_i(0) = 1$ *(Drinfeld polynomials).*

Quantum affine algebra

Theorem (Chari-Pressley, 92)

Simple finite-dimensional representations of $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ (of type 1) are parameterized by n-tuple of complex polynomials

$$(P_i(z))_{i\in I}$$

with $P_i(0) = 1$ (Drinfeld polynomials).

- Analogs of highest weight for a large commutative subalgebra $\mathcal{U}_q(\mathcal{L}\mathfrak{h}) \subset \mathcal{U}_q(\mathcal{L}\mathfrak{g})$.

Theorem (Chari-Pressley, 92)

Simple finite-dimensional representations of $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ (of type 1) are parameterized by n-tuple of complex polynomials

$$(P_i(z))_{i\in I}$$

with $P_i(0) = 1$ (Drinfeld polynomials).

- Analogs of highest weight for a large commutative subalgebra $\mathcal{U}_q(\mathcal{L}\mathfrak{h}) \subset \mathcal{U}_q(\mathcal{L}\mathfrak{g})$.
- Classification of undecomposable modules not known (even for $\mathfrak{sl}_2$).

Quantum affine algebra

Theorem (Chari-Pressley, 92)

Simple finite-dimensional representations of $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ (of type 1) are parameterized by n-tuple of complex polynomials

$$(P_i(z))_{i\in I}$$

with $P_i(0) = 1$ (Drinfeld polynomials).

- Analogs of highest weight for a large commutative subalgebra $\mathcal{U}_q(\mathcal{L}\mathfrak{h}) \subset \mathcal{U}_q(\mathcal{L}\mathfrak{g})$.
- Classification of undecomposable modules not known (even for $\mathfrak{sl}_2$).
- Dimension (or characters) of simple modules not known in general.

Fundamental representations

Example

For $i \in I$ and $a \in \mathbb{C}^*$, the fundamental representation $V_i(a)$ in $\mathcal{C}$ defined by

$$(1, \cdots, 1, \underbrace{1 - za}_{\text{position } i}, 1, \cdots, 1).$$

Fundamental representations

Example

For $i \in I$ and $a \in \mathbb{C}^*$, the fundamental representation $V_i(a)$ in $\mathcal{C}$ defined by

$$(1, \cdots, 1, \underbrace{1 - za}_{\text{position } i}, 1, \cdots, 1).$$

Theorem (Chari, Varagnolo-Vasserot, Kashiwara)

Any simple in $\mathcal{C}$ is a quotient of a tensor product of fundamental representations.

Grothendieck ring

- The Grothendieck ring of $\mathcal{C}$ is the free abelian group

$$K(\mathcal{C}) = \bigoplus_{V \text{ simple in } \mathcal{C}} \mathbb{Z}[V].$$

Grothendieck ring

- The Grothendieck ring of $\mathcal{C}$ is the free abelian group

$$K(\mathcal{C}) = \bigoplus_{V \text{ simple in } \mathcal{C}} \mathbb{Z}[V].$$

- The multiplication induced by tensor product : structure constants = multiplicities of simples in tensor products of simples.

Grothendieck ring

- The Grothendieck ring of $\mathcal{C}$ is the free abelian group

$$K(\mathcal{C}) = \bigoplus_{V \text{ simple in } \mathcal{C}} \mathbb{Z}[V].$$

- The multiplication induced by tensor product : structure constants = multiplicities of simples in tensor products of simples.

Theorem (Frenkel-Reshetikhin, 99)

The fundamental classes $[V_i(a)]$ commute and freely generate the Grothendieck ring $K(\mathcal{C})$.

Grothendieck ring

- The Grothendieck ring of $\mathcal{C}$ is the free abelian group

$$K(\mathcal{C}) = \bigoplus_{V \text{ simple in } \mathcal{C}} \mathbb{Z}[V].$$

- The multiplication induced by tensor product : structure constants = multiplicities of simples in tensor products of simples.

Theorem (Frenkel-Reshetikhin, 99)

The fundamental classes $[V_i(a)]$ commute and freely generate the Grothendieck ring $K(\mathcal{C})$.

- In particular, $K(\mathcal{C})$ is commutative !

T-systems

- Not much is known on the decomposition of tensor products of simples.

T-systems

- Not much is known on the decomposition of tensor products of simples.
- Crucial example : Kirillov-Reshetikhin modules $W_{k,a}^{(i)}$ associated to

$$(1, \cdots, 1, \underbrace{(1 - za)(1 - zaq^{-2}) \cdots (1 - zaq^{2-2k})}_{\text{position } i}, 1, \cdots, 1)$$

Here $k \geq 0$, $a \in \mathbb{C}^*$, $i \in I$.

T-systems

- Not much is known on the decomposition of tensor products of simples.
- Crucial example : Kirillov-Reshetikhin modules $W_{k,a}^{(i)}$ associated to

$$(1, \cdots, 1, \underbrace{(1-za)(1-zaq^{-2})\cdots(1-zaq^{2-2k})}_{\text{position } i}, 1, \cdots, 1)$$

Here $k \geq 0$, $a \in \mathbb{C}^*$, $i \in I$.

Theorem (Kirillov-Reshetikhin conjecture, Nakajima 2004, H. 2006)

We have the T-system in $K(\mathcal{C})$:

$$[W_{k,a}^{(i)}][W_{k,aq^2}^{(i)}] = [W_{k+1,a}^{(i)}][W_{k-1,aq^2}^{(i)}] + \prod_{j | C_{i,j}=-1} [W_{k,aq}^{(j)}].$$

T-systems

- Not much is known on the decomposition of tensor products of simples.
- Crucial example : Kirillov-Reshetikhin modules $W_{k,a}^{(i)}$ associated to

$$(1, \cdots, 1, \underbrace{(1 - za)(1 - zaq^{-2}) \cdots (1 - zaq^{2-2k})}_{\text{position } i}, 1, \cdots, 1)$$

Here $k \geq 0$, $a \in \mathbb{C}^*$, $i \in I$.

Theorem (Kirillov-Reshetikhin conjecture, Nakajima 2004, H. 2006)

We have the T-system in $K(\mathcal{C})$:

$$[W_{k,a}^{(i)}][W_{k,aq^2}^{(i)}] = [W_{k+1,a}^{(i)}][W_{k-1,aq^2}^{(i)}] + \prod_{j | C_{i,j} = -1} [W_{k,aq}^{(j)}].$$

- This is the decomposition of $W_{k,a}^{(i)} \otimes W_{k,aq^2}^{(i)}$ into simples in $K(\mathcal{C})$.

Quantum integrable model

$$\begin{tikzcd}
 & \text{Cluster Algebras} \arrow[dr, dashed, leftrightarrow, "\text{Mutation relations}"] & \\
\text{Quantum Affine Algebras} \arrow[ur, dashed, "\text{Grothendieck ring}"] \arrow[dr, dashed, "\text{Transfer-matrices}"'] & & \text{T/TQ/QQ-relations} \\
 & \text{QIM} \arrow[ur, dashed, "\text{Bethe Ansatz}"'] &
\end{tikzcd}$$

Quantum integrable model

- A quantum affine algebra has a universal R-matrix

$$\mathcal{R}(z) \in (\mathcal{U}_q(\mathcal{L}\mathfrak{g})\hat{\otimes}\mathcal{U}_q(\mathcal{L}\mathfrak{g}))[[z]]$$

solution of the quantum Yang-Baxter equation (braid relation).

Quantum integrable model

- A quantum affine algebra has a universal R-matrix

$$\mathcal{R}(z) \in (\mathcal{U}_q(\mathcal{L}\mathfrak{g})\hat{\otimes}\mathcal{U}_q(\mathcal{L}\mathfrak{g}))[[z]]$$

solution of the quantum Yang-Baxter equation (braid relation).

- For V finite-dimensional representation, the transfer-matrix is

$$\mathcal{T}_V(z) = (Tr_V \otimes \mathrm{Id})(\mathcal{R}(z)) \in \mathcal{U}_q(\mathcal{L}\mathfrak{g})[[z]].$$

Quantum integrable model

- A quantum affine algebra has a universal R-matrix

$$\mathcal{R}(z) \in (\mathcal{U}_q(\mathcal{L}\mathfrak{g})\hat{\otimes}\mathcal{U}_q(\mathcal{L}\mathfrak{g}))[[z]]$$

  solution of the quantum Yang-Baxter equation (braid relation).

- For V finite-dimensional representation, the transfer-matrix is

$$\mathcal{T}_V(z) = (Tr_V \otimes \mathrm{Id})(\mathcal{R}(z)) \in \mathcal{U}_q(\mathcal{L}\mathfrak{g})[[z]].$$

- Integrability : the transfer-matrices commute ! For V, V' in $\mathcal{C}$:

$$[\mathcal{T}_V(z), \mathcal{T}_{V'}(w)] = 0.$$

Quantum integrable model

- Let W be a representation of $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$: the quantum space.

Quantum integrable model

- Let W be a representation of $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$: the quantum space.
- The transfer-matrices induce commuting operators on $W[[z]]$:

$$\mathcal{T}_V(z) \curvearrowright W[[z]].$$

Quantum integrable model

- Let W be a representation of $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$: the quantum space.
- The transfer-matrices induce commuting operators on $W[[z]]$:

$$\mathcal{T}_V(z) \curvearrowright W[[z]].$$

- Crucial example : $\mathfrak{g} = \mathfrak{sl}_2$, W tensor product of fundamental modules : original XXZ-model spin chain.

Spectra of quantum integrable models

- In general: eigenvalues = spectrum of the model. Description ?

Spectra of quantum integrable models

- In general: eigenvalues = spectrum of the model. Description ?
- Representation theoretical method : can be described in terms of q-characters.

Spectra of quantum integrable models

- In general: eigenvalues = spectrum of the model. Description ?
- Representation theoretical method : can be described in terms of q-characters.
- Frenkel-Reshetikhin : V in $\mathcal{C}$ has a q-character

$$\chi_q(V) \in \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^*}.$$

Spectra of quantum integrable models

- In general: eigenvalues = spectrum of the model. Description ?
- Representation theoretical method : can be described in terms of q-characters.
- Frenkel-Reshetikhin : V in $\mathcal{C}$ has a q-character

$$\chi_q(V) \in \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^*}.$$

- Encodes common generalized eigenspaces for $\mathcal{U}_q(\mathcal{L}\mathfrak{h}) \subset \mathcal{U}_q(\mathcal{L}\mathfrak{g})$.

Spectra of quantum integrable models

- In general: eigenvalues = spectrum of the model. Description ?
- Representation theoretical method : can be described in terms of q-characters.
- Frenkel-Reshetikhin : V in $\mathcal{C}$ has a q-character

$$\chi_q(V) \in \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^*}.$$

- Encodes common generalized eigenspaces for $\mathcal{U}_q(\mathcal{L}\mathfrak{h}) \subset \mathcal{U}_q(\mathcal{L}\mathfrak{g})$.

Example

For $\mathfrak{g} = \mathfrak{sl}_2$

$$\chi_q(V_1(a)) = Y_{1,a} + Y_{1,aq^2}^{-1}.$$

Spectra of quantum integrable models

Theorem (Frenkel-H. 15, proof of the Frenkel-Reshetikhin QIM conjecture)

Each eigenvalue of $\mathcal{T}_V(z)$ *on* $W[[z]]$ *is obtained from* $\chi_q(V)$ *by replacing*

$$Y_{i,a} \mapsto \frac{f_i(azq^{-1})Q_i(zaq^{-1})}{f_i(azq)Q_i(zaq)}$$

where Q_i *is polynomial and* f_i *is universal (it depends only on* W*).*

Spectra of quantum integrable models

Theorem (Frenkel-H. 15, proof of the Frenkel-Reshetikhin QIM conjecture)

Each eigenvalue of $\mathcal{T}_V(z)$ on $W[[z]]$ is obtained from $\chi_q(V)$ by replacing

$$Y_{i,a} \mapsto \frac{f_i(azq^{-1})Q_i(zaq^{-1})}{f_i(azq)Q_i(zaq)}$$

where Q_i is polynomial and f_i is universal (it depends only on W).

- Extends the original results of Baxter for the XXZ-model : eigenvalues of the form

$$\frac{f(azq^{-1})Q(zaq^{-1})}{f(azq)Q(zaq)} + \frac{f(azq^3)Q(zaq^3)}{f(azq)Q(zaq)}.$$

Prefundamental representations

- The proof relies on the existence of prefundamental representations (H.-Jimbo, 12) : Infinite-dimensional representations of the Borel

$$\mathcal{U}_q(\hat{\mathfrak{b}}) \subset \mathcal{U}_q(\mathcal{L}\mathfrak{g}).$$

Prefundamental representations

- The proof relies on the existence of prefundamental representations (H.-Jimbo, 12) : Infinite-dimensional representations of the Borel

$$\mathcal{U}_q(\hat{\mathfrak{b}}) \subset \mathcal{U}_q(\mathcal{L}\mathfrak{g}).$$

- Fix $i \in I$ and $a \in \mathbb{C}^*$.

  $L_{i,a}^{\mathfrak{b},+}$ positive prefundamental.

  $L_{i,a}^{\mathfrak{b},-}$ negative prefundamental.

Prefundamental representations

- The proof relies on the existence of prefundamental representations (H.-Jimbo, 12) : Infinite-dimensional representations of the Borel

$$\mathcal{U}_q(\hat{\mathfrak{b}}) \subset \mathcal{U}_q(\mathcal{L}\mathfrak{g}).$$

- Fix $i \in I$ and $a \in \mathbb{C}^*$.

  $L_{i,a}^{\mathfrak{b},+}$ positive prefundamental.

  $L_{i,a}^{\mathfrak{b},-}$ negative prefundamental.

- Play the role of fundamental representations for a category $\mathcal{O}$ of $\mathcal{U}_q(\hat{\mathfrak{b}})$-modules.

Prefundamental representations

Theorem (Frenkel-H. 15, Generalized Baxter TQ-relations)

Let V in $\mathcal{C}$. Replace the q-character $\chi_q(V)$ by the class $[V]$ and each

$$Y_{i,a} \mapsto [L^{\mathfrak{b},+}_{i,aq^{-1}}]/[L^{\mathfrak{b},+}_{i,aq}].$$

We obtain a relation in the fraction field of $K(\mathcal{O})$.

Prefundamental representations

Theorem (Frenkel-H. 15, Generalized Baxter TQ-relations)

Let V in $\mathcal{C}$. Replace the q-character $\chi_q(V)$ by the class $[V]$ and each

$$Y_{i,a} \mapsto [L^{\mathfrak{b},+}_{i,aq^{-1}}]/[L^{\mathfrak{b},+}_{i,aq}].$$

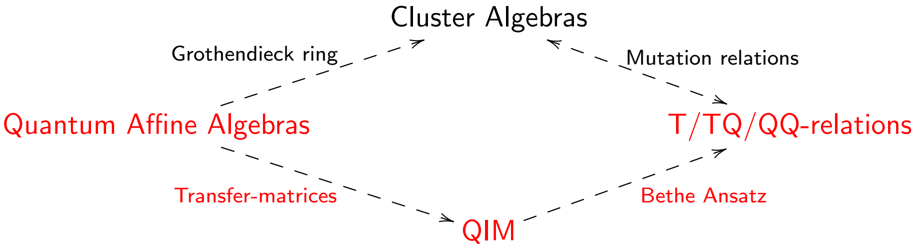
We obtain a relation in the fraction field of $K(\mathcal{O})$.

Example

For $\mathfrak{g} = \mathfrak{sl}_2$, we obtain the original Baxter TQ-relation

$$[V_1(a)] = \frac{[L^{\mathfrak{b},+}_{1,aq^{-1}}] + [L^{\mathfrak{b},+}_{1,aq^3}]}{[L^{\mathfrak{b},+}_{1,aq}]}.$$

Bethe Ansatz and relations



ODE/IM and Bethe Ansatz

- QQ-system for two families of functions $(Q_i(z))_{i\in I}$, $(\tilde{Q}_i(z))_{i\in I}$ is

$$Q_i(zq^{-1})\tilde{Q}_i(zq) - Q_i(zq)\tilde{Q}_i(zq^{-1}) = \prod_{j|C_{i,j}=-1} Q_j(z).$$

ODE/IM and Bethe Ansatz

- QQ-system for two families of functions $(Q_i(z))_{i\in I}$, $(\tilde{Q}_i(z))_{i\in I}$ is

$$Q_i(zq^{-1})\tilde{Q}_i(zq) - Q_i(zq)\tilde{Q}_i(zq^{-1}) = \prod_{j | C_{i,j}=-1} Q_j(z).$$

- From affine opers (differential systems) : Masoero-Raimundo-Valeri.

ODE/IM and Bethe Ansatz

- QQ-system for two families of functions $(Q_i(z))_{i\in I}$, $(\tilde{Q}_i(z))_{i\in I}$ is

$$Q_i(zq^{-1})\tilde{Q}_i(zq) - Q_i(zq)\tilde{Q}_i(zq^{-1}) = \prod_{j | C_{i,j}=-1} Q_j(z).$$

- From affine opers (differential systems) : Masoero-Raimundo-Valeri.
- ODE/QIM correspondence conjectural reformulation by Feigin-Frenkel.

$$\begin{tikzcd}
\boxed{\text{Spectra of } \hat{\mathfrak{g}}\text{-QIM}} \arrow[rr, leftrightarrow, dashed, "Feigin-Frenkel", "\text{ODE/QIM}"'] \arrow[dr, dashed, "?"'] & & \boxed{{}^L\hat{\mathfrak{g}}\text{-affine opers}} \arrow[dl, "MRV"] \\
& \boxed{\begin{array}{c}\text{solutions of} \\ \text{the } QQ\text{-system}\end{array}} &
\end{tikzcd}$$

ODE/IM and Bethe Ansatz

Theorem (Frenkel-H. 18, QQ-system)

There is a family of simple classes in $K(\mathcal{O})$ that satisfy the QQ-system.

ODE/IM and Bethe Ansatz

Theorem (Frenkel-H. 18, QQ-system)

There is a family of simple classes in $K(\mathcal{O})$ that satisfy the QQ-system.

Example

In the sl_2-case, the QQ-system is the quantum Wronskian relation

$$[L_{1,a}^{\mathfrak{b},+}][L_{1,a}^{\mathfrak{b},-}] - [L_{1,aq^2}^{\mathfrak{b},+}][L_{1,aq^{-2}}^{\mathfrak{b},-}] = \text{constant}$$

considered by Bazhanov-Lukyanov-Zalomodchikov.

ODE/IM and Bethe Ansatz

- The QQ-systems imply the Bethe Ansatz equations for (generic) zeroes w of Baxter polynomials Q_j.

ODE/IM and Bethe Ansatz

- The QQ-systems imply the Bethe Ansatz equations for (generic) zeroes w of Baxter polynomials Q_j.

- $$\prod_{j\in I}\frac{Q_j(wq^{C_{i,j}})}{Q_j(wq^{-C_{i,j}})} = -1.$$

Cluster algebra symmetries

- Many remarkable relations in the Grothendieck ring $K(\mathcal{O})$ related to Bethe Ansatz relations :

Cluster algebra symmetries

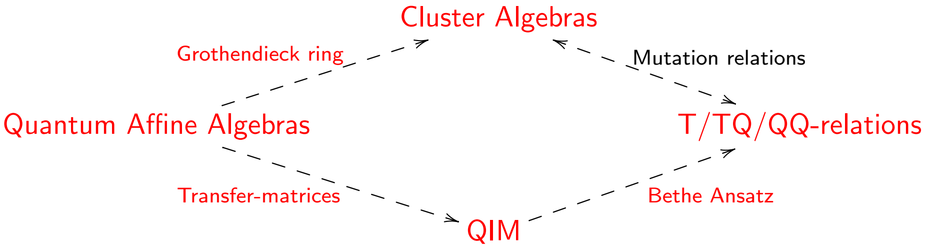
- Many remarkable relations in the Grothendieck ring $K(\mathcal{O})$ related to Bethe Ansatz relations :
-

$$\begin{tikzcd}
TQ\text{-relations} \arrow[dr, dashed, leftrightarrow] & & QQ\text{-relations} \arrow[dl, dashed, leftrightarrow] \\
& \text{Bethe Ansatz for spectra of QIM} \arrow[d, dashed, leftrightarrow] & \\
& T\text{-systems} &
\end{tikzcd}$$

Cluster algebra symmetries

- Many remarkable relations in the Grothendieck ring $K(\mathcal{O})$ related to Bethe Ansatz relations :
- $$\begin{tikzcd}
TQ\text{-relations} \arrow[dr, leftrightarrow, dashed] & & QQ\text{-relations} \arrow[dl, leftrightarrow, dashed] \\
& \text{Bethe Ansatz for spectra of QIM} \arrow[d, leftrightarrow, dashed] & \\
& T\text{-systems} &
\end{tikzcd}$$
- Can be organized as the shadow of *cluster algebras*.

Cluster algebra symmetries



Cluster algebra symmetries

- Theory of cluster algebras : Fomin-Zelevinsky (2000).

Cluster algebra symmetries

- Theory of cluster algebras : Fomin-Zelevinsky (2000).
- Commutative algebra with a distinguished set of generators.

Cluster algebra symmetries

- Theory of cluster algebras : Fomin-Zelevinsky (2000).
- Commutative algebra with a distinguished set of generators.
- Certain Grothendieck rings admit a cluster algebra structure.

Cluster algebra symmetries

- Theory of cluster algebras : Fomin-Zelevinsky (2000).
- Commutative algebra with a distinguished set of generators.
- Certain Grothendieck rings admit a cluster algebra structure.
- D.H. 2024 ECM talk :

  Symmetries of Grothendieck rings in representation theory.

Cluster algebras (quick review)

- Cluster algebra $\mathcal{A}_Q$ attached to a quiver Q (oriented graph) with set of vertices Q_0.

Cluster algebras (quick review)

- Cluster algebra $\mathcal{A}_Q$ attached to a quiver Q (oriented graph) with set of vertices Q_0.
- $\mathcal{A}_Q$: subalgebra of $\mathcal{F} = \mathbb{Q}(X_i)_{i\in Q_0}$ generated by cluster variables.

Cluster algebras (quick review)

- Cluster algebra $\mathcal{A}_Q$ attached to a quiver Q (oriented graph) with set of vertices Q_0.
- $\mathcal{A}_Q$: subalgebra of $\mathcal{F} = \mathbb{Q}(X_i)_{i \in Q_0}$ generated by cluster variables.
- Initial cluster variables : X_i $(i \in Q_0)$.

Cluster algebras (quick review)

- Cluster algebra $\mathcal{A}_Q$ attached to a quiver Q (oriented graph) with set of vertices Q_0.
- $\mathcal{A}_Q$: subalgebra of $\mathcal{F} = \mathbb{Q}(X_i)_{i \in Q_0}$ generated by cluster variables.
- Initial cluster variables : X_i $(i \in Q_0)$.
- New cluster variables : obtained by mutations :

$$X_i X_i^* = \prod_{j \to i} X_j + \prod_{j \leftarrow i} X_j.$$

Cluster algebras (quick review)

- Cluster algebra $\mathcal{A}_Q$ attached to a quiver Q (oriented graph) with set of vertices Q_0.
- $\mathcal{A}_Q$: subalgebra of $\mathcal{F} = \mathbb{Q}(X_i)_{i \in Q_0}$ generated by cluster variables.
- Initial cluster variables : X_i $(i \in Q_0)$.
- New cluster variables : obtained by mutations :

$$X_i X_i^* = \prod_{j \to i} X_j + \prod_{j \leftarrow i} X_j.$$

- The cluster variables grouped into overlapping subsets : the clusters.

Cluster algebras (quick review)

- Cluster algebra $\mathcal{A}_Q$ attached to a quiver Q (oriented graph) with set of vertices Q_0.
- $\mathcal{A}_Q$: subalgebra of $\mathcal{F} = \mathbb{Q}(X_i)_{i \in Q_0}$ generated by cluster variables.
- Initial cluster variables : X_i $(i \in Q_0)$.
- New cluster variables : obtained by mutations :

$$X_i X_i^* = \prod_{j \to i} X_j + \prod_{j \leftarrow i} X_j.$$

- The cluster variables grouped into overlapping subsets : the clusters.
- Cluster monomials : monomials in the cluster variables from one cluster.

Monoidal categorification of cluster algebras

Definition (H.-Leclerc 10, Monoidal categorification)

A monoidal category $\mathcal{M}$ is a *monoidal categorification* of a cluster algebra $\mathcal{A}$ if there exists a ring isomorphism

$$\phi : \mathcal{A} \to K(\mathcal{M})$$

so that the cluster monomials are sent to simple classes.

Monoidal categorification of cluster algebras

Definition (H.-Leclerc 10, Monoidal categorification)

A monoidal category $\mathcal{M}$ is a *monoidal categorification* of a cluster algebra $\mathcal{A}$ if there exists a ring isomorphism

$$\phi : \mathcal{A} \to K(\mathcal{M})$$

so that the cluster monomials are sent to simple classes.

- This can also be seen as a cluster symmetry of the Grothendieck ring.

Monoidal categorification of cluster algebras

Definition (H.-Leclerc 10, Monoidal categorification)

A monoidal category $\mathcal{M}$ is a *monoidal categorification* of a cluster algebra $\mathcal{A}$ if there exists a ring isomorphism

$$\phi : \mathcal{A} \to K(\mathcal{M})$$

so that the cluster monomials are sent to simple classes.

- This can also be seen as a cluster symmetry of the Grothendieck ring.
- It gives useful information on the monoidal category (factorization of representations, geometric character formulas...).

Monoidal categorification of cluster algebras

Definition (H.-Leclerc 10, Monoidal categorification)

A monoidal category $\mathcal{M}$ is a *monoidal categorification* of a cluster algebra $\mathcal{A}$ if there exists a ring isomorphism

$$\phi : \mathcal{A} \to K(\mathcal{M})$$

so that the cluster monomials are sent to simple classes.

- This can also be seen as a cluster symmetry of the Grothendieck ring.
- It gives useful information on the monoidal category (factorization of representations, geometric character formulas...).
- Need a "starting point" : known relations in the Grothendieck ring $->$ then a global understanding of the ring.

Monoidal categorification - examples

- Original examples (H.-Leclerc, 2010) : from the finite-dimensional representations of quantum affine algebras.

Monoidal categorification - examples

- Original examples (H.-Leclerc, 2010) : from the finite-dimensional representations of quantum affine algebras.

- Many developments about monoidal categorifications : Nakajima, Qin, Kashiwara-Kim-Oh-Park, Bittmann, Brito-Chari, Cautis-Williams...

Monoidal categorification - examples

- Original examples (H.-Leclerc, 2010) : from the finite-dimensional representations of quantum affine algebras.
- Many developments about monoidal categorifications : Nakajima, Qin, Kashiwara-Kim-Oh-Park, Bittmann, Brito-Chari, Cautis-Williams...
- State of the art result for finite-dimensional representations can be nicely formulated in terms of *shifted quantum affine algebras*.

Shifted quantum affine algebras

- **Shifted** quantum affine algebras introduced by Finkelberg-Tsymbaliuk in the study of Coulomb branches (of Braverman-Finkelberg-Nakajima).

Shifted quantum affine algebras

- **Shifted** quantum affine algebras introduced by Finkelberg-Tsymbaliuk in the study of Coulomb branches (of Braverman-Finkelberg-Nakajima).
- $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$: variations of quantum affine algebras depending on a shift parameter : $\mu \in P^\vee$ in the integral coweight lattice of $\mathfrak{g}$.

Shifted quantum affine algebras

- **Shifted** quantum affine algebras introduced by Finkelberg-Tsymbaliuk in the study of Coulomb branches (of Braverman-Finkelberg-Nakajima).
- $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$: variations of quantum affine algebras depending on a shift parameter : $\mu \in P^\vee$ in the integral coweight lattice of $\mathfrak{g}$.
- Construction : formal power series of Drinfeld generators are shifted by a power $z^{\alpha_i(\mu)}$ depending on the shift μ.

Shifted quantum affine algebras

- **Shifted** quantum affine algebras introduced by Finkelberg-Tsymbaliuk in the study of Coulomb branches (of Braverman-Finkelberg-Nakajima).
- $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$: variations of quantum affine algebras depending on a shift parameter : $\mu \in P^\vee$ in the integral coweight lattice of $\mathfrak{g}$.
- Construction : formal power series of Drinfeld generators are shifted by a power $z^{\alpha_i(\mu)}$ depending on the shift μ.
- Motivation for us : action of $\mathcal{U}_q(\hat{\mathfrak{b}})$ on $L_{i,a}^{\mathfrak{b},-}$ negative prefundamental module can be extended to $\mathcal{U}_q^{-\omega_i^\vee}(\hat{\mathfrak{g}})$. Notation : $L_{i,a}^-$.

Representations of shifted quantum affine algebras

Theorem (H. 2020)

$\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ *has non-zero finite-dimensional representation if and only if* μ *is codominant.*

Representations of shifted quantum affine algebras

Theorem (H. 2020)

$\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ *has non-zero finite-dimensional representation if and only if* μ *is codominant.*

- Example for $\mu = \omega_i^\vee$ fundamental coweight.
  A continuous family of 1-dimensional representations: $L_{i,a}^+$ ($a \in \mathbb{C}^*$). Positive prefundamental representations.

Representations of shifted quantum affine algebras

Theorem (H. 2020)

$\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ has non-zero finite-dimensional representation if and only if μ is codominant.

- Example for $\mu = \omega_i^\vee$ fundamental coweight.
  A continuous family of 1-dimensional representations: $L_{i,a}^+$ ($a \in \mathbb{C}^*$). Positive prefundamental representations.

- For any coweight μ, $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ has an abelian category $\mathcal{O}^\mu$.

Grothendieck rings

- Individually, each $\mathcal{O}^\mu$ is not monoidal.

Grothendieck rings

- Individually, each $\mathcal{O}^{\mu}$ is not monoidal.
- But the sum of Grothendieck groups

$$K_0(\mathcal{O}^{sh}) = \bigoplus_{\mu \in P^{\vee}} K_0(\mathcal{O}^{\mu})$$

has a ring structure.

Grothendieck rings

- Individually, each $\mathcal{O}^\mu$ is not monoidal.
- But the sum of Grothendieck groups

$$K_0(\mathcal{O}^{sh}) = \bigoplus_{\mu \in P^\vee} K_0(\mathcal{O}^\mu)$$

has a ring structure.

- Finite-dimensional representations of shifted quantum affine algebras.

$$\mathcal{C} \subset \mathcal{C}^{sh} \subset \mathcal{O}^{sh}.$$

Grothendieck rings

- Individually, each $\mathcal{O}^\mu$ is not monoidal.
- But the sum of Grothendieck groups

$$K_0(\mathcal{O}^{sh}) = \bigoplus_{\mu \in P^\vee} K_0(\mathcal{O}^\mu)$$

has a ring structure.

- Finite-dimensional representations of shifted quantum affine algebras.

$$\mathcal{C} \subset \mathcal{C}^{sh} \subset \mathcal{O}^{sh}.$$

- Grothendieck rings

$$K_0(\mathcal{C}) \subset K_0(\mathcal{C}^{sh}) \subset K_0(\mathcal{O}^{sh})$$

Cluster structure - finite-dimensional representations

- H.-Leclerc 16 : an infinite quiver Γ_∞ for any $\mathfrak{g}$.

Cluster structure - finite-dimensional representations

- H.-Leclerc 16 : an infinite quiver Γ_∞ for any $\mathfrak{g}$.
- Vertices : $I \times \mathbb{Z}$ with arrows

$$(i, r) \to (j, s) \text{ iff } (C_{i,j} \neq 0 \text{ and } s = r + C_{i,j}).$$

Cluster structure - finite-dimensional representations

Theorem (H.-Leclerc 16, Kashiwara-Kim-Oh-Park 20, H. 20)

There is a ring isomorphism

$$K_0(\mathcal{C}^{sh}) \simeq \mathcal{A}_{\Gamma_\infty}.$$

Initial cluster variables : classes of positive prefundamental representations.

The cluster monomials correspond to certain classes of simple modules.

Cluster structure - finite-dimensional representations

Theorem (H.-Leclerc 16, Kashiwara-Kim-Oh-Park 20, H. 20)

There is a ring isomorphism

$$K_0(\mathcal{C}^{sh}) \simeq \mathcal{A}_{\Gamma_\infty}.$$

Initial cluster variables : classes of positive prefundamental representations.

The cluster monomials correspond to certain classes of simple modules.

- Remark : T-systems interpreted as mutation relations (starting point).

Cluster structure - finite-dimensional representation

- Example : $\mathfrak{g} = \mathfrak{sl}_2$. Infinite linear quiver :

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

Cluster structure - finite-dimensional representation

- Example : $\mathfrak{g} = \mathfrak{sl}_2$. Infinite linear quiver :

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

- Initial seed of 1-dimensional representations

$$\cdots \longrightarrow L^+_{1,q^{-2}} \longrightarrow L^+_{1,1} \longrightarrow L^+_{1,q^2} \longrightarrow \cdots$$

Cluster structure - finite-dimensional representation

- Example : $\mathfrak{g} = \mathfrak{sl}_3$

$$\begin{tikzcd}
\cdots \arrow[rr] & & L^+_{1,q-2} \arrow[rr] \arrow[dl] & & L^+_{1,1} \arrow[rr] \arrow[dl] & & \cdots \\
\cdots \arrow[r] & L^+_{2,q-3} \arrow[rr] \arrow[ul] & & L^+_{2,q-1} \arrow[rr] \arrow[ul] & & L^+_{2,q} \arrow[r] \arrow[ul] & \cdots
\end{tikzcd}$$

Cluster structure - category $\mathcal{O}^{sh}$

- What about the whole category $\mathcal{O}^{sh}$?

Cluster structure - category $\mathcal{O}^{sh}$

- What about the whole category $\mathcal{O}^{sh}$?
- New families of quivers Γ'_∞ attached to each $\mathfrak{g}$ (Geiss-H.-Leclerc).

Cluster structure - category $\mathcal{O}^{sh}$

- What about the whole category $\mathcal{O}^{sh}$?
- New families of quivers Γ'_∞ attached to each $\mathfrak{g}$ (Geiss-H.-Leclerc).

Theorem (Geiss-H.-Leclerc 23, 26)

The Grothendieck ring $K_0(\mathcal{O}^{sh})$ is isomorphic to (a completion of) $\mathcal{A}_{\Gamma'_\infty}$.

Cluster structure - category $\mathcal{O}^{sh}$

- What about the whole category $\mathcal{O}^{sh}$?
- New families of quivers Γ'_∞ attached to each $\mathfrak{g}$ (Geiss-H.-Leclerc).

Theorem (Geiss-H.-Leclerc 23, 26)

The Grothendieck ring $K_0(\mathcal{O}^{sh})$ is isomorphic to (a completion of) $\mathcal{A}_{\Gamma'_\infty}$.

- Remark : the QQ-systems interpreted as mutation relations (starting point).

Cluster algebra symmetries

$$
\begin{tikzcd}
 & \text{Cluster Algebras} \arrow[dr, dashed, leftrightarrow, "\text{Mutation relations}"] & \\
\text{Quantum Affine Algebras} \arrow[ur, dashed, "\text{Grothendieck ring}"] \arrow[dr, dashed, "\text{Transfer-matrices}"'] & & \text{T/TQ/QQ-relations} \\
 & \text{QIM} \arrow[ur, dashed, "\text{Bethe Ansatz}"'] &
\end{tikzcd}
$$

Cluster structure - category $\mathcal{O}^{sh}$

- Example : $\mathfrak{g} = \mathfrak{sl}_2$

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

Cluster structure - category $\mathcal{O}^{sh}$

- Example : $\mathfrak{g} = \mathfrak{sl}_2$

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

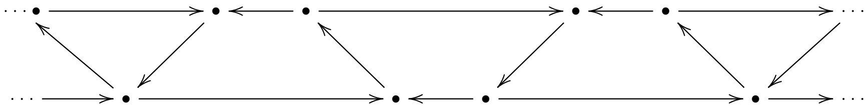
- Initial cluster variables :

$$\cdots \longrightarrow L^+_{1,q^2} \longrightarrow L^+_{1,1} \longleftarrow L^-_{1,q^{-2}} \longrightarrow L^-_{1,q^{-4}} \longrightarrow \cdots$$

Cluster structure - category $\mathcal{O}^{sh}$

- Example $\mathfrak{g} = \mathfrak{sl}_3$.

- The quiver $\Gamma'_{\mathfrak{sl}_3}$ is also built from the periodic quiver $\Gamma_{\mathfrak{sl}_3}$, except that it contains 3 quadrilaterals :



Conjecture

Conjecture (Geiss-H.-Leclerc 23, 26)

The cluster monomials in $\mathcal{A}_{\Gamma'_\infty}$ correspond to simple classes in $\mathcal{O}^{sh}$.

Conjecture

Conjecture (Geiss-H.-Leclerc 23, 26)

The cluster monomials in $\mathcal{A}_{\Gamma'_\infty}$ *correspond to simple classes in* $\mathcal{O}^{sh}$.

Theorem (Geiss-H.-Leclerc 23)

The conjecture is true for $\mathfrak{g} = sl_2$.