3-facets of R-matrices: stable maps, intertwiners and transfer-matrices

David Hernandez

Our first lecture was devoted to present some of the important applications of R-matrices. Recall an R-matrix $\mathcal{R}(z) \in \mathcal{A}^{\otimes 2}((z))$ is a solution of the Yang-Baxter equation

$$\mathcal{R}_{12}(z)\mathcal{R}_{13}(zw)\mathcal{R}_{23}(w) = \mathcal{R}_{23}(w)\mathcal{R}_{13}(zw)\mathcal{R}_{12}(z)$$

for $\mathcal{A}$ an algebra, z, w formal variables,

$$\mathcal{R}_{12} = \mathcal{R} \otimes 1 \ , \ \mathcal{R}_{23} = 1 \otimes \mathcal{R} \ , \ \mathcal{R}_{13} = (\tau \otimes Id)(\mathcal{R}_{23}),$$

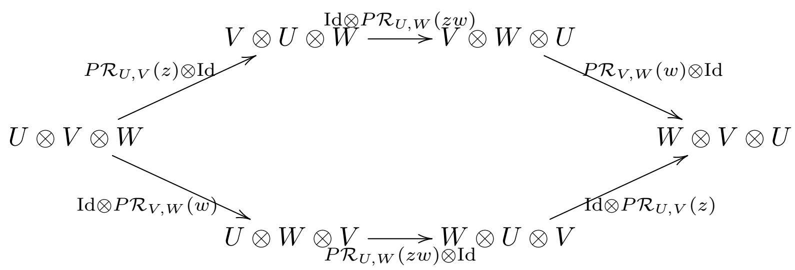
where τ is the flip. We recalled three of the various contexts in which R-matrices appear (see [H1]) :

(i) Intertwiners for quantum groups (representation theory). The specialization of the universal R-matrix $\mathcal{R}(z)$ of a quantum affine algebra to a tensor product of representations $U \otimes V$ gives rise, after composition by the flip, to an isomorphism of (deformed) representations :

$$\mathcal{R}_{U,V}(z) : U \otimes V(z) \to V(z) \otimes U.$$

The Yang-Baxter equation guarantees the hexagonal identity is satisfied :

(1)



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For U, V simple representations, after a proper renormalization and by taking the limit $z \to 1$, it gives rise to a (non necessarily invertible) morphism

$$R_{U,V} : U \otimes V \to V \otimes U.$$

(ii) Construction of transfer-matrices for quantum integrable models (mathematical physics). For V a finite-dimensional representation of a quantum affine algebra, we have the associated transfer-matrix :

$$T_V(z) = (Tr_V \otimes \mathrm{Id})\mathcal{R}(z).$$

The Yang-Baxter equation implies the integrability condition

$$[T_V(z), T_{V'}(w)] = 0,$$

for two such representations V, V'. Then, for another representation W, we obtain a commuting families of operators on W whose spectra constitute the spectrum of the corresponding quantum integrable model.

(iii) Maulik-Okounkov construction of stable maps (symplectic geometry). Maulik-Okounkov [MO] have presented a very general construction of R-matrices from the

action of a pair of tori $A \subset T$ on a symplectic variety X. In its cohomological version, the construction gives the stable maps, morphisms of $H_T^\bullet(pt)$-modules

$$\mathrm{Stab}_{\mathcal{C}} : H_T^\bullet(X^A) \to H_T^\bullet(X)$$

depending in particular on a certain chamber $\mathcal{C}$. In good situations, for two chambers $\mathcal{C}$ and $\mathcal{C}'$, we obtain two stable maps:

$$\begin{array}{ccc} & H_T^\bullet(X) & \\ \nearrow^{\mathrm{Stab}_{\mathcal{C}}} & & \nwarrow^{\mathrm{Stab}_{\mathcal{C}'}} \\ H_T^\bullet(X^A) & \xrightarrow[\mathcal{R}_{\mathcal{C}',\mathcal{C}}]{} & H_T^\bullet(X^A) \end{array},$$

invertible after a proper localization, so that $\mathrm{Stab}_{\mathcal{C}'}^{-1}\mathrm{Stab}_{\mathcal{C}}$ gives an R-matrix.

Our second lecture was entitled,

Baxter polynomials and truncated shifted quantum affine algebras

We explained the application of polynomiality of Q-operators to representations of truncated shifted quantum affine algebras (quantized K-theoretical Coulomb branches). Shifted quantum affine algebras and their truncations arose [FT] in the study of quantized K-theoretic Coulomb branches of 3d $N = 4$ SUSY quiver gauge theories in the sense of Braverman-Finkelberg-Nakajima which are at the center of current important developments. The Q-operators are transfer-matrices associated to prefundamental representations of the Borel subalgebra of a quantum affine algebra, via the standard R-matrix construction. In a joint work [FH] with E. Frenkel, we have proved that, up to a scalar multiple, they act polynomially on simple finite-dimensional representations of a quantum affine algebra. This establishes the existence of Baxter polynomial in a general setting.

We propose a conjectural parameterization of simple modules of a non simply-laced truncation in terms of the Langlands dual quantum affine Lie algebra (this has various motivations, including the symplectic duality relating Coulomb branches and quiver varieties). We prove [H2] that a simple finite-dimensional representation of a shifted quantum affine algebra descends to a truncation as predicted by this conjecture. This is derived from the existence of Baxter polynomials.

References

[FH] E. Frenkel and D. Hernandez, *Baxter's Relations and Spectra of Quantum Integrable Models*, Duke Math. J. **164** (2015), no. 12, 2407–2460.

[FT] M. Finkelberg and A. Tsymbaliuk, *Multiplicative slices, relativistic Toda and shifted quantum affine algebras*, in Progr. Math. **330** (2019), 133–304.

[H1] D. Hernandez, *Avancées concernant les R-matrices et leurs applications*, Sem. Bourbaki **1129**, Astérisque 407 (2019), 297–331.

[H2] D. Hernandez, *Representations of shifted quantum affine algebras*, Preprint arXiv:2010.06996.

[MO] D. Maulik and A. OKounkov, *Quantum Groups and Quantum Cohomology*, Astérisque **408** (2019).