

Cluster algebras and shifted quantum groups

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Shifted quantum affine algebras arose [FT] in the study of quantized K -theoretic Coulomb branches of 3d $N = 4$ SUSY quiver gauge theories. A presentation of shifted quantum affine algebras by generators and relations was given by Finkelberg-Tsymbaliuk.

Let $\mathfrak{g}$ be a simple complex finite-dimensional Lie algebra of rank n , and $\hat{\mathfrak{g}}$ the corresponding untwisted affine Kac-Moody algebra, central extension of the loop algebra $\mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]$. Drinfeld and Jimbo associated to each complex number $q \in \mathbb{C}^*$ the quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ which is a Hopf algebra. Shifted quantum affine algebras $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ can be seen as variations of $\mathcal{U}_q(\hat{\mathfrak{g}})$, but depending on an integral coweight μ of the underlying simple Lie algebra $\mathfrak{g}$. These coweights corresponding to shifts of formal power series in the Cartan-Drinfeld elements (that is quantum analogs of the $t^r h \in \mathcal{L}\mathfrak{g}$, with $r \in \mathbb{Z}$ and $h \in \mathfrak{h}$ in the Cartan subalgebra of $\mathfrak{g}$). In particular $\mathcal{U}_q^0(\hat{\mathfrak{g}})$ is a central extension of the ordinary quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$.

In [H1], we develop the representation theory of shifted quantum affine algebras. We establish several analogies with the representation theory of ordinary quantum affine algebras, but also several new properties, in particular in relations to representations of the ordinary Borel subalgebra $\mathcal{U}_q(\mathfrak{b}) \subset \mathcal{U}_q(\hat{\mathfrak{g}})$. Consider a category $\mathcal{O}_\mu$ of representations of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ which is an analog of the ordinary category $\mathcal{O}$.

Theorem 1 [H1] *The simple representations $L(\Psi)$ in $\mathcal{O}_\mu$ are parametrized by n -tuples $\Psi = (\Psi_i(z))_{1 \leq i \leq n}$ of rational fractions regular at 0 with $\deg(\Psi_i(z)) = \alpha_i(\mu)$.*

For the moment, we assume that $\mathfrak{g}$ is of simply-laced type.

For $1 \leq i \leq n$ and $a \in \mathbb{C}^*$, let $\Psi_{i,a} = (1 - \delta_{i,j}za)_{j \in I}$ and $\omega_i^\vee$ a fundamental coweight. We have the following examples.

- $L_{i,a}^+ = L(\Psi_{i,a})$ is a one-dimensional representation of $\mathcal{U}_q^{\omega_i^\vee}(\mathfrak{g})$ called a positive prefundamental representation,
- $L_{i,a}^- = L(\Psi_{i,a}^{-1})$ is an infinite-dimensional representation of $\mathcal{U}_q^{-\omega_i^\vee}(\mathfrak{g})$ called a negative prefundamental representation
- $V_{i,a} = L(\Psi_{i,aq^{-1}} \Psi_{i,aq}^{-1})$ is a finite-dimensional fundamental representation of the ordinary quantum affine algebra (up to a twist).

Although individually each $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ is not a Hopf algebra, one can define a ring structure on the sum of Grothendieck groups

$$K_0(\mathcal{O}) = \bigoplus_{\mu} K_0(\mathcal{O}_\mu)$$

from a procedure called fusion product. It contains the Grothendieck ring $K_0(\mathcal{C}^{sh})$ of finite-dimensional representations as a subring.

For example, for $\mathfrak{g} = sl_2$ we have the QQ -relation in $K_0(\mathcal{O})$:

$$[L_{1,a}^+][L_{1,a}^-] = 1 + [L(q^{-2})][L_{aq^{-2}}^-][L_{aq^2}^+],$$

with $L(q^{-2})$ one-dimensional representation associated to a constant ℓ -weight.

Theorem 2 [FH] *Analog QQ -relations hold in $K_0(\mathcal{O}^{sh})$ for general $\mathfrak{g}$, involving new elements $Q_{w(\omega_i),a} \in K(\mathcal{O}^{sh})$, where $1 \leq i \leq n$, w is a Weyl group element and $a \in \mathbb{C}^*$.*

The $Q_{w(\omega_i),a}$ will be called Q -variables. We have

$$Q_{\omega_i,a} = [L_{i,a}^+] \text{ and } Q_{-\omega_i,a} = [L_{i,aq^{-h}}^-],$$

with h the Coxeter number of $\mathfrak{g}$. In type A_2 , we have

$$Q_{s_1(\omega_1),a} = Q_{\omega_2-\omega_1,a} = L(\Psi_{2,aq^{-1}} \Psi_{1,aq^{-2}}^{-1}).$$

Recall that the cluster algebra $\mathcal{A}(Q)$ attached to a quiver Q is a commutative ring with a distinguished set of generators called cluster variables and obtained inductively by a procedure (mutations and exchange relations) from initial cluster variables. The cluster variables are grouped into overlapping subsets (the clusters). A cluster monomial is a monomial in cluster variables from a given cluster.

It is very useful to realize Grothendieck rings in terms of cluster algebras (see [H2] for a recent review).

In the following, we will restrict to subcategories $\mathcal{O}_{\mathbb{Z}}^{sh} \subset \mathcal{O}^{sh}$ and $\mathcal{C}_{\mathbb{Z}}^{sh} \subset \mathcal{C}^{sh}$ with an integrality conditions on possible simple constituents (the description of such subcategories is sufficient to understand the whole category).

Theorem 3 [HL, KKOP, H1] *The Grothendieck ring $K_0(\mathcal{C}_{\mathbb{Z}}^{sh})$ has a structure of a cluster algebra $\mathcal{A}(\Gamma_{\mathfrak{g}})$ for an explicit quiver $\Gamma_{\mathfrak{g}}$ with initial cluster variables which are classes of positive prefundamental representations. Moreover, all cluster monomials are classes simple objects.*

In the $\mathfrak{sl}_2$ -case, the quiver $\Gamma_{\mathfrak{sl}_2}$ is the infinite linear quiver :

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

The initial cluster is formed of classes of 1-dimensional positive prefundamental representations

$$\cdots \longrightarrow L_{1,q^{-2}}^+ \longrightarrow L_{1,1}^+ \longrightarrow L_{1,q^2}^+ \longrightarrow \cdots$$

For $\mathfrak{g} = \mathfrak{sl}_3$, the quiver $\Gamma_{\mathfrak{sl}_3}$ and the initial cluster are

$$\begin{array}{ccccccc} \cdots & \longrightarrow & [L_{1,q^{-2}}^+] & \longrightarrow & [L_{1,1}^+] & \longrightarrow & \cdots \\ & \swarrow & & \swarrow & & \swarrow & \\ \cdots & \longrightarrow & [L_{2,q^{-3}}^+] & \longrightarrow & [L_{2,q^{-1}}^+] & \longrightarrow & [L_{2,q}^+] \longrightarrow \cdots \end{array}$$

Theorem 4 [GHL1] *The Grothendieck ring $K_0(\mathcal{O}_{\mathbb{Z}}^{sh})$ is isomorphic to (a completion of) a cluster algebra $\mathcal{A}_{\Gamma'_{\infty}}$, with an explicit quiver Γ'_{∞} . The Q -variables*

are identified with cluster variables and the QQ -system above are identified with exchange relations.

One of the crucial technical step in the proof of this Theorem is the convergence of certain g -vectors of cluster variables for certain infinite sequences of mutations, that we establish. This relies in part on a nice partition of the quiver in slices.

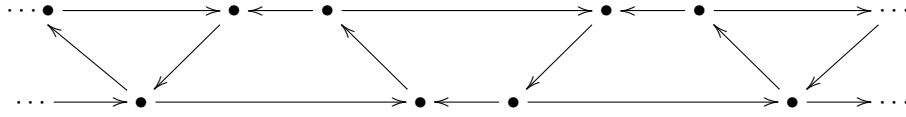
For $\mathfrak{g} = \mathfrak{sl}_2$, the new quiver Γ'_∞ is obtained from the older quiver Γ_∞ by inverting the direction of one arrow :

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

The initial variables are positive/negative prefundamental representations :

$$\cdots \longrightarrow L_{1,q^2}^+ \longrightarrow L_{1,1}^+ \longleftarrow L_{1,q^{-2}}^- \longrightarrow L_{1,q^{-4}}^- \longrightarrow \cdots$$

For $\mathfrak{g} = \mathfrak{sl}_3$, the quiver $\Gamma'_{\mathfrak{sl}_3}$ is also built from the periodic quiver $\Gamma_{\mathfrak{sl}_3}$, except that it contains 3 quadrilaterals :



The initial cluster variables are the $[L_{1,q^{2r}}^+]$, $[L_{2,q^{2r-1}}^+]$, $[L_{1,q^{2s}}^-]$, $[L_{2,q^{2s-1}}^-]$, with $r \geq 0$, $s \leq -2$, as well as $[Q_{s_1(\omega_1),q^{-1}}]$ and $[Q_{s_1(\omega_1),q^{-3}}]$.

In general, we conjecture the following, that we establish for $\mathfrak{g} = \mathfrak{sl}_2$ in [GHL1].

Conjecture [GHL1] *All cluster monomials in $\mathcal{A}_{\Gamma'_\infty}$ correspond to classes of simple objects in $\mathcal{O}$ through our isomorphism.*

In [GHL2] we investigate the non-simply laced types. There are several important differences, for example the nice partition of the quiver into slices does not work here, and we need difference methods to establish the stabilization of g -vectors.

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