QUANTUM PERIODICITY AND KIRILLOV-RESHETIKHIN MODULES

DAVID HERNANDEZ

ABSTRACT. We give a proof of the periodicity of quantum T-systems of type $A_n \times A_\ell$ with certain spiral boundary conditions. Our proof is based on categorification of the T-system in terms of the representation theory of quantum affine algebras, more precisely on relations between classes of Kirillov-Reshetikhin modules and of evaluation modules.

To Nicolai Reshetikhin on his 60th birthday

CONTENTS

1. INTRODUCTION

The Q-system was introduced by Kirillov and Reshetikhin [KR] as a system of relations between characters of certain simple finite-dimensional representations of the quantum affine algebra $\mathcal{U}_q(\hat{sl}_{n+1})$, now called Kirillov-Reshetikhin modules

$$Q_{a,b}^2 = Q_{a-1,b}Q_{a+1,b} + Q_{a,b+1}Q_{a,b-1},$$

where $1 \leq a \leq n$ and b is a non-negative integer. Inspired by this work, the T-system was written in [KNS1] as a refined version of the Q-system depending on a spectral parameter u :

$$T_{a,b}(u-1)T_{a,b}(u+1) = T_{a-1,b}(u)T_{a+1,b}(u) + T_{a,b+1}(u)T_{a,b-1}(u).$$

It was conjectured that it is satisfied by the classes of the Kirillov-Reshetikhin modules.

In a fundamental work [FR], Frenkel and Reshetikhin introduced a character theory for finite-dimensional representations of quantum affine algebras, called the q-characters. Then the T-system is satisfied by q-characters of Kirillov-Reshetikhin modules in all types [N2, H2], and so by their classes as conjectured above.

In another direction, Zamolodchikov initiated in [Z] a long series of work on the periodicity of solutions of T-systems with certain boundary conditions, which culminated in the work of Keller [Kel2] with a very general uniform proof of the periodicity of T-systems associated to a pair (Δ, Δ') of Dynkin diagrams (see [IIKNS, Kel1] for reviews and references).

In this note we propose a simple proof of the periodicity (and half-periodicity) of T-systems of type $A_n \times A_\ell$ and of its quantum version (in the sense of [N2, HL2]), with certain spiral boundary conditions (more general than the unit condition usually considered). We follow the approach in [HL1, Section 12.1] where the proof of the commutative periodicity in type $A_n \times A_1$ is obtained with formulas for solutions in terms of q-characters. Indeed we find solutions in terms of certain evaluation representations, containing Kirillov-Reshetikhin modules but not only, and more precisely in terms of their q,t-characters defined by Nakajima [N1].

The quantum periodicity (and half-periodicity) established in this note should also follow from the analog results in the commutative case (with unit boundary condition) mentioned above and from results in [BZ, CKLP] (indeed the approach in [Kel2] is based on the study of the periodicity of a sequence of mutations in a certain cluster algebra). Our direct method gives an explicit solution in terms of q,t-characters.

The paper is organized as follows. In Section 2 we state the main periodicity and quantum periodicity results and we give several examples. In Section 3 we give the necessary reminders on the representation theory of quantum affine algebras. In Section 4 we recall how the T-system appears in the Grothendieck ring of the category of representations and we prove it has also other incarnations. We conclude in Section 5 with the proof of the quantum periodicity.

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2. Periodicity and quantum periodicity

In this section we state the periodicity and quantum periodicity of T-systems that we establish in this note.

2.1. **Periodicity.** Let us first state the commutative $A_n \times A_\ell$-periodicity. Let

$$I = \{1, \cdots, n\} \text{ and } J = \{1, \cdots, \ell\}.$$

We work on the lattice

$$\Lambda = \{(a, b, u) \in I \times J \times \mathbb{Z} | a + b + u \in 2\mathbb{Z}\}.$$

Let us consider a family of commuting variables $(T_{a,b}(u))_{(a,b,u)\in\Lambda}$ satisfying the T-system (sometimes called octahedron relation) :

$$T_{a,b}(u-1)T_{a,b}(u+1) = T_{a-1,b}(u)T_{a+1,b}(u) + T_{a,b+1}(u)T_{a,b-1}(u),$$

for any $(a, b, u + 1) \in \Lambda$.

So that the system is well-defined, we have to fix the boundary conditions, that is the values of

$$T_{0,b}(u)\ ,\ T_{a,\ell+1}(u)\ ,\ T_{n+1,b}(u) \text{ and } T_{a,0}(u).$$

The first choice is to set all values to 1, this is called the unit boundary condition (see [IIKNS]). The system is already non trivial with such a choice. We will also consider the following boundary condition. Let $(\mathcal{F}_r)_{r\in I}$ be formal variables that we call coefficients. We also set $\mathcal{F}_0 = \mathcal{F}_{n+1} = 1$. We set the following boundary conditions (for u modulo $2(\ell + n + 2)$) :

$$T_{n+1,m}(u) = \begin{cases} \mathcal{F}_{\frac{u+n+m+3}{2}} & \text{for } -n-3 \leq u+m \leq n-1 \\ 1 & \text{for } 0 \leq u+m-n+1 \leq 2\ell+2. \end{cases}$$

$$T_{0,m}(u) = \begin{cases} \mathcal{F}_{\frac{u-m+2}{2}} & \text{for } -2 \leq u-m \leq 2n, \\ 1 & \text{for } 0 \leq u-m-2n \leq 2(\ell+1). \end{cases}$$

$$T_{k,\ell+1}(u) = \begin{cases} \mathcal{F}_{\frac{u-k-\ell+1}{2}} & \text{for } 0 \leq u-\ell-k+1 \leq 2n+2, \\ 1 & \text{for } \ell+1 \leq u-2n-k \leq 3(\ell+1). \end{cases}$$

$$T_{k,0}(u) = \begin{cases} \mathcal{F}_{\frac{u+k+2}{2}} & \text{for } -2 \leq u+k \leq 2n, \\ 1 & \text{for } 0 \leq u-2n+k \leq 2(\ell+1). \end{cases}$$

This is a particular case of the spiral boundary condition (see [IIKNS]).

Remark 2.1. *For $(a, b, u) \in \Lambda$, an induction on $u \geq a + b - 2$ shows that $T_{a,b}(u)$ is a rational fraction in the*

$$X_{k,m} = T_{k,m}(k + m - 2) \text{ for } (k, m) \in I \times J$$

and in the coefficients

$$X_{k,0} = T_{k,0}(k - 2) = \mathcal{F}_k.$$

We have the following periodicity.

Theorem 2.2. *For any $(a, b, u) \in \Lambda$, we have the half-periodicity property :*

$$T_{a,b}(u) = T_{n+1-a,\ell+1-b}(u + n + \ell + 2).$$

It implies that $T_{a,b}(u)$ is $2(n + \ell + 2)$-periodic in u.

Example 2.3. *Let $n = \ell = 1$. The non trivial boundary conditions are*

$$T_{0,1}(1) = T_{1,2}(3) = T_{2,1}(5) = T_{1,0}(7) = \mathcal{F}_1.$$

Set $X = X_{1,1} = T_{1,1}(0)$.
The values of $T_{1,1}(u)$ for $u = 0, 2, 4, 6, 8$ are respectively

$$X\ ,\ \frac{\mathcal{F}_1 + 1}{X}\ ,\ X\ ,\ \frac{\mathcal{F}_1 + 1}{X}\ ,\ X.$$

Example 2.4. *Let $n = 1$ and $\ell = 2$. The non trivial boundary conditions are*

$$T_{0,1}(1) = T_{0,2}(2) = T_{1,3}(4) = T_{2,2}(6) = T_{2,1}(7) = T_{1,0}(9) = \mathcal{F}_1.$$

Set $X_1 = X_{1,1} = T_{1,1}(0)$ and $X_2 = X_{1,2} = T_{1,2}(1)$.
The values of $T_{1,1}(t)$ for $t = 0, 2, 4, 6, 8, 10$ are

$$X_1 \ , \ \frac{\mathcal{F}_1 + X_2}{X_1} \ , \ \frac{X_1 + 1}{X_2} \ , X_2 \ , \ \frac{\mathcal{F}_1 X_1 + \mathcal{F}_1 + X_2}{X_1 X_2} \ , X_1.$$

The values of $T_{1,2}(u)$ for $u = 1, 3, 5, 7, 9, 11$ are respectively

$$X_2 \ , \ \frac{\mathcal{F}_1 X_1 + \mathcal{F}_1 + X_2}{X_1 X_2} \ , X_1 \ , \ \frac{\mathcal{F}_1 + X_2}{X_1} \ , \ \frac{X_1 + 1}{X_2} \ , X_2.$$

Example 2.5. *Let $n = 2$ and $\ell = 1$. The non trivial boundary conditions are*

$$T_{0,1}(1) = T_{1,2}(3) = T_{2,2}(4) = T_{3,1}(6) = T_{2,0}(8) = T_{1,0}(9) = \mathcal{F}_1$$

$$T_{1,0}(1) = T_{0,1}(3) = T_{1,2}(5) = T_{2,2}(6) = T_{3,1}(8) = T_{2,0}(11) = \mathcal{F}_2$$

Set $X_1 = X_{1,1} = T_{1,1}(0)$ and $X_2 = X_{2,1} = T_{2,1}(1)$.
The values of $T_{1,1}(t)$ for $t = 0, 2, 4, 6, 8, 10$ are

$$X_1 \ , \ \frac{\mathcal{F}_1 X_2 + \mathcal{F}_2}{X_1} \ , \ \frac{X_1 + \mathcal{F}_2}{X_2} \ , X_2 \ , \ \frac{X_1 + \mathcal{F}_2 + \mathcal{F}_1 X_2}{X_1 X_2} \ , X_1.$$

The values of $T_{2,1}(u)$ for $u = 1, 3, 5, 7, 9, 11$ are respectively

$$X_2 \ , \ \frac{X_1 + \mathcal{F}_2 + \mathcal{F}_1 X_2}{X_1 X_2} \ , X_1 \ , \ \frac{\mathcal{F}_2 + \mathcal{F}_1 X_2}{X_1} \ , \ \frac{X_1 + \mathcal{F}_2}{X_2} \ , X_2.$$

2.2. **Quantum periodicity.** Let us now the state the quantum version of the $A_n \times A_\ell$-periodicity (see also [KN] and [DFK] for $n = 1$).

We work now with quasi-commuting variables $(X_{a,b})_{(a,b)\in I\times J}$:

$$X_{a,b} * X_{c,d} = t^{\gamma(a,b;c,d)-\gamma(c,d;a,b)} X_{c,d} * X_{a,b}.$$

To define the power of t, we use the inverse $\tilde{C}(z)$ of the quantized Cartan matrix

$$C(z) = ((z + z^{-1})\delta_{i,j} - \delta_{i+1,j} - \delta_{i-1,j})_{i,j\in I}.$$

For $p \in \mathbb{Z}$ and $a, c \in I$, we denote by $\tilde{C}_{a,c}(p)$ the coefficient of z^p in the expansion in z of $\tilde{C}_{a,c}(p)$. We set

$$\gamma(a, b; c, d) = \tilde{C}_{a,c}(2\ell-2b+c-a+1)+\tilde{C}_{a,c}(2\ell-2b+c-a-1)+\cdots+\tilde{C}_{a,c}(2d-2b+c-a+1).$$

The relation is also extended to $b = 0$ or $d = 0$ so that we get the quasi-commutation rule with the coefficients. In particular for $r < r'$, we have

$$\mathcal{F}_r * \mathcal{F}_{r'} = t^{\tilde{C}_{r,r'}(2\ell+r'-r+1)+\tilde{C}_{r,r'}(2\ell+r'-r-1)+\cdots+\tilde{C}_{r,r'}(2\ell+r-r'+3)} \mathcal{F}_{r'} * \mathcal{F}_r \quad (1)$$

and $\mathcal{F}_r * \mathcal{F}_{r+1} = t^{\tilde{C}_{r,r+1}(2\ell+2)} \mathcal{F}_{r+1} * \mathcal{F}_r$. This is derived from $\tilde{C}_{i,j}(k) = 0$ if $k \leq |j - i|$ (which can be observed for example in the formula in [GTL, Appendix A.3]).

The quasi-commuting variables $(X_{a,b})_{(a,b)\in I\times J}$ with the $\mathcal{F}_t$ generate a quantum torus $\mathcal{T}_t$ over $\mathbb{Z}[t^{\pm 1/2}]$. We denote its fraction field by K_t. It has an antimultiplicative bar involution satisfying $\bar{t} = t^{-1}$ and so that the $X_{a,b}$, $\mathcal{F}_t$ are bar-invariant.

For each product m of various $X_{a,b}^{\pm 1}$, $\mathcal{F}_t^{\pm 1}$, there is a unique $\alpha \in \mathbb{Z}$ so that $t^{\alpha/2}m$ is bar invariant. This is called a commutative monomial. The commutative monomials form a $\mathbb{Z}[t^{\pm 1/2}]$-basis of $\mathcal{T}_t$.

Example 2.6. *For $n = \ell = 1$, we have :*

$$X_1 * \mathcal{F}_1 = t^2 \mathcal{F}_1 * X_1.$$

For $n = 1$, $\ell = 2$, we have :

$$X_1 * X_2 = t^{-2} X_2 * X_1 \ , \ X_1 * \mathcal{F}_1 = \mathcal{F}_1 * X_1 \ , \ X_2 * \mathcal{F}_1 = \mathcal{F}_1 * X_2.$$

For $n = 2$, $\ell = 1$, we have :

$$X_1 * X_2 = t X_2 * X_1 \ , \ X_1 * \mathcal{F}_1 = t \mathcal{F}_1 * X_1 \ , \ X_1 * \mathcal{F}_2 = \mathcal{F}_2 * X_1,$$

$$X_2 * \mathcal{F}_2 = t \mathcal{F}_2 * X_2 \ , \ X_2 * \mathcal{F}_1 = t \mathcal{F}_1 * X_2 \ , \ \mathcal{F}_1 * \mathcal{F}_2 = t^{-1} \mathcal{F}_2 * \mathcal{F}_1.$$

We fix the same the boundary conditions as for the commutative setting above.

Theorem 2.7. *Consider a family of bar-invariant $T_{a,b}(u) \in K_t$ satisfying :*

$$T_{a,b}(u-1) * T_{a,b}(u+1) \in t^{\mathbb{Z}/2} T_{a-1,b}(u) * T_{a+1,b}(u) + t^{\mathbb{Z}/2} T_{a,b+1}(u) * T_{a,b-1}(u)$$

for $(a, b, u+1) \in \Lambda$. We assume the same initial conditions as in Remark 2.1 and the same spiral boudary conditions as in the classical setting. Then for $(a, b, u) \in \Lambda$:

$$T_{a,b}(u) = T_{n+1-a,\ell+1-b}(u + n + \ell + 2).$$

It implies that $T_{a,b}(u)$ is $2(n + \ell + 2)$-periodic in u.

The classical periodicity in Theorem 2.2 follows directly from this Theorem. We propose a simple proof based on the representations theory of quantum affine algebras and on their Kirillov-Reshetikhin modules.

Example 2.8. *Let us study the examples 2.3, 2.4, 2.5 above. In these examples, let us just replace each Laurent monomial in the $X_{a,b}$ by the corresponding commutative monomial in the quantum torus. We get bar-invariant elements in $\mathcal{T}_t$ and we keep the notation $T_{a,b}(u)$. Let us verify they satisfy the quantum T-system.*

Let $n = \ell = 1$. We get :

$$T_{1,1}(0) * T_{1,1}(2) = t\mathcal{F}_1 + 1, \quad T_{1,1}(2) * T_{1,1}(4) = t^{-1}\mathcal{F}_1 + 1.$$

Let $n = 1$, $\ell = 2$.

$$\begin{aligned} T_{1,1}(0) * T_{1,1}(2) &= \mathcal{F}_1 + t^{-1} T_{1,2}(1), & T_{1,2}(1) * T_{1,2}(3) &= \mathcal{F}_1 + t^{-1} T_{1,1}(2), \\ T_{1,1}(2) * T_{1,1}(4) &= 1 + t^{-1} T_{1,2}(3), & T_{1,2}(3) * T_{1,2}(5) &= 1 + t^{-1} \mathcal{F}_1 * T_{1,1}(4), \\ T_{1,1}(4) * T_{1,1}(6) &= 1 + t^{-1} T_{1,2}(5), & T_{1,2}(5) * T_{1,2}(7) &= \mathcal{F}_1 + t^{-1} T_{1,1}(6), \\ T_{1,1}(6) * T_{1,1}(8) &= \mathcal{F}_1 + t^{-1} T_{1,2}(7), & T_{1,2}(7) * T_{1,2}(9) &= 1 + t^{-1} T_{1,1}(8), \\ T_{1,1}(8) * T_{1,1}(10) &= 1 + t^{-1} T_{1,2}(9) * \mathcal{F}_1, & T_{1,2}(9) * T_{1,2}(11) &= 1 + t^{-1} T_{1,1}(10). \end{aligned}$$

Let $n=2$, $\ell=1$.

$$
\begin{aligned}
T_{1,1}(0)*T_{1,1}(2) &= t^{\frac{1}{2}}T_{2,1}(1)*\mathcal{F}_1+\mathcal{F}_2, & T_{2,1}(1)*T_{2,1}(3) &= tT_{1,1}(2)+1,\\
T_{1,1}(2)*T_{1,1}(4) &= t^{\frac{3}{2}}T_{2,1}(3)*\mathcal{F}_2+\mathcal{F}_1, & T_{2,1}(3)*T_{2,1}(5) &= t^{\frac{1}{2}}T_{1,1}(4)+t^{-\frac{1}{2}}\mathcal{F}_1,\\
T_{1,1}(4)*T_{1,1}(6) &= t^{\frac{1}{2}}T_{2,1}(5)+t^{-\frac{1}{2}}\mathcal{F}_2, & T_{2,1}(5)*T_{2,1}(7) &= t^{\frac{3}{2}}\mathcal{F}_1*T_{1,1}(6)+\mathcal{F}_2,\\
T_{1,1}(6)*T_{1,1}(8) &= tT_{2,1}(7)+1, & T_{2,1}(7)*T_{2,1}(9) &= t^{\frac{1}{2}}\mathcal{F}_2*T_{1,1}(8)+\mathcal{F}_1,\\
T_{1,1}(8)*T_{1,1}(10) &= t^{\frac{1}{2}}T_{2,1}(9)+t^{-\frac{1}{2}}\mathcal{F}_1, & T_{2,1}(9)*T_{2,1}(11) &= t^{\frac{1}{2}}T_{1,1}(10)+t^{-\frac{1}{2}}\mathcal{F}_2.
\end{aligned}
$$

3. Finite-dimensional representations of quantum affine algebras

We recall the main definitions and properties of finite-dimensional representations of the quantum affine algebra associated to sl_{n+1}.

3.1. **Quantum affine algebras.** All vector spaces, algebras and tensor products are defined over $\mathbb{C}$.

Let $C=(C_{i,j})_{0\leq i,j\leq n}$ be the Cartan matrix of type $A_n^{(1)}$, that is

$$C_{i,j}=2\delta_{i,j}-\delta_{i,j+1}-\delta_{i+1,j}$$

where $n+1$ is identified with 0. Fix $q\in\mathbb{C}^*$ which is not a root of unity.

The *quantum affine algebra* $\mathcal{U}_q(\mathfrak{g})$ is defined by generators $k_i^{\pm 1}$, $x_i^{\pm}$ $(0\leq i\leq n)$ and relations

$$k_ik_j=k_jk_i\ ,\ k_ix_j^{\pm}=q^{\pm C_{i,j}}x_j^{\pm}k_i\ ,\ [x_i^+,x_j^-]=\delta_{i,j}\frac{k_i-k_i^{-1}}{q-q^{-1}},$$

$$\sum_{p=0\cdots 1-C_{i,j}}(-1)^p(x_i^{\pm})^{(1-C_{i,j}-p)}x_j^{\pm}(x_i^{\pm})^{(p)}=0\ (\text{for } i\neq j),$$

where we denote $\left(x_i^{\pm}\right)^{(p)}=\left(x_i^{\pm}\right)^p/[p]_q$ for $0\leq p\leq 2$, where $[p]_q=(q^p-q^{-p})(q-q^{-1})^{-1}$.

It is a Hopf algebra with a coproduct $\Delta:\mathcal{U}_q(\mathfrak{g})\to\mathcal{U}_q(\mathfrak{g})\otimes\mathcal{U}_q(\mathfrak{g})$ defined for $0\leq i\leq n$ by

$$\Delta(k_i)=k_i\otimes k_i\ ,\ \Delta(x_i^+)=x_i^+\otimes 1+k_i\otimes x_i^+\ ,\ \Delta(x_i^-)=x_i^-\otimes k_i^{-1}+1\otimes x_i^-.$$

Let $\overline{\mathfrak{g}}=sl_{n+1}$ be the finite-dimensional simple Lie algebra of Cartan matrix $(C_{i,j})_{i,j\in I}$. We denote respectively by ω_i, α_i, $\alpha_i^\vee$ $(i\in I)$ the fundamental weights, the simple roots and the simple coroots of $\overline{\mathfrak{g}}$. We use the standard partial ordering $\leq$ on the weight lattice P of $\overline{\mathfrak{g}}$.

The algebra $\mathcal{U}_q(\mathfrak{g})$ has another set of generators, the *Drinfeld generators*, denoted by

$$x_{i,m}^{\pm}\ ,\ k_i^{\pm 1}\ ,\ h_{i,r}\ ,\ c^{\pm 1/2}\text{ for } i\in I,\ m\in\mathbb{Z},\ r\in\mathbb{Z}\setminus\{0\}.$$

We have $x_i^{\pm}=x_{i,0}^{\pm}$ for $i\in I$. A complete set of relations for Drinfeld generators was obtained in [B, D]. In particular the multiplication defines a surjective linear morphism

$$\mathcal{U}_q^-(\mathfrak{g})\otimes\mathcal{U}_q(\mathfrak{h})\otimes\mathcal{U}_q^+(\mathfrak{g})\to\mathcal{U}_q(\mathfrak{g}) \tag{2}$$

where $\mathcal{U}_q^{\pm}(\mathfrak{g})$ is the subalgebra generated by the $x_{i,m}^{\pm}$ $(i\in I,\ m\in\mathbb{Z})$ and $\mathcal{U}_q(\mathfrak{h})$ is the subalgebra generated by the $k_i^{\pm 1}$, the $h_{i,r}$ and $c^{\pm 1/2}$ $(i\in I,\ r\in\mathbb{Z}\setminus\{0\})$.

3.2. **Finite-dimensional representations.** We refer to [CH] for generalities on the category $\mathcal{C}$ of finite-dimensional representations of $\mathcal{U}_q(\mathfrak{g})$. For $i \in I$, the action of k_i on any object of $\mathcal{C}$ is diagonalizable with eigenvalues in $\pm q^{\mathbb{Z}}$. Without loss of generality, we can assume that $\mathcal{C}$ is the category of *type 1* finite-dimensional representations (see [CP2]), i.e. we assume that for any object of $\mathcal{C}$, the eigenvalues of k_i are in $q^{\mathbb{Z}}$ for $i \in I$. The simple objects of $\mathcal{C}$ are parametrized by n-tuples of polynomials $(P_i(u))_{i\in I}$ satisfying $P_i(0) = 1$ (they are called *Drinfeld polynomials*) [CP1, CP2].

In type A, there is a family of evaluation morphisms $ev_a : \mathcal{U}_q(\mathfrak{g}) \to \mathcal{U}_q(\overline{\mathfrak{g}})$ parametrized by $a \in \mathbb{C}^*$. Hence for V a simple finite-dimensional representations of $\mathcal{U}_q(\overline{\mathfrak{g}})$, by pull-back we get an evaluation representation $(V)_a$. If the highest weight of V is a multiple of a fundamental weight, then V is a Kirillov-Reshetikhin module. In the particular case of a fundamental weight, we get the fundamental representations $V_i(a) = (V(\omega_i))_a$ of Drinfeld polynomials $(1, \cdots , 1, 1 - za, 1, \cdots , 1)$ with a non-trivial polynomial in position i. Their classes generate the Grothendieck ring $K_0(\mathcal{C})$ of the category $\mathcal{C}$ which is a polynomial ring in the variables $[V_i(a)]$ as proved in [FR]. In general simple finite-dimensional representations are not evaluation modules.

For $\omega \in P$, the *weight space* V_ω of an object V in $\mathcal{C}$ is the set of *weight vectors* of weight ω, i.e. of vectors $v \in V$ satisfying $k_i v = q^{\left(\omega(\alpha_i^\vee)\right)} v$ for any $i \in I$.

The elements $c^{\pm 1/2}$ act by identity on any object V of $\mathcal{C}$, and so the action of the $h_{i,r}$ commute. Since the $h_{i,r}$, $i \in I$, $r \in \mathbb{Z}\backslash\{0\}$, also commute with the k_i, $i \in I$, every object in $\mathcal{C}$ can be decomposed as a direct sum of generalized eigenspaces of the $h_{i,r}$ and k_i. More precisely, by Frenkel-Reshetikhin theory of q-characters [FR], the eigenvalues of the $h_{i,r}$ and k_i can be *encoded* by *monomials* m in formal variables $Y_{i,a}^{\pm 1}$ $(i \in I, a \in \mathbb{C}^*)$. Let $\mathcal{M}$ be the set of such monomials (also called *l-weights*). Given $m \in \mathcal{M}$ and an object V in $\mathcal{C}$, let V_m be the subspace of V of common pseudo-eigenvectors of the $h_{i,r}$, k_i with pseudo-eigenvalues associated to m (also called *l-weight space*). Thus,

$$V = \bigoplus_{m\in\mathcal{M}} V_m.$$

If $v \in V_m$, then v is a weight vector of weight

$$\omega(m) = \sum_{i\in I, a\in\mathbb{C}^*} u_{i,a}(m)\omega_i \in P,$$

where we denote $m = \prod_{i\in I, a\in\mathbb{C}^*} Y_{i,a}^{u_{i,a}(m)}$. For $v \in V_m$, we set $\omega(v) = \omega(m)$.

The *q-character morphism* is an injective ring morphism

$$\chi_q : \mathrm{Rep}(\mathcal{U}_q(\mathfrak{g})) \to \mathcal{Y} = \mathbb{Z}\left[Y_{i,a}^{\pm 1}\right]_{i\in I, a\in\mathbb{C}^*},$$

$$\chi_q(V) = \sum_{m\in\mathcal{M}} \dim(V_m) m.$$

If $V_m \neq \{0\}$ we say that m is an *l-weight of* V.

A monomial $m \in \mathcal{M}$ is said to be *dominant* if $u_{i,a}(m) \geq 0$ for any $i \in I, a \in \mathbb{C}^*$. For V a simple object in $\mathcal{C}$, let $M(V)$ be the *highest weight monomial* of $\chi_q(V)$, that is so that $\omega(M(V))$ is maximal for the partial ordering on P. $M(V)$ is dominant and characterizes the isomorphism class of V (it is equivalent to the data of the Drinfeld polynomials). Hence to a dominant monomial M is associated a simple representation $L(M)$. For $i \in I$ and $a \in \mathbb{C}^*$, we have for example the fundamental representation $V_i(a) = L(Y_{i,a})$. The simple modules of highest weight monomial

$$X_{i,\alpha}^{\beta} = Y_{i,q^\alpha} Y_{i,q^{\alpha+2}} \cdots Y_{i,q^{\alpha+2(\beta-1)}}$$

for some $i \in I, \alpha \in \mathbb{Z}, \beta \geq 1$ are Kirillov-Reshetikhin modules. We will also use the notation $X_{i,\alpha}^{\beta} = 1$ for $\beta \leq 0$.

Example 3.1. *The q-character of the fundamental representation $L(Y_a)$ of $\mathcal{U}_q(\hat{sl}_2)$ is*

$$\chi_q(L(Y_a)) = Y_a + Y_{aq^2}^{-1}.$$

The q-characters of evaluation modules, including Kirillov-Reshetikhin modules and fundamental modules, are known explicitly (see references in the introduction of [H3]). The formulas involve the monomials $A_{i,a}$ defined in [FR] for $i \in I, a \in \mathbb{C}^*$ by

$$A_{i,a} = Y_{i,aq^{-r_i}} Y_{i,aq^{r_i}} \times \prod_{\{j \in I | C_{i,j} = -1\}} Y_{j,a}^{-1}.$$

3.3. **Quantum Grothendieck ring.** The Grothendieck ring $K_0(\mathcal{C})$ has a t-deformation called quantum Grothendieck ring $K_t(\mathcal{C})$ as constructed in [VV, N1, H1] (we use the version of [H1, HL2]). It is a $\mathbb{Z}[t^{\pm 1/2}]$-subalgebra of a quantum torus $\mathcal{Y}_t$ and simple objects $L(m)$ have corresponding classes $[L(m)]_t \in K_t(\mathcal{C})$. A quantum version of a result in [FM] gives the following [N1] :

$$[L(m)]_t \in m * (1 + \mathbb{Z}[t^{\pm 1/2}, A_{i,c}^{-1}]_{i \in I, c \in \mathbb{C}^*}). \tag{3}$$

In other words, m is maximal for the Nakajima partial ordering on monomials, that is $M \preceq M'$ if $M'M^{-1}$ is a product of variables $A_{i,c}$.

If a simple module V is thin, that is if its ℓ-weight spaces are of dimension 1, then $[V]_t$ is a sum of commutative monomials (defined as in section 2.2) and can be identified with its q-character (see [HL2, Corollary 5.3]). In type A, all simple evaluation modules are thin.

4. Relations in the Grothendieck ring

We recall how the T-system originally occurs in the representation theory of quantum affine algebras and we also establish another incarnation of the T-system in the Grothendieck ring $K_0(\mathcal{C})$ (that we call horizontal T-system).

We denote $I = \{1, \cdots, n\}$ and $J = \{1, \cdots, \ell\}$ as above.

4.1. **Original T-systems.** For $1 \leq i \leq n$ and $0 \leq m \leq p \leq \ell$, consider the Kirillov-Reshetikhin module

$$\beta(m,p)^i = L(X_{i,i+2m}^{p-m+1}).$$

We extend the notation to $i = 0$ and $i = n+1$ by setting $\beta(m,p)^i = 1$ in these cases.

For $i \in I$ and $0 \leq m \leq p < \ell$, we have the T-system in $K_0(\mathcal{C})$:

$$\beta(m,p)^i \beta(m+1,p+1)^i = \beta(m,p+1)^i \beta(m+1,p)^i + \beta(m+1,p+1)^{i-1} \beta(m,p)^{i+1}.$$

See the list of references in the introduction of [H3]. It can be deformed into the quantum T-system in $K_t(\mathcal{C})$ (see [N2, HL2]) :

(4)
$$\beta(m,p)^i*\beta(m+1,p+1)^i = t^\lambda\beta(m,p+1)^i*\beta(m+1,p)^i+t^\mu\beta(m+1,p+1)^{i-1}*\beta(m,p)^{i+1}.$$

for some $\lambda,\mu\in\mathbb{Z}/2$ which depend in m,p,i (they can be explicitly computed but this is not relevant for the following).

4.2. **Horizontal T-systems.** The T-system has another incarnation in $K_0(\mathcal{C})$.

For $0\leq i\leq j\leq n+1$ and $0\leq m\leq \ell+1$, consider the evaluation module

$$\alpha(i,j)^m = L(M_{[i,j]}^m) \text{ where } M_{[i,j]}^m = X_{i,i}^m X_{j,j+2m}^{\ell+1-m}.$$

Some of these representations are Kirillov-Reshetikhin modules :

(5)
$$\alpha(i,n+1)^{m+1} = \beta(0,m)^i \text{ and } \alpha(0,i)^m = \beta(m,\ell)^i \text{ for } 0\leq m\leq \ell \text{ and } 0\leq i\leq n+1.$$

For $0\leq i\leq n+1$ and $j\geq 0$, we will denote

$$F_i = L(X_{i,i}^{\ell+1}) = \alpha(i,i)^m = \beta(0,\ell)^i = \alpha(i,i+j)^{\ell+1} = \alpha(i-j,i)^0. \tag{6}$$

Note that $F_0=F_{n+1}=1$.

Theorem 4.1. *For $0\leq i<j\leq n$ and $m\in J$, there are $\lambda,\lambda'\in\mathbb{Z}/2$, so that :*

$$\alpha(i,j)^m*\alpha(i+1,j+1)^m = t^\lambda\alpha(i,j+1)^m*\alpha(i+1,j)^m+t^{\lambda'}\alpha(i,j)^{m+1}*\alpha(i+1,j+1)^{m-1}.$$

Remark 4.2. *(i) This relation is "orthogonal" to the original quantum T-system in the sense that the spectral parameter is replaced by the vertex of the Dynkin diagram.*

(ii) At $t=1$, it can be shown that the relation comes from a non-split exact sequence obtained by a normalized R-matrix, as for the original T-system (see [N2, H2]*). In fact, it can be checked that the two tensor products associated to the right hand terms correspond to simple modules. Using* [C, Theorem 4]*, the proof is analogous to the one for the T-system.*

(iii) At $t=1$, this relation can be seen as an extended T-systems in [MY].

(iv) For the limit values of i,j, the relation involves both the α and the β-modules and so connect the two families. The specialization at $t=1$ reads :

$$\beta(m,\ell)^j\alpha(1,j+1)^m = \beta(m,\ell)^{j+1}\alpha(1,j)^m+\beta(m+1,\ell)^j\alpha(1,j+1)^{m-1} (\textit{ for } i=0)$$
$$\alpha(i,n)^m\beta(0,m-1)^{i+1} = \beta(0,m-1)^i\alpha(i+1,n)^m+\alpha(i,n)^{m+1}\beta(0,m-2)^{i+1} (\textit{ for } j=0).$$

Proof. By [FR, FM], a q-character is determined uniquely by the multiplicity of its dominant monomials. We will use the notation $A_{i,\lambda}$ instead of A_{i,q^λ} for $i\in I$, $\lambda\in\mathbb{Z}$. First we prove that $\alpha(i,j)^m*\alpha(i+1,j+1)^m$ has $2m+1$ dominant monomials :

$$M_1 = M_{[i,j]}^m M_{[i+1,j+1]}^m \text{ , } M_2 = M_1 A_{i+1,i+2m}^{-1}A_{i+2,i+1+2m}^{-1}\cdots A_{j,j+2m-1}^{-1},$$
$$M_{2r} = M_2\prod_{2\leq p\leq r}(A_{i,i+2m-2p+3}A_{i+1,i+2m-2p+2})^{-1} \text{ , } M_{2r+1} = M_{2r}A_{i,i+2m-2r+1}^{-1},$$

where $1\leq r\leq m$. It is clear that these monomials occur. Indeed M_1 is the product of the highest monomials. For $2\leq r\leq 2m+1$, we decompose

$$M_r = (M_{[i+1,j+1]}^m(M_2M_1^{-1}))\times(M_{[i,j]}^m(M_rM_2^{-1})).$$

Now consider $M \neq M_1$ a dominant monomial which occurs. We factorize

$$M = M_1 M' M''$$

where M' (resp. M'') is a monomial of $(M_{[i,j]}^m)^{-1}\alpha(i,j)^m$ (resp. of $(M_{[i+1,j+1]}^m)^{-1}\alpha(i+1,j+1)^m$). As M is dominant, we have $M'M'' \in \mathbb{Z}[A_{k,r}^{-1}]_{k\in I, r\leq j+2\ell}$. Then

$$M' \in \mathbb{Z}[A_{k,r}^{-1}]_{k\leq j-1, r\in\mathbb{Z}}$$

and that there is $R \geq 0$ such that

$$M'' \in (A_{j,j+2m-1}A_{j,j+2m-3}\cdots A_{j,j+2m-1-2R})^{-1}\mathbb{Z}[A_{k,r}^{-1}]_{k\leq j-1, r\in\mathbb{Z}}.$$

The monomial $\tilde{M} = M(X_{j+1,j+1+2m}^{\ell+1-m}X_{j,j+2m}^{\ell+1-m})^{-1}$ is a monomial of $\chi_q(L(X_{i,i}^m X_{i+1,i+1}^m))$ which has a unique dominant monomial. Hence $\tilde{M}Y_{j,j+2m}$ is dominant. So

$$\tilde{M}' = \tilde{M}(A_{j,j+2m-1}\cdots A_{i+1,i+2m}^{-1})$$

is a monomial of $\chi_q(L(X_{i,i}^m X_{i+1,i+1}^m))$. If $\tilde{M}' = X_{i,i}^m X_{i+1,i+1}^m$, then $M = M_2$. Otherwise, $\tilde{M}A_{i,i+2m-1}$ is a monomial of $\chi_q(L(X_{i,i}^m X_{i+1,i+1}^m))$. We continue by induction, and so M is one of the M_r.

This also implies that each M_r occurs with multiplicity which is a power of t.

Similarly, we get that $\alpha(i,j+1)^m * \alpha(i+1,j)^m$ has $m+1$ dominant monomials which are the M_{2r+1} for $0 \leq r \leq m$. We also get that $\alpha(i,j)^{m+1} * \alpha(i+1,j+1)^{m-1}$ has m dominant monomials which are the M_{2r} for $1 \leq r \leq m$.

To conclude, we have to check that the powers of t match : this can be done using positivity in the quantum Grothendieck ring as in [HL2, Section 5.10] or directly as in [HO, section 9]. ∎

5. Proof of periodicity

In this section we finish the proof of the quantum periodicity.

It suffices to identify the $T_{a,b}(t)$ with variables satisfying the T-system, the half-periodicity and such that the variables corresponding to the $X_{a,b}$ are algebraically independent. We will identify the $T_{a,b}(t)$ with certain q,t-characters of minimal affinizations, that is elements of the quantum torus $\mathcal{Y}_t$.

For $0 \leq k \leq n+1$, $0 \leq m \leq \ell+1$ and $u \in \mathbb{Z}$ so that $k+m+u \in 2\mathbb{Z}$, we set :

$$T_{k,m}(u) = \begin{cases} \alpha(\frac{u+2-k-m}{2}, \frac{u+2+k-m}{2})^m & \text{for } 0 \leq u+2-k-m \leq 2(n+1-k), \\ \beta(\frac{u-2n+k-m}{2}, \frac{u-2n-2+k+m}{2})^{n+1-k} & \text{for } m \leq u-2n+k \leq 2\ell-m+2, \\ \alpha(\frac{u-2n-2\ell+k+m-2}{2}, \frac{u-k-2\ell+m}{2})^{\ell+1-m} & \text{for } 0 \leq u-2n-2\ell-2+m+k \leq 2k, \\ \beta(\frac{u-2-2n-k+m-2\ell}{2}, \frac{u-2n-2-k-m}{2})^k & \text{for } -m \leq u-2\ell-2n-k-2 \leq m. \end{cases}$$

This defines $T_{k,m}(u)$ for $0 \leq u-m-k+2 \leq 2n+2\ell+4$, and we extend the definition for any u by $2(n+\ell+2)$-periodicity.

Remark 5.1. *(i) The formulas in all cases are compatible thanks to relations (5).*

(ii) Identifying the class F_r defined in (6) with $\mathcal{F}_r$, we recover boundary conditions of Section 2.

The $X_{k,m}$ quasi-commute, with the same rules as in Section 2. The relations (4) and Theorem 4.1 imply that the $T_{a,b}(u)$ satisfy the quantum T-system for a distinguished choice of the powers of t (let us call it the distinguished powers).

The $(X_{k,m})_{(k,m)\in I\times(J\cup\{0\})}$ form a family of algebraically independent variables. We may argue as in [HL4]. Let us explain this point for completeness : all the representations we consider belong to the monoidal category $\mathcal{C}_\ell^o$ of representations whose classes belong to the subring of the Grothendieck ring $K_0(\mathcal{C})$ generated by the classes of fundamental representations $[L(Y_{k,k+2m})]$ for $(k,m)\in I\times(J\cup\{0\})$. Then there is an injective ring morphism

$$\chi_q^T : K_0(\mathcal{C}_\ell^o) \to \mathcal{Y}$$

called truncated q-character morphism [HL3] : it is defined so that for $L(m)$ a simple module in $\mathcal{C}_\ell^o$, $\chi_q^T(L(m))$ is obtained from $\chi_q(L(m))$ by removing the monomials m' so that in $m'm^{-1}$ contains a factor of the form $A_{k,k+2\ell+1}^{-1}$, $k\in I$. Now by [H2], the $\chi_q^T(X_{k,m}) = X_{k,k+2m}^{\ell+1-m}$ are just monomials which are clearly algebraically independent.

As by construction we have $T_{a,b}(u) = T_{n+1-a,\ell+1-b}(u+n+\ell+2)$, we get the result for the quantum T-system with the distinguished powers of t.

To conclude it suffices to check that the powers of t correspond automatically to the distinguished choice. We consider a solution and we prove by induction on $u\geq a+b-2$ that the $T_{a,b}(u)$ correspond to the q,t-characters and that the powers of t are given by the distinguished choice. As discussed above, the $X_{a,b}$ are algebraically independent so we can identify the $T_{a,b}(u)$ for $u=a+b-2$, with the corresponding q,t-characters. In general, we have a relation

$$T_{a,b}(U+1)*T_{a,b}(U-1) = t^\alpha T_{a-1,b}(U)*T_{a+1,b}(U) + t^\beta T_{a,b+1}(U)*T_{a,b-1}(U),$$

for some $\alpha,\beta\in\mathbb{Z}/2$. For $u\leq U$, we have $T_{a,b}(u) = M_{a,b}(u)\chi_{a,b}(u)$ where $M_{a,b}(u)$ is a monomial in the quantum torus and $\chi_{a,b}(u)$ is a polynomial in the $A_{i,c}^{-1}$ with coefficients in $\mathbb{Z}[t^{\pm1}]$ and with constant term 1 (see (3)). Then $(\chi_{a,b}(u))^{-1}$ is a formal power series in the $A_{i,c}^{-1}$. Each term of the sum

$$T_{a,b}(U+1) = t^\alpha T_{a-1,b}(U)*T_{a+1,b}(U)*(T_{a,b}(U-1))^{-1} + t^\beta T_{a,b+1}(U)*T_{a,b-1}(U)*(T_{a,b}(U-1))^{-1}$$

is a monomial multiplied by such a formal power series. The highest monomial is

$$t^\alpha M_{a-1,b}(U)*M_{a+1,b}(U)*(M_{a,b}(U-1))^{-1}$$

which only appears in the first term. As $T_{a,b}(U+1)$ is bar invariant, it imposes that α is the power of the distinguished choice. Then one may consider

$$T_{a,b}(U+1) - t^\alpha T_{a-1,b}(U)*T_{a+1,b}(U)*(T_{a,b}(U-1))^{-1}.$$

The same arguments identifies β with the distinguished choice. Hence $T_{a,b}(U+1)$ satisfies the equation as the corresponding q,t-character and so is equal to it.

Example 5.2. *Let us study the examples 2.3, 2.4, 2.5 above. Let $n=\ell=1$. We get :*

$$L(Y_3)*L(Y_1) = tL(Y_1Y_3)+1, \quad L(Y_1)*L(Y_3) = t^{-1}L(Y_1Y_3)+1.$$

Let $n = 1$, $\ell = 2$.

$$L(Y_3Y_5) * L(Y_1) = L(Y_1Y_3Y_5) + t^{-1}L(Y_5), \quad L(Y_5) * L(Y_1Y_3) = L(Y_1Y_3Y_5) + t^{-1}L(Y_1),$$
$$L(Y_1) * L(Y_3) = 1 + t^{-1}L(Y_1Y_3), \quad L(Y_1Y_3) * L(Y_3Y_5) = 1 + t^{-1}L(Y_1Y_3Y_5) * L(Y_3),$$
$$L(Y_3) * L(Y_5) = 1 + t^{-1}L(Y_3Y_5), \quad L(Y_3Y_5) * L(Y_1) = L(Y_1Y_3Y_5) + t^{-1}L(Y_5),$$
$$L(Y_5) * L(Y_1Y_3) = L(Y_1Y_3Y_5) + t^{-1}L(Y_1), \quad L(Y_1) * L(Y_3) = 1 + t^{-1}L(Y_1Y_3),$$
$$L(Y_1Y_3) * L(Y_3Y_5) = 1 + t^{-1}L(Y_3) * L(Y_1Y_3Y_5), \quad L(Y_3) * L(Y_5) = 1 + t^{-1}L(Y_3Y_5).$$

Let $n = 2$, $\ell = 1$.

$$L(Y_{1,3}) * L(Y_{1,1}Y_{2,4}) = t^{\frac{1}{2}}L(Y_{2,4}) * L(Y_{1,1}Y_{1,3}) + L(Y_{2,2}Y_{2,4}),$$
$$L(Y_{2,4}) * L(Y_{1,1}) = tL(Y_{1,1}Y_{2,4}) + 1,$$
$$L(Y_{1,1}Y_{2,4}) * L(Y_{2,2}) = t^{\frac{3}{2}}L(Y_{1,1}) * L(Y_{2,2}Y_{2,4}) + L(Y_{1,1}Y_{1,3}),$$
$$L(Y_{1,1}) * L(Y_{1,3}) = t^{\frac{1}{2}}L(Y_{2,2}) + t^{-\frac{1}{2}}L(Y_{1,1}Y_{1,3}),$$
$$L(Y_{2,2}) * L(Y_{2,4}) = t^{\frac{1}{2}}L(Y_{1,3}) + t^{-\frac{1}{2}}L(Y_{2,2}Y_{2,4}),$$
$$L(Y_{1,3}) * L(Y_{1,1}Y_{2,4}) = t^{\frac{3}{2}}L(Y_{1,1}Y_{1,3}) * L(Y_{2,4}) + L(Y_{2,2}Y_{2,4}),$$
$$L(Y_{2,4}) * L(Y_{1,1}) = tL(Y_{1,1}Y_{2,4}) + 1,$$
$$L(Y_{1,1}Y_{2,4}) * L(Y_{2,2}) = t^{\frac{1}{2}}L(Y_{2,2}Y_{2,4}) * L(Y_{1,1}) + L(Y_{1,1}Y_{1,3}),$$
$$L(Y_{1,1}) * L(Y_{1,3}) = t^{\frac{1}{2}}L(Y_{2,2}) + t^{-\frac{1}{2}}L(Y_{1,1}Y_{1,3}),$$
$$L(Y_{2,2}) * L(Y_{2,4}) = t^{\frac{1}{2}}L(Y_{1,3}) + t^{-\frac{1}{2}}L(Y_{2,2}Y_{2,4}).$$

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Université de Paris, Sorbonne Université, CNRS, Institut de Mathématiques de Jussieu-Paris Rive Gauche, F-75013 Paris, France

Email address: `david.hernandez@imj-prg.fr`