

Monoidal categorifications of cluster algebras of type A and D

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To M. Jimbo on his 60th birthday

Abstract

In this note, we introduce monoidal subcategories of the tensor category of finite-dimensional representations of a simply-laced quantum affine algebra, parametrized by arbitrary Dynkin quivers. For linearly oriented quivers of types A and D , we show that these categories provide monoidal categorifications of cluster algebras of the same type. The proof is purely representation-theoretical, in the spirit of [HL1].

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1 Introduction

The theory of cluster algebras has received a lot of attention in the recent years because of its numerous connections with many fields, in particular Lie theory and quiver representations.

One important problem is to categorify cluster algebras. In recent years, many examples of *additive* categorifications of cluster algebras have been constructed. The concept of a *monoidal* categorification of a cluster algebra, which is quite different, was introduced in [HL1, Definition 2.1]. If a cluster algebra has a monoidal categorification, we get informations on its structure (positivity, linear independence of cluster monomials). Conversely, if a monoidal category is a monoidal categorification of a cluster algebra of finite type, we can calculate the factorization of any simple object as a tensor product of finitely many prime objects, as well as the composition factors of a tensor product of simple objects.

In [HL1] we have introduced a certain monoidal subcategory \mathcal{C}_1 of the category \mathcal{C} of finite-dimensional representations of a simply-laced quantum affine algebra, and we have conjectured that \mathcal{C}_1 is a monoidal categorification of a cluster algebra of the same type. This conjecture was proved in [HL1] for types A and D_4 , and in [N] for all A, D, E types. The proof in [HL1] relies on representation theory, and on the well-developed combinatorics of cluster algebras of finite type. Nakajima's proof is different and uses additional geometric tools: a tensor category of perverse sheaves on quiver varieties, and the Caldero-Chapoton formula for cluster variables.

The categories \mathcal{C}_1 of [HL1] are associated with bipartite Dynkin quivers. In this note, we introduce monoidal subcategories \mathcal{C}_ξ of \mathcal{C} associated with arbitrary Dynkin quivers. For types A and D , we show that the categories \mathcal{C}_ξ corresponding to linearly oriented quivers provide new monoidal categorifications of cluster algebras of the same type. The proof is similar to [HL1]. However, the main calculations are much simpler because, for these choices of ξ , the irreducibility criterion for products of prime representations is more accessible than for the categories \mathcal{C}_1 . This is why we can also treat in this note the cases D_n ($n \geq 5$).

In his PhD thesis, Fan Qin [Q] has recently generalized the geometric approach of Nakajima (partly in collaboration with Kimura), and obtained monoidal categorifications of cluster algebras associated with an arbitrary acyclic quiver (not necessarily bipartite) using perverse sheaves on quiver varieties.

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2 Cluster algebras and their monoidal categorifications

We refer to [FZ2, K] for excellent surveys on cluster algebras.

2.1 Let $0 \leq n < r$ be some fixed integers. If $\tilde{B} = (b_{ij})$ is an $r \times (r-n)$ -matrix with integer entries, then the *principal part* B of \tilde{B} is the square matrix obtained from \tilde{B} by deleting the last n rows. Given some $k \in [1, r-n]$ define a new $r \times (r-n)$ -matrix $\mu_k(\tilde{B}) = (b'_{ij})$ by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise,} \end{cases} \quad (1)$$

where $i \in [1, r]$ and $j \in [1, r-n]$. One calls $\mu_k(\tilde{B})$ the *mutation* of the matrix \tilde{B} in direction k . If \tilde{B} is an integer matrix whose principal part is skew-symmetric, then it is easy to check that $\mu_k(\tilde{B})$ is also an integer matrix with skew-symmetric principal part. We will assume from now on that \tilde{B} has skew-symmetric principal part. In this case, one can equivalently encode \tilde{B} by a quiver Γ with vertex set $\{1, \dots, r\}$ and with b_{ij} arrows from j to i if $b_{ij} > 0$ and $-b_{ij}$ arrows from i to j if $b_{ij} < 0$.

Now Fomin and Zelevinsky define a cluster algebra $\mathcal{A}(\tilde{B})$ as follows. Let $\mathcal{F} = \mathbb{Q}(x_1, \dots, x_r)$ be the field of rational functions in r commuting indeterminates $\mathbf{x} = (x_1, \dots, x_r)$. One calls (\mathbf{x}, \tilde{B}) the *initial seed* of $\mathcal{A}(\tilde{B})$. For $1 \leq k \leq r-n$ define

$$x_k^* = \frac{\prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}}}{x_k}. \quad (2)$$

The pair $(\mu_k(\mathbf{x}), \mu_k(\tilde{B}))$, where $\mu_k(\mathbf{x})$ is obtained from \mathbf{x} by replacing x_k by x_k^* , is the *mutation* of the seed (\mathbf{x}, \tilde{B}) in direction k . One can iterate this procedure and obtain new seeds by mutating $(\mu_k(\mathbf{x}), \mu_k(\tilde{B}))$ in any direction $l \in [1, r-n]$. Let \mathcal{S} denote the set of all seeds obtained from (\mathbf{x}, \tilde{B}) by any finite sequence of mutations. Each seed of \mathcal{S} consists of an r -tuple of elements of \mathcal{F} called a *cluster*, and of a matrix. The elements of a cluster are its *cluster variables*. One does not mutate the last n elements of a cluster; they are called *frozen variables* and belong to every cluster. We then define the *cluster algebra* $\mathcal{A}(\tilde{B})$ as the subring of \mathcal{F} generated by all the cluster variables of the seeds of \mathcal{S} . A *cluster monomial* is a monomial in the cluster variables of a *single* cluster. Two cluster variables are said to be *compatible* if they occur in the same cluster.

The first important result of the theory is that every cluster variable z of $\mathcal{A}(\widetilde{B})$ is a Laurent polynomial in \mathbf{x} with coefficients in \mathbb{Z} . It is conjectured that the coefficients are positive.

The second main result is the classification of *cluster algebras of finite type*, i.e. with finitely many different cluster variables. Fomin and Zelevinsky proved that this happens if and only if there exists a seed $(\mathbf{z}, \widetilde{C})$ such that the quiver attached to the principal part of \widetilde{C} is a Dynkin quiver (that is, an arbitrary orientation of a Dynkin diagram of type A, D, E).

In [FZ3], Fomin and Zelevinsky have shown that the cluster variables of a cluster algebra \mathcal{A} have a nice expression in terms of certain polynomials called the *F-polynomials*. In type A and D , explicit formulas for F -polynomials are known.

2.2 The concept of a monoidal categorification of a cluster algebra was introduced in [HL1, Definition 2.1]. We say that a simple object S of a monoidal category is *real* if $S \otimes S$ is simple.

Definition 2.1. *Let \mathcal{A} be a cluster algebra and let \mathcal{M} be an abelian monoidal category. We say that \mathcal{M} is a monoidal categorification of \mathcal{A} if there is an isomorphism between \mathcal{A} and the Grothendieck ring of \mathcal{M} such that the cluster monomials of \mathcal{A} are the classes of all the real simple objects of \mathcal{M} .*

A non trivial simple object S of \mathcal{M} is *prime* if there exists no non trivial factorization $S \cong S_1 \otimes S_2$. By [GLS2, Section 8.2], the cluster variables of \mathcal{A} are the classes of all the real prime simple objects of \mathcal{M} . So Definition 2.1 coincides with the definition in [HL1].

As an application, we get information on the cluster algebra, as shown by the following result.

Proposition 2.2. [HL1] *If a cluster algebra \mathcal{A} has a monoidal categorification, then*

- (i) *every cluster variable of \mathcal{A} has a Laurent expansion with positive coefficients with respect to any cluster;*
- (ii) *the cluster monomials of \mathcal{A} are linearly independent.*

Assertion (ii) can also be proved by using additive categorification, see the recent [CKLP].

Conversely, if \mathcal{M} is a monoidal categorification of a finite type cluster algebra, we can calculate the factorization of any simple object of \mathcal{M} as a tensor product of finitely many prime objects, as well as the composition factors of a tensor product of simple objects of \mathcal{M} . Moreover, every simple object in \mathcal{M} is real.

3 Categories of finite-dimensional representations of $U_q(\mathfrak{Lg})$

For recent surveys on the representation theory of quantum loop algebras, we invite the reader to consult [CH] or [L].

3.1 Let \mathfrak{g} be a simple Lie algebra of type A, D, E . We denote by I the set of vertices of its Dynkin diagram, and we put $n = |I|$. The *Cartan matrix* of \mathfrak{g} is the $I \times I$ matrix $C = (C_{ij})_{i,j \in I}$. We denote by α_i ($i \in I$) and ϖ_i ($i \in I$) the simple roots and fundamental weights of \mathfrak{g} , respectively.

Let $\xi: I \rightarrow \mathbb{Z}$ be a *height function*, that is $|\xi_j - \xi_i| = 1$ if $C_{ij} = -1$. It induces an orientation Q of the Dynkin diagram of \mathfrak{g} such that we have an arrow $i \rightarrow j$ if $C_{ij} = -1$ and $\xi_j = \xi_i - 1$. Define

$$\widehat{I} := \{(i, p) \in I \times \mathbb{Z} \mid p - \xi_i \in 2\mathbb{Z}\}.$$

3.2 Let $L\mathfrak{g}$ be the loop algebra attached to \mathfrak{g} , and let $U_q(L\mathfrak{g})$ be the associated quantum enveloping algebra. We assume that the deformation parameter $q \in \mathbb{C}^*$ is not a root of unity.

The simple finite-dimensional irreducible $U_q(L\mathfrak{g})$ -modules (of type 1) are usually labeled by Drinfeld polynomials. Here we shall use an alternative labeling by dominant monomials (see [FR]). Moreover, as in [HL1], we shall restrict our attention to a certain tensor subcategory $\mathcal{C}_{\mathbb{Z}}$ of the category of finite-dimensional $U_q(L\mathfrak{g})$ -modules. The simple modules in $\mathcal{C}_{\mathbb{Z}}$ are labeled by the dominant monomials in $\mathcal{Y} = \mathbb{Z} \left[Y_{i,p}^{\pm 1} \mid (i,p) \in \widehat{I} \right]$, that is monomials $m = \prod_{(i,p) \in \widehat{I}} Y_{i,p}^{u_{i,p}(m)}$ such that $u_{i,p}(m) \geq 0$ for every $(i,p) \in \widehat{I}$.

We shall denote by $L(m)$ the simple module labeled by the dominant monomial m .

By [FR], every object M in $\mathcal{C}_{\mathbb{Z}}$ has a q -character $\chi_q(M) \in \mathcal{Y}$. These q -characters generate a commutative ring \mathcal{K} isomorphic to the Grothendieck ring of $\mathcal{C}_{\mathbb{Z}}$.

By [FR, FM], we have $\chi_q(L(m)) \in m\mathbb{Z}[A_{i,p+1}^{-1}]_{(i,p) \in \widehat{I}}$ where for $(i,p) \in \widehat{I}$ we denote

$$A_{i,p+1} = Y_{i,p} Y_{i,p+2} \prod_{j \in I, C_{ij} = -1} Y_{j,p+1}^{-1} \in \mathcal{Y}.$$

In particular, an element in \mathcal{K} is characterized by the multiplicity of its dominant monomials. When m is the only dominant monomial occurring in $\chi \in \mathcal{Y}$, χ is said to be *minuscule*. We say that M is minuscule if $\chi_q(M)$ is minuscule. This implies that M is simple.

3.3 Define

$$\widehat{I}_{\xi} := \{(i, \xi_i) \mid i \in I\} \cup \{(i, \xi_i + 2) \mid i \in I\} \subset \widehat{I},$$

and let \mathcal{Y}_{ξ} be the subring of \mathcal{Y} generated by the variables $Y_{i,p}$ ($(i,p) \in \widehat{I}_{\xi}$).

Definition 3.1. \mathcal{C}_{ξ} is the full subcategory of $\mathcal{C}_{\mathbb{Z}}$ whose objects have all their composition factors of the form $L(m)$ where m is a dominant monomial in \mathcal{Y}_{ξ} .

When Q is a sink-source orientation, we recover the subcategories \mathcal{C}_1 introduced in [HL1]. Since \widehat{I}_{ξ} is a ‘‘convex slice’’ of \widehat{I} , we get as in [HL2, Lemma 5.8] :

Lemma 3.2. \mathcal{C}_{ξ} is closed under tensor products, hence is a monoidal subcategory of $\mathcal{C}_{\mathbb{Z}}$.

We denote by \mathcal{K}_{ξ} the subring of \mathcal{K} spanned by the q -characters $\chi_q(L(m))$ of the simple objects $L(m)$ in \mathcal{C}_{ξ} . Then \mathcal{K}_{ξ} is isomorphic to the Grothendieck ring \mathcal{R}_{ξ} of \mathcal{C}_{ξ} . Note that this ring is a polynomial ring over \mathbb{Z} with generators the classes of the $2n$ fundamental modules

$$L(Y_{i,\xi_i}), \quad L(Y_{i,\xi_i+2}), \quad (1 \leq i \leq n).$$

The q -character of a simple object $L(m)$ of \mathcal{C}_{ξ} contains in general many monomials m' which do not belong to \mathcal{Y}_{ξ} . By discarding these monomials we obtain a *truncated q -character* [HL1]. We shall denote by $\widetilde{\chi}_q(L(m))$ the truncated q -character of $L(m)$. One checks that for a simple object $L(m)$ of \mathcal{C}_{ξ} , all the dominant monomials occurring in $\chi_q(L(m))$ belong to the truncated q -character $\widetilde{\chi}_q(L(m))$ (the proof is similar to that of [HL1] for the category \mathcal{C}_1 , as for the proof of Lemma 3.2 above). Therefore the truncation map $\chi_q(L(m)) \mapsto \widetilde{\chi}_q(L(m))$ extends to an injective algebra homomorphism from \mathcal{K}_{ξ} to \mathcal{Y}_{ξ} .

It is sometimes convenient to renormalize the (truncated) q -character of $L(m)$ by dividing it by m , so that its leading term becomes 1. The element of \mathcal{Y} thus obtained is called a *renormalized (truncated) q -character*.

Define a partial ordering \preceq on \mathcal{Y} by $\chi \preceq \chi'$ if $\chi' - \chi$ is an \mathbb{N} -linear combination of monomials. In particular, we have $\widetilde{\chi}_q(M) \preceq \chi_q(M)$ for M in \mathcal{C}_{ξ} .

3.4 Let $J \subset I$ and $\mathfrak{g}_J \subset \mathfrak{g}$ be the corresponding Lie subalgebra. Let $\widehat{I}_J = \widehat{I} \cap (J \times \mathbb{Z})$. For m a monomial, let $m_J = \prod_{(i,p) \in \widehat{I}_J} Y_{i,p}^{u_{i,p}(m)}$. If m_J is dominant, one says that m is J -dominant. In this case, let $L_J(m)$ be the sum (with multiplicities) of the monomials m' occurring in $mm_J^{-1} \chi_q(L(m_J))$ such that $m(m')^{-1}$ is a product of $A_{i,p+1}^{-1}$, $(i,p) \in \widehat{I}_J$. The image of $L_J(m)$ in $\mathbb{Z}[Y_{i,p}]_{(i,p) \in \widehat{I}_J}$, obtained by sending the $Y_{i,p}$ to 1 if $(i,p) \notin \widehat{I}_J$, is the q -character of the simple $U_q(\mathfrak{Lg}_J)$ labeled by m_J [H2, Lemma 5.9]. In particular we have the following:

Lemma 3.3. *Let m and m' be two dominant monomials such that $L(m) \otimes L(m')$ is simple. Then $L_J(m)L_J(m') = L_J(mm')$.*

For m a dominant monomial one has a decomposition [H1, Proposition 3.1]

$$L(m) = \sum_{m'} \lambda_J(m') L_J(m') \quad (3)$$

where the sum runs over J -dominant monomials m' . The $\lambda_J(m') \in \mathbb{N}$ are unique. This corresponds to the decomposition of $L(m)$ in the Grothendieck ring of $U_q(\mathfrak{Lg}_J)$ -modules. This decomposition gives an inductive process to construct monomials occurring in $\chi_q(L(m))$. Let us start with $m_0 = m$. Then the monomials m_1 of $L_J(m_0)$ occur in $\chi_q(L(m))$. If m_1 is J_1 -dominant ($J_1 \subset J$) and if $L_{J_1}(m_1)$ occurs in the decomposition (3), then the monomials m_2 of $L_{J_1}(m_1)$ occur in $\chi_q(L(m))$, and we continue. See [H2, Remark 3.16] for details.

3.5 In this note, we follow the proof of [HL1] to establish that for certain choices of ξ the category \mathcal{C}_ξ is a monoidal categorification of a cluster algebra \mathcal{A} . Let us recall the main steps (see [HL1] for details) :

- (1) We define a family \mathcal{P} of prime simple modules in \mathcal{C}_ξ and we label the cluster variables of an acyclic initial seed Σ of \mathcal{A} with a subset of \mathcal{P} .
- (2) We prove that the renormalized truncated q -characters of the representations of \mathcal{P} coincide with the F -polynomials with respect to Σ of all the cluster variables of \mathcal{A} .
- (3) We prove an irreducibility criterion for tensor products of two representations in \mathcal{P} .
- (4) By using the following general result, we factorize every simple module in \mathcal{C}_ξ as a tensor product of representations in \mathcal{P} .

Theorem 3.4. [H4] *Let S_1, \dots, S_N be simple objects in \mathcal{C} . Then $S_1 \otimes S_2 \otimes \dots \otimes S_N$ is simple if and only if $S_i \otimes S_j$ is simple for any $1 \leq i < j \leq N$.*

In the next sections, we follow these steps for a good choice of ξ in types A and D . We conjecture that for arbitrary choices of ξ and for every type A, D, E , \mathcal{C}_ξ is the monoidal categorification of a cluster algebra of the same type. For type A , this can be proved in the same way as explained in Remark 4.3 (b). For other types, this can be probably established by using the methods in [N].

4 Type A

4.1 Let \mathcal{A} be a cluster algebra of type A_n in the Fomin-Zelevinsky classification. As is well-known, the combinatorics of \mathcal{A} is conveniently recorded in a regular polygon \mathbf{P} with $n+3$ vertices labeled from 0 to $n+2$, see [FZ1, §12.2]. Here, each cluster variable x_{ab} ($0 \leq a < b \leq n+2$) is labeled by the segment joining vertex a to vertex b . The cluster variables x_{ab} for which the segment $[a, b]$ is a side of the polygon are frozen. Moreover we specialize

$$x_{01} = x_{n+1, n+2} = x_{0, n+2} = 1.$$

The exchange relations (Ptolemy relations) are of the form

$$x_{ac}x_{bd} = x_{ab}x_{cd} + x_{ad}x_{bc}, \quad (a < b < c < d). \quad (4)$$

The clusters of \mathcal{A} correspond to the triangulations of \mathbf{P} . The variables x_{0i} ($2 \leq i \leq n+1$) together with the n frozen variables $x_{i,i+1}$ ($1 \leq i \leq n$) form a cluster, whose associated quiver is

$$\begin{array}{ccccccc} x_{02} & \rightarrow & x_{03} & \rightarrow & x_{04} & \rightarrow & \cdots & \rightarrow & x_{0,n+1} \\ \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & & \swarrow & \downarrow \\ x_{12} & & x_{23} & & x_{34} & & \cdots & & x_{n,n+1} \end{array}$$

Note that the principal part of this quiver (*i.e.* the subquiver with vertices the non-frozen variables) is a quiver of type A_n with linear orientation. We denote by Σ this particular seed of \mathcal{A} .

4.2 Let \mathfrak{g} be of type A_n . We will write $Y_{0,p} = Y_{n+1,p} = 1$ for $p \in \mathbb{Z}$. We choose the height function

$$\xi(i) := i, \quad (i \in I),$$

corresponding to a quiver Q of type A_n with linear orientation. We define the following family of irreducible representations in \mathcal{C}_ξ :

$$\mathcal{P} := \{L(i, j) := L(Y_{i,i}Y_{j,j+2}) \mid 0 \leq i \leq j \leq n+1\}.$$

The simple modules $L(i, j)$ are evaluation representations whose q -characters are known (see references in [CH]). In particular they are prime. We have $\tilde{\chi}_q(L(0, j)) = Y_{j,j+2}$ and if $i \neq 0$ we have

$$\tilde{\chi}_q(L(i, j)) = Y_{i,i}Y_{j,j+2}(1 + A_{i,i+1}^{-1} + (A_{i,i+1}A_{i+1,i+2})^{-1} + \cdots + (A_{i,i+1}A_{i+1,i+2} \cdots A_{j-1,j})^{-1}).$$

Dividing both sides by $Y_{i,i}Y_{j,j+2}$ and setting $t_i := A_{i,i+1}^{-1}$, we see that this formula for the renormalized truncated q -characters coincides with the formula for F -polynomials computed in [YZ, Example 1.14]. It is easy to deduce from this that we have the following relations in \mathcal{R}_ξ (also obtained in [MY]):

$$[L(i, k)][L(j, l)] = [L(i, l)][L(j, k)] + [L(i, j-1)][L(k+1, l)] \quad \text{if } 0 \leq i < j \leq k < l \leq n+1. \quad (5)$$

Therefore, comparing with (4), we see that the assignment

$$x_{ab} \mapsto [L(a, b-1)], \quad (0 \leq a < b \leq n+2)$$

extends to an isomorphism from the cluster algebra \mathcal{A} to the Grothendieck ring \mathcal{R}_ξ . This isomorphism maps the seed Σ to

$$\begin{array}{ccccccc} L(0, 1) & \rightarrow & L(0, 2) & \rightarrow & \cdots & \rightarrow & L(0, n) \\ \downarrow & \swarrow & \downarrow & \swarrow & & \swarrow & \downarrow \\ L(1, 1) & & L(2, 2) & & \cdots & & L(n, n) \end{array}$$

where the $L(i, i)$ ($1 \leq i \leq n$) correspond to frozen variables.

We say that (i, k) and (j, l) are *crossing* if and only if $i < j \leq k < l$ or $j < i \leq l < k$. Otherwise, we say that (i, k) and (j, l) are *noncrossing*. The next proposition is similar to the classical irreducibility criterion for prime representations of $U_q(L\mathfrak{sl}_2)$, except that here, spectral parameters are replaced by nodes of the Dynkin diagram.

Proposition 4.1. *The module $L(i, j) \otimes L(k, l)$ is simple if and only if (i, j) and (k, l) are noncrossing.*

Proof — The “only if” part follows from (5). We prove the “if” part. Let $M = Y_{i,i}Y_{j,j+2}Y_{k,k}Y_{l,l+2}$. We have $\tilde{\chi}_q(L(M)) \preceq \chi = \tilde{\chi}_q(L(i, j) \otimes L(k, l))$. We prove the other inequality. By symmetry, we are reduced to the following two cases:

- (a) if $j < k$ or $(k = 0 \text{ and } i, j \leq l)$ or $(1 \leq k \leq i, j = l)$, then χ contains a unique dominant monomial, namely M , so $L(i, j) \otimes L(k, l)$ is simple.
- (b) if $1 \leq k \leq i \leq j < l$, then χ contains exactly two dominant monomials, namely M and

$$M' = M(A_{k,k+1}A_{k+1,k+2} \cdots A_{j,j+1})^{-1}.$$

So it suffices to prove that M' occurs in $\tilde{\chi}(L(M))$. First, by §3.4, the monomial

$$M'' = M(A_{k,k+1}A_{k+1,k+2} \cdots A_{i-1,i})^{-1}$$

occurs in $\tilde{\chi}(L(M))$. Hence $L_J(M'')$ occurs in the decomposition (3) for $J = \{i, \dots, n\}$. But $L(Y_{i,-i}Y_{j,-j-2}) \otimes L(Y_{i,-i})$ is minuscule and simple. Hence, by [H3, Corollary 4.11], the tensor product $L(Y_{i,i}Y_{j,j+2}) \otimes L(Y_{i,i})$ is simple, isomorphic to $L(Y_{i,i}^2Y_{j,j+2})$. So $Y_{i,i}^2Y_{j,j+2}(A_{i,i+1} \cdots A_{j,j+1})^{-1}$ occurs in $\tilde{\chi}(L(Y_{i,i}^2Y_{j,j+2}))$ and M' occurs in $L_J(M'')$. \square

Therefore, as explained in §3.5, we get the following:

Theorem 4.2. \mathcal{C}_ξ is a monoidal categorification of the cluster algebra \mathcal{A} of type A_n .

Remark 4.3. (a) It follows from Theorem 4.2 that when $\xi_i = i$, every simple module in \mathcal{C}_ξ can be factorized as a tensor product of evaluation representations.

(b) For an arbitrary ξ , a theorem similar to Theorem 4.2 can be proved in an analog but slightly more complicated way. A subset $J = [i, j] \subset I$ ($1 \leq i \leq j \leq n$) has a natural orientation induced by ξ . Let J_+ (resp. J_-) be the set of sources (resp. sinks) of J . The prime objects in \mathcal{C}_ξ are the simple modules

$$L(J) := L\left(\prod_{k \in J_-} Y_{k,\xi_k} \prod_{k \in J_+} Y_{k,\xi_{k+2}}\right), \quad L(i) := L(Y_{i,\xi_i}), \quad L'(i) := L(Y_{i,\xi_{i+2}}).$$

Note that $L(J)$ is not an evaluation representation if J has several sources or several sinks.

(c) Different choices of ξ yield different subcategories \mathcal{C}_ξ . These subcategories seem to be quite similar, but they are not equivalent in general. For example, in type A_3 , consider the categories \mathcal{C}_ξ with $\xi_i = i$ and \mathcal{C}_ϕ with $\phi_1 = 1, \phi_2 = 2, \phi_3 = 1$. Both categories are monoidal categorifications of a cluster algebra of type A_3 with 3 coefficients. The category \mathcal{C}_ϕ was studied in [HL1]. In particular, we have the following relation in the Grothendieck ring of \mathcal{C}_ϕ :

$$[L(Y_{1,1}Y_{2,4}Y_{3,1})][L(Y_{2,2})] = [L(Y_{1,1})][L(Y_{3,1})][L(Y_{2,2}Y_{2,4})] + [L(Y_{1,1}Y_{1,3})][L(Y_{3,1}Y_{3,3})].$$

But by (5), in the Grothendieck ring of \mathcal{C}_ξ , a simple constituent of the tensor product of two simple prime representations can be factorized as a tensor product of at most 2 non trivial representations. Hence, \mathcal{C}_ξ and \mathcal{C}_ϕ are not equivalent.

5 Type D

5.1 Let \mathcal{A} be a cluster algebra of type D_n in the Fomin-Zelevinsky classification. The clusters of \mathcal{A} are now encoded by the *centrally symmetric* triangulations of a regular polygon \mathbf{P} with $2n$ vertices, labeled by $a = 0, 1, \dots, 2n - 1$ [FZ1, §12.4] (note that a more modern way to record the combinatorics of a cluster algebra of type D_n would be by means of a once-punctured n -gone and tagged arcs [FST]). A segment $[a, b]$ joining two vertices is called a *diagonal* if it meets the interior of \mathbf{P} , and a *side* otherwise. Let Θ be the 180° rotation of \mathbf{P} , and for a vertex a , write $\bar{a} = \Theta(a)$. Each *non frozen* cluster variable is labeled by a Θ -orbit on the set of diagonals of \mathbf{P} . More precisely, to each non trivial Θ -orbit $([a, b], [\bar{a}, \bar{b}])$ (with $b \neq \bar{a}$) we attach a single cluster variable

$$x_{ab} = x_{\bar{a}\bar{b}}.$$

But we associate with every Θ -fixed diagonal $[a, \bar{a}]$ (or *diameter*) two *different* cluster variables

$$x_{a\bar{a}} \neq x_{\widetilde{a\bar{a}}}.$$

We may think of $[a, \bar{a}]$ and $[\widetilde{a, \bar{a}}]$ as two different Θ -orbits, supported on the same segment but with two different colors. Given two Θ -orbits, one of which at least being non trivial, we say that they are *noncrossing* if they do not meet in the interior of \mathbf{P} . We also declare that two Θ -fixed diagonals are *noncrossing* if and only if they have the same support or the same color. A *centrally symmetric triangulation* of \mathbf{P} is then a maximal subset of pairwise noncrossing Θ -orbits of diagonals. Such a triangulation always consists of n different Θ -orbits. For instance, for $n = 4$, the following are two distinct triangulations

$$\left\{ ([1, \bar{3}], [\bar{1}, 3]), ([2, \bar{3}], [\bar{2}, 3]), [3, \bar{3}], [\widetilde{3, \bar{3}}] \right\}, \quad \left\{ ([1, \bar{3}], [\bar{1}, 3]), ([2, \bar{3}], [\bar{2}, 3]), [3, \bar{3}], [2, \bar{2}] \right\}.$$

To the Θ -orbits of the sides $[a, b]$ of \mathbf{P} we can also attach some frozen variables $x_{ab} = x_{\bar{a}\bar{b}}$. We specialize

$$x_{01} = x_{\overline{n-1}, 0} = 1.$$

Our initial seed for the cluster algebra \mathcal{A} will correspond to the triangulation

$$\left\{ \Theta([a, \overline{n-1}]) \mid 1 \leq a \leq n-2 \right\} \cup \left\{ [n-1, \overline{n-1}], [\widetilde{n-1, \overline{n-1}}] \right\}.$$

More precisely, it is described by the following quiver

$$\begin{array}{ccccccccccc}
 & & & & & & & & & x_{\widetilde{n-1, \overline{n-1}}} & \leftarrow & f_{n-1} \\
 & & & & & & & & & \uparrow & & \searrow \\
 x_{1, \overline{n-1}} & \rightarrow & x_{2, \overline{n-1}} & \rightarrow & \cdots & \rightarrow & x_{n-3, \overline{n-1}} & \rightarrow & x_{n-2, \overline{n-1}} & \leftarrow & x_{n-2, n-1} \\
 \uparrow & \swarrow & \uparrow & \swarrow & & \swarrow & \uparrow & \swarrow & \downarrow & & \nearrow \\
 x_{12} & & x_{23} & & \cdots & & x_{n-3, n-2} & & x_{n-1, \overline{n-1}} & \leftarrow & f_n
 \end{array}$$

where f_n and f_{n-1} are two additional frozen variables, which can not be encoded by sides of \mathbf{P} . The principal part of the quiver (obtained by removing the frozen vertices $x_{i, i+1}$ ($1 \leq i \leq n-2$), f_{n-1} , f_n , and the arrows incident to them) is a Dynkin quiver Q of type D_n , hence \mathcal{A} is indeed a cluster algebra of type D_n in the Fomin-Zelevinsky classification.

One can easily check that, because of this particular choice of frozen variables, \mathcal{A} belongs to the class of cluster algebras studied in [GLS1]. More precisely, let us label the vertices of Q by

$\{1, \dots, n\}$ so that $x_{i, \overline{n-1}}$ lies at vertex i for $i \leq n-1$, and $x_{\widetilde{n-1, n-1}}$ lies at vertex n . Then \mathcal{A} is the same as the algebra attached in [GLS1] to Q and the Weyl group element

$$w = c^2 = (s_n s_{n-1} s_{n-2} \cdots s_1)^2.$$

It follows from [GLS1, Theorem 16.1 (i)] that \mathcal{A} is a polynomial ring in $2n$ generators. These generators are the initial cluster variables

$$z_i := x_{i, \overline{n-1}} \quad (1 \leq i \leq n-1), \quad z_n := x_{\widetilde{n-1, n-1}},$$

together with the new cluster variables z_i^\dagger ($1 \leq i \leq n$) produced by the sequence of mutations

$$\mu_n \circ \mu_{n-1} \circ \mu_{n-2} \circ \cdots \circ \mu_2 \circ \mu_1. \quad (6)$$

Recall from [FZ1] that, our initial cluster being fixed, the cluster variables of \mathcal{A} also have a natural labelling by almost positive roots. The correspondence is as follows. First, the Θ -orbits of the initial triangulation are labeled by negative simple roots:

$$\Theta([i, \overline{n-1}]) \mapsto -\alpha_i, \quad (1 \leq i \leq n-2), \quad [n-1, \overline{n-1}] \mapsto -\alpha_{n-1}, \quad [n-1, \widetilde{n-1}] \mapsto -\alpha_n.$$

Any other Θ -orbit x is mapped to the positive root $\sum_i c_i \alpha_i$, where the diagonals representing x cross the diagonals representing $-\alpha_i$ at c_i pairs of centrally symmetric points (counting an intersection of two diameters of different colors and support as one such pair).

In [YZ, Y], a different labelling for the cluster variables is used. First the choice of an *acyclic* initial seed is encoded by the choice of a Coxeter element c . For our choice of initial seed, this Coxeter element is

$$c = s_n s_{n-1} s_{n-2} \cdots s_1.$$

Next the cluster variables are labeled by weights of the form

$$c^m \varpi_i, \quad (i \in I, 0 \leq m \leq h(i, c)),$$

where $h(i, c)$ is the smallest integer such that $c^{h(i, c)} \varpi_i = w_0 \varpi_i$. The correspondence between the two labellings is as follows. To the fundamental weight ϖ_i corresponds $-\alpha_i$, and to the weight $c^m \varpi_i$ ($m \geq 1$) corresponds the positive root $\beta = c^{m-1} \varpi_i - c^m \varpi_i$.

Example 5.1. We illustrate all these definitions in the case $n = 4$. Here \mathbf{P} is a regular octagon, with vertices labeled by $0, 1, 2, 3, \overline{0}, \overline{1}, \overline{2}, \overline{3}$. Our choice of initial triangulation is

$$\left\{ ([1, \overline{3}], [\overline{1}, 3]), ([2, \overline{3}], [\overline{2}, 3]), [3, \overline{3}], [\widetilde{3, \overline{3}}] \right\},$$

which corresponds to the Coxeter element $c = s_4 s_3 s_2 s_1$. The sixteen Θ -orbits of diagonals (represented by one of their elements), and the corresponding indexings by almost positive roots, and by weights, are given in the table below:

$[1, \overline{3}]$	$-\alpha_1$	ϖ_1	$[0, 3]$	$\alpha_1 + \alpha_2$	$c^3 \varpi_2$
$[2, \overline{3}]$	$-\alpha_2$	ϖ_2	$[1, \overline{1}]$	$\alpha_2 + \alpha_3$	$c^2 \varpi_4$
$[\widetilde{3, \overline{3}}]$	$-\alpha_3$	ϖ_3	$[1, \overline{1}]$	$\alpha_2 + \alpha_4$	$c^2 \varpi_3$
$[3, \overline{3}]$	$-\alpha_4$	ϖ_4	$[0, \overline{0}]$	$\alpha_1 + \alpha_2 + \alpha_3$	$c^3 \varpi_3$
$[0, 2]$	α_1	$c^3 \varpi_1$	$[\widetilde{0, \overline{0}}]$	$\alpha_1 + \alpha_2 + \alpha_4$	$c^3 \varpi_4$
$[1, 3]$	α_2	$c^2 \varpi_1$	$[0, \overline{0}]$	$\alpha_1 + \alpha_2 + \alpha_4$	$c^3 \varpi_4$
$[2, \overline{2}]$	α_3	$c \varpi_3$	$[2, \overline{1}]$	$\alpha_2 + \alpha_3 + \alpha_4$	$c \varpi_2$
$[\widetilde{2, \overline{2}}]$	α_4	$c \varpi_4$	$[2, \overline{0}]$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$	$c \varpi_1$
			$[1, \overline{0}]$	$\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$	$c^2 \varpi_2$

5.2 Let \mathfrak{g} be of type D_n . We will write $Y_{0,p} = Y_{n+1,p} = 1$ for $p \in \mathbb{Z}$. We choose the height function $\xi_i = n - 1 - i$ if $i < n$ and $\xi_n = 0$. This induces a partial order \preceq on $\{1, \dots, n\}$ defined by

$$i \prec j \iff \xi_i < \xi_j.$$

Note that $n - 1$ and n are not comparable for \preceq . Moreover, for convenience, we extend this to $\{0, \dots, n + 1\}$ by declaring that 0 is a maximal element and $n + 1$ a minimal element for \preceq .

We define the following family \mathcal{P} of representations in \mathcal{C}_ξ :

$$\begin{aligned} L(i, j) &= L(Y_{i, \xi_i} Y_{j, \xi_{j+2}}), & (n+1 \preceq i \preceq j \preceq 0), \\ L(i, j)^\dagger &= L(Y_{n,0} Y_{n-1,0} Y_{i, \xi_i+2} Y_{j, \xi_j+2}), & (n-2 \preceq j \prec i \preceq 0). \end{aligned}$$

Since \mathcal{A} and \mathcal{R}_ξ are both polynomial rings over \mathbb{Z} with $2n$ generators, the assignment

$$z_i \mapsto [L(n+1, i)] = [L(Y_{i, \xi_i+2})], \quad z_i^\dagger \mapsto [L(i, 0)] = [L(Y_{i, \xi_i})], \quad (1 \leq i \leq n),$$

extends to a ring isomorphism $\iota: \mathcal{A} \xrightarrow{\sim} \mathcal{R}_\xi$. Thus \mathcal{R}_ξ is endowed with the structure of a cluster algebra. Moreover, using the T -system equations for calculating the products

$$[L(Y_{i, \xi_i})][L(Y_{i, \xi_i+2})] = z_i z_i^\dagger,$$

and comparing them with the exchange relations involved in the sequence of mutations (6), we can easily check that the frozen variables of \mathcal{A} are mapped by ι to the classes $[L(i, i)] = [L(Y_{i, \xi_i} Y_{i, \xi_i+2})]$. More precisely,

$$\iota(f_{n-1}) = [L(n-1, n-1)], \quad \iota(f_n) = [L(n, n)], \quad \iota(x_{i, i+1}) = [L(i, i)], \quad (1 \leq i \leq n-2).$$

Therefore ι maps the initial seed of \mathcal{A} to

$$\begin{array}{ccccccc} & & & & L(n+1, n-1) & \leftarrow & L(n-1, n-1) \\ & & & & \uparrow & & \searrow \\ L(n+1, 1) & \rightarrow & L(n+1, 2) & \rightarrow & \cdots & \rightarrow & L(n+1, n-2) & \leftarrow & L(n-2, n-2) \\ \uparrow & \swarrow & \uparrow & \swarrow & \cdots & \swarrow & \downarrow & \nearrow & \\ L(1, 1) & & L(2, 2) & & \cdots & & L(n+1, n) & \leftarrow & L(n, n) \end{array}$$

5.3 Let us compute the truncated q -characters of the representations in \mathcal{P} . As in §4.2, the modules $L(i, j)$ are prime minimal affinizations. We have

$$\begin{aligned} \tilde{\chi}_q(L(n+1, j)) &= Y_{j, \xi_{j+2}}, \\ \tilde{\chi}_q(L(i, j)) &= Y_{i, \xi_i} Y_{j, \xi_{j+2}} (1 + A_{i, \xi_i+1}^{-1} + \cdots + (A_{i, \xi_i+1} \cdots A_{j-1, \xi_j})^{-1}), \quad (n-1 \preceq i), \\ \tilde{\chi}_q(L(n, j)) &= Y_{n,0} Y_{j, \xi_{j+2}} (1 + A_{n,1}^{-1} \chi_j), \quad (0 \preceq j \preceq n-2), \end{aligned}$$

where $\chi_j := 1 + A_{n-2,2}^{-1} + \cdots + (A_{n-2,2} \cdots A_{j+1, \xi_j})^{-1}$. In general the $L(i, j)^\dagger$ are not minimal affinizations. However, we have:

Lemma 5.2. *For $n-2 \preceq j \prec i \preceq 0$, the representation $L(i, j)^\dagger$ is prime and*

$$\tilde{\chi}_q(L(i, j)^\dagger) = Y_{n,0} Y_{n-1,0} Y_{i, \xi_i+2} Y_{j, \xi_j+2} (1 + (A_{n-1,1}^{-1} + A_{n,1}^{-1}) \chi_j + A_{n-1,1}^{-1} A_{n,1}^{-1} \chi_i \chi_j).$$

Proof — As $\tilde{\chi}_q(L(i, j)^\dagger) \preceq \tilde{\chi}_q(L(n, j) \otimes L(n-1, i))$ and $\tilde{\chi}_q(L(i, j)^\dagger) \preceq \tilde{\chi}_q(L(n, i) \otimes L(n-1, j))$ there are $A, B \preceq \chi_j$ and $C \preceq \chi_i \chi_j$ such that

$$\tilde{\chi}_q(L(i, j)^\dagger) = Y_{n,0} Y_{n-1,0} Y_{i,\xi_i+2} Y_{j,\xi_j+2} (1 + A_{n-1,1}^{-1} A + A_{n,1}^{-1} B + A_{n,1}^{-1} A_{n-1,1}^{-1} C).$$

From Proposition 4.1 with $J = \{1, \dots, n-1\}$, we have

$$Y_{n,0} L_J(Y_{n-1,0} Y_{j,n-j+1}) L_J(Y_{i,n-i+1}) = L_J(Y_{n,0} Y_{n-1,0} Y_{i,n-i+1} Y_{j,n-j+1}).$$

Hence, by §3.4, we have $A = \chi_j$. The proof that $B = \chi_j$ is analog. Similarly, from Proposition 4.1 with $J = \{1, \dots, n-2\}$, we have

$$L_J(Y_{n-2,1} Y_{i,n-i+1}) L_J(Y_{n-2,1} Y_{j,n-j+1}) = L_J(Y_{n-2,1}^2 Y_{i,n-i+1} Y_{j,n-j+1}).$$

So

$$C = (Y_{n-2,1}^2 Y_{i,n-i+1} Y_{j,n-j+1})^{-1} \tilde{\chi}_q(L(Y_{n-2,1}^2 Y_{i,n-i+1} Y_{j,n-j+1})) = \chi_i \chi_j.$$

This explicit formula shows that $\tilde{\chi}_q(L(i, j)^\dagger)$ can not be factorized and so $L(i, j)^\dagger$ is prime. \square

Let $\mathcal{P}' := \mathcal{P} \setminus \{L(i, i) \mid 1 \leq i \leq n\}$. We introduce the following bijection between the non frozen cluster variables of \mathcal{A} and the representations in \mathcal{P}' .

$$\begin{aligned} x_{ij} &\mapsto L(j-1, i), & (0 \leq i \leq j-2 \leq n-3), \\ x_{i\bar{j}} &\mapsto L(j, i)^\dagger, & (0 \leq j < i \leq n-2), \\ x_{i\bar{i}} &\mapsto L(n-1, i), & (0 \leq i \leq n-2), \\ x_{i\bar{i}}^\sim &\mapsto L(n, i), & (0 \leq i \leq n-2), \\ x_{i, \overline{n-1}} &\mapsto L(n+1, i), & (1 \leq i \leq n-2), \\ x_{\overline{n-1, n-1}} &\mapsto L(n+1, n-1), \\ x_{n-1, \overline{n-1}} &\mapsto L(n+1, n). \end{aligned}$$

One can check that under this correspondence, the renormalized truncated q -characters for the representations in \mathcal{P}' coincide with the F -polynomials of the cluster variables of \mathcal{A} calculated in [YZ, Y]. One then deduces that this bijection is the restriction of the ring automorphism ι to the set of non frozen cluster variables.

Example 5.3. We continue Example 5.1. The table below gives the list of cluster variables of \mathcal{A} together with the corresponding representations of \mathcal{P}' and their truncated q -characters. Here $t_i = A_{i, \xi_i+1}^{-1}$.

x_{02}	$L(1, 0)$	$Y_{1,2}(1 + t_1)$
x_{03}	$L(2, 0)$	$Y_{2,1}(1 + t_2 + t_2 t_1)$
x_{13}	$L(2, 1)$	$Y_{1,4} Y_{2,1}(1 + t_2)$
$x_{1\bar{0}}$	$L(0, 1)^\dagger$	$Y_{1,4} Y_{3,0} Y_{4,0}(1 + t_3 + t_3 t_2 + t_4 + t_4 t_2 + t_3 t_4 + 2t_3 t_4 t_2 + t_3 t_4 t_2^2 + t_3 t_4 t_2 t_1 + t_3 t_4 t_2^2 t_1)$
$x_{2\bar{0}}$	$L(0, 2)^\dagger$	$Y_{2,3} Y_{3,0} Y_{4,0}(1 + t_3 + t_4 + t_3 t_4 + t_3 t_4 t_2 + t_3 t_4 t_2 t_1)$
$x_{2,\bar{1}}$	$L(1, 2)^\dagger$	$Y_{1,4} Y_{2,3} Y_{3,0} Y_{4,0}(1 + t_3 + t_4 + t_3 t_4 + t_3 t_4 t_2)$
$x_{0\bar{0}}$	$L(3, 0)$	$Y_{3,0}(1 + t_3 + t_3 t_2 + t_3 t_2 t_1)$
$x_{1\bar{1}}$	$L(3, 1)$	$Y_{1,4} Y_{3,0}(1 + t_3 + t_3 t_2)$
$x_{2\bar{2}}$	$L(3, 2)$	$Y_{2,3} Y_{3,0}(1 + t_3)$
$x_{\bar{0}\bar{0}}$	$L(4, 0)$	$Y_{4,0}(1 + t_4 + t_4 t_2 + t_4 t_2 t_1)$
$x_{\bar{1}\bar{1}}$	$L(4, 1)$	$Y_{1,4} Y_{4,0}(1 + t_4 + t_4 t_2)$
$x_{\bar{2}\bar{2}}$	$L(4, 2)$	$Y_{2,3} Y_{4,0}(1 + t_4)$
$x_{1\bar{3}}$	$L(5, 1)$	$Y_{1,4}$
$x_{2\bar{3}}$	$L(5, 2)$	$Y_{2,3}$
$x_{\bar{3}\bar{3}}$	$L(5, 3)$	$Y_{3,2}$
$x_{\bar{3}\bar{3}}$	$L(5, 4)$	$Y_{4,2}$

5.4 We now describe which tensor products of representations of \mathcal{P} are simple.

Proposition 5.4. *We have the following :*

- (a) *Suppose $\{i, k\} \neq \{n-1, n\}$. Then $L(i, j) \otimes L(k, l)$ is not simple if and only if $i \prec k \preceq j \prec l$ or $k \prec i \preceq l \prec j$.*
- (b) *Suppose $\{i, k\} = \{n-1, n\}$. Then $L(i, j) \otimes L(k, l)$ is simple if and only if $j = l$ or $i = j$ or $k = l$.*
- (c) *Suppose $j \prec i$ and $l \prec k$. Then $L(i, j)^\dagger \otimes L(k, l)^\dagger$ is simple if and only if $j \preceq l \prec k \preceq i$ or $l \preceq j \prec i \preceq k$.*
- (d) *Suppose $i \succeq n-2$ and $l \prec k$. Then $L(i, j) \otimes L(k, l)^\dagger$ is simple if and only if $i = j$ or $i \prec j \preceq l \prec k$ or $l \prec k \prec i \prec j$ or $l \prec i \prec j \preceq k$.*
- (e) *Suppose $i \prec n-2$ and $l \prec k$. Then $L(i, j) \otimes L(k, l)^\dagger$ is simple if and only if $i = j$ or $((i \neq n+1)$ and $l \preceq j \preceq k)$ or $(i = n+1$ and $k \preceq j)$.*

Proof — In each case, the proof of non simplicity follows from the identification of truncated q -characters with F -polynomials in the last section. So we treat only the proof of the simplicity.

(a) The irreducibility is proved as in type A , except for the tensor product

$$L(n+1, n) \otimes L(n+1, n-1)$$

which is minuscule and so is simple.

(b) If $n-2 \preceq j = l$ or $i = j$ or $k = l$, $L(n, j) \otimes L(n-1, j)$ is minuscule and so is simple.

(c) By symmetry, we can assume $j \preceq l$. Suppose that $j \preceq l \prec k \preceq i$ and let us prove that $L(i, j)^\dagger \otimes L(k, l)^\dagger$ is simple. Let M be its highest weight monomial. It suffices to prove that any

dominant monomial m occurring in $\tilde{\chi}(L(i, j)^\dagger)\tilde{\chi}(L(k, l)^\dagger)$ occurs in $\tilde{\chi}_q(L(M))$. If $A_{n-1,1}^{-1}$ or $A_{n,1}^{-1}$ is not a factor of mM^{-1} , this is proved as for type A. If $A_{n-1,1}^{-2}$ is a factor of mM^{-1} , first from §3.4 $MA_{n-1,1}^{-2}$ occurs and $L_J(MA_{n-1,1}^{-2})$ occurs in the decomposition (3) for $J = \{1, \dots, n-2, n\}$. But from type A

$$L_J(MA_{n-1,1}^{-2}) = Y_{n-1,2}^{-2}L_J(Y_{n,0}Y_{n-2,1}Y_{i,n-i+1}Y_{j,n-j+1})L_J(Y_{n,0}Y_{n-2,1}Y_{k,n-k+1}Y_{l,n-l+1})$$

and we can conclude by §3.4. This is analog if $A_{n,1}^{-2}$ is a factor. So we can assume that $A_{n,1}^{-1}$ and $A_{n-1,1}^{-1}$ are factors with power 1. Then m is one of the following monomials

$$MA_{n,1}^{-1}A_{n-1,1}^{-1}A_{n-2,2}^{-1}\cdots A_{j,n-j}^{-1} \quad \text{with multiplicity } 5,$$

$$MA_{n,1}^{-1}A_{n-1,1}^{-1}A_{n-2,2}^{-1}\cdots A_{l,n-l}^{-1} \quad \text{with multiplicity } 2,$$

$$MA_{n,1}^{-1}A_{n-1,1}^{-1}A_{n-2,2}^{-1}\cdots A_{k,n-k}^{-1} \quad \text{with multiplicity } 1,$$

$$MA_{n,1}^{-1}A_{n-1,1}^{-1}A_{n-2,2}^{-2}\cdots A_{j,n-j}^{-2}A_{j+1,n-j-1}^{-1}\cdots A_{l,n-l}^{-1} \quad \text{with multiplicity } 1.$$

Then we conclude as above. For example for the last monomial of the list,

$$M' := MA_{n,1}^{-1}A_{n-2,2}^{-1}\cdots A_{j,n-j}^{-1}$$

occurs in $L_{\{n,n-2,\dots,j\}}(M)$ from type A. Hence $M'A_{j,n-j}$ occurs in $L_{\{n-1,n-2,\dots,j\}}(MA_{n,1}^{-1})$, but M' does not. So $L_{\{n-1,n-2,\dots,j\}}(M')$ occurs in the decomposition (3). Since M is a monomial in $L_{\{n-1,n-2,\dots,1\}}(M')$, we get the result.

(d) and (e) : The proof is analog.

□

Proposition 5.4 implies that the tensor products of representations of \mathcal{P} corresponding to compatible cluster variables are simple. Indeed, two cluster variables are compatible if and only if the corresponding diagonals in \mathbf{P} do not cross (with the convention that diameters of the same color do not cross each other) [FZ1, §12.4]. This coincides with the conditions of Proposition 5.4.

Example 5.5. We continue Example 5.1. The following table lists the compatible pairs of non frozen variables of \mathcal{A} , and indicates in which case of Proposition 5.4 the corresponding pairs of simple modules fall.

(x_{02}, x_{03})	(a)	$(x_{02}, x_{0\bar{0}})$	(a)	$(x_{02}, x_{0\bar{0}})$	(a)	$(x_{02}, x_{2\bar{2}})$	(a)
$(x_{02}, x_{2\bar{2}})$	(a)	$(x_{02}, x_{2\bar{3}})$	(a)	$(x_{02}, x_{3\bar{3}})$	(a)	$(x_{02}, x_{3\bar{3}})$	(a)
$(x_{03}, x_{0\bar{0}})$	(a)	$(x_{03}, x_{0\bar{0}})$	(a)	(x_{03}, x_{13})	(a)	$(x_{03}, x_{3\bar{3}})$	(a)
$(x_{03}, x_{3\bar{3}})$	(a)	$(x_{0\bar{0}}, x_{13})$	(a)	$(x_{0\bar{0}}, x_{1\bar{1}})$	(a)	$(x_{0\bar{0}}, x_{2\bar{2}})$	(a)
$(x_{0\bar{0}}, x_{3\bar{3}})$	(a)	$(x_{0\bar{0}}, x_{13})$	(a)	$(x_{0\bar{0}}, x_{1\bar{1}})$	(a)	$(x_{0\bar{0}}, x_{2\bar{2}})$	(a)
$(x_{0\bar{0}}, x_{3\bar{3}})$	(a)	$(x_{13}, x_{1\bar{1}})$	(a)	$(x_{13}, x_{1\bar{1}})$	(a)	$(x_{13}, x_{1\bar{3}})$	(a)
$(x_{13}, x_{3\bar{3}})$	(a)	$(x_{13}, x_{3\bar{3}})$	(a)	$(x_{1\bar{1}}, x_{1\bar{3}})$	(a)	$(x_{1\bar{1}}, x_{2\bar{2}})$	(a)
$(x_{1\bar{1}}, x_{3\bar{3}})$	(a)	$(x_{1\bar{1}}, x_{1\bar{3}})$	(a)	$(x_{1\bar{1}}, x_{2\bar{2}})$	(a)	$(x_{1\bar{1}}, x_{3\bar{3}})$	(a)
$(x_{1\bar{3}}, x_{2\bar{2}})$	(a)	$(x_{1\bar{3}}, x_{2\bar{2}})$	(a)	$(x_{1\bar{3}}, x_{2\bar{3}})$	(a)	$(x_{1\bar{3}}, x_{3\bar{3}})$	(a)
$(x_{1\bar{3}}, x_{3\bar{3}})$	(a)	$(x_{2\bar{2}}, x_{2\bar{3}})$	(a)	$(x_{2\bar{2}}, x_{2\bar{3}})$	(a)	$(x_{2\bar{3}}, x_{3\bar{3}})$	(a)
$(x_{2\bar{3}}, x_{3\bar{3}})$	(a)	$(x_{3\bar{3}}, x_{3\bar{3}})$	(a)	$(x_{0\bar{0}}, x_{0\bar{0}})$	(b)	$(x_{1\bar{1}}, x_{1\bar{1}})$	(b)
$(x_{2\bar{2}}, x_{2\bar{2}})$	(b)	$(x_{1\bar{0}}, x_{2\bar{0}})$	(c)	$(x_{2\bar{0}}, x_{2\bar{1}})$	(c)	$(x_{13}, x_{1\bar{0}})$	(d)
$(x_{02}, x_{2\bar{0}})$	(d)	$(x_{2\bar{2}}, x_{2\bar{1}})$	(e)	$(x_{2\bar{2}}, x_{2\bar{0}})$	(e)	$(x_{1\bar{1}}, x_{2\bar{1}})$	(e)
$(x_{1\bar{1}}, x_{2\bar{0}})$	(e)	$(x_{1\bar{1}}, x_{1\bar{0}})$	(e)	$(x_{0\bar{0}}, x_{2\bar{0}})$	(e)	$(x_{0\bar{0}}, x_{1\bar{0}})$	(e)
$(x_{2\bar{2}}, x_{2\bar{1}})$	(e)	$(x_{2\bar{2}}, x_{2\bar{0}})$	(e)	$(x_{1\bar{1}}, x_{2\bar{1}})$	(e)	$(x_{1\bar{1}}, x_{2\bar{0}})$	(e)
$(x_{1\bar{1}}, x_{1\bar{0}})$	(e)	$(x_{0\bar{0}}, x_{2\bar{0}})$	(e)	$(x_{0\bar{0}}, x_{1\bar{0}})$	(e)	$(x_{3\bar{1}}, x_{2\bar{1}})$	(e)

Now, as explained in §3.5, we may conclude that:

Theorem 5.6. \mathcal{C}_ξ is a monoidal categorification of the cluster algebra \mathcal{A} of type D_n .

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