REPRESENTATIONS OF SHIFTED QUANTUM AFFINE ALGEBRAS

DAVID HERNANDEZ

Abstract. We develop the representation theory of shifted quantum affine algebras $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ and of their truncations which appeared in the study of quantized K-theoretic Coulomb branches of 3d $N = 4$ SUSY quiver gauge theories. Our direct approach is based on relations that we establish with the category $\mathcal{O}$ of representations of the quantum affine Borel subalgebra $\mathcal{U}_q(\hat{\mathfrak{b}})$ of the quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ and on associated quantum integrable models we have previously studied. We introduce the category $\mathcal{O}^\mu$ of representations of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ and we classify its simple objects. For $\mathfrak{g} = sl_2$ we prove the existence of evaluation morphisms to q-oscillator algebras. We establish the existence of a fusion product for shifted quantum affine algebras and we get a ring structure on the sum of the Grothendieck groups $K_0(\mathcal{O}^\mu)$. We introduce induction and restriction functors to the category $\mathcal{O}$ of $\mathcal{U}_q(\hat{\mathfrak{b}})$. As a by product we classify simple finite-dimensional representations of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ and we obtain a cluster algebra structure on the Grothendieck ring of finite-dimensional representations. We establish a necessary condition for a simple representation to descend to a truncation, which is also sufficient for $\mathfrak{g} = sl_2$. We introduce a related partial ordering on simple modules and we prove a truncation has only a finite number of simple representations. We state a conjecture on the parametrization of simple modules of a non simply-laced truncation in terms of the Langlands dual Lie algebra. We have several evidences, including a general result for simple finite-dimensional representations proved by using the Baxter polynomiality of quantum integrable models.

Contents

1. Introduction

Shifted quantum affine algebras and their truncations arose [FT] in the study of quantized K-theoretic Coulomb branches of 3d $N = 4$ SUSY quiver gauge theories in the sense of Braverman-Finkelberg-Nakajima [BFN1] which are at the center of current important developments (see for instance [N4, F] and references therein). A presentation of (truncated) shifted quantum affine algebras by generators and relations was given by Finkelberg-Tsymbaliuk [FT]. Their rational analogs, the shifted Yangians, and their truncations, appeared in type A in the context of the representation theory of finite W-algebras [BrK], then in the study of quantized affine Grassmannian slices [KWWY] for general types and in the study of quantized Coulomb branches of 3d $N = 4$ SUSY quiver gauge theories [BFN1] for simply-laced types and [NW] for non simply-laced types.

Shifted quantum affine algebras $\mathcal{U}_q^{\mu_+,\mu_-}(\hat{\mathfrak{g}})$ can be seen as variations of Drinfeld-Jimbo quantum affine algebras $\mathcal{U}_q(\hat{\mathfrak{g}})$ in their Drinfeld presentation, but depending on two coweights μ_+, μ_- of the underlying simple Lie algebra $\mathfrak{g}$. In the particular case $\mu_+ = \mu_- = 0$, the algebra $\mathcal{U}_q^{0,0}(\hat{\mathfrak{g}})$ is a central extension of the ordinary quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$. The algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ and its representations have been under intense study in recent years, the reader may refer to the recent ICM talks [Kas3, O] for recent important developments.

The truncations depend on additional parameters, including a dominant coweight λ. In this context, these parameters λ and $\mu = \mu_+ + \mu_-$ can be interpreted as parameters for generalized slices of the affine Grassmannian $\overline{\mathcal{W}}_\mu^\lambda$ (usual slices when μ is dominant) or its symplectic dual in the sense of [BLPW], a Nakajima quiver variety $\mathcal{M}_{\lambda,\mu}$. Up to isomorphism, $\mathcal{U}_q^{\mu_+,\mu_-}(\hat{\mathfrak{g}})$ only depends on μ and will be denoted simply by $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$.

For simply-laced types, representations of shifted Yangians and related Coulomb branches have been intensively studied, see [BrK, KTWWY1, KTWWY2] and references therein. For non simply-laced types, representations of quantizations of Coulomb branches have been studied by Nakajima and Weekes [NW] by using the method originally developed in [N5] for simply-laced types.

In the present paper, we develop the representation theory of shifted quantum affine algebras with a direct approach based on relations that we establish with the category $\mathcal{O}$ of representations of the quantum affine Borel subalgebra $\mathcal{U}_q(\hat{\mathfrak{b}})$ of the quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ and on associated quantum integrable models. We establish several analogies with the representation theory of ordinary quantum affine algebras, but also several new features which require new technical developments.

We first relate these representations to q-oscillator algebras and to the quantum affine Borel algebra $\mathcal{U}_q(\hat{\mathfrak{b}})$. It is known since [BLZ] that certain representations of the q-oscillator algebra give rise to representations of the quantum affine Borel algebra $\mathcal{U}_q(\hat{\mathfrak{b}})$ of the quantum affine algebra $\mathcal{U}_q(\hat{sl}_2)$. For general untwisted types, the category $\mathcal{O}$ of representations of the quantum affine Borel algebra $\mathcal{U}_q(\hat{\mathfrak{b}})$ was introduced and studied in [HJ]. Some representations in this category extend to a representation of the whole quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$, but many do not, including the prefundamental representations constructed in [HJ] and whose transfer-matrices have remarkable properties for the corresponding quantum integrable systems [FH2]. However, it was first observed in [HJ] that for some of the simple representations in $\mathcal{O}$, the structure of representation of $\mathcal{U}_q(\hat{\mathfrak{b}})$ can be extended to a larger algebra. It is called the asymptotical algebra $\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})$ and we will prove it is also related to certain shifted quantum affine algebras. In the Yangian case, it was first observed in [Z] that for certain simple representations in an analog of the category $\mathcal{O}$ (the analog of the prefundamental representations), the action can be extended to a shifted Yangian.

This picture motivated us to introduce a category $\mathcal{O}^\mu$ of representations of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ which is an analog of the ordinary category $\mathcal{O}$. In particular, the category $\mathcal{O}^0$ contains the category of finite-dimensional representations of the quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$, but for other μ, there are much more finite-dimensional representations in the categories $\mathcal{O}^\mu$ which seem to be a natural extension of the ordinary representation theory of quantum affine algebras. These categories $\mathcal{O}^\mu$ are the main categories studied in the present paper.

Our main results are the following :

(1) The classification of simple representations in the category $\mathcal{O}^\mu$.
(2) The classification of simple finite-dimensional representations of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$.
(3) A ring structure on the sum of Grothendieck groups $K_0(\mathcal{O}^\mu)$ from fusion products.
(4) A cluster algebra structure on the Grothendieck ring of finite-dimensional representations of shifted quantum affine algebras.
(5) The construction of induction and restriction functors to the category $\mathcal{O}$ of $\mathcal{U}_q(\hat{\mathfrak{b}})$.
(6) A q-characters formula for simple finite-dimensional representations of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ in terms of the q-character of certain simple representations of $\mathcal{U}_q(\hat{\mathfrak{b}})$.
(7) The rationality and polynomiality of remarkable Cartan-Drinfeld series on simple representations in $\mathcal{O}^\mu$, using Baxter polynomiality of quantum integrable models.
(8) The proof of the finiteness of the number of simple isomorphism classes for truncations and their complete classification for $\mathfrak{g} = sl_2$.
(9) A new related partial ordering on simple representations of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$.
(10) The statement of a conjecture to explicitly parametrize simple representations of non simply-laced truncated shifted quantum affine algebras involving interpolating (q,t)-characters and the Langlands dual twisted quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}}^L)$.
(11) The proof, using Baxter polynomiality of quantum integrable models, that simple finite-dimensional representations descend to a truncation as in the conjecture.

Let us comment on related structures and on previous results.

For simply-laced types, simple representations of truncated shifted Yangians have been parametrized in terms of Nakajima monomial crystals [KTWWY2]. Combining with [N5], this implies an analog statement for simply-laced shifted quantum affine algebras. This is a fundamental motivation for our conjecture (10) in non simply-laced types. We have several strong evidences, including a complete result in type B_2, and a general result for finite-dimensional representations as mentioned above.

Based on [N5], Nakajima-Weekes [NW] gave a bijection between more general simple representations of a non simply-laced quantized Coulomb branch and those for simply-laced types (they are parametrized by the same canonical base). Nakajima explained to the author this bijection preserves finiteness of dimension and category $\mathcal{O}$. Thus, combining with [KTWWY2], this gives an explicit parametrization of simple representations in category $\mathcal{O}$

of truncated non simply-laced shifted Yangians and quantum affine algebras. After using the comparison between simply-laced and twisted q-characters [He4], one can consider a possible relation between the two parametrizations. In small examples discussed in a correspondence between Nakajima and the author, this different method seems to give the same parametrization as our result. Note also that results (8) above can be obtained by this method from simply-laced types and an equivalence of representations of truncated shifted Yangians/quantum affine algebras. For this last point, in the formulation of [N5, NW], once the spectral parameters are specialized, the algebras relevant to homological and K-theoretic Coulomb branches become isomorphic by Riemann-Roch theorem (there should be also other approaches to this last problem, in the spirit of [GTL]).

One of the aims of the last part of the paper is to understand truncations and their representations, uniformly, from the direct methods developed in the first parts. We see it is also relevant to use the theory of quantum integrable models we have previously studied. We derive an explicit parametrization using the direct algebraic and transfer-matrices approach.

Let us recall that to each finite-dimensional representation V of $\mathcal{U}_q(\hat{\mathfrak{g}})$, and more generally to each representation of $\mathcal{U}_q(\hat{\mathfrak{b}})$ in the category $\mathcal{O}$, is attached a transfer-matrix $t_V(z)$ which is a formal power series in a formal parameter z with coefficients in $\mathcal{U}_q(\hat{\mathfrak{g}})$ (the transfer-matrix is defined via the R-matrix construction). Given another simple finite-dimensional representation W of $\mathcal{U}_q(\hat{\mathfrak{g}})$, we get a family of commuting operators on $W[[z]]$. This is a quantum integrable model generalizing the XXZ-model. As conjectured in [FR2] and established in [FH2], the spectrum of this system, that is the eigenvalues of the transfer-matrices, can be described in terms of certain polynomials, generalizing Baxter's polynomials associated to the XXZ-model. These Baxter's polynomials are obtained from the eigenvalues of transfer-matrices associated to prefundamental representations of $\mathcal{U}_q(\hat{\mathfrak{b}})$. Moreover, this Baxter polynomiality implies the polynomiality of certain series of Cartan-Drinfeld elements acting on finite-dimensional representations [FH2]. We relate this result to the structures of representations of truncated shifted quantum affine algebras (results (7), (9), (11) above).

In non-simply-laced types, the parametrization (10) does not involve directly the a monomial crystal or a q-character of a standard module, but a "mixture" between the q-characters of Langlands dual standard modules obtained from interpolating (q,t)-characters. The latter were defined in [FH1] as an incarnation of Frenkel-Reshetikhin deformed W-algebras interpolating between q-characters of a non simply-laced quantum affine algebra and its Langlands dual. They lead [FH1] to the definition of an interpolation between the Grothendieck ring $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ of finite-dimensional representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ (at $t=1$) and the Grothendieck ring $\mathrm{Rep}(\mathcal{U}_t(\hat{\mathfrak{g}}^L))$ of finite-dimensional representations of the Langlands dual algebra quantum affine algebra $\mathcal{U}_t(\hat{\mathfrak{g}}^L)$ (at $q=\epsilon$ a certain root of 1) :

$$\begin{tikzcd} & \mathfrak{K}_{q,t} \arrow[dl, dashed, "t=1"'] \arrow[dr, dashed, "q=\epsilon"] & \\ \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) & & \mathrm{Rep}(\mathcal{U}_t(\hat{\mathfrak{g}}^L)) \end{tikzcd}$$

Here $\mathfrak{K}_{q,t}$ is the ring of interpolating (q,t)-characters. To describe our parametrization (10), we found it is relevant to introduce different specializations of interpolating (q,t)-characters that we call Langlands dual q-characters (with $t=1$ for variables but $q=\epsilon$ for coefficients).

The interpolating (q,t)-characters are closely related to the deformed W-algebras which appeared in [FR1] in the context of the geometric Langlands program. Note also that the parametrization in [KTWWY2] for simply-laced types can be understood in the context of symplectic duality (more precisely from the equivariant version of the Hikita conjecture [Hi] for the symplectic dual form by a pair affine Grassmannian slice and a quiver variety). Hence the statement of our conjecture has also as main motivations the symplectic duality and the Langlands duality.

Let us discuss another application of our approach in the context of cluster algebra theory (result (4) above). The cluster algebra $\mathcal{A}(Q)$ attached to a quiver Q is a commutative ring with a distinguished set of generators called cluster variables and obtained inductively via the Fomin-Zelevinsky procedure of mutation [FZ]. A monoidal category $\mathcal{C}$ is said to be a monoidal categorification of $\mathcal{A}(Q)$ if there exists a ring isomorphism $\mathcal{A}(Q) \overset{\sim}{\to} K_0(\mathcal{C})$, with some additional properties, in particular with cluster variables corresponding to classes of certain (prime) simple modules (see [HL1]). Various examples of monoidal categorifications have been established. It was proved in [HL3] that the Grothendieck rings of certain categories of representations of $\mathcal{U}_q(\hat{\mathfrak{b}})$ have a cluster algebra structure. We prove, using results (3) and (5) above, that this leads to a cluster algebra structure on the Grothendieck ring of finite-dimensional representations of shifted quantum affine algebras (result (4) above).

Let us note we expect the results and methods in this paper to extend uniformly to twisted shifted quantum affine algebras.

The paper is organized as follows.

In Section 2, we introduce finite-type analogs of shifted quantum affine algebras and we underline the relation with the q-oscillator algebras.

In Section 3, we recall the definition of shifted quantum affine algebras $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$. In the sl_2-case, we introduce evaluation morphisms to the q-oscillator algebra (Proposition 3.7) which give the first examples of evaluation representations. For general types, we recall the definition of shift homomorphism and we prove its injectivity (Proposition 3.11). We prove that for μ anti-dominant, $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ contains a subalgebra isomorphic to the quantum affine Borel algebra $\mathcal{U}_q(\hat{\mathfrak{b}})$ (Proposition 3.3).

In Section 4, we introduce the category $\mathcal{O}_\mu$ of representations of the shifted quantum affine algebra $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ and we classify its simple objects (Theorem 4.11). The category $\mathcal{O}^{sh} = \bigoplus_{\mu\in\Lambda} \mathcal{O}_\mu$ is the sum of the abelian categories $\mathcal{O}_\mu$. We study shift functors induced by shift homomorphisms.

In Section 5, we construct the fusion product of representations of shifted quantum affine algebras by using the deformed Drinfeld coproduct and the renomalization procedure introduced in [He2] (Theorem 5.4). This leads to the definition of a ring structure on the Grothendieck groups $K_0(\mathcal{O}^{sh})$. We establish a simple module in $\mathcal{O}^{sh}$ is a quotient of a fusion product of various prefundamental (Corollary 5.6). Along the way we consider analogs of Frenkel-Reshetikhin q-characters of representations of shifted quantum affine algebras. We establish q-characters of simple representations satisfy a triangularity property with respect to Nakajima partial ordering (Theorem 5.11).

In Section 6, we classify the simple finite-dimensional representations of shifted quantum affine algebras (Theorem 6.4).

In Section 7, we define and study induction and restriction functors relating the category $\mathcal{O}$ of representations of the quantum affine Borel algebra $\mathcal{U}_q(\hat{\mathfrak{b}})$ and the categories $\mathcal{O}_\mu$ of representations of shifted quantum affine algebras.

In Section 8, we establish a q-characters formula for simple finite-dimensional representations of shifted quantum affine algebras in terms of the q-character of certain simple representations of $\mathcal{U}_q(\hat{\mathfrak{b}})$ in the category $\mathcal{O}$ (Theorem 8.1). Then, we prove the results in [HL3] imply a description of simple finite-dimensional representations of $\mathcal{U}_q(\hat{sl}_2)$ (Theorem 8.3), isomorphisms of Grothendieck rings between categories of representations of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ associated to dominant and anti-dominant μ (Theorem 8.6), and a cluster algebra structure on the Grothendieck ring of finite-dimensional representations of shifted quantum affine algebras (Theorem 8.9).

In Section 9, we recall Cartan-Drinfeld series $Y_i^\pm(z)$, $T_i^\pm(z)$ introduced respectively in [FR2] and in [HJ] in the study of transfer-matrices of representations of quantum affine algebras. We establish (Theorem 9.12) the rationality of $Y_i^\pm(z)$ (resp. the polynomiality of $(T_i^\pm(z))^{\mp 1}$) on a simple representation of a shifted quantum affine algebra in the category $\mathcal{O}_\mu$, using Baxter polynomiality of quantum integrable models [FH2].

In Section 10, we recall the definition of truncated shifted quantum affine algebras and we explain how it appears naturally in terms of the Cartan-Drinfeld series $Y_i^\pm(z)$, $T_i^\pm(z)$. We establish (Proposition 10.7) a necessary and sufficient condition for the defining series to have a rational action on a simple representation.

In Section 11, we start investigating which simple representations descend to truncated shifted quantum affine algebras. We establish a necessary condition (Proposition 11.11). It implies that a truncated shifted quantum affine algebra has only a finite number of isomorphism classes of simple representations (Theorem 11.15). Then we introduce a partial ordering $\preceq_{\mathcal{Z}}$ on simple modules, related to the Langlands dual Lie algebra ${}^L\mathfrak{g}$. We prove a related triangularity property for simple representations of truncated shifted quantum affine algebra (Theorem 11.9). In the sl_2-case we characterize simple representations of a shifted quantum affine algebra (Theorem 11.17).

In Section 12, we state a conjecture (Conjecture 12.3) on the parametrization of simple modules of non simply-laced truncated shifted quantum affine algebras. It is given in terms Langlands dual q-characters that we introduce. We establish in type B_2 that our parametrization gives representations of the truncated shifted quantum affine algebra (Proposition 12.8). In general, we establish that a simple finite-dimensional representation of a shifted quantum affine algebra descends to a truncation as in Conjecture 12.3 (Theorem 12.9). The proof of this last result is based on Baxter polynomiality of quantum integrable models.

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2. q-oscillator algebras

We first introduce finite-type analogs of shifted quantum affine algebras and we underline the relation with the q-oscillator algebras.

2.1. **Definition.** Let $\mathfrak{g}$ be a simple finite-dimensional of rank n and $I = \{1, \cdots, n\}$. We denote by $(\omega_i)_{i\in I}$, $\{\alpha_i\}_{i\in I}$, $\{\alpha_i^\vee\}_{i\in I}$, $\{\omega_i^\vee\}_{i\in I}$ the fundamental weights, the simple roots, simple coroots and fundamental coweights respectively. P is integral weight lattice and $P_\mathbb{Q} = P \otimes \mathbb{Q}$. The set of dominant weights is denoted by P^+. The Cartan matrix is $C = (\alpha_j(\alpha_i^\vee))_{i,j\in I}$ and $r_1, \cdots, r_n > 0$ integers are minimal so that we have a symmetric matrix

$$B = \mathrm{diag}(r_1, \cdots, r_n)C.$$

Let $q \in \mathbb{C}^*$ not a root of unity. For $i \in I$, we set $q_i = q^{r_i}$. The quantum Cartan matrix $C(q)$ is defined for $i \neq j \in I$ by

$$C_{i,i}(q) = [2]_{q_i} \text{ and } C_{i,j}(q) = [C_{i,j}]_q,$$

with the standard q-number notation $[m]_u = \frac{u^m - u^m}{u - u^{-1}}$ for $m \in \mathbb{Z}$, $u \in \mathbb{C}^* \setminus \{-1, 1\}$.

Set $\mathfrak{t}^* = \left(\mathbb{C}^\times\right)^I$, and endow it with a group structure by pointwise multiplication. We define a group morphism $^{-} : P_\mathbb{Q} \longrightarrow \mathfrak{t}^*$ by setting $\overline{\omega}(i) = q_i^{\omega(\alpha_i^\vee)}$. We shall use the standard partial ordering on $\mathfrak{t}^*$:

$$\omega \preceq \omega' \quad \text{if } \omega'\omega^{-1} \text{ is a product of } \{\overline{\alpha}_i\}_{i\in I}.$$

We introduce the following generalization of q-oscillator algebras. For $J, K \subset I$, the algebra $\mathcal{U}_q^{J,K}(\mathfrak{g})$ is defined with same generators $k_i^{\pm 1}$, e_i, f_i as for the quantum group $\mathcal{U}_q(\mathfrak{g})$ but with the modified relation

$$[e_i, f_i] = \frac{\delta_{i\notin I} k_i - \delta_{i\notin J} k_i^{-1}}{q_i - q_i^{-1}} \text{ for } i \in I.$$

The other relations are the same, that is for $i, j \in I$

$$k_i k_j = k_j k_i \ , \ k_i e_j = q_i^{C_{i,j}} e_j k_i \ , \ k_i f_f = q_i^{-C_{i,j}} f_j k_i,$$

and if $i \neq j$, $[e_i, f_j] = 0$ and

$$\sum_{0\leq r\leq 1-C_{i,j}} (-1)^r \begin{bmatrix} 1 - C_{i,j} \\ r \end{bmatrix}_{q_i} e_i^{1-C_{i,j}-r} e_j e_i^r = 0 = \sum_{0\leq r\leq 1-C_{i,j}} (-1)^r \begin{bmatrix} 1 - C_{i,j} \\ r \end{bmatrix}_{q_i} f_i^{1-C_{i,j}-r} f_j f_i^r.$$

We used here the standard q-binomials from the standard q-factorial notations.

2.2. **q-oscillator algebras in the sl_2-case.** We recover the usual q-oscillator algebras

$$\mathcal{U}_q^+(sl_2) = \mathcal{U}_q^{\emptyset,\{1\}}(sl_2) \text{ and } \mathcal{U}_q^-(sl_2) = \mathcal{U}_q^{\{1\},\emptyset}(sl_2).$$

$\mathcal{U}_q^\pm(sl_2)$ is generated by e, f, k, k^{-1} with relations

$$ke = q^2 ek \ , \ kf = q^{-2} fk \ , \ kk^{-1} = k^{-1}k = 1 \ , \ [e, f] = \frac{\pm k^{\pm 1}}{q - q^{-1}}.$$

Note that exchanging e and f, k and k^{-1}, we have an isomorphism

$$\mathcal{U}_q^+(sl_2) \simeq \mathcal{U}_q^-(sl_2). \tag{2.1}$$

Remark 2.1. *(i) The quantum Boson algebra $\mathcal{B}_q(sl_2)$ of Kashiwara* [Kas1] *is isomorphic to the subalgebra of $\mathcal{U}_q^+(sl_2)$ generated by f and*

$$e' = (q - q^{-1})k^{-1}e \text{ as } e'f - q^2 fe' = 1.$$

(ii) We denote $\mathcal{U}_{q,loc}^{\pm}(sl_2)$ the algebra $\mathcal{U}_q^{\pm}(sl_2)$ localized at the Casimir central element

$$C_\pm = ef + \frac{q^{\mp 1}k^{\pm 1}}{(q-q^{-1})^2} = fe + \frac{q^{\pm 1}k^{\pm 1}}{(q-q^{-1})^2}.$$

(iii) The algebra $\mathcal{U}_q^{\pm}(sl_2)$ has a natural triangular decomposition. In particular the Borel subalgebra $\mathcal{U}_q(\mathfrak{b}) \subset \mathcal{U}_q(sl_2)$ generated by e, k, k^{-1} is a subalgebra of $\mathcal{U}_q^{\pm}(sl_2)$.

Proposition 2.2. *There is an anti-isomorphism $S : \mathcal{U}_q^+(sl_2) \to \mathcal{U}_q^-(sl_2)$ defined by*

$$S(e) = -ek \ , \ S(f) = -k^{-1}f \ , \ S(k) = k^{-1} \ , \ S(k^{-1}) = k.$$

Composing with the isomorphism (2.1), it gives also an anti-automorphisms

$$S^{\pm} : \mathcal{U}_q^{\pm}(sl_2) \to \mathcal{U}_q^{\pm}(sl_2).$$

Proof. It suffices to check the relations are preserved. For the first three relations, it follows from the fact that the usual antipode is well-defined. For the last one, one has :

$$[S(f), S(e)] = [f, e] = \frac{k^{-1}}{q - q^{-1}} = S([e, f]).$$

□

Proposition 2.3. *There are algebra morphisms*

$$\Delta_\pm : \mathcal{U}_q(sl_2) \to \mathcal{U}_q^{\pm}(sl_2) \otimes \mathcal{U}_q^{\mp}(sl_2),$$

$$\Delta_\pm(e) = e \otimes 1 + k^{\mp 1} \otimes e \ , \ \Delta_\pm(f) = f \otimes k^{\pm 1} + 1 \otimes f \ , \ \Delta_\pm(k) = k \otimes k \ , \ \Delta_\pm(k^{-1}) = k^{-1} \otimes k^{-1}.$$

The same formulas define left-comodule and right-comodule structures :

$$\mathcal{U}_q^{\pm}(sl_2) \to \mathcal{U}_q(sl_2) \otimes \mathcal{U}_q^{\pm}(sl_2) \ , \ \mathcal{U}_q^{\mp}(sl_2) \to \mathcal{U}_q^{\mp}(sl_2) \otimes \mathcal{U}_q(sl_2).$$

Proof. It suffices to check the compatibility with the defining relations of $\mathcal{U}_q(sl_2)$. For the first three standard relations which are the same as for the shifted quantum groups, it follows from the fact that the usual coproduct is well-defined. For the last one, one has :

$$[\Delta_\pm(e), \Delta_\pm(f)] = [e \otimes 1 + k^{\mp 1} \otimes e, f \otimes k^{\pm 1} + 1 \otimes f] = [e \otimes 1, f \otimes k^{\pm 1}] + [k^{\mp 1} \otimes e, 1 \otimes f]$$

$$= \frac{\pm k^{\pm 1}}{q - q^{-1}} \otimes k^{\pm 1} \mp k^{\mp 1} \otimes \frac{k^{\mp 1}}{q - q^{-1}} = \frac{k \otimes k - k^{-1} \otimes k^{-1}}{q - q^{-1}} = \Delta_\pm([e, f]).$$

□

Note that composing with the isomorphim (2.1), we also get an algebra morphism

$$\mathcal{U}_q(sl_2) \to \mathcal{U}_q^+(sl_2) \otimes \mathcal{U}_q^+(sl_2)$$

which can be considered as an analog of a coproduct $\mathcal{U}_q^+(sl_2)$. The author believes this map is known but could not find it in the literature.

2.3. **Representations of $\mathcal{U}_q^{\pm}(sl_2)$.** Let $\mathcal{C}_0$ be the category $\mathcal{U}_q(sl_2)$-module and $\mathcal{C}_1$ be the category of $\mathcal{U}_q^{+}(sl_2)$-modules. It is well known $\mathcal{C}_0$ has a monoidal structure

$$\otimes : \mathcal{C}_0 \times \mathcal{C}_0 \to \mathcal{C}_0.$$

From the algebra morphisms above we get bi-functors

$$\mathcal{C}_0 \times \mathcal{C}_1 \to \mathcal{C}_1 \ , \ \mathcal{C}_1 \times \mathcal{C}_0 \to \mathcal{C}_1 \ , \ \mathcal{C}_1 \times \mathcal{C}_1 \to \mathcal{C}_0,$$

which make the category $\mathcal{C}_0 \oplus \mathcal{C}_1$ into a $\mathbb{Z}/2\mathbb{Z}$-graded monoidal category. In particular we have a ring structure on

$$K_0(\mathcal{C}_0) \oplus K_0(\mathcal{C}_1).$$

Remark 2.4. *(i) $\mathcal{C}_0$ admits a left and a right action by the category $\mathcal{C}_1$.*

(ii) For $V^{\pm}$ a representation of $\mathcal{U}_q^{\pm}(sl_2)$, $V^+ \otimes V^-$ and $V^- \otimes V^+$ are $\mathcal{U}_q(sl_2)$-modules.

(iii) We expect to get a $(\mathbb{Z}/2\mathbb{Z})^n$-graded monoidal category for a general $\mathfrak{g}$, although we will not use it in this paper. We plan to come back to it in a forthcoming paper.

From the triangular decomposition (see (iii) in Remark 2.1), the algebra $\mathcal{U}_q^{\pm}(sl_2)$ has well a defined Verma modules associated to each $\gamma \in \mathbb{C}^*$ eigenvalue of k. Precisely, these representations $V_\gamma = \mathrm{Vect}(v_r)_{r\geq 0}$, $W_\gamma = \mathrm{Vect}(w_r)_{r\geq 0}$ can be explicitly described :

$$e.v_r = \delta_{r>0} v_{r-1} \ , \ f.v_r = \gamma q^{-r} \frac{[r+1]_q}{q-q^{-1}} v_{r+1} \ , \ k.v_r = \gamma q^{-2r} v_r,$$

$$e.w_r = \delta_{r>0} w_{r-1} \ , \ f.w_r = -\gamma^{-1} q^{r} \frac{[r+1]_q}{q-q^{-1}} w_{r+1} \ , \ k.w_r = \gamma q^{-2r} w_r.$$

Remark 2.5. *V_γ is also a representation of $\mathcal{U}_{q,loc}^{+}(sl_2)$ as the action of C_+ is $\frac{q\gamma Id}{(q-q^{-1})^2}$.*

For V a representation of $\mathcal{U}_q^{\pm}(sl_2)$, we can define its weight space $V_\gamma = \{v \in V | k.v = \gamma\}$ for $\gamma \in \mathbb{C}^*$. We have a corresponding category $\mathcal{O}^{\pm}$ of $\mathcal{U}_q^{\pm}(sl_2)$-modules (defined as for the ordinary category $\mathcal{O}$, see Section 4.1) and the corresponding character morphisms :

$$\chi : K_0(\mathcal{O}^{\pm}) \to \mathcal{E} \ , \ \chi(V) = \sum_\gamma \dim(V_\gamma)[\gamma],$$

where $\mathcal{E} \subset \mathbb{Z}^{\mathbb{C}^*}$ is a ring of formal power series as for the category $\mathcal{O}$ of $\mathcal{U}_q(sl_2)$ (see [HJ, Section 3.4] for instance).

Proposition 2.6. *The Verma modules of $\mathcal{U}_q^{\pm}(sl_2)$ are irreducible and exhaust all simple modules of the category $\mathcal{O}^{\pm}$. The non-zero representations of $\mathcal{U}_q^{\pm}(sl_2)$ are infinite-dimensional.*

Proof. In V_γ (resp. W_γ), the highest weight vectors $\mathbb{C}.v_0$ (resp. $\mathbb{C}.w_0$) are the only primitive vectors. This implies V_γ and W_γ are simple.

By standard arguments, a simple finite-dimensional representation of $\mathcal{U}_q^{\pm}(sl_2)$ is a quotient of a Verma module, the second point follows. ☐

Example 2.7. *(i) For $\gamma, \beta \in \mathbb{C}^*$, the $\mathcal{U}_q(sl_2)$-modules $V_\gamma \otimes W_\beta$ and $W_\beta \otimes V_\gamma$ have character*

$$\sum_{r,r'\geq 0} [\gamma\beta q^{-2(r+r')}] = \sum_{r\geq 0} \chi(M(\gamma\beta q^{-2r}))$$

where $M(\lambda)$ is the Verma module of $\mathcal{U}_q(sl_2)$ of highest weight λ. For γ, β so that $\gamma\beta \notin q^{\mathbb{Z}_{\geq 0}}$, these representations are semi-simple and

$$V_\gamma \otimes W_\beta \simeq W_\beta \otimes V_\gamma \simeq \bigoplus_{r\geq 0} M(\gamma\beta q^{-2r}).$$

Indeed, the weight space associated to $\gamma\beta q^{-2r}$ has dimension $r+1$. Hence e is not injective on this weight space which contains a primitive vector generating $M(\gamma\beta q^{-2r})$.

(ii) For $\gamma \in \mathbb{C}^*$ and V fundamental $\mathcal{U}_q(sl_2)$-module of highest weight $[q]$, we have

$$V \otimes V_\gamma \simeq V_{\gamma q} \oplus V_{\gamma q^{-1}}.$$

3. Shifted quantum affine algebras

We recall, for μ in the coweight lattice, the definition of the shifted quantum affine algebra $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ in the sense of Finkelberg-Tsymbaliuk [FT] and its first properties. In the sl_2-case, we introduce evaluation morphisms to the q-oscillator algebra of the previous section which give the first examples of evaluation representations (Proposition 3.7). For general types, we recall the definition of the shift homomorphism and we prove its injectivity (Proposition 3.11). We prove that for μ anti-dominant, the shifted quantum affine algebra $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ contains a subalgebra isomorphic to the quantum affine Borel subalgebra $\mathcal{U}_q(\hat{\mathfrak{b}})$ of the quantum affine algebra (Proposition 3.3).

3.1. **Definition and first properties.** For $\hat{\mathfrak{g}}$ an untwisted affine algebra, recall the Drinfeld presentation of the quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ (established in [Dr, Be, Da2], see for instance [He1, Definition 3.1]).

Let $\Lambda = \bigoplus_{i\in I} \mathbb{Z}\omega_i^\vee$ be the coroot lattice (that is the weight lattice of the Langlands dual Lie algebra ${}^L\mathfrak{g}$). It contains the set Λ^+ of dominant coweights $\omega^\vee \in \Lambda$ so that $\alpha_i(\omega^\vee) \geq 0$ for $i \in I$.

Let $\mu_+, \mu_- \in \Lambda$. The shifted quantum affine algebra $\mathcal{U}_q^{\mu_+,\mu_-}(\hat{\mathfrak{g}})$ is defined in [FT, Section 5] by Drinfeld generators $x_{i,m}^\pm$, $\phi_{i,m}^\pm$, $h_{i,r}$ with $i \in I, m \in \mathbb{Z}, r \in \mathbb{Z}\setminus\{0\}$ and relations [He1, (7), (8), (9), (10)], where

$$\sum_{r\in\mathbb{Z}} \phi_{i,\pm r}^\pm z^{\pm r} = \phi_i^\pm(z) = z^{\mp\alpha_i(\mu_\pm)}\phi_{i,\mp\alpha_i(\mu_\pm)}^\pm \exp(\pm(q_i - q_i^{-1})\sum_{r>0} h_{i,\pm r} z^{\pm r}),$$

and $\phi_{i,\mp\alpha_i(\mu_\pm)}^\pm$ are invertible and satisfy the same relations [He1, (5, 6)] as $k_i^{\pm 1}$.

Explicitly, for $i, j \in I$, $r, r' \in \mathbb{Z}$, $m \in \mathbb{Z}\setminus\{0\}$, we have

$$[\phi_{i,r}^\pm, \phi_{j,r'}^\pm] = [\phi_{i,r}^\pm, \phi_{j,r'}^\mp] = 0, \tag{3.2}$$

$$\phi_{i,-\alpha_i(\mu_+)}^+ x_{j,r}^\pm = q_i^{\pm C_{i,j}} x_{j,r}^\pm \phi_{i,-\alpha_i(\mu_+)}^+ \text{ and } \phi_{i,\alpha_i(\mu_-)}^- x_{j,r}^\pm = q_i^{\mp C_{i,j}} x_{j,r}^\pm \phi_{i,\alpha_i(\mu_-)}^-, \tag{3.3}$$

$$[h_{i,m}, x_{j,r}^\pm] = \pm\frac{1}{m}[mC_{i,j}]_{q_i} x_{j,m+r}^\pm, \tag{3.4}$$

$$[x_{i,r}^+, x_{j,r'}^-] = \delta_{i,j}\frac{\phi_{i,r+r'}^+ - \phi_{i,r+r'}^-}{q_i - q_i^{-1}}, \tag{3.5}$$

$$x^{\pm}_{i,r+1}x^{\pm}_{j,r'} - q^{\pm B_{i,j}}x^{\pm}_{j,r'}x^{\pm}_{i,r+1} = q^{\pm B_{i,j}}x^{\pm}_{i,r}x^{\pm}_{j,r'+1} - x^{\pm}_{j,r'+1}x^{\pm}_{i,r}, \tag{3.6}$$

and for $i \neq j$, $r_1, \cdots, r_s \in \mathbb{Z}$ where $s = 1 - C_{i,j}$,

$$\sum_{\pi \in \Sigma_s} \sum_{0 \leq r \leq s} (-1)^r \begin{bmatrix} s \\ r \end{bmatrix}_{q_i} x^{\pm}_{i,r_{\pi(1)}} \cdots x^{\pm}_{i,r_{\pi(r)}} x^{\pm}_{j,r'} x^{\pm}_{i,r_{\pi(r+1)}} \cdots x^{\pm}_{i,r_{\pi(s)}} = 0. \tag{3.7}$$

The relations may be written in terms of currents $x_i^{\pm}(z) = \sum_{r\in\mathbb{Z}} x^{\pm}_{i,r}z^r$, $\delta(z) = \sum_{r\in\mathbb{Z}} z^r$:

$$[\phi_i^{\pm}(z), \phi_j^{\pm}(w)] = [\phi_i^{\pm}(z), \phi_j^{\mp}(w)] = 0,$$

$$\phi_i^{\epsilon}(z)x_j^{\pm}(w) = \frac{q^{\pm B_{i,j}}w - z}{w - q^{\pm B_{i,j}}z} x_j^{\pm}(w)\phi_i^{\epsilon}(z) \text{ for } \epsilon = + \text{ or } -, \tag{3.8}$$

$$[x_i^+(z), x_j^-(w)] = \frac{\delta_{i,j}}{q_i - q_i^{-1}} \left[\delta\left(\frac{w}{z}\right)\phi_i^+(z) - \delta\left(\frac{z}{w}\right)\phi_i^-(z)\right], \tag{3.9}$$

$$(w - q^{\pm B_{i,j}}z)x_i^{\pm}(z)x_j^{\pm}(w) = (q^{\pm B_{i,j}}w - z)x_j^{\pm}(w)x_i^{\pm}(z),$$

$$\sum_{\pi \in \Sigma_s} \sum_{0 \leq r \leq s} (-1)^r \begin{bmatrix} s \\ r \end{bmatrix}_{q_i} x_i^{\pm}(w_{\pi(1)}) \cdots x_i^{\pm}(w_{\pi(r)}) x_j^{\pm}(z) x_i^{\pm}(z_{\pi(r+1)}) \cdots x_i^{\pm}(z_{\pi(s)}) = 0 \text{ for } i \neq j.$$

Remark 3.1. *(i) Up to isomorphism, $\mathcal{U}_q^{\mu_+,\mu_-}(\hat{\mathfrak{g}})$ depends only on $\mu = \mu_+ + \mu_-$, see* [FT, Section 5.(i)]. *We will simply denote $\mathcal{U}_q^{\mu}(\hat{\mathfrak{g}}) = \mathcal{U}_q^{0,\mu}(\hat{\mathfrak{g}})$.*

(ii) For $i \in I$, the product

$$\phi^+_{i,-\alpha_i(\mu_+)}\phi^-_{i,\alpha_i(\mu_-)}$$

is central. The quantum loop algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ is the quotient of $\mathcal{U}_q^{0,0}(\hat{\mathfrak{g}})$ by identifying $\phi^+_{i,0}\phi^-_{i,0}$ with 1 for $i \in I$, see [FT].

(iii) The algebra $\mathcal{U}_q^{\mu_+,\mu_-}(\hat{\mathfrak{g}})$ has a triangular decomposition analog to the Drinfeld triangular decomposition of $\mathcal{U}_q(\hat{\mathfrak{g}})$ (see [FT, Proposition 2] *and* [He1, Theorem 2]*). Each triangular factor can be presented by their Drinfeld generators and the relations of $\mathcal{U}_q^{\mu_+,\mu_-}(\hat{\mathfrak{g}})$ involving these generators (there are no hidden relations). The subalgebra generated by the $\phi^{\pm}_{i,m}$, $(\phi^{\pm}_{i,\alpha_i(\mu_{\pm})})^{\mp 1}$, with $i \in I$, $m \in \mathbb{Z}$, is commutative and called the Cartan-Drinfeld subalgebra.*

(iv) Consequently, for $\mu_+, \mu_- \in -\Lambda^+$ and $J_{\pm} = \{i \in I | \alpha_i(\mu_{\mp}) \neq 0\}$, the q-oscillator algebra $\mathcal{U}_q^{J_+,J_-}(\mathfrak{g})$ of the previous section is a subalgebra of

$$\mathcal{U}_q^{\mu_+,\mu_-}(\hat{\mathfrak{g}}) / < \phi^+_{i,0}\phi^-_{i,0} = 1, i \notin J_+ \cup J_- > .$$

(v) The algebra $\mathcal{U}_q^{\mu_+,\mu_-}(\hat{\mathfrak{g}})$ has a natural $\mathbb{Z}$-grading defined so that

$$deg(x^{\pm}_{i,m}) = deg(\phi^{\pm}_{i,m}) = m \text{ and } deg(h_{i,r}) = r \text{ for } i \in I,\ m \in \mathbb{Z},\ r \in \mathbb{Z} \setminus \{0\}.$$

In particular, for $a \in \mathbb{C}^*$, there is an algebra automorphism τ_a of $\mathcal{U}_q^{\mu_+,\mu_-}(\hat{\mathfrak{g}})$ such that for $i \in I$, $m \in \mathbb{Z}$ and $r \in \mathbb{Z} \setminus \{0\}$:

$$\tau_a(x^{\pm}_{i,m}) = a^m x^{\pm}_{i,m} \ ,\ \tau_a(\phi^{\pm}_{i,m}) = a^m \phi^{\pm}_{i,m} \ ,\ \tau_a(h_{i,r}) = a^r h_{i,r}.$$

(vi) z^{-1} in the notations of [FT] *is z here.*

(vii) In type A, RTT realizations have been established [FT, FPT] *when $\mu \in -\Lambda^+$.*

Example 3.2. *Let $i \in I$. For $\mathcal{U}_q^{-\omega_i^\vee,0}(\hat{\mathfrak{g}})$ (resp. $\mathcal{U}_q^{0,-\omega_i^\vee}(\hat{\mathfrak{g}})$), the modified relations are :*

$$[x_{i,r}^+, x_{i,-r}^-] = \frac{-\phi_{i,0}^-}{q_i - q_i^{-1}} \text{ (resp. } [x_{i,r}^+, x_{i,-r}^-] = \frac{\phi_{i,0}^+}{q_i - q_i^{-1}}\text{) for } r \in \mathbb{Z},$$

with the definition of the $\phi_{i,r}^+$ (resp. $\phi_{i,-r}^-$) modified to :

$$\phi_i^+(z) = z\phi_{i,1}^+ exp((q_i - q_i^{-1})\sum_{r>0} h_{i,r} z^r)$$

$$\text{(resp. } \phi_i^-(z) = z^{-1}\phi_{i,-1}^- exp(-(q_i - q_i^{-1})\sum_{r>0} h_{i,-r} z^{-r})).$$

Let $\mathcal{U}_q(\hat{\mathfrak{b}})$ be the quantum affine Borel subalgebra of $\mathcal{U}_q(\hat{\mathfrak{g}})$, in the sense of Drinfeld-Jimbo. It is generated by a subset e_i, $k_i^{\pm 1}$, $0 \leq i \leq n$, of the Chevalley generators of $\mathcal{U}_q(\hat{\mathfrak{g}})$.

Proposition 3.3. *If $\mu \in -\Lambda^+$, then $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ contains a subalgebra isomorphic to $\mathcal{U}_q(\hat{\mathfrak{b}})$.*

Proof. Let us consider the triangular decomposition of $\mathcal{U}_q(\hat{\mathfrak{g}})$ as in (iii) of Remark 3.1. It induces a triangular decomposition of the quantum affine Borel algebra $\mathcal{U}_q(\hat{\mathfrak{b}})$ (this follows from [Be, Da1], see [HJ, Section 2.3] for instance) :

$$\mathcal{U}_q(\hat{\mathfrak{b}}) \simeq \mathcal{U}_q^-(\hat{\mathfrak{b}}) \otimes \mathcal{U}_q^0(\hat{\mathfrak{b}}) \otimes \mathcal{U}_q^+(\hat{\mathfrak{b}}).$$

Now let us consider the analog triangular decomposition of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$:

$$\mathcal{U}_q^\mu(\hat{\mathfrak{g}}) \simeq \mathcal{U}_q^{\mu,-}(\hat{\mathfrak{g}}) \otimes \mathcal{U}_q^{\mu,0}(\hat{\mathfrak{g}}) \otimes \mathcal{U}_q^{\mu,+}(\hat{\mathfrak{g}})$$

as in (iii) of Remark 3.1. The triangular decomposition statement is not only a linear isomorphism with the tensor product of the three algebras. In addition, it states that each of the three algebras can be presented by Drinfeld generators and the relations involving these generators only (no additional hidden relations are necessary).

Hence each triangular factor, $\mathcal{U}_q^{\mu,-}(\hat{\mathfrak{g}})$, $\mathcal{U}_q^{\mu,0}(\hat{\mathfrak{g}})$, $\mathcal{U}_q^{\mu,+}(\hat{\mathfrak{g}})$, is isomorphic, as an algebra, to the corresponding triangular factor in $\mathcal{U}_q^0(\hat{\mathfrak{g}})$. Hence $\mathcal{U}_q^{\mu,-}(\hat{\mathfrak{g}})$ (resp. $\mathcal{U}_q^{\mu,0}(\hat{\mathfrak{g}})$, $\mathcal{U}_q^{\mu,+}(\hat{\mathfrak{g}})$) contains a subalgebra isomorphic to $\mathcal{U}_q^-(\hat{\mathfrak{b}})$ (resp. $\mathcal{U}_q^0(\hat{\mathfrak{b}})$, $\mathcal{U}_q^+(\hat{\mathfrak{b}})$). Let us prove these three subalgebras generate a subalgebra isomorphic to $\mathcal{U}_q(\hat{\mathfrak{b}})$. As μ is anti-dominant, the commutations relations between the Drinfeld generators of positive degrees are the same in $\mathcal{U}_q^0(\hat{\mathfrak{g}})$ and in $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$. Hence we have a surjective ring morphism from $\mathcal{U}_q(\hat{\mathfrak{b}})$ to the subalgebra of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ generated by the three subalgebras. Then it is injective as it is injective on each triangular factor. $\square$

We will denote by e_i, $0 \leq i \leq n$, the corresponding Chevalley generators in $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$.

Example 3.4. *For $\mu \in -\Lambda^+$, the subalgebra of $\mathcal{U}_q^\mu(\hat{sl}_2)$ generated by $e_1 = x_{1,0}^+$, $e_0 = (\phi_{1,0}^+)^{-1} x_{1,1}^-$, $(\phi_{1,0}^+)^{\pm 1}$ is isomorphic to $\mathcal{U}_q(\hat{\mathfrak{b}}) \subset \mathcal{U}_q(\hat{sl}_2)$.*

Remark 3.5. *The subalgebra isomorphic to $\mathcal{U}_q(\hat{\mathfrak{b}})$ contains the $x_{i,m}^+$, $x_{i,r}^-$, $\phi_{i,m}^+$, $(\phi_{i,0}^+)^{-1}$ for $i \in I$, $m \geq 0$, $r > 0$, but is not generated by these elements (except in the sl_2-case).*

The same argument as for the proof of Proposition 3.3 implies the following.

Proposition 3.6. *Let $\mu \in \Lambda$. The subalgebras of $\mathcal{U}_q(\hat{\mathfrak{b}})$ and of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ generated by :*

$$x_{i,r}^+ \ , \ x_{i,s}^- \ , \ \phi_{i,r}^+ \ , \ (\phi_{i,0}^+)^{-1}$$

for $i \in I$, $r \geq 0$, $s > Max(0, \alpha_i(\mu))$, are isomorphic.

We will denote this algebra by $\mathcal{U}_q^\mu(\hat{\mathfrak{b}})$.

3.2. **Example - the algebras $\mathcal{U}_q^\pm(\hat{sl}_2)$.** In the sl_2-case, we will simply denote

$$\mathcal{U}_q^+(\hat{sl}_2) = \mathcal{U}_q^{0,-\omega_1^\vee}(\hat{\mathfrak{g}}) \text{ and } \mathcal{U}_q^-(\hat{sl}_2) = \mathcal{U}_q^{-\omega_1^\vee,0}(\hat{\mathfrak{g}}).$$

Note that we have $e_1 = x_{1,0}^+$ and $e_0 = k_1^{-1} x_{1,0}^-$.

It is well known there exist evaluation morphisms $\mathcal{U}_q(\hat{\mathfrak{b}}) \to \mathcal{U}_q^\pm(sl_2)$ (see [BLZ]) but they can not be extended to $\mathcal{U}_q(\hat{sl}_2)$. We prove it can be extended to a shifted quantum affine algebra.

Proposition 3.7. *For $a \in \mathbb{C}^*$, we have evaluation morphisms*

$$ev_a^\pm : \mathcal{U}_q^\pm(\hat{sl}_2) \to \mathcal{U}_{q,loc}^\pm(sl_2)$$

defined for $m \in \mathbb{Z}$, $r > 0$ by :

$$ev_a(x_m^+) = a^m e k^m \ , \ ev_a(x_m^-) = a^m k^m f,$$

$$ev_a(\phi_r^+) = a^r [ek^r, f](q - q^{-1}) \ , \ ev_a(\phi_{-r}^-) = a^{-r}[ek^{-r}, f](q^{-1} - q),$$

$$ev_a(\phi_0^+) = \delta_{\pm,+} k \ , \ ev_a(\phi_0^-) = \delta_{\pm,-} k^{-1}.$$

Remark 3.8. *(i) Only the last two formulas differ from the Jimbo evaluation morphism* [J]

$$\mathcal{U}_q(\hat{sl}_2) \to \mathcal{U}_q(sl_2).$$

(ii) We expect analog evaluation morphism exist for $\mathfrak{g} = sl_{n+1}$.

Proof. First we have

$$ev_a^\pm(\phi_{\mp 1}^\mp) = -q^{\pm 1} a^{\mp 1} (q - q^{-1})^2 C_\pm k^{\mp 1}$$

which is invertible in $\mathcal{U}_{q,loc}^\pm(sl_2)$. We have to check the other defining relations of $\mathcal{U}_q^\pm(\hat{sl}_2)$ are compatible with the defining formulas of ev_a. From the result for the standard evaluation morphism, this is clear for the formulas (3.2), (3.3), (3.4), (3.5), (3.6) except of $r + r' = 0$ (note that are no Drinfeld-Serre relations (3.7) in the sl_2-case). In the last case, we have for $r \in \mathbb{Z}$

$$[x_r^+, x_{-r}^-] = \frac{\phi_0^+ - \phi_0^-}{q - q^{-1}} \mapsto \frac{\delta_{\pm,+} k - \delta_{\pm,-} k^{-1}}{q - q^{-1}} = [e, f] = [a^e k^r, a^{-r} k^{-r} f] = [ev_a(x_r^+), ev_a(x_{-r}^-)].$$

$\square$

Example 3.9. *Recall the Verma module V_γ of $\mathcal{U}_q^+(sl_2)$. Its evaluation at $q^2\gamma^{-1}$ satisfies*

$$e_0.v_r = q^{-r+2} \frac{[r+1]_q}{q - q^{-1}} v_{r+1} \ , \ f_0.v_r = \gamma q^{-2} v_{r-1}.$$

By twisting by the automorphisms τ_a, we get a continuous family of representations that we denote by $L^-_{\gamma,a}$. The action of $\phi_\pm(z) \in \mathcal{U}_q^+(\hat{sl_2})[[z^{\pm 1}]]$ is given by

$$\phi^\pm(z).v_j = \gamma q^{-2j}\frac{1-q^2za}{(1-q^{2-2j}az)(1-q^{-2j}az)}.v_j \in L^-_{\gamma,a}[[z^{\pm 1}]].$$

This matches formulas in [HJ, Section 4.1] *: the restriction to $\mathcal{U}_q(\hat{\mathfrak{b}})$ is $[\gamma]\otimes L_a^{\mathfrak{b},-}$ where $L_a^{\mathfrak{b},-}$ is a prefundamental representation and $[\gamma]$ is 1-dimensional (see Section 4.1). Note that for $m \geq 0$, $r > 0$, we recover the action on $L^-_{\gamma,a}$ from the action on $L^-_{1,a}$ by the twist*

$$x_r^- \mapsto \gamma x_r^- \ ,\ x_m^+ \mapsto x_m^+ \ ,\ \phi_m^\pm \mapsto \gamma \phi_m^\pm.$$

Remark 3.10. *The action of $\mathcal{U}_q(\hat{\mathfrak{b}})$ on $L_{1,a}^{\mathfrak{b},-}$ can not be extended to $\mathcal{U}_q(\hat{sl_2})$ (see* [HJ]*). This shows there is no subalgebra of $\mathcal{U}_q^+(\widehat{sl_2})$ isomorphic to $\mathcal{U}_q(\widehat{\mathfrak{sl}_2})$ so that the standard quantum affine Borel subalgebras $\mathcal{U}_q(\hat{\mathfrak{b}})$ coincide.*

3.3. **Shift homomorphism.** A shift homomorphism is introduced in [FT, Section 10.(vii)] as an analog of the morphism defined for shifted Yangians in [FKPRW]. For $\mu \in \Lambda$, $\mu' \in -\Lambda^+$ and $a \in \mathbb{C}^*$ there is an algebra morphism

$$\iota_{\mu,\mu',a} : \mathcal{U}_q^\mu(\hat{\mathfrak{g}}) \to \mathcal{U}_q^{\mu+\mu'}(\hat{\mathfrak{g}})$$

defined by the following (for $i \in I$) :

$$x_i^+(z) \mapsto x_i^+(z)\ ,\ x_i^-(z) \mapsto (1-az)^{-\mu'(\alpha_i^\vee)}x_i^-(z)\ ,\ \phi_i^\pm(z) \mapsto (1-az)^{-\mu'(\alpha_i^\vee)}\phi_i^\pm(z). \tag{3.10}$$

This is obtained from the shift homomorphism in [FT] after conjugating by the change of variables $z \mapsto az$.

Proposition 3.11. *The shift homomorphism $\iota_{\mu,\mu',a}$ is injective.*

Remark 3.12. *(i) In type A, this is proved by another method in* [FT].

(ii) Consequently, for $\mu \in -\Lambda^+$, $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ contains a subalgebra isomorphic to $\mathcal{U}_q^0(\hat{\mathfrak{g}})$ and so a subalgebra isomorphic to $\mathcal{U}_q(\hat{\mathfrak{b}})$. It is not equal to the subalgebra constructed in the proof of Proposition 3.3 if $\mu \neq 0$ (this follows from Remark 3.10).

Proof. The algebra $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ admits a completion $\tilde{\mathcal{U}}_q^\mu(\hat{\mathfrak{g}})$ relatively to its natural $\mathbb{Z}$-grading (as defined in (v) in Remark 3.1) : each element in $\tilde{\mathcal{U}}_q^\mu(\hat{\mathfrak{g}})$ is a (possibly infinite) sum $\sum_{r\leq R} X_r$ where $R \geq 0$ and $X_r \in \mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ is homogeneous of degree r. For $x \in \mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ homogeneous of degree m, $\iota_{\mu,\mu',a}(x)$ is a sum of homogeneous elements of degree lower than m. Hence $\iota_{\mu,\mu',a}$ extends to an algebra morphism

$$\tilde{\iota}_{\mu,\mu',a} : \tilde{\mathcal{U}}_q^\mu(\hat{\mathfrak{g}}) \to \tilde{\mathcal{U}}_q^{\mu+\mu'}(\hat{\mathfrak{g}}),$$

by the defining formula (3.10). Then, the formulas

$$x_i^+(z) \mapsto x_i^+(z)\ ,\ x_i^-(z) \mapsto (1-az)^{\mu'(\alpha_i^\vee)}x_i^-(z)\ ,\ \phi_i^\pm(z) \mapsto (1-az)^{\mu'(\alpha_i^\vee)}\phi_i^\pm(z),$$

give a well-defined algebra morphism

$$\tilde{\iota}^{-1}_{\mu,\mu',a} : \tilde{\mathcal{U}}_q^{\mu+\mu'}(\hat{\mathfrak{g}}) \to \tilde{\mathcal{U}}_q^\mu(\hat{\mathfrak{g}}).$$

Hence we get an isomorphism

$$\tilde{\mathcal{U}}_q^{\mu}(\hat{\mathfrak{g}}) \simeq \tilde{\mathcal{U}}_q^{\mu+\mu'}(\hat{\mathfrak{g}}).$$

In particular, $\tilde{\iota}_{\mu,\mu',a}$ is injective and so is $\iota_{\mu,\mu',a}$ □

4. Category $\mathcal{O}^{sh}$

It is known since the work in [BLZ] that certain representations of the q-oscillator algebra associated to sl_2 give rise to representations of the quantum affine Borel subalgebra $\mathcal{U}_q(\hat{\mathfrak{b}})$ of the quantum affine algebra $\mathcal{U}_q(\hat{sl}_2)$ (see Section 3.2). For general untwisted types, the category $\mathcal{O}$ of representations of the quantum affine Borel subalgebra of a quantum affine algebra was introduced and studied in [HJ]. Some representations in this category extend to a representation of the full quantum affine algebra (so belong to the category $\hat{\mathcal{O}}$ of [He2]), but many do not.

It was first observed in [HJ] that for some of these representations, the structure of representation of the quantum affine Borel algebra can be extended to a larger algebra. It is called the asymptotical algebra $\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})$ and we will see this algebra is related to certain shifted quantum affine algebras. In the Yangian case, it was first observed in [Z] that for certain simple representations in an analog of the category $\mathcal{O}$ (the prefundamental representations), the action can be extended to a shifted Yangian.

In this section we introduce categories $\mathcal{O}_\mu$ of representations of shifted quantum affine algebras and classify their simple objects (Theorem 4.11). The category $\mathcal{O}^{sh}$ is the sum of these abelian categories. We also study shift functors induced by shift homomorphisms.

4.1. **Reminder - the category $\mathcal{O}$ for the quantum affine Borel algebra.** The category $\mathcal{O}$ of representations of $\mathcal{U}_q(\hat{\mathfrak{b}})$ is defined in [HJ] as an analog of the ordinary category $\mathcal{O}$ (see [Kac]). It is the category of $\mathcal{U}_q(\hat{\mathfrak{b}})$-modules V sum of their weight spaces

$$V_\omega = \{v \in V | k_i.v = \omega(i)v, \forall i \in I\} \text{ where } \omega \in \mathfrak{t}^*,$$

such that the V_ω are finite-dimensional and there are a finite number of $\omega_1, \cdots, \omega_s \in \mathfrak{t}^*$ so that the weight of V, that is the ω so that $V_\omega \neq 0$, belong to $D(\omega_1) \cup \cdots \cup D(\omega_s)$ where

$$D(\omega_i) = \{\omega \in \mathfrak{t}^* | \omega \preceq \omega_i\}.$$

A series $\mathbf{\Psi} = (\Psi_{i,m})_{i\in I, m\geq 0}$ of complex numbers such that $\Psi_{i,0} \neq 0$ for all $i \in I$ is called an ℓ-weight. We also denote

$$\mathbf{\Psi} = (\Psi_i(z))_{i\in I} \text{ where } \Psi_i(z) = \sum_{m\geq 0} \Psi_{i,m} z^m.$$

A $\mathcal{U}_q(\mathfrak{b})$-module V is of highest ℓ-weight $\mathbf{\Psi}$ if there is $v \in V$ with $V = \mathcal{U}_q(\mathfrak{b}).v$ and

$$e_i\, v = 0 \quad (i \in I)\,, \qquad \phi_{i,m}^+ v = \Psi_{i,m} v \quad (i \in I,\ m \geq 0)\,.$$

For any $\mathbf{\Psi}$ ℓ-weight, there exists a unique simple highest ℓ-weight $\mathcal{U}_q(\hat{\mathfrak{b}})$-module $L^{\mathfrak{b}}(\mathbf{\Psi})$ of highest ℓ-weight $\mathbf{\Psi}$. For example, for $i \in I$ and $a \in \mathbb{C}^\times$, we have the prefundamental representations

$$L_{i,a}^{\mathfrak{b},\pm} = L^{\mathfrak{b}}(\mathbf{\Psi}_{i,a}^{\pm 1}) \quad \text{where} \quad (\mathbf{\Psi}_{i,a})_j(z) = \begin{cases} (1-za) & (j=i)\,, \\ 1 & (j \neq i)\,. \end{cases} \tag{4.11}$$

We denote $L^{\mathfrak{b}}$ to distinguish with representations of shifted quantum affine algebras we will study. For $\omega \in \mathfrak{t}^*$, we have the 1-dimensional representation, called constant representation

$$[\omega] = L^{\mathfrak{b}}(\mathbf{\Psi}_\omega) \quad \text{where} \quad (\mathbf{\Psi}_\omega)_i(z) = \omega(i) \quad (i \in I).$$

Let $\mathfrak{r}$ be the group rational of ℓ-weights $\mathbf{\Psi}$, so that the $\Psi_i(z)$ are rational for $i \in I$.

Theorem 4.1. [HJ] *The simple module in the category $\mathcal{O}$ are the $L^{\mathfrak{b}}(\mathbf{\Psi})$ for $\mathbf{\Psi} \in \mathfrak{r}$.*

For V in the category $\mathcal{O}$ and $\mathbf{\Psi}$ an ℓ-weight, we have the ℓ-weight space

$$V_{\mathbf{\Psi}} = \{v \in V \mid \exists p \geq 0, \forall i \in I, \forall m \geq 0, (\phi_{i,m}^+ - \Psi_{i,m})^p v = 0\}. \tag{4.12}$$

A representation in the category $\mathcal{O}$ is the direct sum of its ℓ-weight spaces. Moreover :

Theorem 4.2. [HJ] *For V in category $\mathcal{O}$, $V_{\mathbf{\Psi}} \neq 0$ implies $\mathbf{\Psi} \in \mathfrak{r}$.*

4.2. **Reminder - the category $\hat{\mathcal{O}}$ for the quantum affine algebra.** The category $\hat{\mathcal{O}}$ of $\mathcal{U}_q(\hat{\mathfrak{g}})$-modules which are in the category $\mathcal{O}$ as $\mathcal{U}_q(\hat{\mathfrak{b}})$-modules was introduced in [He1]. The categories $\hat{\mathcal{O}}$ and $\mathcal{O}$ are monoidal and there is a forgetful functor

$$\hat{f} : \hat{\mathcal{O}} \to \mathcal{O}.$$

The following was established by Bowman [Bo] and Chari-Greenstein [CG] for finite-dimensional representations. The proof in [HJ, Proposition 3.5] can be adapted to the category $\hat{\mathcal{O}}$.

Proposition 4.3. *For V a simple representation in $\hat{\mathcal{O}}$, $\hat{f}(V)$ is simple.*

Remark 4.4. *(i) It is proved in* [He1, Lemma 14] *and* [HJ, Lemma 3.9] *that for $i \in I$, the action of $\phi_i^+(z)$ and $\phi_i^-(z)$ coincide on a representation V in $\hat{\mathcal{O}}$, seen as rational operators on each weight space (it follows directly from the existence of a polynomial $P(z)$ so that $P(z)(\phi_i^+(z) - \phi_i^-(z)) = 0$; this is also proved in* [GTL, Section 3.6]*).*

(ii) As $\phi_i^+(z)$ (resp. $\phi_i^-(z)$) is regular at 0 (resp. ∞), this also implies that $\phi_i^+(z)$ has degree 0 and that $\phi_i^+(0)\phi_i^+(\infty) = Id$. In particular, the simple representations V in $\hat{\mathcal{O}}$ are parametrized by the highest ℓ-weight $\mathbf{\Psi}$ of $\hat{f}(V)$: it is rational of degree 0 with $\Psi_i(0)\Psi_i(\infty) = 1$ for $i \in I$. The converse statement is true by [MY] *: these rational ℓ-weights parametrize the simple representations in $\hat{\mathcal{O}}$.*

4.3. **The category $\tilde{\mathcal{O}}$ for the asymptotical algebra.** The asymptotical algebra $\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})$ is defined in [HJ] as the subalgebra of $\mathcal{U}_q(\hat{\mathfrak{g}})$ generated by the

$$x_{i,m}^+ \,,\, k_i^{-1} x_{i,m}^- \,,\, k_i^{-1}\phi_{i,m}^\pm \,,\, k_i^{-1} \text{ for } m \in \mathbb{Z},\, i \in I.$$

These elements are denoted respectively by $\tilde{x}_{i,m}^+$, $\tilde{x}_{i,m}^-$, $\tilde{\phi}_{i,m}^\pm$, κ_i. Note that $\tilde{\phi}_{i,0}^+ = 1$ and $\tilde{\phi}_{i,0}^- = \kappa_i^2$. We will denote

$$\tilde{\phi}_i^\pm(z) = \sum_{m \geq 0} \tilde{\phi}_{i,m}^\pm z^{\pm m}.$$

For our purposes, we will consider representations of $\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})$ with 0 as a possible eigenvalue of κ_i. Hence we have to modify the axioms to introduce the proper notion of category $\mathcal{O}$ for this algebra.

A representation V of $\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})$ is said to be $\mathfrak{t}^*$-graded if there is a decomposition into a direct sum of linear subspaces $V = \bigoplus_{\omega \in \mathfrak{t}^*} V^{(\omega)}$ such that

$$\tilde{x}_{i,r}^{\pm} V^{(\omega)} \subset V^{(\omega \overline{\alpha_i}^{\pm 1})}\ ,\ \tilde{\phi}_{i,\pm m}^{\pm} V^{(\omega)} \subset V^{(\omega)}\ ,\ \kappa_i V^{(\omega)} \subset V^{(\omega)} \text{ for any } \omega \in \mathfrak{t}^*,\ i \in I,\ r \in \mathbb{Z}, m \geq 0.$$

The weights of V are the ω so that $V^{(\omega)} \neq 0$.

Definition 4.5. *A $\mathfrak{t}^*$-graded representation V of $\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})$ is said to be in the category $\tilde{\mathcal{O}}$ if :*

(i) the $V^{(\omega)}$ are finite-dimensional,

(ii) for any $i \in I$, there is $b_i \leq$ so that $\tilde{\phi}_{i,b_i}^-$ is invertible, diagonalizable on V and is the non-zero coefficient of lower degree in z^{-1} of $\tilde{\phi}_i^-(z)$ seen in $End(V)[[z^{-1}]]$,

(iii) For each $\omega \in \mathfrak{t}^*$, the space $V_\omega = \{v \in V | \tilde{\phi}_{i,b_i}^-.v = \omega(i)v, \forall i \in I\}$ is finite-dimensional,

(iv) there are a finite number of elements $\omega_1, \cdots, \omega_s \in \mathfrak{t}^*$, so that $V^{(\omega)} \neq 0$ or $V_\omega \neq 0$ implies $\omega \in \bigcup_{1 \leq j \leq s} D(\omega_j)$.

The same argument as in Remark 4.4 (i) shows that for each $i \in I$, the action of $\tilde{\phi}_i^{\pm}(z)$ are rational on weight spaces and coincide as rational operators.

As for the category $\mathcal{O}$, a representation in $\tilde{\mathcal{O}}$ is the direct sum of its ℓ-weight spaces corresponding to pseudo-eigenvalues of the $\tilde{\phi}_i^+(z)$. A simple representation is determined up to isomorphism by its highest ℓ-weight $\mathbf{\Psi}$. It is rational, with $\deg(\Psi_i) \geq 0$ and $\Psi_i(0) = 1$ for any $i \in I$. We will prove that such $\mathbf{\Psi}$ parametrize the simple representations in $\tilde{\mathcal{O}}$ (see Theorem 7.3).

It is proved in [HJ, Section 2.4] that a $\mathfrak{t}^*$-graded[1] representation V gives a representation of $\mathcal{U}_q(\hat{\mathfrak{b}})$ such that

$$e_i\, v = \tilde{x}_{i,0}^+ v, \quad e_0\, v = y\, v, \quad k_i v = \omega(i) v \quad (i \in I,\ v \in V^{(\omega)})\,, \tag{4.13}$$

where $y \in \tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})$ is a certain distinguished element defined as iterated quantum brackets. This defines a functor

$$\tilde{f} : \tilde{\mathcal{O}} \to \mathcal{O}.$$

This is how the prefundamental representations $L_{i,a}^{\mathfrak{b},-}$ are constructed in [HJ].

Example 4.6. *Monoidal subcategories $\mathcal{O}^{\pm}$ of $\mathcal{O}$ were introduced in* [HL3] *in the context of monoidal categorification of cluster algebras. $\mathcal{O}^{\pm}$ is the subcategory of representations in $\mathcal{O}$ whose simple constituents have highest ℓ-weight which is a product of various $\mathbf{\Psi}_{i,a}\mathbf{\Psi}_{i,aq_i^2}^{-1}$, $\mathbf{\Psi}_{i,a}^{\mp 1}$, $[\omega]$ for various $i \in I$, $a \in \mathbb{C}^*$, $\omega \in \mathfrak{t}^*$. By* [HL3, Section 7.2]*, the simple representations in $\mathcal{O}^-$ are in the image of the functor $\tilde{f}$.*

[1]It is proved in [HJ] for Q-graded representations, but the proof also works for $\mathfrak{t}^*$-graded representations.

4.4. **The category $\mathcal{O}_\mu$ for the shifted quantum affine algebra.** For $\mu_+, \mu_- \in \Lambda$, we introduce the following category.

Definition 4.7. *The category $\mathcal{O}_{\mu_+,\mu_-}$ is the category of $\mathcal{U}_q^{\mu_+,\mu_-}(\hat{\mathfrak{g}})$-modules*

$$V = \bigoplus_{\omega\in\mathfrak{t}^*} V_\omega^+ = \bigoplus_{\omega\in\mathfrak{t}^*} V_\omega^-$$

where for $\omega \in \mathfrak{t}^*$, the weight space

$$V_\omega^\pm = \{v \in V | \phi^\pm_{i,\mp\alpha_i(\mu_\pm)}.v = (\omega(i))^{\pm 1} v, \forall i \in I\},$$

are finite-dimensional and there are a finite number of $\omega_1, \cdots, \omega_s \in \mathfrak{t}^*$ so that $V_\omega^\pm \neq \{0\}$ implies

$$\omega \in D(\omega_1) \cup \cdots \cup D(\omega_s).$$

Remark 4.8. *(i) By (i) in Remark 3.1, this category depends only on $\mu = \mu_+ + \mu_-$. We will be simply denote $\mathcal{O}_\mu = \mathcal{O}_{0,\mu}$.*

(ii) By (ii) in Remark 3.1, $\hat{\mathcal{O}}$ is a full subcategory of $\mathcal{O}_0$.

Proposition 4.9. *Let V be a representation in $\mathcal{O}_\mu$ (or in $\tilde{\mathcal{O}}$). For each weight space of V, there is a non-zero polynomial $P(z)$ so that for any $i \in I$, $P(z)(\phi_i^+(z) - \phi_i^-(z))$ and $P(z)x_i^\pm(z)$ are zero on this weight space. The action of $\phi_i^+(z)$ and $\phi_i^-(z)$ are rational of degree $\alpha_i(\mu)$ on this weight space and coincide as rational operators.*

Proof. The same argument as in Remark 4.4 (i) shows that for each $i \in I$, the action of $\phi_i^+(z)$ and $\phi_i^-(z)$ are rational on weight spaces of a representation in $\mathcal{O}_\mu$ and coincide as rational operators. In particular on each weight space $\phi_i^+(z)$ is equivalent to $\phi^-_{i,\alpha_i(\mu)} z^{\alpha_i(\mu)}$ when $z \to \infty$, which implies the degree is $\alpha_i(\mu)$ as $\phi^-_{i,\alpha_i(\mu)}$ is invertible. The statement for the $x_i^\pm(z)$ is proved as in [He2, Proposition 3.8] (it is proved there under the assumption the representation is integrable, but the fact that weight spaces are finite-dimensional is only used there; an analog result was also obtained in [BeK]). □

As above, the representations in $\mathcal{O}_\mu$ are the direct sum of their ℓ-weight spaces corresponding to pseudo-eigenvalues of the $\phi_i^+(z)$. The simple representations are determined up to isomorphism by their highest ℓ-weight $\mathbf{\Psi}$ which is rational with $\deg(\Psi_i) = \alpha_i(\mu)$. We will denote by $\mathfrak{r}_\mu$ the set of such ℓ-weights :

$$\mathfrak{r}_\mu = \{\mathbf{\Psi} = (\Psi_i(z))_{i\in I} \in \mathfrak{r} | \deg(\Psi_i(z)) = \alpha_i(\mu)\}.$$

A representation in $\mathcal{O}_\mu$ is said to be of highest ℓ-weight $\mathbf{\Psi} \in \mathfrak{r}_\mu$ if it is generated by a vector v such that

$$x^+_{i,m}.v = 0 \text{ and } \phi^\pm_{i,m}.v = \Psi^\pm_{i,m} v \text{ for } i \in I,\, m \in \mathbb{Z},$$

where

$$\Psi_i(z) = \sum_{m\geq 0} \Psi^+_{i,m} z^m = \sum_{m\geq \alpha_i(\mu)} \Psi^-_{i,-m} z^{-m} \in \mathbb{C}(z).$$

Corollary 4.10. *A simple representation in the category $\mathcal{O}_\mu$, $\mu \in -\Lambda^+$ (resp. in the category $\tilde{\mathcal{O}}$) is simple as a representation of $\mathcal{U}_q(\hat{\mathfrak{b}})$.*

Proof. It suffices to treat the case of a simple representation V in a category $\mathcal{O}_\mu$. From Proposition 4.9, we have established that the action of the operators $x^+_{i,m}$ (resp. $x^-_{j,s}$) for $m \in \mathbb{Z}$ are determined by the action of these operators for $m \geq 1$, which are in $\mathcal{U}_q(\hat{\mathfrak{b}})$. Hence, V is generated by its highest weight vectors and has no primitive vector as a representation of $\mathcal{U}_q(\hat{\mathfrak{b}})$. So it is simple. □

Theorem 4.11. *For $\mu \in \Lambda$, the simple modules in the category $\mathcal{O}_\mu$ are parametrized by $\mathfrak{r}_\mu$.*

Proof. We have seen in Section 4.4 that for $i \in I$, a simple representation in the category $\mathcal{O}_\mu$ has a highest ℓ-weight $\mathbf{\Psi}$ satisfying $\deg(\Psi_i) = \alpha_i(\mu)$ and that $\phi^+_i(z)$, $\phi^-_i(z)$ coincide as rational operators on this representation. So we have to prove that there exists a simple representation in the category $\mathcal{O}_\mu$ for each such ℓ-weight $\mathbf{\Psi}$. In opposition to the case of the category $\mathcal{O}$, we have all Drinfeld generators in $\mathcal{U}^\mu_q(\hat{\mathfrak{g}})$. So we do not have to us the strategy in [HJ] (asymptotical representation theory), but we can use arguments as for finite-dimensional representations in [CP] (see also [MY, Theorem 3.6]). We consider a representation L of highest ℓ-weight $\mathbf{\Psi}$ (such a representation can be constructed from a Verma module of highest ℓ-weight as in [HJ] for instance). It suffices to prove its weight spaces $L_{\omega'}$, $\omega' \in \mathfrak{t}^*$, are finite-dimensional. Let $\omega = \mathbf{\Psi}(0) \in \mathfrak{t}^*$ be the highest weight of L. As in [CP, Section 5, PROOF of (b)], this is proved by induction on the height of $\omega'\omega^{-1}$ factorized as a product of simple roots. The first step in the proof is to establish that for any $j \in I$, $L_{\omega\overline{\alpha_j}^{-1}}$ is finite-dimensional (the induction starts on weights ω' so that $\omega'\omega^{-1}$ has height 2). By the properties of $\mathbf{\Psi}$, there is a non-zero polynomial $P(z)$ such that for any $i \in I$, the operator $P(z)(\phi^+_i(z) - \phi^-_i(z))$ is 0 on L_ω. For $i, j \in I$ and $s \in \mathbb{Z}$, by the relation (3.9),

$$x^+_{i,s}(P(z)x^-_j(z)) = 0$$

on L_ω. As L is simple, we get $P(z)x^-_j(z) = 0$ on L_ω. This implies that $L_{\omega\overline{\alpha_j}^{-1}}$ is finite-dimensional. We finish the proof word by word as in [CP, Section 5, PROOF of (b)]. □

Example 4.12. *For $i \in I, a \in \mathbb{C}^*$, we have the positive and negative prefundamental representations $L^\pm_{i,a} = L(\mathbf{\Psi}^{\pm 1}_{i,a})$ in the category $\mathcal{O}_{\pm\omega_i^\vee}$. The representation $L(\mathbf{\Psi}_{i,a})$ is one-dimensional, with the action of the $x^\pm_{j,m}$ equal to 0 for $j \in I$, $m \in \mathbb{Z}$, and*

$$\phi^+_j(z) = 1 - z\delta_{i,j}\ ,\ \phi^-_i(z) = z(z^{-1} - \delta_{i,j}).$$

The structure of $\mathcal{U}_q^{-\omega_i^\vee}(\hat{\mathfrak{g}})$-module of $\tilde{L}^-_{i,a}$ extends the structure of $\mathcal{U}_q(\hat{\mathfrak{b}})$-module. This generalizes the result in the sl_2-case obtained in terms of evaluation morphisms in Example 3.9 ($L^-_{\gamma,a}$ there is $L(\gamma(1-za)^{-1})$). In the case of shifted Yangians, these examples were first discussed in [Z]. *It was first noted in* [HJ] *the action of $\mathcal{U}_q(\hat{\mathfrak{b}})$ can be extended to the asymptotical algebra, but not to the whole quantum affine algebra : we understand now that the correct framework is given by shifted quantum affine algebras.*

We now define the direct sum of the abelian categories

$$\mathcal{O}^{sh} = \bigoplus_{\mu \in \Lambda} \mathcal{O}_\mu.$$

By Theorem 4.11, the simple objects in $\mathcal{O}^{sh}$ are parametrized by $\mathfrak{r}$.

4.5. **Shift functors.** Let $\mu \in \Lambda$ and $\mu' \in -\Lambda^+$. Then the shift homomorphism $\iota_{\mu,\mu',a}$ defines a functor

$$\mathcal{R}_{\mu,\mu',a} : \mathcal{O}_{\mu+\mu'} \to \mathcal{O}_{\mu}.$$

Conversely, for a representation V of $\mathcal{I}_q(\hat{\mathfrak{g}})$, let us consider the $\mathcal{U}_q^{\mu+\mu'}(\hat{\mathfrak{g}})$-module :

$$\mathcal{I}_{\mu,\mu',a}(V) = \mathcal{U}_q^{\mu+\mu'}(\hat{\mathfrak{g}}) \otimes_{\mathcal{U}_q^{\mu}(\hat{\mathfrak{g}})} V.$$

This gives a functor

$$\mathcal{I}_{\mu,\mu',a} : \mathrm{Mod}_{\mu} \to \mathrm{Mod}_{\mu+\mu'},$$

from the category of representations Mod_{μ} of $\mathcal{U}_q^{\mu}(\hat{\mathfrak{g}})$ to the category $\mathrm{Mod}_{\mu+\mu'}$ of representations of $\mathcal{U}_q^{\mu+\mu'}(\hat{\mathfrak{g}})$.

From the defining formulas of $\iota_{\mu,\mu',a}$, we get the following.

Proposition 4.13. *Let $L(\mathbf{\Psi})$ be a simple module in $\mathcal{O}_{\mu+\mu'}$. Then $\mathcal{R}_{\mu,\mu',a}(L(\mathbf{\Psi}))$ is a highest ℓ-weight module of highest ℓ-weight*

$$\mathbf{\Psi}' = \mathbf{\Psi} \prod_{i \in I} \mathbf{\Psi}_{i,a}^{-\mu'(\alpha_i^\vee)}$$

and so admits $L(\mathbf{\Psi}')$ as a subquotient. Conversely, $\mathcal{I}_{\mu,\mu',a}(L(\mathbf{\Psi}'))$ is a representation of $\mathcal{U}_q^{\mu+\mu'}(\hat{\mathfrak{g}})$ of highest ℓ-weight $\mathbf{\Psi}$ which admits $L(\mathbf{\Psi})$ as a simple quotient.

Example 4.14. *For $i \in I$ and $a \in \mathbb{C}^*$ we have the functor*

$$\mathcal{R}_{\omega_i^\vee, -2\omega_i^\vee, a} : \mathcal{O}_{-\omega_i^\vee} \to \mathcal{O}_{\omega_i^\vee}.$$

Then, $L_{i,a}^+$ is a 1-dimensional subquotient of $\mathcal{R}_{\omega_i^\vee,-2\omega_i^\vee,a}(L_{i,a}^-)$ which is not simple.

5. Fusion product and Grothendieck ring

We construct the fusion product of representations of shifted quantum affine algebras in the category $\mathcal{O}^{sh}$ by using the deformed Drinfeld coproduct and the renomalization procedure in [He2] (Theorem 5.4). This leads to the definition of a ring structure on the Grothendieck group $K_0(\mathcal{O}^{sh})$. We establish a simple module in $\mathcal{O}^{sh}$ is a quotient of a fusion product of various prefundamental and constant representations (Corollary 5.6). Note that the Drinfeld coproduct is not an analog of the shifted Yangian coproduct of [FKPRW] (see Remark 5.3).

Along the way we consider analogs of Frenkel-Reshetikhin q-characters of representations of shifted quantum affine algebras. We establish q-characters of simple representations satisfy a triangularity property with respect to Nakajima partial ordering (Theorem 5.11).

5.1. **Characters.** Following [FR2, HJ], there is a linear q-character morphism

$$\chi_q : K_0(\mathcal{O}_{\mu}) \to \mathcal{E}_{\ell,\mu}$$

where $K_0(\mathcal{O}_{\mu})$ is the Grothendieck group of the abelian category $\mathcal{O}_{\mu}$ and $\mathcal{E}_{\ell,\mu} \subset \mathbb{Z}^{\mathfrak{r}_{\mu}}$ is a group of formal series with coefficients in $\mathfrak{r}_{\mu}$ as in [HJ]. It is defined by

$$\chi_q(V) = \sum_{\mathbf{\Psi} \in \mathfrak{r}_{\mu}} \dim(V_{\mathbf{\Psi}})[\mathbf{\Psi}]$$

where $V_{\mathbf{\Psi}}$ is the ℓ-weight space of ℓ-weight $\mathbf{\Psi}$ as above and $[\mathbf{\Psi}]$ is the map $\delta_{\mathbf{\Psi},\cdot}$. We recover the standard character

$$\chi(V) = \varpi(\chi_q(V)) = \sum_{\omega\in\mathfrak{t}^*} \dim(V_\omega)[\omega]$$

where $\varpi(\mathbf{\Psi}) = \mathbf{\Psi}(0) \in \mathfrak{t}^*$.

From Theorem 4.11, the q-characters of simple modules are linearly independent and so we get the following.

Corollary 5.1. *The q-character morphism χ_q is injective.*

Example 5.2. *(i) For $i \in I$ and $a \in \mathbb{C}^*$, the prefundamental representation $L^+_{i,a}$ in $\mathcal{O}^{\omega_i^\vee}$ satisfies*

$$\chi_q(L^+_{i,a}) = [\mathbf{\Psi}_{i,a}].$$

It is different from the q-character of the corresponding simple $\mathcal{U}_q(\hat{\mathfrak{b}})$-module in $\mathcal{O}$ which is infinite-dimensional.

(ii) When restricted to the category $\hat{\mathcal{O}} \subset \mathcal{O}_0$, we recover the q-character of $\mathcal{U}_q(\hat{\mathfrak{g}})$-modules.

(iii) For $\mathfrak{g} = sl_2$, we have

$$\chi_q(L^-_{1,a}) = \sum_{m\geq 0} [q^{-2m} \mathbf{\Psi}^{-1}_{1,aq^{-2m}} \mathbf{\Psi}_{1,aq^2} \mathbf{\Psi}^{-1}_{1,aq^{2(1-m)}}].$$

(iv) More generally, for $i \in I$, $a \in \mathbb{C}^*$, we get an analog of the q-character formula established in [FH3] *:*

$$\tilde{\mathbf{\Psi}}_{i,a} = \mathbf{\Psi}^{-1}_{i,a} \left(\prod_{j, C_{i,j}=-1} \mathbf{\Psi}_{j,aq_i}\right) \left(\prod_{j, C_{i,j}=-2} \mathbf{\Psi}_{j,a}\mathbf{\Psi}_{j,aq^2}\right) \left(\prod_{j, C_{i,j}=-3} \mathbf{\Psi}_{j,aq^{-1}}\mathbf{\Psi}_{j,aq}\mathbf{\Psi}_{j,aq^3}\right),$$

$$\chi_q(L(\tilde{\mathbf{\Psi}}_{i,a})) = \sum_{m\geq 0} [\overline{-m\alpha_i}][\tilde{\mathbf{\Psi}}_{i,aq_i^{-2m}} \mathbf{\Psi}_{i,aq_i^2} \mathbf{\Psi}^{-1}_{i,q_i^{2(1-m)}}].$$

Indeed, this representation can be realized with a basis of $(v_m)_{m\geq 0}$ of ℓ-weight vectors corresponding to the terms in this sum. For $r \in \mathbb{Z}$, the $x^\pm_{j,r}$ have a 0 action if $j \neq i$,

$$x^+_{i,r}.v_m = a^r q^{2r(1-m)} \delta_{m>0} v_{m-1} \ ,\ x^-_{i,r}.v_m = a^r q^{-(2r+1)m} \frac{[m+1]_q}{q-q^{-1}} v_{m+1}.$$

Let $\mathcal{E}_\ell = \bigoplus_{\mu\in\Lambda} \mathcal{E}_{\ell,\mu}$. We get an injective linear morphism.

$$\chi_q : K_0(\mathcal{O}^{sh}) = \bigoplus_{\mu\in\Lambda} K_0(\mathcal{O}_\mu) \to \mathcal{E}_\ell.$$

We have a natural bilinear product

$$\mathcal{E}_{\ell,\mu_1} \otimes \mathcal{E}_{\ell,\mu_2} \to \mathcal{E}_{\ell,\mu_1+\mu_2},$$

which induces a ring structure on $\mathcal{E}_\ell$. Hence, we can multiply q-characters. Let us explain the categorical meaning of this product.

5.2. **Deformed Drinfeld coproduct.** The Drinfeld coproduct, and its deformed version [He2, Section 3.1], can be defined for shifted quantum affine algebras by using the same formula as for quantum affine algebras (see [FT, Section 10.1]).

For u a formal parameter and $\mu_1, \mu_2 \in \Lambda$,

$$(\mathcal{U}_q^{\mu_1}(\hat{\mathfrak{g}}) \otimes \mathcal{U}_q^{\mu_2}(\hat{\mathfrak{g}}))((u))$$

is the algebra of formal Laurent series with coefficients in $\mathcal{U}_q^{\mu_1}(\hat{\mathfrak{g}}) \otimes \mathcal{U}_q^{\mu_2}(\hat{\mathfrak{g}})$. The deformed Drinfeld coproduct is the algebra morphism defined :

$$\Delta_u : \mathcal{U}_q^{\mu_1+\mu_2}(\hat{\mathfrak{g}}) \to (\mathcal{U}_q^{\mu_1}(\hat{\mathfrak{g}}) \otimes \mathcal{U}_q^{\mu_2}(\hat{\mathfrak{g}}))((u))$$

by the formulas

$$\Delta_u(x_i^+(z)) = x_i^+(z) \otimes 1 + \phi_i^-(z) \otimes x_i^+(zu) \ , \ \Delta_u(x_i^-(z)) = 1 \otimes x_i^-(zu) + x_i^-(z) \otimes \phi_i^+(zu),$$

$$\Delta_u(\phi_i^\pm(z)) = \phi_i^\pm(z) \otimes \phi_i^\pm(zu).$$

Remark 5.3. *(i) The specialization at $u = 1$ of Δ_u is well-defined only in a completion of the tensor product $\mathcal{U}_q^{\mu_1}(\hat{\mathfrak{g}}) \otimes \mathcal{U}_q^{\mu_2}(\hat{\mathfrak{g}})$.*

(ii) Another coproduct, analog to the Drinfeld-Jimbo coproduct of quantum affine algebras and to the coproduct for shifted Yangians in [FKPRW]*, is defined in* [FT] *for certain shifted quantum affine algebras (it is conjectured to exist for any types).*

5.3. **Fusion product.** Consider V_1, V_2 respectively in $\mathcal{O}^{\mu_1}$, $\mathcal{O}^{\mu_2}$. We get a structure of $\mathcal{U}_q^{\mu_1+\mu_2}(\hat{\mathfrak{g}})$-module on the space of Laurent formal power series with coefficients in $V_1 \otimes V_2$:

$$(V_1 \otimes V_2)((u)).$$

This representation is the sum of its weight spaces which may be infinite-dimensional. But let us study how to get a representation in the category $\mathcal{O}_{\mu_1+\mu_2}$ from this representation.

We use the fusion procedure introduced in [He1, He2] for quantum affine algebras (and quantum affinizations). In general, the formal parameter u can not be specialized directly to a specific complex number. However, one can prove as in [He1, Lemma 3.10] that for $V = V_1 \otimes V_2$, the subspace of rational Laurent formal power series

$$V(u) \subset V((u))$$

is a stable submodule.

Let $\mathcal{A} \subset \mathbb{C}(u)$ the subring of rational fraction without pole at 1. An $\mathcal{A}$-form $\tilde{V} \subset V(u)$ is a $\mathcal{A} \otimes \mathcal{U}_q^{\mu_1+\mu_2}(\hat{\mathfrak{g}})$-submodule generating $V(u)$ as a $\mathbb{C}(u)$-vector space and so that its intersection with a weight space of $V(u)$ is a finitely generated $\mathcal{A}$-module.

Suppose that V_1, V_2 are of highest ℓ-weights. Then it is proved as in [He2, Theorem 6.2] that $V(u)$ is cyclic for the action of $\mathcal{U}_q^{\mu_1+\mu_2}(\hat{\mathfrak{g}}) \otimes \mathbb{C}(u)$ generated by a tensor product $v_1 \otimes v_2$ of highest weight vectors v_1, v_2. Then we obtain as in [He2, Lemma 4.8] that the $\mathcal{A} \otimes \mathcal{U}_q^{\mu}(\hat{\mathfrak{g}})$-submodule generated by $v_1 \otimes v_2$ is an $\mathcal{A}$-form that we denote $\tilde{V}$. Then

$$V_1 * V_2 = \tilde{V}/(u-1)\tilde{V}$$

is a $\mathcal{U}_q^{\mu_1+\mu_2}(\hat{\mathfrak{g}})$-module in the category $\mathcal{O}^{\mu_1+\mu_2}$ called the fusion product of V_1 and V_2.

Theorem 5.4. *The fusion product $V_1 * V_2$ is a well-defined highest ℓ-weight module in $\mathcal{O}^{\mu_1+\mu_2}$ satisfying*

$$\chi_q(V_1 * V_2) = \chi_q(V_1)\chi_q(V_2).$$

For $V_1, \cdots, V_r$ a family of highest ℓ-weight representations V_i in $\mathcal{O}^{\mu_i}$, the same procedure gives a fusion module

$$V_1 * V_2 * \cdots * V_r$$

in $\mathcal{O}^{\mu_1+\cdots+\mu_r}$ with

$$\chi_q(V_1 * V_2 * \cdots * V_r) = \chi_q(V_1)\cdots\chi_q(V_r).$$

A first example is the following fusion of positive (resp. negative) prefundamental representations.

Theorem 5.5. *A fusion product of positive (resp. negative) prefundamental representations is simple :*

$$L^{\pm}_{i_1,a_1} * L^{\pm}_{i_2,a_2} * \cdots * L^{\pm}_{i_N,a_N} \simeq L((\mathbf{\Psi}_{i_1,a_1}\mathbf{\Psi}_{i_2,a_2}\mathbf{\Psi}_{i_N,a_N})^{\pm 1})$$

for any $i_1, \cdots, i_N \in I$, $a_1, \cdots, a_N \in \mathbb{C}^*$.

Proof. For positive prefundamental representations, it is clear as these representations are one-dimensional. It is proved in [FH2] that a tensor product of negative prefundamental representations of $\mathcal{U}_q(\hat{\mathfrak{b}})$ is simple. So the result follows from Corollary 4.10. □

The following confirms prefundamental representations play the role of fundamental representations in the category $\mathcal{O}^{sh}$.

Corollary 5.6. *A simple module in $\mathcal{O}^{sh}$ is a quotient of a fusion product of various prefundamental representations by a simple constant representation.*

Proof. For $L(\mathbf{\Psi})$ such as simple representation, it suffices to write $\mathbf{\Psi} = \mathbf{\Psi}(0)\mathbf{\Psi}^+\mathbf{\Psi}^-$ where $\mathbf{\Psi}^\pm$ is a product of various $\mathbf{\Psi}_{i,a}^{\pm 1}$. Then $L(\mathbf{\Psi})$ is a subquotient of $L(\mathbf{\Psi}(0)) * L(\mathbf{\Psi}^+) * L(\mathbf{\Psi}^-)$. □

Corollary 5.7. *A simple module in $\mathcal{O}^{sh}$ is a subquotient of a fusion product of 1-dimensional module by a simple module of $\mathcal{U}_q(\hat{\mathfrak{b}})$.*

Proof. Let $L(\mathbf{\Psi})$ be a simple representation in $\mathcal{O}^\mu$. Then we can factorize $\mathbf{\Psi} = \lambda\mathbf{\Psi}^+\mathbf{\Psi}^-$ where λ is constant, $\mathbf{\Psi}^\pm$ is a product of various $\mathbf{\Psi}_{i,a}^{\pm 1}$. Then $L(\lambda\mathbf{\Psi}^+)$ is 1-dimensional and by Corollary 4.10, $L(\mathbf{\Psi}^-)$ is simple when restricted $\mathcal{U}_q(\hat{\mathfrak{b}})$. Then $L(\mathbf{\Psi})$ is a quotient of

$$L(\lambda\mathbf{\Psi}^+) * L(\mathbf{\Psi}^-).$$

□

Remark 5.8. *(i) For V_2 a tensor product of positive prefundamental representations and V_1 a representation in $\mathcal{O}_{\mu_1}$, $(V_1 \otimes V_2) \otimes \mathcal{A}$ is a $\mathcal{A}$-lattice. Indeed V_2 is 1-dimensional and $x^\pm_{i,r}$ act by 0 and $\phi^+_{i,r}$ act by 0 for r large enough. Then the image of $\mathcal{U}_q^{\mu_1+\mu_2}(\hat{\mathfrak{g}})$ by Δ_u, after composing by the representation morphisms, gives a Laurent polynomial in End$(V_1 \otimes V_2)[u^{\pm 1}]$.*

(ii) For $V_2 = L^+_{i,a}$, one gets a functor

$$*_{i,a} : \mathcal{O}_\mu \to \mathcal{O}_{\mu+\omega_i^\vee}$$

which preserves the dimension and the character and so that $\chi_q(*_{i,a}(V)) = [\Psi_{i,a}]\chi_q(V)$. It coincides with the functors from Section 4.5.

5.4. **The Grothendieck ring $K_0(\mathcal{O}^{sh})$.** As the q-character morphism is injective by Corollary 5.1, it follows from the last Theorem 5.4 that the image

$$\chi_q(K_0(\mathcal{O}^{sh})) \subset \mathcal{E}_\ell$$

is a subring of $\mathcal{E}_\ell$. This induces a ring structure on $K_0(\mathcal{O}^{sh})$ with positive constant structures on the basis of simple modules. By construction

$$\chi_q : K_0(\mathcal{O}^{sh}) \to \mathcal{E}_\ell$$

is an injective ring morphism. Clearly, $K_0(\mathcal{O}_0)$ is a subring of $K_0(\mathcal{O}^{sh})$.

Example 5.9. *(i) For $\mathfrak{g} = sl_2$, we have*

$$[L^-_{1,a}][L^+_{1,a}] = 1 + [\overline{-\alpha_1}][L^-_{1,aq^{-2}}][L^+_{1,aq^2}]).$$

(ii) More generally, recall the representations $L(\tilde{\mathbf{\Psi}}_{i,a})$ from Example 5.2. We have an analog of the $Q\tilde{Q}$-system established in [FH3] *in $K_0(\mathcal{O})$:*

$$[L(\tilde{\mathbf{\Psi}}_{i,a})][L^+_{i,a}] = [\overline{-\alpha_i}][L(\tilde{\mathbf{\Psi}}_{i,aq_i^{-2}})][L^+_{i,aq_i^2}]$$

$$+\left(\prod_{j,C_{i,j}=-1}[L^+_{j,aq_i}]\right)\left(\prod_{j,C_{i,j}=-2}[L^+_{j,a}][L^+_{j,aq^2}]\right)\left(\prod_{j,C_{i,j}=-3}[L^+_{j,aq^{-1}}][L^+_{j,aq}][L^+_{j,aq^3}]\right).$$

5.5. **Root monomials and Nakajima partial ordering.** Following [FR2], we introduce for $i \in I$, $a \in \mathbb{C}^*$ the following ℓ-weight which is a monomial analog of a simple root :
(5.14)

$$A_{i,a} = Y_{i,aq_i^{-1}}Y_{i,aq_i}\left(\prod_{j\in I,C_{j,i}=-1}Y^{-1}_{j,a}\prod_{j\in I,C_{j,i}=-2}Y^{-1}_{j,aq^{-1}}Y^{-1}_{j,aq}\prod_{j\in I,C_{j,i}=-3}Y^{-1}_{j,aq^{-2}}Y^{-1}_{j,a}Y^{-1}_{j,aq^2}\right),$$

where

$$Y_{i,a} = \overline{\omega_i}\mathbf{\Psi}^{-1}_{i,aq_i}\mathbf{\Psi}_{i,aq_i^{-1}}.$$

Note this ℓ-weight can also be written simply as

$$A_{i,a} = \overline{\alpha_i}\prod_{j\in I}\mathbf{\Psi}^{-1}_{j,aq^{B_{i,j}}}\mathbf{\Psi}_{j,aq^{-B_{i,j}}}.$$

Remark 5.10. *Note that the Langlands dual Cartan matrix $(C_{j,i})_{i,j}$ occurs in the definition of $A_{i,a}$ in opposition to the definition of the ℓ-weights $\tilde{\mathbf{\Psi}}_{i,a}$ in Example 5.2. However, we can rewrite the formula therein*

$$\chi_q(L(\tilde{\mathbf{\Psi}}_{i,a})) = [\tilde{\mathbf{\Psi}}_{i,a}]\sum_{k\geq 0}A^{-1}_{i,a}A^{-1}_{i,aq_i^{-2}}\cdots A^{-1}_{i,aq_i^{-2(k-1)}}.$$

We extend Nakajima partial ordering [N3] to ℓ-weights : we set $\mathbf{\Psi} \preceq \mathbf{\Psi'}$ if and only if $\mathbf{\Psi'}\mathbf{\Psi}^{-1}$ is a monomial in the $A_{i,a}$.

Theorem 5.11. *For $\mathbf{\Psi}$ an ℓ-weight, we have*

$$\chi_q(L(\mathbf{\Psi})) \in [\mathbf{\Psi}] + \sum_{\mathbf{\Psi}' \prec \mathbf{\Psi}} \mathbb{N}[\mathbf{\Psi}'].$$

Proof. By Corollary 5.7, it suffices to prove the statement for prefundamental representations. This is clear for positive prefundamental representations as they are 1-dimensional. For negative prefundamental representations $L^-_{i,a}$, it follows from Corollary 4.10 that the q-character coincides with the q-character of the negative prefundamental representation $L^{\mathfrak{b},-}_{i,a}$ of $\mathcal{U}_q(\hat{\mathfrak{b}})$. In this case the result is proved in [HJ]. □

Remark 5.12. *(i) The statement was proved for finite-dimensional representations of quantum affine algebras in* [FM].

(ii) The analog statement in not satisfied in general for the representations of the quantum affine Borel algebra (for example for positive prefundamental representations $L^{\mathfrak{b},+}_{i,a}$).

6. Finite-dimensional representations

In this section we classify the simple finite-dimensional representations of shifted quantum affine algebras (Theorem 6.4).

For $\mathfrak{g}$ simply-laced, a classification of simple finite-dimensional representations of simply-laced shifted Yangians is given in [KTWWY1, Theorem 1.4] (see (iii) in Remark 6.3).

The standard Theorem of Chari-Pressley [CP] classifying finite-dimensional representations of quantum affine algebras in terms of Drinfeld polynomials can be formulated in the following form (see also [HJ, Examples in Section 3.2]).

Theorem 6.1. *The simple finite dimensional representations of $\mathcal{U}_q^0(\hat{\mathfrak{g}})$ are the $L(\mathbf{\Psi})$ where $\mathbf{\Psi}(z)(\mathbf{\Psi}(0))^{-1}$ is a monomial in the*

$$\tilde{Y}_{i,a} = \overline{\omega_i}^{-1} Y_{i,a} = \mathbf{\Psi}_{i,aq_i^{-1}} (\mathbf{\Psi}_{i,aq_i})^{-1} \text{ for } i \in I,\ a \in \mathbb{C}^*.$$

Proof. By [CP], the simple finite dimensional representations of the quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ are parametrized by the $\mathbf{\Psi}$ which are monomial in the $Y_{i,a}$.

Recall that by (ii) in Remark 3.1, $\mathcal{U}_q^0(\hat{\mathfrak{g}})$ is a central extension of $\mathcal{U}_q(\hat{\mathfrak{g}})$. Let $L(\mathbf{\Psi})$ be a simple finite-dimensional representation of $\mathcal{U}_q^0(\hat{\mathfrak{g}})$. Then for $\lambda \in (\mathbb{C}^*)^n$ a square root of $(\mathbf{\Psi}(0)\mathbf{\Psi}(\infty))^{-1}$, $L(\lambda\mathbf{\Psi})$ is a simple finite-dimensional representation of $\mathcal{U}_q(\hat{\mathfrak{g}})$ and so $\mathbf{\Psi}$ has the correct form. Conversely, if $\mathbf{\Psi} = \lambda^{-1}\mathbf{\Psi}'$ where $L(\mathbf{\Psi}')$ is a simple finite-dimensional representation of $\mathcal{U}_q(\hat{\mathfrak{g}})$, then

$$L(\mathbf{\Psi}) \simeq L(\lambda^{-1}) \otimes L(\mathbf{\Psi}')$$

is finite-dimensional with the same dimension as $L(\mathbf{\Psi}')$. □

To generalize this for all shifted quantum affine algebras, first the following follows easily from the previous results.

Proposition 6.2. *For $\mu \in \Lambda$, the algebra $\mathcal{U}_\mu(\hat{\mathfrak{g}})$ admits non-zero finite-dimensional representations if and only if μ is dominant.*

Let $\mu \in \Lambda^+$ dominant and $\mathbf{\Psi} \in \mathfrak{r}_\mu$ so that $\mathbf{\Psi}(z)(\mathbf{\Psi}(0))^{-1}$ is a product of various

$$\tilde{Y}_{i,a} \text{ and } \mathbf{\Psi}_{i,a} \text{ for } i \in I,\ a \in \mathbb{C}^*.$$

Then $L(\mathbf{\Psi})$ is a simple finite-dimensional representation of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$.

Proof. If there is $i \in I$ so that $\alpha_i(\mu) < 0$, then $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ contains a subalgebra isomorphic to $\mathcal{U}_{q_i}^+(sl_2)$. So it follows from Proposition 2.6 that zero is the only finite-dimensional representation of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$. This implies the "only if" part of the first point. For the "if" part, it suffices to establish the second point. So consider $\mathbf{\Psi}(z)$ as in the statement. We can write $\mathbf{\Psi}(z) = \mathbf{\Psi}(0)M_1M_2$ where M_1 is a product of various $\tilde{Y}_{i,a}$ and

$$M_2 = \mathbf{\Psi}_{i_1,a_1}\mathbf{\Psi}_{i_2,a_2}\cdots\mathbf{\Psi}_{i_N,a_N}$$

for various $i_1, \cdots, i_N \in I$, $a_1, \cdots, a_N \in \mathbb{C}^*$. Then $L(\mathbf{\Psi}(0)M_1)$ is a simple finite-dimensional representation of $\mathcal{U}_q^0(\hat{\mathfrak{g}})$ by Proposition 6.1. Using inductively the functors

$$\mathcal{R}_{\omega_{i_1}^\vee+\cdots+\omega_{i_{j-1}}^\vee,-\omega_{i_j}^\vee,a_j} : \mathcal{O}_{\omega_{i_1}^\vee+\cdots+\omega_{i_{j-1}}^\vee} \to \mathcal{O}_{\omega_{i_1}^\vee+\cdots+\omega_{i_j}^\vee}$$

from Section 4.5, we get from Proposition 4.13 that $L(\mathbf{\Psi})$ is finite-dimensional (with dimension lower than the dimension of $L(M_1)$). $\square$

Remark 6.3. *(i) The condition in this Proposition 6.2 appeared in* [HL3] *and in* [FJMM] *for the ℓ-weights of the simple modules of a category $\mathcal{O}^+$ of representations of the quantum affine Borel algebra (see Remark 4.6).*

(ii) If $\mathbf{\Psi}$ satisfies the condition of the statement and in addition is a Laurent monomial in the $\tilde{Y}_{i,a}$, then the powers of the $\tilde{Y}_{i,a}$ are positive. So, following [FR2, FJMM]*, let us call a general ℓ-weight satisfying this condition a dominant ℓ-weight.*

(iii) Let $\mathbf{\Psi}(z)$ be an ℓ-weight whose poles and zeros are in $q^{\mathbb{Z}}$. It is a Laurent monomial in the $\mathbf{\Psi}_{i,q^r}^{\pm 1}$. There is a structure of crystal on the set of such Laurent monomials [N2, Kas2] *(the variables Y_{i,q^r} in* [N2, Kas2] *are the Ψ_{i,q^r} here; this should not be confused with the Y_{i,q^r} above). The ℓ-weight $\mathbf{\Psi}(z)$ is dominant if and only if it is highest weight for this crystal structure. For $\mathfrak{g}$ simply-laced, it is the condition found in* [KTWWY1, Theorem 1.4] *where a classification of simple finite-dimensional representations of simply-laced shifted Yangians was given (the proof therein is based on type A results in* [BrK, Section 7.2]*).*

We will prove the converse statement which gives a complete classification of finite-dimensional representations of shifted quantum affine algebras.

Theorem 6.4. *The simple finite-dimensional representations of shifted quantum affine algebras are the $L(\mathbf{\Psi})$ where $\mathbf{\Psi}$ is dominant.*

Proof. From Proposition 6.2, it suffices to prove for $\mathfrak{g} = sl_2$ that $L(\mathbf{\Psi})$ finite-dimensional implies $\mathbf{\Psi}$ dominant.

Let $\mu \in \Lambda^+$ and suppose that $L(\mathbf{\Psi})$ is a simple finite-dimensional representation of $\mathcal{U}_q^\mu(\hat{sl_2})$. As discussed above, $\mathbf{\Psi}(z)$ is a rational fraction of non-negative degree. Without loss of generality, we may assume $\mathbf{\Psi}(0) = 1$. There is a (non-unique) factorization $\mathbf{\Psi} = \mathbf{\Psi}^+\mathbf{\Psi}^0$ where

$$\mathbf{\Psi}^0 = (\mathbf{\Psi}_{a_1}\mathbf{\Psi}_{b_1}^{-1})\cdots(\mathbf{\Psi}_{a_N}\mathbf{\Psi}_{b_N}^{-1}) \text{ for certain } N \geq 0,\ a_1, \cdots a_N, b_1, \cdots, b_N \in \mathbb{C}^*,$$

$$\mathbf{\Psi}^+ = \mathbf{\Psi}_{c_1} \cdots \mathbf{\Psi}_{c_M} \text{ where } M = \deg(\mathbf{\Psi}) \geq 0 \text{ and } c_1, \cdots, c_M \in \mathbb{C}^*.$$

Moreover, we assume that for any $1 \leq j \leq M$, $1 \leq j' \leq N$:

$$(c_j \notin b_{j'}q^{-2\mathbb{Z}} \text{ if } a_{j'} \notin b_{j'}q^{-2\mathbb{Z}}) \text{ and } (c_j \notin \{b_{j'}, b_{j'}q^{-2}, \cdots, a_{j'}q^2\} \text{ otherwise}).$$

Then we prove that

$$L(\mathbf{\Psi}) \simeq L(\mathbf{\Psi}_0) * L(\mathbf{\Psi}^+).$$

As $\chi_q([L(\mathbf{\Psi}^+)] = [\mathbf{\Psi}^+]$, we want to prove that the multiplicities of ℓ-weights are the same in $\chi_q(L(\mathbf{\Psi}))$ and in $[\mathbf{\Psi}^+]\chi_q(L(\mathbf{\Psi}_0))$. First $L(\mathbf{\Psi})$ is a quotient of $L(\mathbf{\Psi}_0) * L(\mathbf{\Psi}^+)$, so the multiplicities are lower in $\chi_q(L(\mathbf{\Psi}))$ than in $[\mathbf{\Psi}^+]\chi_q(L(\mathbf{\Psi}_0))$. But $L(\mathbf{\Psi}_0)$ is a quotient of $L(\mathbf{\Psi}) * L((\mathbf{\Psi}^+)^{-1})$. We have precise informations on the q-character of $L(\mathbf{\Psi}_0)$ and $L((\mathbf{\Psi}^+)^{-1})$:

$$\chi_q(L((\mathbf{\Psi}^+)^{-1}) \in [(\mathbf{\Psi}^+)^{-1}](1 + \sum_j A_{c_j}^{-1}\mathbb{Z}[A_a^{-1}]_{a\in\mathbb{C}^*}),$$

$$\chi_q(L(\mathbf{\Psi}_0)) \in [\mathbf{\Psi}_0]\mathbb{Z}[A^{-1}_{b_jq^{-2m}}]_{j,m\geq 0}.$$

(see (iii) in Example 5.2 for the first one; as $L(\mathbf{\Psi}_a\mathbf{\Psi}_b^{-1})$ is a quotient of the fusion $L(\mathbf{\Psi}_a) * L(\mathbf{\Psi}_b^{-1})$, the second also follows from this Example). As $c_j \neq b_{j'}$ for any j, j', we get that the multiplicities of ℓ-weights are lower in $\chi_q(L(\mathbf{\Psi}_0))$ than in $[(\mathbf{\Psi}^+)^{-1}]\chi_q(L(\mathbf{\Psi}))$. We have proved the isomorphism. To conclude, this implies that $L(\mathbf{\Psi}_0)$ is finite-dimensional, hence the result. □

Remark 6.5. *(i) The result implies that $L(\mathbf{\Psi})$ is finite-dimensional if and only if the simple module $L^{\mathfrak{b}}(\mathbf{\Psi})$ of $\mathcal{U}_q(\hat{\mathfrak{b}})$ is in the category $\mathcal{O}^+$ (see Remarks 4.6, 6.3).*

(ii) The factorization of $\mathbf{\Psi} = \mathbf{\Psi}^+\mathbf{\Psi}^0$ appeared in [FJMM] *in the classification of "finite-type" simple representations of $\mathcal{U}_q(\hat{\mathfrak{b}})$. The proof that $L^{\mathfrak{b}}(\mathbf{\Psi}) \simeq L^{\mathfrak{b}}(\mathbf{\Psi}^+) \otimes L^{\mathfrak{b}}(\mathbf{\Psi}^0)$ in this context is given in* [FJMM, Lemma 5.9] *and is more complicated.*

(iii) In type A, a classification of simple finite-dimensional representations of shifted quantum current algebras is obtained in [KW] *in terms of Drinfeld polynomials. These are subalgebras of a shifted quantum affine algebra of type A generated by positive mode Drinfeld generators. Their motivation comes from representations of cyclotomic q-Schur algebras.*

Example 6.6. *In addition to finite-dimensional representations of quantum affine algebras, there are many new examples of finite-dimensional representations. For example the positive prefundamental representations $L(\mathbf{\Psi}_{i,a})$ have dimension 1 and for*

$$\mathbf{\Psi}^*_{i,a} = \mathbf{\Psi}^{-1}_{i,a} \prod_{j, C_{i,j}\neq 0} \mathbf{\Psi}_{j, aq_j^{-C_{j,i}}}$$

the ℓ-weight in [HL3, Section 6.1.3]*, $L(\mathbf{\Psi}^*_{i,a})$ has dimension 2 with*

$$\chi_q(L(\mathbf{\Psi}^*_{i,a})) = [\mathbf{\Psi}^*_{i,a}] + \overline{[-\alpha_i]}[\mathbf{\Psi}^{-1}_{i,a} \prod_{j, C_{i,j}\neq 0} \mathbf{\Psi}_{j, aq_j^{C_{j,i}}}] = [\mathbf{\Psi}^*_{i,a}](1 + A^{-1}_{i,a}).$$

Indeed this representation can be realized with a basis v_0, v_1 of ℓ-weight vectors corresponding to the terms in this sum. For $r \in \mathbb{Z}$, the $x^{\pm}_{j,r}$ have a zero action if $j \neq i$ and

$$x^+_{i,r}.v_0 = x^-_{i,r}.v_1 = 0 \ , \ x^-_{i,r}.v_0 = a^r v_1 \ , \ x^+_{i,r}.v_1 = a^r q^{-1} v_0.$$

We get an analog in $K_0(\mathcal{O}^{sh})$ of the QQ^*-system established in [HL3] *in $K_0(\mathcal{O})$:*

$$[L(\mathbf{\Psi}_{i,a}^*)][L_{i,a}^+] = \overline{[-\alpha_i]} \prod_{j, C_{i,j}\neq 0} [L^+_{j,aq_j^{C_{j,i}}}] + \prod_{j, C_{i,j}\neq 0} [L^+_{j,aq_j^{-C_{j,i}}}].$$

7. Induction and restriction functors

We define and study induction and restriction functors relating the category $\mathcal{O}$ of representations of the quantum affine Borel algebra $\mathcal{U}_q(\hat{\mathfrak{b}})$ and the categories $\mathcal{O}_\mu$ of representations of shifted quantum affine algebras $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$. These functors will be useful tools for our study in the following.

7.1. Functors for antidominant weights.

Proposition 7.1. *We have an equivalence of categories*

$$\tilde{\mathcal{O}} \xrightarrow{\sim} \bigoplus_{\mu\in -\Lambda^+} \mathcal{O}_\mu.$$

Proof. For $\mu \in -\Lambda^+$, let $\tilde{\mathcal{O}}_\mu$ be the subcategory of representations in $\tilde{\mathcal{O}}$ on which for $i \in I$

$$\kappa_i = \phi^-_{i,-1} = \cdots = \phi^-_{i,\alpha_i(\mu)+1} = 0$$

and $\phi_{i,\alpha_i(\mu)}$ is invertible. Such representations V are representations of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$. Indeed, $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ is obtained from the quotient

$$\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})/(\kappa_i = \phi^-_{i,-1} = \cdots = \phi^-_{i,\alpha_i(\mu)+1} = 0, i \in I)$$

by localization at the $\phi_{i,\alpha_i(\mu)}$ for all $i \in I$, and for the $i \in I$ so that $\alpha_i(\mu) < 0$, by adding $k_i^{\pm 1}$ with the quasi-commutation relations (3.2), (3.3). Then for a representation V as above, we set $k_i.v = \omega(i)v$ for $v \in V^{(\omega)}$. We obtain an equivalence of categories

$$\tilde{\mathcal{O}}_\mu \xrightarrow{\sim} \mathcal{O}_\mu.$$

It suffices to check that $\tilde{\mathcal{O}} = \bigoplus_{\mu\in-\Lambda^+} \tilde{\mathcal{O}}_\mu$.

For V a representation in $\tilde{\mathcal{O}}$, and $\mu \in -\Lambda^+$, let $V_{(\mu)}$ the sum of ℓ-weight spaces of ℓ-weight $\mathbf{\Psi} = (\Psi_i(z))_{i\in I}$ satisfying $\deg(\Psi_i) = \alpha_i(\mu)$ for $i \in I$. This means the corresponding pseudo-eigenvalues $\Psi^-_{i,m}$ of $\tilde{\phi}^-_{i,m}$ satisfy

$$\Psi^-_{i,0} = \Psi^-_{i,-1} = \cdots = \Psi^-_{i,\alpha_i(\mu)+1} = 0 \text{ and } \Psi^-_{i,\alpha_i(\mu)} \neq 0.$$

Then V is the direct sum of the $V_{(\mu)}$ and each of them is a submodule by the condition (ii) in Definition 4.5 and by the defining relations of $\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})$. In particular there are no extensions between such submodules. □

For $\mu \in -\Lambda^+$, composing the equivalence $\tilde{\mathcal{O}}_\mu \xrightarrow{\sim} \mathcal{O}_\mu$ with $\tilde{f}$, we get a functor

$$f_\mu : \mathcal{O}_\mu \to \mathcal{O}.$$

Any simple module in $\tilde{\mathcal{O}}$ is in one of the categories $\tilde{\mathcal{O}}_\mu$.

Example 7.2. *For $i \in I$, $a \in \mathbb{C}^*$, the prefundamental representation $L^{\mathfrak{b},-}_{i,a}$ of $\mathcal{U}_q(\hat{\mathfrak{b}})$ is in the image by the functor $\tilde{f}$ of a module $\tilde{L}^-_{i,a}$ in $\tilde{\mathcal{O}}$ by* [HJ].

As a consequence and from the result of the previous, one gets the following.

Theorem 7.3. *The simple modules in the category $\tilde{\mathcal{O}}$ are parametrized by rational ℓ-weights of non-positive degree and constant term* 1.

7.2. **Induction functors.** To generalize the results of the previous section to $\mu \in \Lambda$, we have to proceed differently.

Recall the algebra $\mathcal{U}_q^\mu(\hat{\mathfrak{b}})$ in Proposition 3.6. It is isomorphic to a subalgebra of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ and of $\mathcal{U}_q(\hat{\mathfrak{b}})$.

For V a $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$-module in the category $\mathcal{O}_\mu$, one can consider its restriction to $\mathcal{U}_q^\mu(\hat{\mathfrak{b}})$ and then its induction to $\mathcal{U}_q(\hat{\mathfrak{b}})$:

$$\mathcal{I}_\mu(V) = \mathcal{U}_q(\hat{\mathfrak{b}}) \otimes_{\mathcal{U}_q^\mu(\hat{\mathfrak{b}})} V.$$

As $\mathcal{U}_q^+(\hat{\mathfrak{b}})$ and $\mathcal{U}_q^0(\hat{\mathfrak{b}})$ are both contained in $\mathcal{U}_q^\mu(\hat{\mathfrak{b}})$, the weight spaces of $\mathcal{I}_\mu(V)$ are finite-dimensional and we get a representation in the category $\mathcal{O}$. So this defines a functor

$$\mathcal{I}_\mu : \mathcal{O}_\mu \to \mathcal{O}.$$

Remark 7.4. *For $V = L(\mathbf{\Psi})$ in $\mathcal{O}_\mu$, the $\mathcal{U}_q(\hat{\mathfrak{b}})$-module $\mathcal{I}_\mu(V)$ is of highest ℓ-weight generated by a highest weight vector of V. It admits $L^{\mathfrak{b}}(\mathbf{\Psi})$ as a quotient.*

Example 7.5. *(i) If $\mu \in -\Lambda^+$, then $\mathcal{I}_\mu$ preserves the dimension and simple modules (see Corollary 4.10) .*

(ii) For $i \in I$, $a \in \mathbb{C}^*$, let $\mu = \omega_i^\vee$ and $V = L_{i,a}^+$ in $\mathcal{O}_{\omega_i^\vee}$. It has dimension 1*. Then $\mathcal{I}_{\omega_i^\vee}(V)$ admits the simple infinite-dimensional $\mathcal{U}_q(\hat{\mathfrak{b}})$-module $L_{i,a}^{\mathfrak{b},+}$ as a quotient. In the sl_2-case, we have $\mathcal{U}_q(\hat{\mathfrak{b}}) = \mathcal{U}_q^{\omega_1^\vee}(\hat{\mathfrak{b}}) \otimes \mathbb{C}[x_{1,1}^-]$ and so $\mathcal{I}_{\omega_i^\vee}(V) = \sum_{m\geq 0}(x_{1,1}^-)^m.V = L_{1,a}^{\mathfrak{b},+}$.*

7.3. **Restriction functors.** Let $\mu \in \Lambda$. For $i \in I$, we set $\mu_i = \mathrm{Max}(1, \alpha_i(\mu))$.

For V a representation in $\mathcal{O}$, we consider its subspace V_μ (resp. $V_{<\mu}$) of vectors $v \in V$ so that for any $i \in I$, $\phi_i^+(z).v \in V(z)$ has degree lower than $\alpha_i(\mu)$ (respectively strictly lower than $\alpha_i(\mu)$).

Remark 7.6. *V_μ is not a submodule of V in general. Let $\mathfrak{g} = sl_2$ and $V = L((1-z)^3)$ with highest weight vector v. Let*

$$x^{+,+}(z) = \sum_{m\geq 0} x_m^+ z^m.$$

For $w = x_1^-.v$ one has $x^{+,+}(z).w = \frac{z^2-3z+3}{q-q^{-1}}v$ and

$$-q^2(1-z)^3 x_{i,0}^+(\phi_1^+(z))^{-1}.w = -x_{i,0}^+.w + (1-q^4)x_i^{+,+}(zq^2).w = \frac{(1-q^4)(q^4z^2-3q^2z+3)-3}{q-q^{-1}}.v$$

In particular, $(\phi_1^+(z))^{-1}.w$ has degree larger than -1. But on the weight space of V of weight $-\alpha_1$, $\phi_1^+(z) = q^{-2}(1-z)^3 Id + N(z)$ with $N(z)^3 = 0$. Its inverse is

$$q^2(1-z)^{-3}Id - q^4(1-z)^{-6}N(z) + q^6(1-z)^{-9}N^2(z).$$

Hence $N(z)$ has degree larger than 4*, and so is $\phi_1^+(z)$.*

Proposition 7.7. *For $v \in V_\mu$ (resp. $V_{<\mu}$), there is $M \geq 0$ so that $\phi_{i,s}^+.v$, $x_{i,m}^+.v$, $x_{i,m}^-.v$ are in V_μ (resp. $V_{<\mu}$) for any $i \in I$, $s \geq 0$, $m \geq M$.*

Proof. It suffices to show the statement for V_μ as

$$V_{<\mu} = \sum_{i \in I} V_{\mu - \omega_i^\vee}.$$

Let $v \in V_\mu$ and $i, j \in I$. Let $m > 0$ so that

$$\deg(\phi_i^+(z) x_{j,m}^+.v) \geq \deg(\phi_i^+(z) x_{j,m-1}^+.v).$$

Then the relation

$$\phi_i^+(z)(x_{j,m-1}^+ - q^{B_{i,j}} z x_{j,m}^+).v = (q^{B_{i,j}} x_{j,m-1}^+ - z x_{j,m}^+)\phi_i^+(z).v$$

implies that $\deg(\phi_i^+(z) x_{j,m}^+.v) \leq \alpha_i(\mu)$. So we are reduced to the case when for any $m \geq 0$, the maximum of the $\deg(\phi_i^+(z) x_{j,m'}^+.v)$, $m' \geq m$ is realized at $m' = m$ only. This means that $\deg(\phi_i^+(z) x_{j,m}^+.v)$ is strictly decreasing. Contradiction as weight spaces are finite-dimensional.

This is analog for the $x_{j,m}^- v$. And it is clear for the $\phi_{j,s}^+$ which commute with $\phi_i^+(z)$. $\square$

We see the elements of $\mathcal{U}_q(\hat{\mathfrak{b}})$ as operators on V. For $v \in V_\mu$, we consider M as in the previous Proposition. Let $i \in I$. The rational $\sum_{m \geq M} x_{i,m}^+.v \in V_\mu(z)$ has a degree d. Let $M' > \mathrm{Max}(M, d)$. Then $\sum_{m > M'} x_{i,m}^+ z^m.v$ has degree M'. We expand it in z^{-1} and we get a series $-\sum_{m \leq M'} v_m^{(M')} z^m$. We also set $v_m^{(M')} = x_{i,m}^+.v$ for $m > M'$. We get a new family $(v_m(M'))_{m \in \mathbb{Z}}$. It does not depend on the choice of M'. Indeed, for $M'' > M'$, we have

$$-\sum_{m < M''} v_m^{(M'')} z^m = \sum_{m > M''} v_m^{(M'')} z^m = \sum_{m > M'} v_m^{(M')} z^m - \sum_{M'' \geq m > M'} v_m^{(M')} z^m$$

$$= -\sum_{m \leq M'} v_m^{(M')} z^m - \sum_{M'' \geq m > M} v_m^{(M')} z^m = -\sum_{m < M''} v_m^{(M')} z^m.$$

Identifying the developments in z^{-1}, we get $v_m^{(M'')} = v_m^{(M')}$ for $m < M''$, and the developments in z $v_m^{(M'')} = v_m^{(M')}$ for $m > M''$. So we define we can set $\tilde{x}_{i,m}^+.v = v_m^{(M')}$ which is well defined for $m \in \mathbb{Z}$. In the same way, one defines operators $\tilde{x}_{i,m}^-$ on V_μ for $i \in I$, $m \in \mathbb{Z}$.

These operators are well defined on $\tilde{V}_\mu = V_\mu / V_{<\mu}$. Also $\phi_i^+(z)$ is rational of degree $\alpha_i(\mu)$ on this quotient. We expand in z^{-1} and we get an operator in

$$\phi_i^-(z) \in z^{\alpha_i(\mu)} \mathrm{End}(\tilde{V}_\mu)[[z^{-1}]].$$

Proposition 7.8. *The operators $\tilde{x}_{i,m}^\pm$, $\phi_i^\pm(z)$ constructed above on $\tilde{V}_\mu$ define a structure of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$-module on $\tilde{V}_\mu$ which is in the category $\mathcal{O}_\mu$.*

We obtain a restriction functor

$$\mathcal{R}_\mu : \mathcal{O} \to \mathcal{O}_\mu.$$

Proof. We have to check the defining relations of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ are satisfied. For example let us study the relation

$$[x_i^+(z), x_i^-(w)] = \frac{\delta(wz^{-1})\phi_i^+(w) - \delta(zw^{-1})\phi_i^-(z)}{q_i - q_i^{-1}},$$

which is the most complicated to handle as it involves all operators. We work on a vector v for which we can consider M large enough so that $\tilde{x}_{i,m}^\pm.v = x_{i,m}^\pm.v$ for $i \in I$ and $m \geq M$. First, we prove exactly as in [FT, Lemma B.2 (c)] that for

$$x_i^{\pm,M}(z) = \sum_{m\geq M} x_{i,m}^\pm z^m \ , \ \phi_i^{\pm,M}(z) = \sum_{m\geq M} \phi_{i,\pm m} z^{\pm m},$$

we have in $\mathcal{U}_q(\hat{\mathfrak{b}})[[z,w]]$:

$$[x_i^{+,M}(z), x_i^{-,M}(w)] = \frac{w^M z^{1-M}\phi_i^{+,2M}(w) - z^M w^{1-M}\phi_i^{+,2M}(z)}{(w-z)(q_i - q_i^{-1})}.$$

Now we expand this in z^{-1} and we get

$$[-\tilde{x}_i^{+,M}(z), x_i^{-,M}(w)] = \frac{w^M z^{1-M}\phi_i^{+,2M}(w) - z^M w^{1-M}\phi_i^-(z) + z^M w^{1-M}\sum_{0\leq m<2M}\phi_{i,m}^+ z^m}{(w-z)(q_i - q_i^{-1})},$$

where we denote

$$\tilde{x}_i^{\pm,M}(z) = \sum_{m<M} \tilde{x}_{i,m}^\pm z^m.$$

This implies

$$[\tilde{x}_i^+(z), x_i^{-,M}(w)] = z^{M-1}w^{1-M}\frac{\phi_i^+(z) - \phi_i^-(z)}{(q_i - q_i^{-1})}\sum_{r\geq 0}(wz^{-1})^r.$$

In the same way, one gets

$$[\tilde{x}_i^+(z), \tilde{x}_i^{-,M}(w)] = z^M w^{-M}\frac{\phi_i^+(z) - \phi_i^-(z)}{(q_i - q_i^{-1})}\sum_{r\geq 0}(zw^{-1})^r.$$

The sum of the two relations give the correct relation. $\square$

Example 7.9. *(i) Let V be a simple finite-dimensional representation in $\mathcal{O}$. Then, up to a twist, it is a representation of the quantum affine algebra and the action of the $\phi_i^+(z)$ are of degree 0. This gives the restricted action and $\mathcal{R}_\mu(V)$ is simple of dimension dim(V).*

(ii) Consider the prefundamental representation $V = L_{1,1}^{\mathfrak{b},-}$ in the sl_2-case. Then $V = V_{-\omega_1^\vee}$ and $\mathcal{R}_{-\omega_1^\vee}(V)$ is simple in $\mathcal{O}_{-\omega_1^\vee}$.

(iii) Consider the prefundamental representation $V = L_{1,1}^{\mathfrak{b},+}$ in the sl_2-case. Then $V = V_{\omega_1^\vee}$. By construction, $\tilde{x}_1^+(z) = \tilde{x}_1^-(z) = 0$ on $\mathcal{R}_{\omega_1^\vee}(V)$ which is semi-simple in $\mathcal{O}_{\omega_1^\vee}$ equal to an infinite direct sum of simple modules of dimension 1.

Remark 7.10. *We get functors*

$$\mathcal{I} : \mathcal{O}^{sh} \to \mathcal{O} \text{ and } \mathcal{R} : \mathcal{O} \to \mathcal{O}^{sh}.$$

We may wonder if these functors are biadjoint.

8. Characters and cluster algebra structures

In this section we establish a q-characters formula for simple finite-dimensional representations of shifted quantum affine algebras in terms of the q-character of certain simple representations of the quantum affine Borel algebra $\mathcal{U}_q(\hat{\mathfrak{b}})$ in the category $\mathcal{O}$ (Theorem 8.1). Then, we prove the results in [HL3] imply a description of simple finite-dimensional representations of $\mathcal{U}_q(\hat{sl}_2)$ (Theorem 8.3), isomorphisms of Grothendieck rings between categories of representations of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ associated to dominant and anti-dominant μ (Theorem 8.6), and a cluster algebra structure on the Grothendieck ring of finite-dimensional representations of shifted quantum affine algebras (Theorem 8.9).

Note that the structure of K-theoretic Coulomb branches of a 3d $N = 4$ quiver gauge theory has been studied in the context of cluster theory in [SS]. Here we consider cluster structures emerging from their representation theory.

8.1. **q-characters of finite-dimensional representations.** For $i \in I$, let χ_i be the character of $L^{\mathfrak{b}}(\mathbf{\Psi}_{i,1})$. It is proved in [HJ, FH2] that for $a \in\in \mathbb{C}^*$:

$$\chi_q(L^{\mathfrak{b},+}_{i,a}) = [\mathbf{\Psi}_{i,a}]\chi_i. \tag{8.15}$$

Theorem 8.1. *Let $L(\mathbf{\Psi})$ be a simple finite-dimensional representation of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$. The q-character of the simple $\mathcal{U}_q(\hat{\mathfrak{b}})$-module $L^{\mathfrak{b}}(\mathbf{\Psi})$ is :*

$$\chi_q(L^{\mathfrak{b}}(\mathbf{\Psi})) = \chi_q(L(\mathbf{\Psi}))\prod_{i\in I}\chi_i^{\alpha_i(\mu)}.$$

Remark 8.2. *This generalizes the q-characters formulas (8.15) for $L^{\mathfrak{b}}(\mathbf{\Psi}_{i,a})$ established in* [FH2]*) and the formula in* [HL3] *for $L^{\mathfrak{b}}(\mathbf{\Psi}^*_{i,a})$ (see Example 6.6) :*

$$\chi_q(L^{\mathfrak{b}}(\mathbf{\Psi}^*_{i,a})) = [\mathbf{\Psi}^*_{i,a}](1 + A_{i,a}^{-1})\prod_{j,C_{j,i}<0}\chi_j = \chi_q(L(\mathbf{\Psi}^*_{i,a}))\prod_{j,C_{j,i}<0}\chi_j.$$

Proof. From Theorem 6.4, $L(\mathbf{\Psi})$ is a quotient of $L(\mathbf{\Psi}_0) * L(\mathbf{\Psi}^+)$ where $L(\mathbf{\Psi}_0)$ a finite-dimensional representation of $\mathcal{U}_q(\hat{\mathfrak{g}})$ and $L(\mathbf{\Psi}^+)$ is one-dimensional.

Consider the $\mathcal{U}_q(\hat{\mathfrak{b}})$-module $L^{\mathfrak{b}}(\mathbf{\Psi}_0)\otimes L^{\mathfrak{b}}(\mathbf{\Psi}^+)$ in the category $\mathcal{O}$. Let v_0 and v_+ be corresponding highest weight vectors. By [FJMM], the Drinfeld coproduct gives a well-defined action on this tensor product. Let us denote by V this representation. It is established in [FJMM, Lemma 5.7] that any non-zero submodule of V is of the form $W \otimes L^{\mathfrak{b}}(\mathbf{\Psi}^+)$ where $W \subset L^{\mathfrak{b}}(\mathbf{\Psi}_0)$ is a subspace containing v_0. In particular, the submodule V' of V generated by $v_0 \otimes v_+$ is simple isomorphic to $L^{\mathfrak{b}}(\mathbf{\Psi})$. There is a corresponding $W \subset L^{\mathfrak{b}}(\mathbf{\Psi}_0)$. By construction, W is stable by the action of the $x^+_{i,m}$, $\phi^\pm_{i,m}$ for $i \in I$, $m \geq 0$. Then by Formula (8.15), We have

$$\chi_q(L^{\mathfrak{b}}(\mathbf{\Psi})) = \chi_q(L(W))\chi_q(L^{\mathfrak{b}}(\mathbf{\Psi}_0)) = \chi_q(L(W))[\mathbf{\Psi}_0]\prod_{i\in I}\chi_i^{\alpha_i(\mu)}.$$

Consider the restricted representation $\mathcal{R}_\mu(V')$ in the category $\mathcal{O}_\mu$. It admits $L(\mathbf{\Psi})$ as a subquotient.

By construction, $W\otimes v_0$ is stable for the action of the $\tilde{x}^+_{i,m}$. Now, for $m>\alpha_i(\mu)$, we have $x^-_{i,m}.v_0=0$ (this follows from $x^+_{j,r}x^-_{i,m}.v_0=\delta_{i,j}(q_i-q_i^{-1})^{-1}\phi^+_{i,m+r}.v_0$ for any $r\geq 0$, $j\in I$). Hence

$$x^-_{i,m}.(w\otimes v_0)=(\sum_{0\leq r\leq\alpha_i(\mu)}\Psi^+_{i,r}x^-_{i,m-r})w\otimes v_0. \tag{8.16}$$

This implies that $W\otimes v_0$ is stable for the action of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$. To conclude, it suffices to prove this module is simple isomorphic to $L(\mathbf{\Psi})$.

Recall that the there is a non-zero polynomial $P(z)$ so that $P(z).x_i^\pm(z)=0$ on the finite-dimensional representation $L(\mathbf{\Psi}_0)$ (see Proposition 4.9 for instance).

If there is $w\otimes v_0\in W\otimes v_0$ so that $\tilde{x}^+_{i,m}(w\otimes v_0)=0$ for any $m\in\mathbb{Z}$, $i\in I$, then $x^+_{i,m}.w=0$ for any $m\geq\alpha_i(\mu)$ and w is highest weight. So, we just have to prove that the $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$-module $W\otimes v_0$ is generated by $v_+\otimes v_0$. Let $w\otimes v_0\in W$. There is x in the negative subalgebra of $\mathcal{U}_q(\hat{\mathfrak{b}})$ so that $x.(v_+\otimes v_0)=w\otimes v_0$. From the Drinfeld coproduct formula, we have

$$x.(v_+\otimes v_0)\in(x'.v_+\otimes v_0)+\text{ additional terms}$$

where x' is obtained from x by replacing each $x^-_{i,m}$ by $x^{-,'}_{i,m}=\sum_{0\leq r\leq\alpha_i(\mu)}\Psi^+_{i,r}x^-_{i,m-r}$ (see formula (8.16)) and the additional terms have a right factor of weight strictly lower than the weight of v_0. Hence

$$w\otimes v_0=x.(v_+\otimes v_0)=(x'.v_+)\otimes v_0.$$

This implies that

$$W\otimes v_0\subset(< x^{-,'}_{i,m}>_{i\in I,m\in\mathbb{Z}}.v_+)\otimes v_0=(< x^{-,'}_{i,m}>_{i\in I,m\geq\alpha_i(\mu)}.v_+)\otimes v_0\subset\mathcal{U}_q^\mu(\hat{\mathfrak{g}}).(v_+\otimes v_0).$$

□

8.2. **Description of simple finite-dimensional representations of $\mathcal{U}_q^\mu(\hat{sl}_2)$.** We get a complete description of all simple finite dimensional representations of shifted quantum affine algebras $\mathcal{U}_q^\mu(\hat{sl}_2)$, that is simple objects in the category $\mathcal{C}^{sh}\subset\mathcal{O}^{sh}$ of finite-dimensional representations.

Suppose that $\mathfrak{g}=sl_2$. A description of simple modules of the category $\mathcal{O}^+$ was given in [HL3, Section 7.3].

For $k\geq 0$ and $a\in\mathbb{C}^*$, we have the Kirillov-Reshetikhin (KR) module

$$W_{k,a}=L(\overline{k\omega}\mathbf{\Psi}_{aq^{-1}}\mathbf{\Psi}^{-1}_{aq^{2k-1}})=L(Y_aY_{aq^2}\cdots Y_{aq^{2(k-1)}}).$$

It is a representation of $\mathcal{U}_q(\hat{sl}_2)$ of dimension $k+1$ obtained by evaluation from a $\mathcal{U}_q(sl_2)$-module.

A q-set is a subset of $\mathbb{C}^*$ of the form $\{aq^{2r}\mid R_1\leq r\leq R_2\}$ for some $a\in\mathbb{C}^*$ and $R_1\leq R_2\in\mathbb{Z}\cup\{-\infty,+\infty\}$. The modules $W_{k,a}$, $W_{k',b}$ are said to be in *special position* if the union of $\{a,aq^2,\cdots,aq^{2(k-1)}\}$ and $\{b,bq^2,\cdots,bq^{2(k'-1)}\}$ is a q-set which contains both properly. The module $W_{k,a}$ and the prefundamental representation L_b^+ are said to be in special position if the union of $\{a,aq^2,aq^4,\cdots,aq^{2(k-1)}\}$ and $\{bq,bq^3,bq^5,\cdots\}$ is a q-set which contains both properly. Two positive prefundamental representations are never in special position. Two representations are in *general position* if they are not in special position.

The invertible elements in the category $\mathcal{C}^{sh}$ are the 1-dimensional constant simple representations $[\omega]$.

From Theorem 8.1, we have now the following direct consequence of [HL3, Theorem 7.9].

Theorem 8.3. *Suppose that $\mathfrak{g} = sl_2$. The prime simple objects in the category $\mathcal{C}^{sh}$ are the positive prefundamental representations and the KR-modules. Any simple object in $\mathcal{C}^{sh}$ can be factorized in a unique way as a fusion product of prefundamental representations and KR-modules (up to permutation of the factors and to invertibles). Moreover, such a fusion product is simple if and only all its factors are pairwise in general position.*

Remark 8.4. (i) This is a generalization of the factorization of simple representations in the category $\mathcal{C}$ of finite-dimensional representations of $\mathcal{U}_q(\hat{sl}_2)$ by Chari-Pressley [CP].

(ii) All simple finite-dimensional representations can be factorized in a unique way into a fusion product of a simple finite-dimensional representation of $\mathcal{U}_q^0(\hat{sl}_2)$ and a one-dimensional representation.

(iii) This result for $\mathfrak{g} = sl_2$ implies that all simple finite-dimensional representations are real and that their factorization into prime representations is unique.

8.3. **Grothendieck ring isomorphisms.** Let us consider completed tensor products $\hat{\otimes}_{\mathbb{Z}}$ as in [HL3, Section 4.1]. We have the following consequence of Theorem 8.1.

Corollary 8.5. *There is a ring isomorphism*

$$K_0(\mathcal{C}^{sh})\hat{\otimes}_{\mathbb{Z}}\mathcal{E} \simeq K_0(\mathcal{O}^+)$$

which preserves the classes of simple objects.

Proof. For L a finite-dimensional representation of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$, we assign to the class of L in $K_0(\mathcal{C}^{sh})$ the class in $K_0(\mathcal{O}^+)$ of q-character

$$\chi_q(L)\prod_{i\in I}\chi_i^{\alpha_i(\mu)}.$$

This defines an injective ring morphism from $K_0(\mathcal{C}^{sh})\hat{\otimes}_{\mathbb{Z}}\mathcal{E}$ to $K_0(\mathcal{O}^+)$ which sends a simple class to a simple class. As $K_0(\mathcal{O}^+)$ is topologically generated by the $[L^{\mathfrak{b}}(\mathbf{\Psi})]$ where $\mathbf{\Psi}$ is an dominant ℓ-weight, the morphism is surjective. $\square$

It is proved in [HL3] that there is an isomorphism of Grothendieck rings

$$D : K_0(\mathcal{O}^+) \to K_0(\mathcal{O}^-)$$

which preserves dimensions, characters and so that $D([L^{\mathfrak{b}}(\mathbf{\Psi})]) = [L^{\mathfrak{b}}(\mathbf{\Psi}^{-1})]$. Note however that it is not compatible with q-characters.

Let $\mathcal{O}^{sh,+}$ (resp. $\mathcal{O}^{sh,-}$) be the subcategory of representations in $\mathcal{O}^{sh}$ whose simple constituents have a highest ℓ-weight $\mathbf{\Psi}$ so that $\mathbf{\Psi}$ is dominant (resp. $\mathbf{\Psi}^{-1}$ is dominant). This is motivated by analog categories of $\mathcal{U}_q(\hat{\mathfrak{b}})$-modules (see Remarks 4.6, 6.3).

Note that all simple modules in $\mathcal{O}^{sh,+}$ are finite-dimensional and that $\mathcal{C}^{sh} \subset \mathcal{O}^{sh,+}$.

Theorem 8.6. *The categories $\mathcal{O}^{sh,+}$, $\mathcal{O}^{sh,-}$ are stable by fusion product and we have a ring isomorphism which preserves simple classes*

$$\bigoplus_{\mu\in\Lambda^+} K_0(\mathcal{O}_\mu) \supset K_0(\mathcal{O}^{sh,+}) \simeq K_0(\mathcal{O}^{sh,-}) \subset \bigoplus_{\mu\in-\Lambda^+} K_0(\mathcal{O}_\mu).$$

Proof. The stability of $\mathcal{O}^{sh,+}$ by fusion product follows from the stability of the category $\mathcal{C}^{sh}$ of finite-dimensional representations as both categories have the same simple objects. We have

$$K_0(\mathcal{O}^{sh,+}) = K_0(\mathcal{C}^{sh})\hat{\otimes}_{\mathbb{Z}}\mathcal{E}.$$

The stability of $\mathcal{O}^{sh,-}$ by fusion product is clear as the simple objects in $\mathcal{O}^-$ and $\mathcal{O}^{sh,-}$ have the same q-character (Corollary 4.10). Hence we have an isomorphism

$$K_0(\mathcal{O}^{sh,-}) \simeq K_0(\mathcal{O}^-).$$

Then we can use the isomorphism in Corollary 8.5 :

$$K_0(\mathcal{O}^{sh,+}) = K_0(\mathcal{C}^{sh})\hat{\otimes}_{\mathbb{Z}}\mathcal{E} \simeq K_0(\mathcal{O}^+).$$

□

8.4. **Cluster algebra structure.** A certain monoidal subcategory

$$\mathcal{O}^+_{2\mathbb{Z}} \subset \mathcal{O}^+ \subset \mathcal{O}$$

of representations of $\mathcal{U}_q(\hat{\mathfrak{b}})$ is introduced in [HL3]. It is defined as the subcategory of representations in $\mathcal{O}^+$ whose simple constituents have a highest ℓ-weight $\mathbf{\Psi}$ such that the roots and the poles of $\Psi_i(z)$ are of the form q^r where (i,r) belong to certain remarkable $V \subset I \times \mathbb{Z}$.

Its Grothendieck $K_0(\mathcal{O}^+_{2\mathbb{Z}})$ captures the combinatorics of $K_0(\mathcal{O}^+)$. Moreover the main Theorem of [HL3] is a ring isomorphism

$$K_0(\mathcal{O}^+_{2\mathbb{Z}}) \simeq \mathcal{A}\hat{\otimes}_{\mathbb{Z}}\mathcal{E},$$

where $\mathcal{A}$ is a cluster algebra and the classes of prefundamental representations $[L^{+,\mathfrak{b}}_{i,q^r}]_{(i,r)\in V}$ in $\mathcal{O}^+_{2\mathbb{Z}}$ form an initial seed.

Now consider the subcategory

$$\mathcal{C}^{sh}_{2\mathbb{Z}} \subset \mathcal{C}^{sh} \subset \mathcal{O}^{sh}$$

of finite-dimensional representations whose simple constituents have a highest ℓ-weight $\mathbf{\Psi}$ such that the roots and the poles of $\Psi_i(z)$ are of the form q^r where $(i,r) \in V$.

Similarly, we have also corresponding categories $\mathcal{O}^{sh,\pm}_{2\mathbb{Z}} \subset \mathcal{O}^{sh,\pm}$.

Theorem 8.7. *We have ring isomorphisms*

$$K_0(\mathcal{O}^{sh,+}_{2\mathbb{Z}}) \simeq \mathcal{A}\hat{\otimes}_{\mathbb{Z}}\mathcal{E} \simeq K_0(\mathcal{O}^{sh,-}_{2\mathbb{Z}}),$$

with classes of prefundamental representations corresponding to an initial seed.

Let us recall a simple object is said to be real it its fusion square is simple.

Conjecture 8.8. *The classes of real simple objects in $K_0(\mathcal{O}^{sh,+}_{2\mathbb{Z}})$ (resp. $K_0(\mathcal{O}^{sh,-}_{2\mathbb{Z}})$) get identified with cluster monomials.*

By the results in the present paper, in particular Corollary 8.5, this Conjecture 8.8 is equivalent to [HL3, Conjecture 7.12]. Moreover, by [HL3, Theorem 7.12], [HL3, Conjecture 7.12] is equivalent to [HL2, Conjecture 5.2]. Then a part of this Conjecture 8.8 was established in [Q] for ADE types, for general types recently in the announcement [KKOP]. Combining these results, one gets the following.

Theorem 8.9. *(i) We have an algebra isomorphism*

$$\mathcal{A} \simeq K_0(\mathcal{C}_{2\mathbb{Z}}^{sh}).$$

(ii) The cluster monomials in $\mathcal{A}$ are real simple objects in $K_0(\mathcal{C}_{2\mathbb{Z}}^{sh})$.

Proof. By the discussion above, (ii) is known. The arguments in [HL3, Proposition 6.1] imply that

$$K_0(\mathcal{C}_{2\mathbb{Z}}^{sh}) \subset \mathcal{A}.$$

As cluster monomials generate a cluster algebra, now (i) follows from (ii). $\square$

Remark 8.10. *In the sl_2-case, Conjecture 8.8 is proved in* [HL3, Theorem 7.11].

We will study again these and other cluster algebras structure related to the representation theory of shifted quantum affine algebras in another work.

9. Cartan-Drinfeld series and Baxter polynomiality

Adjoint versions of shifted quantum affine algebras are defined as the usual adjoint versions of quantum affine algebras by adding Cartan generators corresponding to fundamental weights. We discuss series of Cartan-Drinfeld elements $Y_i^{\pm}(z)$ and $T_i^{\pm}(z)$ ($i \in I$) introduced respectively in [FR2] in the study of transfer-matrices of finite-dimensional representations of quantum affine algebras and in [HJ] as limits of transfer-matrices of prefundamental representations of quantum affine Borel algebras.

As a main result of this section (Theorem 9.12), we establish the rationality of $Y_i^{\pm}(z)$ (resp. the polynomiality of $(T_i^{\pm}(z))^{\mp 1}$) on a simple representation in the category $\mathcal{O}_\mu$ (up to the highest eigenvalue). The proof is partly based on the Cartan-Drinfeld polynomiality established in [FH2] as a limit of Baxter polynomiality of quantum integrable models. We also obtain the equality of the developments in $z^{\pm 1}$ up to a constant factor.

9.1. **Adjoint versions.** Adjoint versions for quantum affine algebras are used in the literature (see [FR2] for instance where additional elements are denoted by $\tilde{k}_i$).

Fix $\mu \in \Lambda$. The adjoint version $\mathcal{U}_q^{\mu,ad}(\hat{\mathfrak{g}})$ of the shifted quantum affine algebra is [FT] which is a slight extension $\mathcal{U}_q^{\mu}(\hat{\mathfrak{g}})$. New generators $(\overline{\phi}_i^+)^{\pm 1}$, $(\overline{\phi}_i^-)^{\pm 1}$ are added satisfying

$$\prod_{j\in I}(\overline{\phi}_j^+)^{C_{j,i}} = \phi_{i,0}^+ \text{ and } \prod_{j\in I}(\overline{\phi}_j^-)^{C_{j,i}} = \phi_{i,\alpha_i(\mu)}^-,$$

and satisfying the obvious analogs of the quasi-commutations relations (3.2), (3.3).

Remark 9.1. *(i) For $i \in I$, $\overline{\phi}_i^+\overline{\phi}_i^-$ is central in $\mathcal{U}_q^{\mu,ad}(\hat{\mathfrak{g}})$.*

(ii) There is a group of automorphisms of $\mathcal{U}_q^{\mu,ad}(\hat{\mathfrak{g}})$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$: for $(\epsilon_1, \cdots, \epsilon_n) \in \{\pm 1\}^n$, there is a unique automorphism so that for $i \in I$ and $m \in \mathbb{Z}$:

$$\overline{\phi}_i^{\pm} \mapsto \epsilon_i \overline{\phi}_i^{\pm} \ , \ \phi_{i,m}^{\pm} \mapsto \eta_i \phi_{i,m}^{\pm} \ , \ x_{i,m}^- \mapsto \eta_i x_{i,m}^- \ , \ x_{i,m}^+ \mapsto x_{i,m}^+ \text{ where } \eta_i = \prod_{j\in I} \epsilon_j^{C_{j,i}} \in \{\pm 1\}.$$

This induces a group of automorphisms of $\mathcal{U}_q^{\mu}(\hat{\mathfrak{g}})$ that we call sign-twist (the order is 2^n, except 2^{n-1} in type B_n and 1 in type A_1).

The representation theory of $\mathcal{U}_q^{\mu,ad}(\hat{\mathfrak{g}})$ is a slight modification of the representation theory of $\mathcal{U}_q^{\mu}(\hat{\mathfrak{g}})$. Indeed, a representation in $\mathcal{O}_\mu$ has a structure of $\mathcal{U}_q^{\mu,ad}(\hat{\mathfrak{g}})$-module, but it is not unique. A representation of $\mathcal{U}_q^{\mu,ad}(\hat{\mathfrak{g}})$ is said to be in the category $\mathcal{O}_{\mu,ad}$ if it is in the category $\mathcal{O}_\mu$ as a $\mathcal{U}_q^{\mu}(\hat{\mathfrak{g}})$-module. A module in $\mathcal{O}_{\mu,ad}$ is simple if and only if it is simple as a $\mathcal{U}_q^{\mu}(\hat{\mathfrak{g}})$-module. Hence the simple module in $\mathcal{O}_{\mu,ad}$ are parametrized by triples

$$(\mathbf{\Psi}, \overline{\omega}^+, \overline{\omega}^-) \in \mathfrak{r}_\mu \times \mathfrak{t}^* \times \mathfrak{t}^*$$

satisfying :

$$\prod_{j\in I}(\overline{\omega}^+(j))^{C_{j,i}} = \Psi_i(0) \text{ and } \prod_{j\in I}(\overline{\omega}^-(j))^{C_{j,i}} = (z^{-\alpha_i(\mu)}\Psi_i(z))(\infty) \text{ for } i\in I.$$

Such $\overline{\omega}^\pm$ are said to be compatible with $\mathbf{\Psi}$. The corresponding simple representation is denoted by $L(\mathbf{\Psi}, \overline{\omega}^+, \overline{\omega}^-)$. The dimensions of its weight spaces are the same as those of $L(\mathbf{\Psi})$ and its structure can be completely described from the structure of $L(\mathbf{\Psi})$, and $\overline{\omega}^\pm$.

Remark 9.2. *$\overline{\omega}^+$ (resp. $\overline{\omega}^-$) is uniquely determined by $\mathbf{\Psi}$ up to an element $(c_1,\cdots,c_n)$ in the group $K\subset(\mathbb{C}^*)^n$ of solutions of the equations $\prod_{j\in I}(\overline{\omega}^+(j))c_j^{C_{j,i}} = 1$.*

Example 9.3. *The algebra $\mathcal{U}_q^{-\omega_1^\vee,ad}(\hat{sl}_2)$ has additional generator $\overline{\phi}^\pm$ with $(\overline{\phi}^+)^2 = \phi_0^+$ and $(\overline{\phi}^-)^2 = \phi_{-1}^-$. For $a,b\in\mathbb{C}^*$, the representation $L(b(1-az)^{-1})$ is in the category $\mathcal{O}_{-\omega_1^\vee}$. For α (resp. β) a square root of a (resp. b), we have compatible $\overline{\omega}^+ = \beta$ and $\overline{\omega}^- = i\beta\alpha^{-1}$ which give a structure of $\mathcal{U}_q^{-\omega_1^\vee,ad}(\hat{sl}_2)$-module on $L(b(1-az)^{-1})$.*

9.2. **Fundamental Cartan-Drinfeld series.** We consider series of Cartan-Drinfeld series which appear naturally from R-matrices and transfer-matrices in [FR2]. For $i\in I$, set

$$Y_i^\pm(z) = \overline{\phi}_i^\pm \exp\left(\pm(q-q^{-1})\sum_{m>0}\tilde{h}_{i,\pm m}z^{\pm m}\right),$$

$$\tilde{h}_{i,m} = \sum_{j\in I}[r_j]_q\tilde{C}_{j,i}(q^m)h_{j,m} \text{ for } m\neq 0,$$

where $\tilde{C}(z)$ is the inverse of the quantum Cartan matrix $C(q)$ (invertible for a generic q).

By [FR2, Formula (4.9)] (see formula (5.14) above), we have

$$z^{-\alpha_i(\mu)\delta_{\pm,-}}\phi_i^\pm(z) = H_i(Y_1^\pm(z),\cdots,Y_n^\pm(z)) \tag{9.17}$$

where $H_i(Y_1^\pm(z),\cdots,Y_n^\pm(z))$ is set to be equal to

$$\frac{Y_i^\pm(zq_i^{-1})Y_i^\pm(zq_i)}{\prod_{j\in I,C_{j,i}=-1}Y_j^\pm(z)\prod_{j\in I,C_{j,i}=-2}Y_j^\pm(zq^{-1})Y_j^\pm(zq)\prod_{j\in I,C_{j,i}=-3}Y_j^\pm(q^{-2}z)Y_j^\pm(z)Y_j^\pm(q^2z)}.$$

Remark 9.4. *Note that for Laurent formal power series $d_i(z)\in A((z))$ with coefficients in a commutative algebra A, if the system of n-functional equations*

$$d_i(z) = H_i(s_1(z),\cdots,s_n(z)) \;,\; i\in I,$$

has a solution as a formal Laurent power series, it is unique up to constant factors.

The following Cartan-Drinfeld series introduced in [FH2] as limits of transfer-matrices associated to prefundamental representations :

$$T_i^{\pm}(z) = \exp\left(\mp \sum_{m>0} z^{\mp m} \frac{\tilde{h}_{i,\pm m}}{[r_i]_q [m]_{q_i}}\right).$$

We have

$$Y_i^{\pm}(z) = \overline{\phi}_i^{\pm} \frac{T_i^{\pm}(z^{-1}q_i^{\pm 1})}{T_i^{\pm}(z^{-1}q_i^{\mp 1})}. \tag{9.18}$$

For $i \in I$ and $a \in \mathbb{C}^*$, recall the ℓ-weight $\tilde{\mathbf{\Psi}}_{i,a}$ in Example 5.2. Motivated by the next result, we set

$$\Lambda_{i,a} = \tilde{\mathbf{\Psi}}_{i,aq_i^{-1}}^{-1} \mathbf{\Psi}_{i,aq_i}$$

$$= \mathbf{\Psi}_{i,aq_i^{-1}} \mathbf{\Psi}_{i,aq_i} \left(\prod_{j, C_{i,j}=-1} \mathbf{\Psi}_{j,a}^{-1}\right) \left(\prod_{j, C_{i,j}=-2} \mathbf{\Psi}_{j,aq^{-1}}^{-1} \mathbf{\Psi}_{j,aq}^{-1}\right) \left(\prod_{j, C_{i,j}=-3} \mathbf{\Psi}_{j,aq^{-2}}^{-1} \mathbf{\Psi}_{j,a}^{-1} \mathbf{\Psi}_{j,aq^2}^{-1}\right).$$

In particular we have

$$A_{i,a} = \overline{\alpha_i} \Lambda_{i,aq_i^{-1}} / \Lambda_{i,aq_i}. \tag{9.19}$$

Remark 9.5. *The degrees of the coordinates of $\Lambda_{i,a}$ form the simple roots $\alpha_i^\vee$ of the Langlands dual Lie algebra ${}^L\mathfrak{g}$, in opposition to the powers of the monomial $A_{i,a}$ in terms of the $Y_{j,b}$ which give the simple root of $\mathfrak{g}$. This is an indication of the important role played by the Langlands dual Lie ${}^L\mathfrak{g}$ in the following (see Section 12).*

Lemma 9.6. *Consider a rational ℓ-weight $\mathbf{\Psi} = \mathbf{\Psi}(0) \prod_{i \in I, a \in \mathbb{C}^*} \mathbf{\Psi}_{i,a}^{\nu_{i,a}}$. For $i \in I$, the corresponding eigenvalue of $(\overline{\phi}_i^{\pm})^{\mp} Y_i^{\pm}(z)$ is equal to*

$$\overline{Y}_{i,\mathbf{\Psi}}^{\pm}(z) = exp\left(\sum_{j \in I, m>0, a \in \mathbb{C}^*} \tilde{C}_{j,i}(q^m) \nu_{j,a} \frac{a^{\pm m}}{-m} z^{\pm m}\right).$$

The following are equivalent :

(i) for any $i \in I$, $\overline{Y}_{i,\mathbf{\Psi}}^{+}(z)$ is rational.

(ii) for any $i \in I$, $\overline{Y}_{i,\mathbf{\Psi}}^{-}(z)$ is rational.

(iii) $\mathbf{\Psi}(\mathbf{\Psi}(0))^{-1}$ is a Laurent monomial in the $\Lambda_{i,a}$, $i \in I, a \in \mathbb{C}^*$.

Then $\overline{Y}_{i,\mathbf{\Psi}}^{+}(z)$ and $z^{-\omega_i(\mu)} \overline{Y}_{i,\mathbf{\Psi}}^{-}(z)$ coincide as rational fractions up to a constant.

Proof. The formula for $\overline{Y}_{i,\mathbf{\Psi}}^{\pm}(z)$ is clear as the eigenvalue of $h_{i,m}$ associated to $\mathbf{\Psi}$ is

$$-\sum_{a \in \mathbb{C}^*} \frac{\nu_{i,a} a^m}{m(q_i - q_i^{-1})} \text{ for } m \in \mathbb{Z} \setminus \{0\}.$$

Now suppose that (i) is satisfied. There are $v_{i,b} \in \mathbb{Z}$ so that for any $i \in I$, $m \in \mathbb{Z} \setminus \{0\}$:

$$\sum_{j \in I, a \in \mathbb{C}^*} \tilde{C}_{j,i}(q^m) \nu_{j,a} \frac{a^m}{m} = \sum_{b \in \mathbb{C}^*} \frac{v_{i,b} b^m}{m}.$$

We obtain for any $k \in I$:

$$\sum_{a\in\mathbb{C}^*}\nu_{k,a}a^m = \sum_{i\in I,b\in\mathbb{C}^*} C_{i,k}(q^m)v_{i,b}b^m,$$

$$\mathbf{\Psi}(z)(\mathbf{\Psi}(0))^{-1} = \prod_{k\in I,b\in\mathbb{C}^*}\Lambda_{k,a}^{v_{k,a}}.$$

Hence we get (iii). The same computation gives that (ii) implies (iii), and that (iii) implies (i) or (ii).

To conclude, let us suppose that the conditions are satisfied. From (9.17), we have

$$\deg(\overline{Y}^+_{i,\mathbf{\Psi}}(z)) = -\omega_i(\mu) \text{ and } \deg(\overline{Y}^-_{i,\mathbf{\Psi}}(z)) = 0.$$

The $v_{i,b}$ are well-defined from (iii) as the powers of the $\Lambda_{i,aq}$ in the factorization of $\mathbf{\Psi}(\mathbf{\Psi}(0))^{-1}$. Then from the computations above :

$$\overline{Y}^+_{i,\mathbf{\Psi}}(z) = \prod_{b\in\mathbb{C}^*}(1-zb)^{v_{i,b}} = z^{-\omega_i(\mu)}\prod_{b\in\mathbb{C}^*}b^{v_{i,b}}(1-z^{-1}b^{-1})^{v_{i,b}} = (z^{-\omega_i(\mu)}\prod_{b\in\mathbb{C}^*}b^{v_{i,b}})\overline{Y}^-_{i,\mathbf{\Psi}}(z).$$

□

Remark 9.7. *This statement can be seen as a generalization of* [FR2, Lemma 5] *where the case when* $\mathbf{\Psi}$ *is a Laurent monomial in the* $Y_{i,a}$ *is considered.*

With the same notations as in Lemma 9.6, the eigenvalue[2] of $T_i^\pm(z)$ associated to $\mathbf{\Psi}$ is

$$T^\pm_{i,\mathbf{\Psi}}(z) = \exp\left(\sum_{j\in I,m>0,a\in\mathbb{C}^*}\frac{z^{\mp m}\tilde{C}_{j,i}(q^m)}{(q_i^m - q_i^{-m})m}a^{\pm m}\nu_{j,a}\right).$$

9.3. **Rationality and polynomiality.** Consider $W = L(\mathbf{\Psi})$ simple in the category $\mathcal{O}_\mu$. Let $\omega = \mathbf{\Psi}(0)$ be its highest weight and w be a highest weight vector of W.

Let $\mathbf{\Psi'}$ be an ℓ-weight space of W. We have proved in Theorem 5.11 that there are $i_1,\cdots,i_R \in I$, $a_1,\cdots,a_R\in\mathbb{C}^*$ so that

$$\mathbf{\Psi'} = \mathbf{\Psi}A^{-1}_{i_1,a_1}\cdots A^{-1}_{i_R,a_R}.$$

The same computation as for [FH2, Proposition 5.8] gives the following.

Proposition 9.8. *The eigenvalue of* $T_i^\pm(z)$ *on* $W_{\mathbf{\Psi'}}$ *is*

$$T^\pm_{i,\mathbf{\Psi'}}(z) = T^\pm_{i,\mathbf{\Psi}}(z)\times\prod_{1\leq k\leq R, i_k=i}(1-(za_k^{-1})^{\mp1})^{\mp1}.$$

Remark 9.9. *The formula in* [C, Proposition 1.6] *to define an involution of* $\mathcal{U}_q(\hat{\mathfrak{g}})$ *also defines an involution*

$$\sigma : \mathcal{U}_q^{\mu,ad}(\hat{\mathfrak{g}}) \to \mathcal{U}_q^{-\mu,0,ad}(\hat{\mathfrak{g}}) \simeq \mathcal{U}_q^{\mu,ad}(\hat{\mathfrak{g}})$$

so that for $i\in I$, $m\in\mathbb{Z}$, $r\in\mathbb{Z}\setminus\{0\}$ *:*

$$\sigma(x^\pm_{i,r}) = x^\mp_{i,-r-\alpha_i(\mu)}\ ,\ \sigma(h_{i,m}) = -h_{i,-m}\ ,\ \sigma(\overline{\phi}_i^\pm) = \overline{\phi}_i^\mp.$$

[2] This is consistent with the eigenvalue computed in [FH2, (5.20)] when $\mathbf{\Psi}$ is a Laurent monomial in the $Y_{j,b}$, except that there is a misprint in that paper : $\tilde{C}_{i,j}(q^m)$ there should be $\tilde{C}_{j,i}(q^m)$).

For W a representation of $\mathcal{U}_q^{\mu,ad}(\hat{\mathfrak{g}})$, we denote by W^σ its twist by σ. Note that we have :

$$\sigma(\phi_i^+(z)) = z^{\alpha_i(\mu)}\phi_i^-(z^{-1}) \ , \ \sigma(Y_i^+(z)) = z^{\omega_i(\mu)}Y_i^-(z^{-1}) \ , \ \sigma(T_i^+(z)) = T_i^-(z^{-1}).$$

Example 9.10. *The representation $W = L(Y_1^2)$ of $\mathcal{U}_q(\hat{sl}_2)$ was studied in the Example of* [FH2, Section 5.8]. *It is a simple representation of $\mathcal{U}_q^{0,ad}(\hat{sl}_2)$ with parameter $\left(q^2\frac{(1-zq^{-1})^2}{(1-zq)^2}, q, q^{-1}\right)$. It has a weight space of weight 0 of dimension 2. In a slight modification of the basis in* [FH2], *the matrix of $T^-(z)/T^-_{\mathbf{\Psi}}(z)$ and of $Y^-(z)/Y^-_{\mathbf{\Psi}}(z)$ are respectively*

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1-zq^{-1} & z & 0\\ 0 & 0 & 1-zq^{-1} & 0\\ 0 & 0 & 0 & (1-zq^{-1})^2\end{pmatrix} \ , \ \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \frac{q-z^{-1}q^{-1}}{1-z^{-1}} & \frac{z^{-1}(1-q^2)}{(1-z^{-1})^2} & 0\\ 0 & 0 & \frac{q-z^{-1}q^{-1}}{1-z^{-1}} & 0\\ 0 & 0 & 0 & \frac{(q-z^{-1}q^{-1})^2}{(1-z^{-1})^2}\end{pmatrix},$$

$$\text{for } \frac{T^-_{\mathbf{\Psi}}(z^{-1}q^{-2})}{T^-_{\mathbf{\Psi}}(z^{-1}q^2)} = \left(\frac{1-z^{-1}q}{1-z^{-1}q^{-1}}\right)^2 \ , \ Y^-_{\mathbf{\Psi}}(z) = \frac{q^{-1}T^-_{\mathbf{\Psi}}(z^{-1}q^{-1})}{T^-_{\mathbf{\Psi}}(z^{-1}q)}.$$

As $W^\sigma \simeq L(Y^2_{q^{-2}})$, the matrix of $T^+(z)/T^+_{\mathbf{\Psi}}(z)$ and $Y^+(z)/Y^+_{\mathbf{\Psi}}(z)$ are respectively

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \frac{1}{1-z^{-1}q} & \frac{z^{-1}q^2}{(1-z^{-1}q)^2} & 0\\ 0 & 0 & \frac{1}{1-z^{-1}q} & 0\\ 0 & 0 & 0 & \frac{1}{(1-z^{-1}q)^2}\end{pmatrix} \ , \ \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \frac{q^{-1}-zq}{1-z} & \frac{z(1-q^2)}{(1-z)^2} & 0\\ 0 & 0 & \frac{q^{-1}-zq}{1-z} & 0\\ 0 & 0 & 0 & \frac{(q^{-1}-zq)^2}{(1-z)^2}\end{pmatrix},$$

$$\text{for } T^+_{\mathbf{\Psi}}(z) = T^-_{\mathbf{\Psi}}(z^{-1}q^2)(1-z^{-1}q)^2 \ , \ Y^+_{\mathbf{\Psi}}(z) = q\frac{T^+_{\mathbf{\Psi}}(z^{-1}q)}{T^+_{\mathbf{\Psi}}(z^{-1}q^{-1})} = q\frac{T^-_{\mathbf{\Psi}}(zq)(1-z)^2}{T^-_{\mathbf{\Psi}}(zq^3)(1-zq^2)^2}.$$

These operators are rational. The action of $Y^+(z)/Y^+_{\mathbf{\Psi}}(z)$ and $Y^-(z)/Y^-_{\mathbf{\Psi}}(z)$ coincide. On a weight space of weight $\overline{\omega}^2\overline{\alpha}^{-h}$, the action of the following does not depend on z :

$$z^{-h}\frac{T^+(z)T^-(z)}{T^+_{\mathbf{\Psi}}(z)T^-_{\mathbf{\Psi}}(z)} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -q^{-1} & 1 & 0\\ 0 & 0 & -q^{-1} & 0\\ 0 & 0 & 0 & q^{-2}\end{pmatrix}.$$

We note that $(T^\pm(z)/T^\mp_{\mathbf{\Psi}}(z^{-1}q^2))^{\pm 1}$ and $(T^\pm_{\mathbf{\Psi}}(z)/T^\pm(z))^{\pm 1}$ are polynomials in $z^{\mp 1}$.

Example 9.11. *Consider the prefundamental representation $W = L(\mathbf{\Psi}_1^{-1})$ of $\mathcal{U}_q^{-\omega_1^\vee}(\hat{sl}_2)$ as in Example 3.9. Then we have for $j \geq 0$:*

$$\frac{Y^\pm(z)}{Y^\pm_{\mathbf{\Psi}}(z)}.v_j = \frac{q^{-j}(1-zq)}{1-q^{1-2j}z}.v_j \ , \ \frac{T^\pm(z)}{T^\pm_{\mathbf{\Psi}}(z)}.v_j = (1-z^{\mp 1})^{\mp 1}(1-z^{\mp 1}q^{\mp 2})^{\mp 1}\cdots(1-z^{\mp 1}q^{\mp 2(j-1)})^{\mp 1}.v_j,$$

$$\text{for } Y^\pm_{\mathbf{\Psi}} = (i)^{\delta_{\pm,-}}\, exp\left(-\sum_{m>0}\frac{z^{\pm m}}{m(q^m+q^{-m})}\right) \ , \ T^\pm_{\mathbf{\Psi}}(z) = exp\left(\sum_{m>0}\frac{z^{\mp m}}{m(q^{2m}-q^{-2m})}\right).$$

We have $(T^+(z)T^-(z)/(T^+_{\mathbf{\Psi}}(z)T^-_{\mathbf{\Psi}}(z))).v_j = (-z)^j q^{j(j-1)}v_j$.

The following was partly established in [FH2, Theorem 5.17] for simple finite-dimensional representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$. Let ω' be a weight of W. For $i \in I$, we denote by $ht_i(\omega(\omega')^{-1})$ the multiplicity of $\overline{\alpha_i}$ in the factorization of $\omega(\omega')^{-1}$ as a product of simple roots.

Theorem 9.12. *(i) The operators*

$$\frac{Y_i^-(z)}{Y_{i,\mathbf{\Psi}}^-(z)} \text{ and } \frac{Y_i^+(z)}{Y_{i,\mathbf{\Psi}}^+(z)} \in (End(W))(z)$$

are rational of degree 0 *on* W *and coincide.*

(ii) On $W_{\omega'}$ *the operators*

$$\frac{T_i^-(z)}{T_{i,\mathbf{\Psi}}^-(z)} \text{ and } z^{ht_i(\omega(\omega')^{-1})}\frac{T_{i,\mathbf{\Psi}}^+(z)}{T_i^+(z)} \in (End(W_{\omega'}))[z]$$

are polynomial in z *of degree* $ht_i(\omega(\omega')^{-1})$ *and coincide up to a constant operator factor.*

Remark 9.13. *This constant operator is not necessarily diagonalizable, see Example 9.10.*

Proof. For $m \in \mathbb{Z} \setminus \{0\}$, $r \in \mathbb{Z}$, $\epsilon = 1$ or $\epsilon = -1$, we have

$$[\tilde{h}_{i,m}, x_{j,r}^\pm] = \delta_{i,j}\frac{[mr_i]_q}{m}x_{j,m+r}^\pm \text{ and } \overline{\phi}_i^\pm x_{j,r}^\epsilon = q_i^{\pm\delta_{i,j}\epsilon}x_{j,r}^\epsilon\overline{\phi}_i^\pm.$$

In particular $[x_j^\epsilon(w), Y_i^\pm(z)] = 0$ for $j \neq i$ and $\epsilon = 1$ or -1. For $i = j$, the relation (3.4) is

$$[(q_i - q_i^{-1})h_{i,m}, x_{i,r}^\pm] = \pm\frac{q_i^{2m} - q_i^{-2m}}{m}x_{j,m+r}^\pm.$$

It the same relation as

$$[(q - q^{-1})\tilde{h}_{i,m}, x_{i,r}^\pm] = \frac{q_i^{m} - q_i^{-m}}{m}x_{j,m+r}^\pm,$$

except that in the right side we have q_i replaced by q_i^2, the term $(q_i - q_i^{-1})h_{i,m}$ being replaced by $(q - q^{-1})\tilde{h}_{i,m}$ as in the definition of $Y_i^\pm(z)$ in comparison to the definition of $\phi_i^\pm(z)$ (we have the same substitution for the analog of the relations (3.3)). Hence we get as for the relation (3.8) :

$$Y_i^\epsilon(z)x_j^\pm(w) = \frac{q_i^{\pm 1}w - z}{w - q_i^{\pm 1}z}x_j^\pm(w)Y_i^\epsilon(z) \text{ for } \epsilon = + \text{ or } -,$$

First assume that $T_i^\pm(z)/T_{i,\mathbf{\Psi}}^\pm(z)$ has a rational action on W. Then by (9.18) $Y_i^\pm(z)/Y_{i,\mathbf{\Psi}}^\pm(z)$ has a rational action on W of degree 0. By the remarks above, the rational operator

$$\frac{Y_i^+(z)Y_{i,\mathbf{\Psi}}^-(z)}{Y_i^-(z)Y_{i,\mathbf{\Psi}}^+(z)}$$

commutes with all operators in the image of $\mathcal{U}_q$ in the endomorphism ring of W. Hence, by Schur Lemma, $Y_i^+(z)/Y_{i,\mathbf{\Psi}}^+(z)$ and $Y_i^-(z)/Y_{i,\mathbf{\Psi}}^-(z)$ coincide as rational operators on W.

Hence (i) is proved. Besides, consider the rational operator on $W_{\omega'}$:

$$U(z) = z^{-ht_i(\omega(\omega')^{-1})}\frac{T_i^-(z)T_i^+(z)}{T_{i,\mathbf{\Psi}}^-(z)T_{i,\mathbf{\Psi}}^+(z)}$$

Then by (9.18) and (i), we get that $U(z) = U(zq^2)$ and so $U(z)$ does not depend on z, that is $\frac{T_i^-(z)}{T_{i,\mathbf{\Psi}}^-(z)}$, $z^{ht_i(\omega(\omega')^{-1})}\frac{T_{i,\mathbf{\Psi}}^+(z)}{T_i^+(z)}$ coincide up to a constant operator factor.

Now, the statement for $T_i^-(z)/T_{i,\mathbf{\Psi}}^-(z)$ is known for finite-dimensional simple $\mathcal{U}_q(\hat{\mathfrak{g}})$-modules by [FH2, Theorem 5.17]. But, using the involution σ of $\mathcal{U}_q^{ad}(\hat{\mathfrak{g}})$ as in Remark 9.9, we get that $T_i^+(z)$ is rational on a finite-dimensional W up to a scalar map, and so that $T_i^+(z)/T_{i,\mathbf{\Psi}}^+(z)$ is rational. By the discussion in the first part of this proof, this implies that result for $T_i^+(z)$ in this case.

Now let W be a tensor product of various negative prefundamental representations. By Corollary 4.10, it is simple as a $\mathcal{U}_q(\hat{\mathfrak{b}})$-module isomorphic to $L^{\mathfrak{b}}(\mathbf{\Psi})$. This representation can be constructed in [HJ, HL3] as an inductive limit of a linear inductive system of simple tensor products of Kirillov-Reshetikhin modules which are simple finite-dimensional representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$. In this inductive system, the highest weight vectors are preserved and the action of $\phi_i^+(z)$ is stationary up to a scalar function factor. Hence the result follows for $T_i^+(z)$ for W from the result for the finite-dimensional $\mathcal{U}_q(\hat{\mathfrak{b}})$-modules. But not only the inductive construction gives the action of $\mathcal{U}_q(\hat{\mathfrak{b}})$, but also of the whole asymptotical algebra $\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})$ from which the action of $T_i^-(z)$ on W is obtained. As above, it is stationary up to a scalar function factor. The rationality of $T_i^-(z)/T_{i,\mathbf{\Psi}}^-(z)$ on W, and the result, follow.

The result is also clear for a tensor product of various positive prefundamental representations as it is one-dimensional. Now, as

$$\Delta_u(T_i^\pm(z)) = T_i^\pm(z)\otimes T_i^\pm(zu^{-1}),$$

it follows from Corollary 5.6 and (i) in Remark 5.8 that the result true for a tensor product of negative prefundamental representations by a tensor product of positive prefundamental representations. The result follows. $\square$

10. Truncated shifted quantum affine algebras

Truncation of shifted quantum affine algebras are defined in [FT, Section 8.(iii)] in the study of quantized K-theoretic Coulomb branches of $3d$ $N=4$ SUSY quiver gauge theories (see the Introduction).

We recall the definition of truncated shifted quantum affine algebras in terms of series $A_i^{\mathcal{Z},\pm}(z)$ of Cartan-Drinfeld generators. We explain how these series appear naturally in terms of the Cartan-Drinfeld series derived from transfer-matrices in the previous section.

We establish (Proposition 10.7) a necessary and sufficient condition for the defining series $A_i^{\mathcal{Z},\pm}(z)$ to have a rational action on a simple representation.

10.1. **Truncation series.** We consider variation of these series $Y_i^{\pm}(z)$. We fix

$$\lambda = \sum_{i\in I} N_i \omega_i^\vee \in \Lambda^+ \text{ and } \lambda \succeq \mu = \lambda - \sum_{i\in I} a_i\alpha_i^\vee \in \Lambda,$$

with the N_i and a_i non-negative. We consider a family of polynomials Z_i of degree N_i :

$$Z_i(z) = (1 - q_i z z_{i,1})(1 - q_i z z_{i,2}) \cdots (1 - q_i z z_{i,N_i}),$$

and we get an ℓ-weight $\mathbf{Z} = (Z_i(z))_{i\in I}$. We also fix additional parameters $z_i' \in \mathbb{C}^*$ so that

$$\prod_{j\in I} (z_j')^{C_{j,i}} = (-q_i)^{N_i} z_{i,1} \cdots z_{i,N_i}.$$

There are unique up to the group K of Remark 9.2. The collection of these data is denoted

$$\mathcal{Z} = (\mathbf{Z}, z_1', \cdots, z_n').$$

Then we define

$$A_i^{\mathcal{Z},\pm}(z) = \sum_{r\geq 0} A_{i,\pm r}^{\mathcal{Z},\pm} z^{\pm r} = (z_i')^{\delta_{\pm,-}} \frac{\overline{Y}_{i,\mathbf{Z}}^{\pm}(zq_i^{-1})}{Y_i^{\pm}(zq_i^{-1})} \in \mathcal{U}_q^{\mu,ad}(\hat{\mathfrak{g}})[[z^{\pm 1}]]. \tag{10.20}$$

Note that by definition we have

$$A_{i,0}^{\mathcal{Z},+} = (\overline{\phi}_i^+)^{-1} \ ,\ A_{i,0}^{\mathcal{Z},-} = z_i'(\overline{\phi}_i^-)^{-1}. \tag{10.21}$$

From (9.17), we recover the defining formula in [FT], that for $i \in I$:

$$z^{\alpha_i(\lambda-\mu)\delta_{\pm,-}} \phi_i^{\pm}(z)(Z_i(z))^{-1} = (H_i(A_1^{\mathcal{Z},\pm}(zq_1), \cdots, A_n^{\mathcal{Z},\pm}(zq_n)))^{-1}.$$

Remark 10.1. *(i) The series $A_i^{\mathcal{Z},\pm}(z)$ are characterized by this property and by (10.21), see Remark 9.4.*

(ii) The notations could be misleading as the series $A_i^{-,\mathcal{Z}}(z)$ in [FT] *are variations of the $Y_i(z)$ in* [FR2]*, not of the $A_i(z)$ therein.*

(iii) The subalgebra generated by the $A_{i,\pm r}^{\mathcal{Z},\pm}$ is called the Gelfand-Tsetlin subalgebra in [BrK]*. It equals the Cartan-Drinfeld subalgebra generated by the $\phi_{i,\pm m}^{\pm}$ and the $A_{i,0}^{\mathcal{Z},\pm 1}$.*

Example 10.2. *Assume that $\mathfrak{g}$ is of type B_2 with $r_1 = 2$ and $r_2 = 1$. The formula give*

$$\phi_1^+(z)(Z_1(z))^{-1} = \frac{A_2^{\mathcal{Z},+}(z)A_2^{\mathcal{Z},+}(zq^2)}{A_1^{\mathcal{Z},+}(z)A_1^{\mathcal{Z},+}(zq^4)} \text{ and } \phi_2^+(z)(Z_2(z))^{-1} = \frac{A_1^{\mathcal{Z},+}(zq^2)}{A_2^{\mathcal{Z},+}(z)A_2^{\mathcal{Z},+}(zq^2)}.$$

10.2. **Definition.**

Definition 10.3. *The truncated shifted quantum affine algebra $\mathcal{U}_{q,\lambda}^{\mu,\mathcal{Z}}(\hat{\mathfrak{g}})$ is the quotient of $\mathcal{U}_q^{\mu,ad}(\hat{\mathfrak{g}})$ by the relations that for $i \in I$, $A_i^{\mathcal{Z},\pm}(z)$ is a polynomial of degree a_i in $z^{\pm 1}$ and :*

$$A_{i,0}^{\mathcal{Z},+} A_{i,0}^{\mathcal{Z},-} = (-q_i)^{a_i} \ ,\ A_i^{\mathcal{Z},+}(z) = (zq_i^{-1})^{a_i} A_i^{\mathcal{Z},-}(z) \text{ for } i \in I.$$

Remark 10.4. *(i) The relations imply* $\phi^+_{i,0}\phi^-_{i,\alpha_i(\mu)} = \phi_{i,\mathcal{Z}}$ *for*

$$\phi_{i,\mathcal{Z}} = (-1)^{N_i+\sum_j C_{j,i}a_j} q_i^{\alpha_i(\mu)} z_{i,1}z_{i,2}\cdots z_{i,N_i}.$$

(ii) We do not write the relations [FT, (8.11)] *which are redundant in our notations.*
(iii) The relations are not preserved by twist of spectral parameter $z \mapsto az$.

Example 10.5. $\mathcal{U}^{-\omega_1^\vee,\mathcal{Z}}_{q,\omega_1^\vee}(\hat{sl}_2)$ *is the quotient of* $\mathcal{U}_q^{-\omega_1^\vee,ad}(\hat{sl}_2)$ *by the relations :*

$$A^{\mathcal{Z},\pm}_{\pm s} = 0 \text{ for } s > 1,\ A_0^{\mathcal{Z},+}A_1^{\mathcal{Z},+} = -1\ ,\ A_0^{\mathcal{Z},-} = A_1^{\mathcal{Z},+}q\ ,\ A_{-1}^{\mathcal{Z},-} = A_0^{\mathcal{Z},+}q.$$

In the following, when $\mathcal{Z}$ is fixed without ambiguity, we will simply denote $A^\pm(z)$ and $\mathcal{U}^\mu_{q,\lambda}(\hat{\mathfrak{g}})$. The defining relations of $\mathcal{U}^\mu_{q,\lambda}(\hat{\mathfrak{g}})$ can be interpreted in the following way.

Proposition 10.6. *For each* $i \in I$*, the images of* $\phi_i^\pm(z)$ *in* $\mathcal{U}^\mu_{q,\lambda}(\hat{\mathfrak{g}})[[z^{\pm 1}]]$ *are rational of degree* $\alpha_i(\mu)$ *and coincide in* $\mathcal{U}^\mu_{q,\lambda}(\hat{\mathfrak{g}})(z)$*. They satisfy* $\phi_i^+(0)(\phi_i^+(z)z^{-\alpha_i(\mu)})(\infty) = \phi_{i,\mathcal{Z}}$.

Proof. The rationality is clear as the $A_j^\pm(z)$ are polynomials. Then we get in $\mathcal{U}^\mu_{q,\lambda}(\hat{\mathfrak{g}})(z)$:

$$\frac{\phi_i^+(z)}{Z_i(z)} = \frac{\prod_{j,C_{j,i}=-1} z^{a_j}\prod_{j,C_{j,i}=-2}(zq^{-1}zq)^{a_j}\prod_{j,C_{j,i}=-1}(zq^{-2}zzq^2)z^{a_j}}{(zq_i^{-1}zq_i)^{a_i}}\frac{z^{\alpha_i(\lambda-\mu)}\phi_i^-(z)}{Z_i(z)}$$

$$= \prod_{j\in I} z^{-a_jC_{j,i}}\frac{z^{\alpha_i(\lambda-\mu)}\phi_i^-(z)}{Z_i(z)} = z^{\sum_{j\in I}C_{j,i}\omega_j(\mu-\lambda)}\frac{z^{\alpha_i(\lambda-\mu)}\phi_i^-(z)}{Z_i(z)} = \frac{\phi_i^-(z)}{Z_i(z)}.$$

This implies the equality as rational fractions. The degree is

$$N_i - \sum_{j\in I} C_{j,i}a_j = \alpha_i(\lambda) + \sum_{j\in I} C_{j,i}\omega_j(\mu-\lambda) = \alpha_i(\mu).$$

$\square$

10.3. **Rationality of truncation series.** Let $W = L(\mathbf{\Psi})$ be a simple in $\mathcal{O}_\mu$.

Proposition 10.7. *(1) The following are equivalent :*
(i) for any $i \in I$, $A_i^{\mathcal{Z},+}(z)$ *is rational on* W.
(ii) for any $i \in I$, $A_i^{\mathcal{Z},-}(z)$ *is rational on* W.
(iii) $\mathbf{\Psi}(0)\mathbf{Z}\mathbf{\Psi}^{-1}$ *is a Laurent monomial in the* $\Lambda_{i,a}$, $i \in I$, $a \in \mathbb{C}^*$.
(2) When these conditions are satisfied, $A_i^{\mathcal{Z},+}(z)$ *and* $(zq_i^{-1})^{a_i}A_i^{\mathcal{Z},-}(z)$ *coincide on* W *as rational fractions (up to an element of the group* K *of Remark 9.2) and have degree* a_i.

Proof. (1) From Theorem 9.12, $A_i^{\mathcal{Z},\pm}(z)$ is rational on W if and only the eigenvalue on a highest weight vector is rational. From Lemma 9.6, this is equivalent to (i) or (ii).

(2) When the condition are satisfied, it follows from Theorem 9.12 that $Y_i^\pm(z)/Y^\pm_{i,\mathbf{\Psi}}(z)$ coincide as rational fractions, so it suffices to prove that $(z_i'z^{a_i})^{\delta_{\pm,-}}\overline{Y}^\pm_{i,\mathbf{Z}}(z)/Y^\pm_{i,\mathbf{\Psi}}(z)$ coincide as rational fractions. From Lemma 9.6, they coincide up to a constant c_i :

$$\overline{Y}^+_{i,\mathbf{Z}}(z)/Y^+_{i,\mathbf{\Psi}}(z) = c_iz_i'z^{a_i}\overline{Y}^-_{i,\mathbf{Z}}(z)/Y^-_{i,\mathbf{\Psi}}(z).$$

As

$$H_i(\overline{Y}_{1,\mathbf{Z}}^{-}(z),\cdots,\overline{Y}_{n,\mathbf{Z}}^{-}(z)) = (-q_i z)^{-N_i}(z_{i,1}\cdots z_{i,N_i})^{-1}Z_i(z),$$

we get

$$Z_i(z)\Psi_i^{-1}(z) = H_i(c_1,\cdots,c_n)z^{\alpha_i(-\mu)}Z_i(z)/(z^{-\alpha_i(\mu)}\Psi_i(z)),$$

and so $(c_1,\cdots,c_n)\in K$. □

Example 10.8. *We continue Example 9.10. Let us consider the polynomial operators :*

$$A^-(zq) = \frac{T^-(z^{-1}q)}{T^-_{\mathbf{\Psi}}(z^{-1}q)}\frac{T^-_*(z^{-1}q^{-1})}{T^-(z^{-1}q^{-1})} = \frac{q^3(1-q^{-2}z^{-1})^2 Y^-_{\mathbf{\Psi}}(z)}{Y^-(z)}$$

$$= \begin{pmatrix} q^3(1-q^{-2}z^{-1})^2 & 0 & 0 & 0 \\ 0 & (q^2-z^{-1})(1-z^{-1}) & z^{-1}(q^3-q) & 0 \\ 0 & 0 & (q^2-z^{-1})(1-z^{-1}) & 0 \\ 0 & 0 & 0 & q(1-z^{-1})^2 \end{pmatrix}.$$

$$A^+(zq) = \frac{T^+(z^{-1}q^{-1})}{T^+_*(z^{-1}q^{-1})}\frac{T^+_{\mathbf{\Psi}}(z^{-1}q)}{T^+(z^{-1}q)} = \frac{q^{-1}(1-zq^2)^2 Y^+_{\mathbf{\Psi}}(z)}{Y^+(z)}$$

$$= \begin{pmatrix} q^{-1}(1-zq^2)^2 & 0 & 0 & 0 \\ 0 & (1-zq^2)(1-z) & z(q^3-q) & 0 \\ 0 & 0 & (1-zq^2)(1-z) & 0 \\ 0 & 0 & 0 & q(1-z)^2 \end{pmatrix}$$

We note that $A^\pm(z) = A^{\mathcal{Z},\pm}(z)$ *with* $\mathcal{Z} = ((1-zq^{-1})^2(1-zq^3)^2, q^2)$ *as we have*

$$A^+(z) = (zq^{-1})^2 A^-(z)\ ,\ A_+(0)A_-(\infty) = q^2 Id,$$

$$(A^+(z)A^+(zq^2))^{-1} = \frac{\phi^+(z)}{(1-zq^{-1})^2(1-zq^3)^2} \text{ and } (A^-(z)A^-(zq^2))^{-1} = \frac{z^2\phi^-(z)}{(1-zq^{-1})^2(1-zq^3)^2}.$$

Remark 10.9. *We see from the proof of Lemma 9.6 that the contribution of the factor* $\Lambda_{i,a}$ *to the eigenvalue of* $A_i^+(z)$ *is* $(1-zaq_i^{-1})^{-1}$.

11. Descent to the truncation

We study which simple representations descend to truncated shifted quantum affine algebras. See the introduction for a discussion on earlier results [BrK, KTWWY1, KTWWY2, N5, NW].

We establish a necessary condition on a simple representation to be a representation of a truncated shifted quantum affine algebra (Proposition 11.11). As a consequence we establish that a truncated shifted quantum affine algebra has only a finite number of isomorphism classes of simple representations (Theorem 11.15). Then we introduce a partial ordering $\preceq_{\mathcal{Z}}$ on ℓ-weights (up to sign) and we prove that the simple representations $L(\mathbf{\Psi})$ of a truncated shifted quantum affine algebra of parameter $\mathbf{Z}$ must satisfy $\mathbf{\Psi} \preceq_{\mathcal{Z}} \mathbf{Z}$ (Theorem 11.9). We note this partial ordering is also related to the Langlands dual Lie algebra ${}^L\mathfrak{g}$, a point which will be crucial in the next Section. In the sl_2-case we establish a complete characterization of simple representations of a shifted quantum affine algebra (Theorem 11.17).

11.1. **Descent.**

Definition 11.1. *A representation in $\mathcal{O}_\mu$ descends to the truncation $\mathcal{U}_{q,\lambda}^{\mu,\mathcal{Z}}$ if there is has structure of $\mathcal{U}_q^{\mu,ad}(\hat{\mathfrak{g}})$-module compatible with the defining relations of the quotient $\mathcal{U}_{q,\lambda}^{\mu,\mathcal{Z}}(\hat{\mathfrak{g}})$.*

This defines an abelian subcategory $\mathcal{O}_{\mu,\mathcal{Z}}^\lambda$ of $\mathcal{O}_\mu$.

Remark 11.2. *The category $\mathcal{O}_{\mu,\mathcal{Z}}^\lambda$ is stable by sign-twist.*

We investigate which simple modules $L(\mathbf{\Psi})$ are in $\mathcal{O}_{\mu,\mathcal{Z}}^\lambda$. It means there is a structure $L(\mathbf{\Psi},\overline{\omega}^+,\overline{\omega}^-)$ of $\mathcal{U}_q^{\mu,ad}(\hat{\mathfrak{g}})$-module on $L(\mathbf{\Psi})$ which is a $\mathcal{U}_{q,\lambda}^{\mu,\mathcal{Z}}(\hat{\mathfrak{g}})$-module. If $\overline{\omega}^\pm$ exist, then

$$(\overline{\omega}^+\overline{\omega}^-)(i) = z_i'(-q_i)^{-a_i} \text{ for any } i \in I. \tag{11.22}$$

By Proposition 10.6, such $\overline{\omega}^\pm$ compatible with $\mathbf{\Psi} \in \mathfrak{r}_\mu$ exist if and only if for any $i \in I$, $\Psi_i(0)(\Psi_i(z)z^{-\alpha_i(\mu)})(\infty) = \phi_{i,\mathcal{Z}}$. We will denote by $\mathfrak{r}_{\mu,\mathcal{Z}}$ the set of such $\mathbf{\Psi}$. We will work with such ℓ-weights (if necessary, although not written explicitly, we will renormalize ℓ-weights by constants to work in this set). So let us consider $\mathbf{\Psi} \in \mathfrak{r}_{\mu,\mathcal{Z}}$ and $\overline{\omega}^\pm \in \mathfrak{t}^*$ compatible with $\mathbf{\Psi}$ satisfying the relations (11.22). Then the central element $\overline{\phi}_i^+\overline{\phi}_i^-$ (resp. $A_{i,0}^{\mathcal{Z},+}A_{i,0}^{\mathcal{Z},-}$) acts as the scalar $z_i'(-q_i)^{-a_i}$ (resp. $(-q_i)^{a_i}$) on $L(\mathbf{\Psi})$.

Example 11.3. *Suppose $\lambda = \mu$. Then for V in $\mathcal{O}_{\mu,\mathcal{Z}}^\mu$, the operator $A_{i,0}^{\mathcal{Z},+} = A_i^{\mathcal{Z},+}(z) = A_i^{\mathcal{Z},-}(z) = A_{i,0}^{\mathcal{Z},-}$ is constant and satisfies $Id = A_{i,0}^{\mathcal{Z},+}A_{i,0}^{\mathcal{Z},-} = (A_{i,0}^{\mathcal{Z},+})^2$ and $\phi_i^+(z)(Z_i(z))^{-1} = \prod_{j\in I}(A_{j,0}^{\mathcal{Z},+})^{-C_{j,i}}$. Hence $\mathcal{O}_{\mu,\mathcal{Z}}^\mu$ is semi-simple, its objects are the 1-dimensional of highest ℓ-weight $\Psi(z) = (\eta_i Z_i(z))_{i\in I}$ with $\eta_i = \prod_j \epsilon_j^{C_{j,i}}$ for a choice of $\epsilon_j = \pm 1$ (see Example 4.12). Up to sign-twist, there is a unique simple representation in $\mathcal{O}_{\mu,\mathcal{Z}}^\mu$.*

Example 11.4. *Let $\mathfrak{g} = sl_2$, $\lambda = 2\omega_1^\vee$, $\mu = 0$. Set $\mathcal{Z} = ((1-aq^3z)(1-aq^{-1}z), aq)$. For $\mathbf{\Psi}(z) = \frac{aq(1-aq^{-1}z)}{(1-aqz)}$, $L(\mathbf{\Psi})$ is a 2-dimensional representation in $\mathcal{O}_{0,\mathcal{Z}}^{2\omega_1^\vee}$. Indeed its other ℓ-weight is $\mathbf{\Psi}'(z) = \frac{aq^{-1}(1-aq^3z)}{(1-aqz)}$. Let α so that $\alpha^2aq = 1$. For $A(z) = (\alpha - \alpha^{-1}z)$ and $A'(z) = (q\alpha - \alpha^{-1}q^{-1}z)$, one has*

$$\Psi(z) = \frac{Z(z)}{A(z)A(zq^2)}\ ,\ \Psi'(z) = \frac{Z(z)}{A'(z)A'(zq^2)}.$$

Remark 11.5. *Let $i \in I$ and V a representation in $\mathcal{O}_\mu$ so that the central element $\overline{\phi}_i^+\overline{\phi}_i^-$ acts by $z_i'(-q_i)^{-a_i}$. As a direct consequence of Proposition 10.7, if $A_i^{\mathcal{Z},+}(z)$ has a rational action on V, then $A_i^+(z) = (zq_i^{-1})^{a_i}A_i^-(z)$ if and only if we have on V*

$$A_i^+(z) \sim_\infty (-z)^{a_i}(A_i^+(0))^{-1}.$$

Besides it suffices that this condition is satisfied on a highest weight vector.

11.2. **Partial ordering.** We have the following refinement of Proposition 10.7.

Lemma 11.6. *If for any $i \in I$, $A_i^{\mathcal{Z},+}(z)$ is polynomial on $L(\mathbf{\Psi})$, then $\mathbf{\Psi}(0)\mathbf{Z}\mathbf{\Psi}^{-1}$ is a monomial in the $\Lambda_{i,a}$, $i \in I$, $a \in \mathbb{C}^*$.*

Proof. By Proposition 10.7, $\mathbf{\Psi}(0)\mathbf{Z}\mathbf{\Psi}^{-1}$ is a Laurent monomial in the $\Lambda_{i,a}$, $i \in I$, $a \in \mathbb{C}^*$. Following the proof of Lemma 9.6, we see that the powers of the $\Lambda_{i,a}$ have to be non-negative so that the eigenvalue of $A_i^{\mathcal{Z},+}(z)$ on a highest weight vector of $L(\mathbf{\Psi})$ is a polynomial. □

This suggest the following for the set of ℓ-weights :

$$\mathfrak{r}_{\mathcal{Z}} = \bigsqcup_{\mu\in\Lambda} \mathfrak{r}_{\mu,\mathcal{Z}}.$$

For $\mathbf{\Psi}, \mathbf{\Psi}' \in \mathfrak{r}_{\mathcal{Z}}$, we set $\mathbf{\Psi}' \preceq_{\mathcal{Z}} \mathbf{\Psi}$ if $(\mathbf{\Psi}(0))^{-1}(\mathbf{\Psi}'(0))\mathbf{\Psi}(\mathbf{\Psi}')^{-1}$ is a monomial in the $\Lambda_{i,a}$, $i \in I$, $a \in \mathbb{C}^*$.

Proposition 11.7. *If $\mathbf{\Psi}' \preceq_{\mathcal{Z}} \mathbf{\Psi}$, then $\mathbf{\Psi}$ is determined by $\mathbf{\Psi}'$ and $(\mathbf{\Psi}(0))^{-1}(\mathbf{\Psi}'(0))\mathbf{\Psi}(\mathbf{\Psi}')^{-1}$ up to a sign $i \in I$.*

In particular, $\preceq_{\mathcal{Z}}$ defines a partial ordering on $\mathfrak{r}_{\mathcal{Z}}$ (up to signs).

Proof. Let $\mathbf{\Psi} \in \mathfrak{r}_{\mu,\mathcal{Z}}$, $\mathbf{\Psi}' \in \mathfrak{r}_{\mu',\mathcal{Z}}$. It suffices to prove that each $\mathbf{\Psi}_i(0)$ is determined up to a sign. The conditions imply that for any $i \in I$, $(\mathbf{\Psi}_i(0))^{-1}(\mathbf{\Psi}_i(z)z^{-\alpha_i(\mu)})(\infty)$ is determined. But $\mathbf{\Psi}_i(0)(\mathbf{\Psi}_i(z)z^{-\alpha_i(\mu)})(\infty)$ is fixed, so $(\mathbf{\Psi}_i(0))^2$ is determined.

For the second point, it suffices to consider $\mathbf{\Psi}$, $\mathbf{\Psi}'$ so that $\mathbf{\Psi} \preceq_{\mathcal{Z}} \mathbf{\Psi}'$ and $\mathbf{\Psi}' \preceq_{\mathcal{Z}} \mathbf{\Psi}$. Then, $(\mathbf{\Psi}(0))^{-1}(\mathbf{\Psi}'(0))\mathbf{\Psi}(\mathbf{\Psi}')^{-1} = 1$ and from the first point we get that $\mathbf{\Psi}$ and $\mathbf{\Psi}'$ are equal (up to a sign for each $i \in I$). □

Remark 11.8. *(i) If we add the data of a $\overline{\omega}^{\pm} \in \mathfrak{t}^*$ compatible with each $\mathbf{\Psi}$ as above, then as for Lemma 9.6 we can replace "up to a sign" by "up to sign twist".*

(ii) This partial ordering is different the extension of Nakajima partial ordering $\preceq$ in Section 5.5.

Now Lemma 11.6 can be reformulated in terms of this partial ordering.

Theorem 11.9. *For $L(\mathbf{\Psi})$ a simple representation in $\mathcal{O}_{\mu,\mathcal{Z}}^{\lambda}$ we have*

$$\mathbf{\Psi} \preceq_{\mathcal{Z}} \mathbf{Z}.$$

Remark 11.10. *For $L(\mathbf{\Psi})$ a representation in $\mathcal{O}_{\mu,\mathcal{Z}}^{\lambda}$, all its ℓ-weights $\mathbf{\Psi}'$ must also satisfy $\mathbf{\Psi}' \preceq_{\mathcal{Z}} \mathbf{Z}$. For instance, for an ℓ-weight $\mathbf{\Psi}' = \mathbf{\Psi} A_{i,a}^{-1}$, Formula (9.19) gives that a factor Λ_{i,aq_i} in $\mathbf{Z}\mathbf{\Psi}^{-1}$ is replaced by $\Lambda_{i,aq_i^{-1}}$ in $\mathbf{Z}(\mathbf{\Psi}')^{-1}$. This is analog for all other ℓ-weights $\mathbf{\Psi}'$.*

11.3. **A necessary condition on highest ℓ-weight and finiteness.** Recall $\overline{Y}_{i,\mathbf{Z}\mathbf{\Psi}^{-1}}(zq_i^{-1})$ the eigenvalue of $A_i^{\mathbf{\Psi},+}(z)$ on a highest weight vector of a simple representation $L(\mathbf{\Psi})$ in $\mathcal{O}_{\mu}$.

Proposition 11.11. *Suppose that $L(\mathbf{\Psi})$ is in $\mathcal{O}_{\mu,\mathcal{Z}}^{\lambda}$. Then for $i \in I$, $\overline{Y}_{i,\mathbf{Z}\mathbf{\Psi}^{-1}}(zq_i^{-1})$ and $\overline{Y}_{i,\mathbf{Z}\mathbf{\Psi}^{-1}}(zq_i^{-1})\Psi_i(z)$ are polynomials with*

$$\overline{Y}_{i,\mathbf{Z}\mathbf{\Psi}^{-1}}(zq_i^{-1}) \sim_{\infty} (-z)^{a_i}(\overline{Y}_{i,\mathbf{Z}\mathbf{\Psi}^{-1}}(0))^{-1}.$$

Remark 11.12. *(i) The property does not depend on the choice of $\overline{\omega}^{\pm}$ but only on $\mathbf{\Psi}$.*

(ii) The degree of the two polynomials are a_i and $a_i + \alpha_i(\mu)$ respectively.

We will use the currents defined for $i \in I$ by

$$x_i^{+,+}(z) = \sum_{r\geq 0} x_{i,r}^+ z^r \ , \ x_i^{-,+}(z) = -\sum_{r>0} x_{i,r}^- z^r \ , \ C_i(z) = (q_i - q_i^{-1}) x_i^{-,+}(z) A_i^+(z) = \sum_{r>0} C_{i,r} z^r.$$

The following relations were proved in [FT] when $\mu \in -\Lambda^+$. But has the commutators $[x_{i,r}^\pm, \phi_{j,s}^+]$ are the same for a general $\mu \in \Lambda$, it holds in general :

$$(w-z)[A_i(z), C_i(w)]_{q_i} = (q_i - q_i^{-1})(zC_i(w)A_i(z) - wC_i(z)A_i(w)). \tag{11.23}$$

Here we use that standard notation $[a,b]_{q_i^{\pm 1}} = ab - q_i^{\pm 1} ba$.

Proof. Let $i \in I$ and v an highest weight vector of $L(\mathbf{\Psi})$. The polynomiality of $\overline{Y}_{i,\mathbf{Z}\mathbf{\Psi}^{-1}}(z)$ is clear. It is already observed in [FT] that the polynomiality of $C_i(z)$ can be deduced in the following way : the coefficient of w in relation (11.23) gives

$$-z[A_i^+(z), C_{i,1}]_{q_i^{-1}} = -(q_i - q_i^{-1}) C_i(z) A_{i,0}^+.$$

As $A_{i,0}^+$ is invertible, $C_i(z)$ is a polynomial on $L(\mathbf{\Psi})$. Now we have also :

$$x_{i,0}^+ . C_i(z).v = q_i^{-1}[x_{i,0}^+, x_i^{-,+}(z)] A_i^+(z).v = (1 - q_i^{-2}) \overline{Y}_{i,\mathbf{Z}\mathbf{\Psi}^{-1}}(zq_i^{-1})(\sum_{r>0}(\phi_{i,r}^+ - \phi_{i,r}^-)z^r).v$$

$$= (1 - q_i^{-2}) \overline{Y}_{i,\mathbf{Z}\mathbf{\Psi}^{-1}}(zq_i^{-1})(\Psi_i(z) - \Psi_{i,0}^+ - \sum_{0<r\leq \alpha_i(\mu)} \Psi_{i,r}^- z^r).v,$$

As $\overline{Y}_{i,\mathbf{Z}\mathbf{\Psi}^{-1}}(zq_i^{-1})$ and $x_{i,0}^+.C_i(z).v$ are polynomial in z, this implies that $\overline{Y}_{i,\mathbf{Z}\mathbf{\Psi}^{-1}}(zq_i^{-1})\Psi_i(z)$ is a polynomial. For the second point, the eigenvalue of A_{i,a_i}^+ on v is $(-1)^{a_i}(\overline{Y}_{i,\mathbf{Z}\mathbf{\Psi}^{-1}}(0))^{-1}$. □

This condition is not sufficient in general.

Example 11.13. *Let* $\mathfrak{g} = sl_3$, $\lambda = \omega_1^\vee$, $\mu = \lambda - 2\alpha_1^\vee - \alpha_2^\vee = -2\omega_1^\vee$, $Z_1(z) = 1 - z$, $Z_2(z) = 1$,

$$\Psi(z) = \left(\frac{q^{-4}}{(1-zq^{-4})(1-zq^{-2})}, 1\right).$$

We fix $\mathcal{Z}$ *with compatible* z_1', z_2'. *We have*

$$\overline{Y}_{1,\mathbf{Z}\mathbf{\Psi}^{-1}}(zq^{-1}) = \alpha^2(1-zq^{-4})(1-zq^{-2}) \text{ and } \overline{Y}_{2,\mathbf{Z}\mathbf{\Psi}^{-1}}(zq^{-1}) = \alpha(1-zq^{-3})$$

with $\alpha^3 = q^4$. *The condition of Proposition 11.11 are satisfied. However,* $L(\mathbf{\Psi})$ *is not in* $\mathcal{O}_{\mu,\mathcal{Z}}^\lambda$. *Indeed, by Theorem 5.5, this representation is the fusion product of two negative prefundamental representations (with a* 1*-dimensional constant representation). The* q*-character of each of these factor is known as it is the same as for the corresponding negative prefundamental representation of* $\mathcal{U}_q(\hat{\mathfrak{b}})$ *(in the* sl_3*-case these representations are explicitly described in* [HJ]*). The following is an* ℓ*-weight of* $L(\mathbf{\Psi})$ *and not of the correct form :*

$$\left(\frac{q^{-6}}{(1-zq^{-4})(1-zq^{-6})}, \frac{q(1-zq^{-5})}{1-zq^{-3}}\right).$$

However in some cases the conditions are enough to determine the simple representations in $\mathcal{O}_{\mu,\mathcal{Z}}^\lambda$. This is the case for the $\mathfrak{g} = sl_2$ in the next section. We have also the following.

Example 11.14. *Let $\lambda = \omega_i^\vee$ and $\mu = \omega_i^\vee - \alpha_i^\vee$. Set $Z_j(z) = 1 - zaq_i^2\delta_{i,j}$ and associated z_j'. Up to sign twist $\mathcal{O}_\mu^{\mu,\mathcal{Z}}$ contains the unique simple representation $L(\tilde{\mathbf{\Psi}}_{i,a})$ (see Example 5.2). The uniqueness follows from Proposition 11.11. From Example 5.2, the ℓ-weight spaces are of dimension 1 parametrized by $m \geq 0$ with the corresponding eigenvalue of $A_j^{\mathcal{Z},+}(z)$ equal to 1 if $j \neq i$ and if $j = i$ equal to $v_i^{-1}q_i^m - v_i q_i^{2-m} z$ where $v_i^2 = q_i$ fixed.*

We have the following consequence of Proposition 11.11.

Theorem 11.15. *There is a finite number of simple representations $L(\mathbf{\Psi})$ in $\mathcal{O}_{\mu,\mathcal{Z}}^\lambda$.*

Proof. Let $L(\mathbf{\Psi})$ be in $\mathcal{O}_{\mu,\mathcal{Z}}^\lambda$. As $\mathbf{\Psi} \preceq_{\mathcal{Z}} \mathbf{Z}$ by Theorem 11.9, there are $v_{i,a} \geq 0$ so that

$$\mathbf{\Psi}(\mathbf{\Psi}(0))^{-1} = \mathbf{Z} \prod_{i \in I, a \in \mathbb{C}^*} \Lambda_{i,a}^{-v_{i,a}}.$$

Moreover, $\mathbf{\Psi}$ is determined by the $v_{i,a}$ up to sign twist. So it suffices to show that there is a finite number of possibilities for the $v_{i,a}$. Besides, $\sum_{a \in \mathbb{C}^*} v_{i,a} = a_i$. So it suffices to prove that there is a finite number of possible a so that $v_{i,a} \neq 0$.

For $i \in I$, $a \in \mathbb{C}^*$, let $z_{i,a}$ be the multiplicity of a^{-1} as a root of $Z_i(z)$.

For each $i \in I$, let

$$\tilde{Z}_i(z) = \prod_{a \in \mathbb{C}^*} (1 - za)^{v_{i,a}}$$

which is equal to $\overline{Y}_{i, \mathbf{Z}\mathbf{\Psi}^{-1}}(zq_i^{-1})$ up to a constant. Then from Proposition 11.11, $\tilde{Z}_i(zq_i^2)$ divides

$$Z_i(z) \left(\prod_{C_{j,i} = -1} \tilde{Z}_j(zq_i) \right) \left(\prod_{C_{j,i} = -2} \tilde{Z}_j(z)\tilde{Z}_j(zq^2) \right) \left(\prod_{C_{j,i} = -3} \tilde{Z}_j(z)\tilde{Z}_j(zq^2)\tilde{Z}_j(zq^4) \right).$$

Suppose that $v_{i,a} \neq 0$. Then $z_{i,aq_i^2} \neq 0$ or there $j \neq i$ so that $v_{j,b} \neq 0$ with $b = aq^r$ for $r = r_i$ if $C_{j,i} = -1$, $r = 4$ or 2 if $C_{j,i} = -2$, $r = 6$ or 4 or 2 if $C_{j,i} = -3$.

Consequently : $v_{i,a} \neq 0$ implies that there is $j \in I$, $0 < r$ so that $z_{j,aq^r} \neq 0$, and also that there exists a finite sequence

$$(i_0, a_0) = (i, a), (i_1, a_1) = (i_1, aq^{r_1}), \cdots, (i_R, a_R) = (j, aq^r)$$

so that for any k, $r_k < r_{k+1} \leq r_k + 2r_{i_k}$ and $v_{i_k, a_k} > 0$.

This implies that $r \leq 6 \sum_{l \in I} a_l$. So there is a finite number of possible a and the result follows. $\square$

Remark 11.16. *(i) It follows from the proof that $\mathbf{\Psi}$ is the product of $\mathbf{Z}$ by various $\Lambda_{i,a}^{-1}$ so that there are $j \in I$, $r > 0$ with $z_{j,aq^r} \neq 0$.*

(ii) The same proof implies that the category

$$\mathcal{O}_{\mathcal{Z}}^\lambda = \bigoplus_{\mu \in \Lambda} \mathcal{O}_{\mu,\mathcal{Z}}^\lambda$$

has a finite number of simple objects.

11.4. **Descent for $\mathfrak{g} = sl_2$.** We suppose in this section $\mathfrak{g} = sl_2$.

The condition of Proposition 11.11 is

$$(\mathbf{Z}(zq^{-2}))^{-1} \preceq_{\mathcal{Z}} \mathbf{\Psi}(z) \preceq_{\mathcal{Z}} \mathbf{Z}(z). \tag{11.24}$$

Indeed $\overline{Y}_{\mathbf{Z}\mathbf{\Psi}^{-1}}(zq^{-1})$ polynomial means $\mathbf{\Psi}(z) \preceq_{\mathcal{Z}} \mathbf{Z}(z)$. Let $\tilde{Z}(z)$ be this polynomial. Then $\Psi(z)\tilde{Z}(z) = Z(z)/\tilde{Z}(zq^2) = Q(z)$ polynomial means $\tilde{Z}(zq^2)$ divides $Z(z)$. But

$$\Psi(z)Z(zq^{-2}) = \frac{Z(z)Z(zq^{-2})}{\tilde{Z}(zq^2)\tilde{Z}(z)} = Q(z)Q(zq^{-2})$$

and so $\mathbf{\Psi}(z)\mathbf{Z}(zq^{-2}) \succeq_{\mathcal{Z}} 1$.

We prove the condition (11.24) is sufficient. Let us denote

$$\Phi(z) = \sum_{r\geq 0} \Phi_r z^r = \sum_{r\geq 0}(\phi_r^+ - \phi_r^-)z^r = \phi^+(z) - \sum_{0\leq r\leq \alpha_1(\mu)} \phi_r^- z^r.$$

Note that if $\mu \in -\Lambda^+ \setminus \{0\}$ is strictly antidominant, $\Phi(z) = \phi^+(z)$. For $m, m' \geq 0$, we have $(q - q^{-1})[x_m^+, x_{m'}^-] = \Phi_{m+m'}$, and so

$$(w - z)[x^{+,+}(z), x^{-,+}(w)] = w\frac{\Phi(w) - \Phi(z)}{q - q^{-1}}. \tag{11.25}$$

This generalizes formulas established in [FT] for $\mu \in -\Lambda^+$. The following relations is established in [FT] (for $\mu \in -\Lambda^+$ but the same proof gives the result in general) :

$$[C(z), C(w)] = 0. \tag{11.26}$$

We prove the converse of the statement in Proposition 11.11 is true in the sl_2-case.

Theorem 11.17. *$L(\mathbf{\Psi})$ in $\mathcal{O}_\mu$ descends to the truncation $\mathcal{U}_{q,\lambda}^{\mu,\mathcal{Z}}(\hat{sl}_2)$ if and only if $\overline{Y}_{\mathbf{Z}\mathbf{\Psi}^{-1}}(zq^{-1})$ and $\overline{Y}_{\mathbf{Z}\mathbf{\Psi}^{-1}}(zq^{-1})\Psi_i(z)$ are polynomials with $\overline{Y}_{\mathbf{Z}\mathbf{\Psi}^{-1}}(zq^{-1}) \sim_\infty (-z)^a(\overline{Y}_{\mathbf{Z}\mathbf{\Psi}^{-1}}(0))^{-1}$.*

Proof. One implication is proved in Proposition 11.11. Let us suppose the condition are satisfied. We prove that the action of $A^+(z)$, but also of $C(z)$, are polynomial on $V = L(\mathbf{\Psi})$. Let ω_0 be the highest weight of V. For $N \geq 0$, let V_N be the sum of weight spaces V_ω with $\omega - \omega_0$ of height N.

First we prove that $C(z)$ is polynomial on V_ω. Let $v \in V_0$. We prove that for $m \geq a + |\alpha_1(\mu)|$, the coefficient of w^{m+1} in $C(w).v$ is zero. As V is simple, it suffices to prove this for $x^+(z)C(w).v$. From Proposition 4.9, it suffices to prove this is true for

$$x^{+,+}(z)C(w).v = (q - q^{-1})[x^{+,+}(z), x^{-,+}(w)]A^+(w).v.$$

Relation (11.25) gives

$$(w - z)x^{+,+}(z)C(w).v = w(\Phi(w) - \Phi(z))A^+(w).v.$$

For $l \geq 0$, considering the coefficients of w^{l+1}, we get

$$x^{+,+}(z)C_l.v - zx^{+,+}(z)C_{l+1}.v = (\Phi A)_l.v - \Phi(z)A_l.v,$$

where $(\Phi A)_l$ is the coefficient of z^l in $\Phi(z)A^+(z)$. We multiply this relation by z^l and we take the sum of the relations for $l=0$ to m. As $C_0=0$, $A^+(z).v$ is a polynomial of degree $a\leq m$ and $\Phi(z)A^+(z).v$ a polynomial of degree lower than m :

$$-x^{+,+}(z)z^{m+1}C_{m+1}.v=(\sum_{0\leq l\leq m}(\Phi A)_l z^l-\Phi(z)A_l z^l).v=(\Phi(z)A^+(z)-\Phi(z)A^+(z)).v=0.$$

This proves the claim, that is $C_{m+1}.v=0$.

Now by the relation (11.26), we get by induction on N that $C(z)$ is a polynomial on V_N of degree lower than $a+|\alpha_1(\mu)|$. Then, using relation (11.23), we obtain that the action of $A^+(z)$ on V is polynomial. We conclude by Remark 11.5. $\square$

Remark 11.18. *As a by product one gets that* $B(z)=(q-q^{-1})A^+(z)x_i^{+,+}(z)$ *and* $D(z)=A^+(z)\phi^+(z)+(q-q^{-1})^2x^{-,+}(z)A(z)x^{+,+}(z)$ *are polynomial on this representation.*

Example 11.19. *Let* $\lambda=\omega_1^\vee$, $\mu=-\omega_1^\vee=\lambda-\alpha_1^\vee$. *We fix* $z_1,z_1'\in\mathbb{C}^*$. *From Example 11.14, up to sign twist there is a unique simple representation* $L(\mathbf{\Psi})$ *in* $\mathcal{O}_{-\omega_1^\vee,\mathcal{Z}}^{\omega_1^\vee}$ *with*

$$\Psi(z)=\frac{q^{-1}z_1}{1-zq^{-1}z_1}.$$

This is Example 9.3 with $a=b=z_1q^{-1}$. *Indeed for* $L(b(1-az)^{-1})$ *we have (here* $\alpha^2=a$*) :*

$$A^+(z)\cdot v_j=(\alpha^{-1}q^j-\alpha q^{-j}z)v_j\ ,\ \phi_+(z).v_j=\frac{a(1-q^2az)q^{-2j}}{(1-q^{2-2j}az)(1-q^{-2j}az)}v_j,$$

$$B(z).v_j=\alpha^{-1}(q-q^{-1})q^{j-1}v_{j-1}\ ,\ C(z).v_j=[j+1]_q q^{-2j}\alpha^3zv_{j+1}\ ,\ D(z).v_j=\alpha q^{-j}v_j.$$

The representation $L(b(1-az)^{-1})$ *descends to a truncation if and only if* $a=b$.

Example 11.20. *Let* $\lambda=2\omega_1^\vee$, $\mu=-2\omega_1^\vee=\lambda-2\alpha_1^\vee$. *We fix* $z_1,z_2\in\mathbb{C}^*$. *Let* $\mathbf{\Psi}$ *of degree* $\alpha(\mu)=-2$ *and* $\Psi(0)(\Psi(z)z^2)(\infty)=\phi_{\mathbf{\Psi}}=q^{-2}z_1z_2$. *We have* $\overline{Y}_{\mathbf{Z\Psi}^{-1}}(z)$ *of degree 2 and*

$$\Psi(z)=\frac{(1-qzz_1)(1-qzz_2)}{\overline{Y}_{\mathbf{Z\Psi}^{-1}}(zq^{-1})\overline{Y}_{\mathbf{Z\Psi}^{-1}}(zq)}. \tag{11.27}$$

The conditions of Proposition 11.11 give that the roots of $\overline{Y}_{\mathbf{Z\Psi}^{-1}}(z)$ *are* z_1^{-1}, z_2^{-1} *and*

$$\Psi(z)=\frac{\Psi(0)}{(1-zq^{-1}z_1)(1-zq^{-1}z_2)}=\frac{q^{-2}z_1z_2}{(1-zq^{-1}z_1)(1-zq^{-1}z_2)}.$$

There is one simple representation in $\mathcal{O}_{-2\omega_1^\vee,\mathcal{Z}}^{2\omega_1^\vee}$. *It is a fusion product of two negative prefundamental representations.*

Example 11.21. *Let* $\lambda=2\omega_1^\vee$, $\mu=0=\lambda-\alpha_1^\vee$. *We fix* $z_1,z_2\in\mathbb{C}^*$. *Let* $\mathbf{\Psi}$ *be an* ℓ*-weight of degree* $\alpha(\mu)=0$ *and* $\Psi(0)(\Psi(z)z^2)(\infty)=z_1z_2$. *We have (11.27) but with* $\overline{Y}_{\mathbf{Z\Psi}^{-1}}(z)$ *of degree 1. The conditions of the Proposition 11.11 give that* $\overline{Y}_{\mathbf{Z\Psi}^{-1}}(zq^{-1})=\overline{Y}_{\mathbf{Z\Psi}^{-1}}(0)-(\overline{Y}_{\mathbf{Z\Psi}^{-1}}(0))^{-1}z$. *Hence the root of* $\overline{Y}_{\mathbf{Z\Psi}^{-1}}(z)$ *is* z_1^{-1} *or* z_2^{-1}, *say it is* z_i^{-1}, *and let* z_j^{-1} *be the other root. Then*

$$\Psi(z)=\frac{\Psi(0)(1-qz_jz)}{(1-q^{-1}z_iz)}=\frac{q^{-1}z_i(1-qz_jz)}{(1-q^{-1}z_iz)}.$$

Hence if $z_1 \neq z_2$, there are two simple representations in $\mathcal{O}^{2\omega_1^\vee}_{0,\mathcal{Z}}$. If $z_1 = z_2$, there is one simple representation in $\mathcal{O}^{2\omega_1^\vee}_{0,\mathcal{Z}}$, its ℓ-weight is $q^{-1}z_1$ and it is of dimension 1.

Corollary 11.22. *If $\mathcal{O}^\lambda_{\mu,\mathcal{Z}}$ is non empty then $-\lambda \preceq \mu \preceq \lambda$. Its simple representations are in bijection with the divisor $\tilde{Z}(z)$ of $Z(zq^2)$ of degree $\omega_1(\lambda - \mu)$ satisfying $\tilde{Z}(0) = 1$. The corresponding ℓ-weight satisfies*

$$\Psi(z) = \Psi(0)\frac{Z(z)}{\tilde{Z}(zq^{-2})\tilde{Z}(z)}.$$

Proof. The conditions of the Theorem 11.17 imply that $\overline{Y}_{\mathbf{Z}\mathbf{\Psi}^{-1}}(zq)$ divides $Z(z)$. This determines $\Psi(z)$ up to the constant $\Psi(0)$. Conversely, for each divisor $\tilde{Z}(z)$ of $Z(zq^2)$ normalized with $\tilde{Z}(0) = 1$, we can fix $\overline{Y}_{\mathbf{Z}\mathbf{\Psi}^{-1}}(zq^{-1}) = \alpha\tilde{Z}(z)$ for a certain $\alpha \in \mathbb{C}^*$ determined up to a sign by the limit condition. □

11.5. **An example for $\mathfrak{g} = sl_3$.** Let $\mathfrak{g} = sl_3$, $\lambda = \omega_1^\vee$ with $Z_1(z) = 1 - zq^3$ and $Z_2(z) = 1$.

$\mu = \lambda = \omega_1^\vee$. By Example 11.3, we get $L(1 - zq^3, 1)$ unique simple in $\mathcal{O}^\lambda_{\mu,\mathcal{Z}}$.

$\mu = \lambda - \alpha_1^\vee = \omega_2^\vee - \omega_1^\vee$. By Example 11.14, we get $L(\frac{q}{1-zq}, v^{-1}(1 - zq^2))$ unique simple in $\mathcal{O}^\lambda_{\mu,\mathbf{\Psi}}$ up to sign twist (here v is a square root of q).

$\mu = -\omega_2^\vee = \lambda - \alpha_1^\vee - \alpha_2^\vee$. By Proposition 11.11, there is at most one simple module $L(q, \frac{v^{-1}}{1-z})$ in $\mathcal{O}^\lambda_{\mu,\mathcal{Z}}$ up to sign twist. The action of $\mathcal{U}_q(\hat{\mathfrak{b}})$ is described in [HJ, Section 4.1]. The ℓ-weight spaces are of dimension 1 with ℓ-weights parametrized by $0 \leq n' \leq n$:

$$\left(q^{1+n-2n'}\frac{(1-q^3z)(1-q^{1-2n}z)}{(1-q^{1-2n'}z)(1-q^{3-2n'}z)}, v^{-1}q^{n'-2n}\frac{1-q^{2-2n'}z}{(1-q^{-2n}z)(1-q^{-2n+2}z)}\right),$$

with $v^{-1}q^{n'} - vq^{-n'}z$, $q^n - q^{-n}z$ respective eigenvalues of $A_1^+(z), A_2^+(z)$.

12. A conjecture : truncation and Langlands dual standard modules

We state a conjecture (Conjecture 12.3) on the parametrization of simple modules of non-simply laced truncated shifted quantum affine algebras. The statement of the conjecture involves the Langlands dual Lie algebra $\mathfrak{g}^L$, more precisely it is given in terms of the structure of a standard module of the twisted quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}}^L)$. We introduce a corresponding Langlands dual q-character to state Conjecture 12.3.

For simply-laced types, simple representations of truncated shifted Yangians have been parametrized in terms of Nakajima monomial crystals [KTWWY2]. Combining with [N5], this implies an analog statement for simply-laced shifted quantum affine algebras. This is a fundamental motivation for our conjecture in non simply-laced types. See the introduction and Remark 12.4 for a discussion on earlier results.

We have several strong evidences for our conjecture. We establish in type B_2 that our parametrization gives representations of the truncated shifted quantum affine algebra (Proposition 12.8). In general, we establish that a simple finite-dimensional representation of a shifted quantum affine algebra descends to a truncation as in Conjecture 12.3 (Theorem 12.9). The proof of this last result is based on Baxter polynomiality of quantum integrable models. As in the previous section, we have fixed $\lambda \in \Lambda^+$ and corresponding $\mathbf{Z}$, $\mathcal{Z}$.

12.1. **Reminder - interpolating (q,t)-characters.** Interpolating (q,t)-characters where introduced in [FH1] as an incarnation of Frenkel-Reshetikhin deformed W-algebras [FR1] to interpolate between q-characters of a non simply-laced quantum affine algebra and its Langlands dual. These interpolating (q,t)-character are tools in [FH1] to study Langlands duality between finite-dimensional representations of quantum affine algebras.

Let $r = \mathrm{Max}_{i\in I}(r_i)$ be the lacing number of $\mathfrak{g}$. We set $\epsilon = e^{2i\pi/r}$.

For $i \in I$, $a \in \mathbb{C}^*$, we set

$$Z_{i,a} = \begin{cases} Y_{i,a} & \text{if } r_i = r, \\ Y_{i,aq^{-1}}Y_{i,aq} & \text{if } r_i = r-1, \\ Y_{i,aq^{-2}}Y_{i,a}Y_{i,aq^2} & \text{if } r_i = r-2. \end{cases} \tag{12.28}$$

Then $W = L(Z_{i,a})$ is a Kirillov-Reshetikhin module of the untwisted quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ (it is a fundamental representation when $r_i = r$).

Let us recall that the interpolating (q,t)-character $\chi_{q,t}(W)$ of W is an element of a quotient of the ring

$$\chi_{q,t}(W) \in \mathbb{Z}[Y_{j,b}^{\pm 1}, \alpha]_{j\in I, b\in aq^{\mathbb{Z}}t^{\mathbb{Z}}\epsilon^{\mathbb{Z}}}$$

where t is an additional formal variable and α is a rational fraction in q, t which can considered as an indeterminate satisfying $\alpha^r = \alpha$ for our purposes (in type G_2, this indeterminate is denoted by β in [FH1]).

The quotient is defined in the following way. It is obtained from the ideal generated for $j, j' \in I$, $b, b' \in az^{\mathbb{Z}}t^{\mathbb{Z}}$ by the :

$$\alpha(\alpha - 1)\ ,\ \alpha(Y_{j,b} - Y_{j,bt})\ ,\ (\alpha - 1)(Z_{j,bq} - Z_{j,b\epsilon})\ ,\ (Z_{j,b} - Z_{j,bt})(Z_{j',b'q} - Z_{j',b'\epsilon}).$$

The interpolating (q,t)-characters have interesting limits :

When $t \to 1$, α specializes to 1 and $\chi_{q,t}(L(Z_{i,a}))$ specializes to the q-character of the $\mathcal{U}_q(\hat{\mathfrak{g}})$-module W.

When $q \to \epsilon$, α specializes to 0 and $\chi_{q,t}(L(Z_{i,a}))$ specializes to the t-character of the fundamental representation $V_i^L(a)$ of highest monomial $Z_{i,a}$ of the Langlands dual twisted quantum affine algebra $\mathcal{U}_t(\hat{\mathfrak{g}}^L)$ (in the sense of [He4]).

12.2. **Langlands dual q-characters.** For $i \in I$, $a \in \mathbb{C}^*$, consider the interpolating q,t-character $\chi_{q,t}(L(Z_{i,a}))$.

To state our conjecture, we are interested in another specialization : we set $t = 1$ but with discard the monomials with coefficient α, that is we set $\alpha = 0$. By [FH2], only variables $Z_{j,aq^m}^{\pm 1}$ with $j \in I$, $m \in \mathbb{Z}$, occur and we get a well-defined element

$$\chi_q^L(V_i^L(a)) \in \mathbb{Z}[Z_{j,aq^m}^{\pm 1}]_{j\in I, m\in\mathbb{Z}},$$

that we call the Langlands dual q-character of $V_i^L(a)$. The number of terms of $\chi_q^L(V_i^L(a))$ is the dimension of the $\mathcal{U}_t({}^L\hat{\mathfrak{g}})$-module $V_i^L(a)$, but the monomials occurring in $\chi_q^L(V_i^L(a))$ are some of the monomials occurring in the q-character of the $\mathcal{U}_q(\hat{\mathfrak{g}})$-module $L(Z_{i,a})$ (which is not a fundamental representation in general, but a Kirillov-Reshetikhin module).

Remark 12.1. *Other interesting different limits are also studied in* [FHR]*, in particular when $q \to 1$, in the context of quantum integrable models.*

Now let V be the standard module of $\mathcal{U}_t(\hat{\mathfrak{g}}^L)$ of highest monomial

$$M_0 = \prod_{i\in I} Z_{i,q_i^{-1}z_{i,1}^{-1}}\cdots Z_{i,q_i^{-1}z_{i,N_i}^{-1}}.$$

It is defined as a tensor product of the fundamental representations $V_i^L(q_i^{-1}z_{i,j})$ (as for standard modules considered in simply-laced cases in [N1, VV]). Its isomorphism class depends on the ordering of the tensor product, but not its class in the Grothendieck ring.

We introduce its Langlands dual q-character

$$\chi_q^L(V) = \prod_{i\in I, 1\leq s\leq N_i} \chi_q^L(V_i^L(q_i^{-1}z_{i,s}^{-1})) \in \mathbb{Z}[Z_{j,b}^{\pm 1}]_{j\in I, b\in\mathbb{C}^*}.$$

It should not be confused with its q-character $\chi_t(V)$ as a representation of $\mathcal{U}_t({}^L\hat{\mathfrak{g}})$, although there is a bijection between the monomials in $\chi_q^L(V)$ and the monomial of $\chi_t(V)$.

12.3. **Statement.** For a monomial $M = \prod_{i\in I, a\in\mathbb{C}^*} Z_{i,a}^{u_{i,a}}$, we define the corresponding ℓ-weight $\mathbf{\Psi}_M = (\Psi_i(z))_{i\in I}$ by

$$\Psi_i(z) = \Psi_i(0)\prod_{a\in\mathbb{C}^*}(1 - za^{-1})^{u_{i,a}},$$

with

$$(\Psi_i(0))^2 = (\prod_{a\in\mathbb{C}^*}(-a)^{u_{i,a}})\phi_{i,\mathcal{Z}}$$

(this ℓ-weight is defined up to sign-twist). The corresponding weight is $\mu_M = \sum_{i\in I, a\in\mathbb{C}^*} u_{i,a}\omega_i^\vee \in \Lambda$. As an example, we have $\mathbf{\Psi}_{M_0} = \mathbf{Z}$.

Recall the partial ordering $\preceq_{\mathcal{Z}}$ in Section 11.2.

Proposition 12.2. *For a monomial M occurring in the Langlands dual q-character of V, we have*

$$\mathbf{\Psi}_M \preceq_{\mathcal{Z}} \mathbf{Z}.$$

Proof. By definition, M is the limit at $t=1$ of a monomial $\tilde{M}$ in the interpolating (q,t)-character of V, without a factor α. By construction, M_0 is the product of $\tilde{M}$ by a product of various

$\tilde{A}_{j,bt^s}$ with $j\in I$ so that $r_j = r$,

$\tilde{A}_{j,bq^{-1}t^s}\tilde{A}_{j,bqt^s}$ with $j\in I$ so that $r_j = 1$ and $r=2$,

$\tilde{A}_{j,bq^{-2}t^s}\tilde{A}_{j,bt^s}\tilde{A}_{j,bq^2t^s}$ with $j\in I$ so that $r_j=1$ and $r=3$,

for certain $b\in q^{\mathbb{Z}}\{q_i^{-1}z_{i,p}^{-1}\}_{i\in I, 1\leq p\leq N_i}$, $s\in\mathbb{Z}$. The limits at $t=1$ are respectively

$A_{j,b} = Z_{j,bq_j^{-1}}Z_{j,bq_j}\prod_{k, C_{j,k}=-1} Z_{k,b}^{-1}$,

$A_{j,bq^{-1}}A_{j,bq} = Z_{j,bq^{-1}}Z_{j,bq}\prod_{k,C_{j,k}=-1} Z_{k,b}^{-1}\prod_{k,C_{j,k}=-2} Z_{j,bq^{-1}}^{-1}Z_{j,bq}^{-1}$,

$A_{j,bq^{-2}}A_{j,b}A_{j,bq^2} = Z_{j,bq^{-1}}Z_{j,bq}\prod_{k,C_{j,k}=-1} Z_{k,b}^{-1}\prod_{k,C_{j,k}=-3} Z_{j,bq^{-2}}^{-1}Z_{j,b}Z_{j,bq^2}^{-1}$.

The corresponding ℓ-weight is $\Lambda_{j,b^{-1}}$, hence the result. □

Recall that by Theorem 11.9, $L(\mathbf{\Psi})$ in $\mathcal{O}_{\mu,\mathcal{Z}}^\lambda$ must satisfy $\mathbf{\Psi}\preceq_{\mathcal{Z}}\mathbf{Z}$, the same condition as in Proposition 12.2.

We state our main Conjecture for $\mathfrak{g}$ of non simply-laced type.

Conjecture 12.3. *(A) For a monomial M occurring in the Langlands dual q-character of V, the simple module $L(\mathbf{\Psi}_M)$ is in the category $\mathcal{O}^\lambda_{\mu,\mathcal{Z}}$.*

(B) This assignment defines a bijection between the classes of simple representations in $\mathcal{O}^\lambda_{\mu,\mathcal{Z}}$ up to sign-twist and the monomials of weight μ occurring in the Langlands dual q-character of V.

Remark 12.4. *(i) As explained above, results already obtained for simply-laced types are fundamental motivations for this conjecture. Simple representations of simply-laced truncated shifted Yangians have been parametrized in terms of Nakajima monomial crystals* [KTWWY2]. *Combining with* [N5], *this implies an analog statement for simply-laced shifted quantum affine algebras. In simply-laced type, V is a standard module of $\mathcal{U}_q(\hat{\mathfrak{g}}^L) = \mathcal{U}_q(\hat{\mathfrak{g}})$. Note that the set of monomials occurring in $\chi_q(V)$ is the product of the set of monomials occurring in the monomial crystals $\mathcal{M}(Y_{q^{-1}z_{i,s}^{-1}})$, see* [KTWWY1, Section 7].

(ii) Conjecture 12.3 does not involve the monomial crystal for non simply-laced types in [Kas2] *(see also* [HN]*) or the q-character of the standard module of $\mathcal{U}_q(\hat{\mathfrak{g}})$ (see also Remark 12.5).*

(iii) According to our conjecture, the ℓ-weights of simple representations can be read from the monomials in the Langlands dual q-character. By interpolation, these monomials correspond to monomials in the q-character of the standard module of the twisted quantum affine algebra, but the ℓ-weights can not be read directly from the latter in general. Also, by folding in [He4], *these monomials correspond to monomials in the q-character of a standard module of a corresponding untwisted simply-laced quantum affine algebra. Hence we get a relation between non simply-laced shifted quantum affine algebras and simply-laced quantum affine algebras (see also (iii) in Remark 12.5).*

(iv) As discussed in the introduction, Nakajima-Weekes [NW], *combining with* [KTWWY2], *gave an explicit parametrization of simple representations in category $\mathcal{O}$ of truncated non simply-laced shifted Yangians and quantum affine algebras. Using the previous point (iii), one can compare and consider a possible relation between the two parametrizations. In small examples this different method seems to give the same parametrization as our result.*

(v) The point (A) in conjecture 12.3 implies that an arbitrary simple representation in $\mathcal{O}_\mu$ is in one of the categories $\mathcal{O}^\lambda_{\mu,\mathcal{Z}}$. Indeed, for any $i \in I$, $a \in \mathbb{C}^*$, the monomial $Z_{i,a}^{-1}$ occurs as the lowest weight monomial of the Langlands dual q-character of the representation $V_{\bar{i}}^L(aq^{-rh^\vee})$ (this follows from the result for $\mathcal{U}_q(\hat{\mathfrak{g}})$ in [FM]*). Here $h^\vee$ is the dual Coxeter of $\mathfrak{g}$ and $\bar{i} \in I$ is set so that $w_0(\alpha_i) = -\alpha_{\bar{i}}$ for w_0 the longest element of the Weyl group of $\mathfrak{g}$. Hence, an arbitrary Laurent monomial in the $Z_{i,a}^{\pm 1}$ occurs in the Langlands dual q-character of a standard module of $\mathcal{U}_t(\hat{\mathfrak{g}}^L)$.*

(vi) Recall the category $\mathcal{O}^\lambda_{\mathcal{Z}}$ has a finite number of simple objects by Remark 11.16. We may expect it is a categorification of a subspace of the Langlands dual standard module V, associated to the monomials of $\chi_q^L(V)$, in the spirit of [KTWWY2].

(vii) Should we extend the construction of this paper to twisted shifted quantum affine algebras, we expect the parametrization of simple representations would involve interpolating (q,t)-characters of finite-dimensional representations of twisted quantum affine algebras as in [FH1, Section 6], *as well as Langlands dual q-character of standard modules of untwisted non-simply laced quantum affine algebras.*

12.4. **Examples in simply-laces types.** See the Introduction and Remark 12.4 for general earlier results in simply-laced types (see also Corollary 11.22 for $\mathfrak{g} = sl_2$).

For $\mathfrak{g} = sl_3$, in the examples in section 11.5, the 3 simple representations in $\mathcal{O}_{\mu,\mathcal{Z}}^{\lambda}$ correspond to the 3 monomials occurring in

$$\chi_q(L(Y_{1,q^{-3}})) = Y_{1,q^{-3}} + Y_{2,q^{-2}}Y_{1,q^{-1}}^{-1} + Y_{2,1}^{-1}.$$

12.5. **Examples in type B_2.** We work in type B_2 with $r_1 = 2$ and $r_2 = 1$.

First let us set $\lambda = \omega_2^\vee$, $Z_1(z) = 1$, $Z_2(z) = 1-z$. The interpolating (q,t)-character of the $\mathcal{U}_q(B_2^{(1)})$-representation $L(Z_{2,1})$ has 11 terms and was computed explicitly in [FH1, Section 3.5] :

$$Y_{2,q^{-1}}Y_{2,q} + \alpha Y_{2,q^{-1}}Y_{2,q^3t^2}^{-1}Y_{1,q^2t} + Y_{2,qt^2}^{-1}Y_{2,q^3t^2}^{-1}Y_{1,t}Y_{1,q^2t} + \alpha Y_{2,q^{-1}}Y_{2,q^5t^2}Y_{1,q^6t^3}^{-1} + Y_{1,q^2t}Y_{1,q^4t^3}^{-1}$$
$$+Y_{2,qt^2}^{-1}Y_{2,q^5t^2}Y_{1,q^6t^3}^{-1}Y_{1,t} + \alpha Y_{2,q^{-1}}Y_{2,q^7t^4}^{-1} + Y_{1,q^4t^3}^{-1}Y_{1,q^6t^3}^{-1}Y_{2,q^3t^2}Y_{2,q^5t^2} + \alpha Y_{2,qt^2}^{-1}Y_{2,q^7t^4}^{-1}Y_{1,t}$$
$$+\alpha Y_{1,q^4t^3}^{-1}Y_{2,q^3t^2}Y_{2,q^7t^4}^{-1} + Y_{2,q^5t^4}^{-1}Y_{2,q^7t^4}^{-1}.$$

We put $\alpha = 0$, $t = 1$, $Z_{2,q^r} = Y_{2,q^{-r-1}}Y_{2,q^{r+1}}$, $Z_{1,q^r} = Y_{1,q^r}$ and we get

$$\chi_q^L(V_2^L(1)) = Z_{2,1} + Z_{2,q^2}^{-1}Z_{1,1}Z_{1,q^2} + Z_{1,1}Z_{1,q^6}^{-1}Z_{2,q^2}^{-1}Z_{2,q^4} + Z_{1,q^2}Z_{1,q^4}^{-1} + Z_{1,q^6}^{-1}Z_{1,q^4}^{-1}Z_{2,q^4} + Z_{2,q^6}^{-1}.$$

We have

$$\phi_1^+(z) = \frac{A_2^+(z)A_2^+(zq^2)}{A_1^+(z)A_1^+(zq^4)} \text{ and } \phi_2^+(z) = \frac{(1-z)A_1^+(zq^2)}{A_2^+(z)A_2^+(zq^2)}.$$

For the following values of μ, one gets a unique simple object in $\mathcal{O}_\lambda^\mu$ up to sign-twist :

$\mu = \omega_2^\vee$: $\mathbf{\Psi} = (1, 1-z)$.

$\mu = \lambda - \alpha_2^\vee = 2\omega_1^\vee - \omega_2^\vee$: $\mathbf{\Psi} = \left(q^2(1-zq^{-2})(1-z), \frac{q^{-2}}{1-zq^{-2}}\right)$.

$\mu = \lambda - \alpha_2^\vee - 2\alpha_1^\vee = \omega_2^\vee - 2\omega_1^\vee$: $\mathbf{\Psi} = \left(\frac{q^{-8}}{(1-zq^{-6})(1-zq^{-4})}, q^3(1-zq^{-4})\right)$.

$\mu = \lambda - 2\alpha_2^\vee - 2\alpha_1^\vee = -\omega_2^\vee$: $\mathbf{\Psi} = \left(q^{-2}, \frac{q^{-3}}{1-zq^{-6}}\right)$.

The first two cases follow respectively from respective Example 11.3 and Example 11.14. The representation associated to $\mu = \omega_2^\vee - 2\omega_1^\vee$ is a subquotient of

$$L\left(\frac{q^{-8}}{1-zq^{-4}}, q^3(1-zq^{-4})\right) \otimes L_{1,q^{-6}}^- \text{ and } L\left(\frac{q^{-8}}{1-zq^{-6}}, q^3(1-zq^{-4})\right) \otimes L_{1,q^{-4}}^-.$$

This implies $[\mathbf{\Psi}^{-1}]\chi_q(L(\mathbf{\Psi}))$ equals

$$\sum_{0\leq \alpha,\beta, 0\leq\gamma\leq \mathrm{Min}(2\alpha, 1+2\beta)} (A_{1,q^{-4}}^{-1}A_{1,q^{-8}}^{-1}\cdots A_{1,q^{-4\alpha}}^{-1})(A_{1,q^{-6}}^{-1}A_{1,q^{-10}}^{-1}\cdots A_{1,q^{-2-4\beta}}^{-1})(A_{2,q^{-2}}^{-1}\cdots A_{2,q^{-2\gamma}}^{-1}).$$

It is a thin representation, that is its ℓ-weight spaces have dimension 1. So to check this representation descends to truncation reduces to a direct combinatorial check on the explicit q-character formula, which can be done directly.

By [He3], Kirillov-Reshetikhin modules are thin in type B and we have an explicit formula for its q-character. So this is also true for negative prefundamental representations which are limits of Kirillov-Reshetikhin modules by [HJ]. Hence we can conclude as for the previous representation.

Let us focus for the details on the most subtle weight $\mu = \lambda - \alpha_2^\vee - \alpha_1^\vee = 0$.

Let $L(\mathbf{\Psi})$ be a simple representation in $\mathcal{O}^\lambda_{\mu,\mathcal{Z}}$. We have

$$\overline{Y}_{1,\mathbf{Z}\mathbf{\Psi}^{-1}}(zq^{-2}) = (\delta^{-1} - z\delta) \ , \ \overline{Y}_{2,\mathbf{Z}\mathbf{\Psi}^{-1}}(zq^{-1}) = (\gamma^{-1} - z\gamma)$$

for some $\beta, \gamma \in \mathbb{C}^*$. As $\Psi_2(z)\overline{Y}_{2,\mathbf{Z}\mathbf{\Psi}^{-1}}(zq^{-1})$ and $\Psi_1(z)\overline{Y}_{1,\mathbf{Z}\mathbf{\Psi}^{-1}}(zq^{-2})$ are polynomials, we have $\overline{Y}_{2,\mathbf{Z}\mathbf{\Psi}^{-1}}(zq^{-1}) = \pm(q - zq^{-1})$ and $\overline{Y}_{1,\mathbf{Z}\mathbf{\Psi}^{-1}}(zq^{-2}) = \pm(q^3 - zq^{-3})$ or $\pm(q^2 - zq^{-2})$. Hence, up to sign-twist, we get two possibilities for $\mathbf{\Psi}(z)$:

$$\mathbf{\Psi}_1 = \left(q^{-2}\frac{1-z}{1-zq^{-6}}, q\frac{1-zq^{-4}}{1-zq^{-2}}\right) \text{ or } \mathbf{\Psi}_2 = \left(q^{-2}\frac{1-zq^{-2}}{1-zq^{-4}}, 1\right).$$

We can check directly these two simple representations descends indeed to the truncation. First

$$\chi_q(L(\mathbf{\Psi}_1)) = \mathbf{\Psi}_1 \sum_{\alpha\beta\geq 0, \beta\leq 2\alpha+1} (A_{1,q^{-6}}A_{1,q^{-10}}\cdots A_{1,q^{-2-4\alpha}})^{-1}(A_{2,q^{-2}}A_{2,q^{-4}}\cdots A_{2,q^{-2\beta}})^{-1},$$

with the action on the 1-dimensional ℓ-weight space corresponding to each term given by

$$A_1^+(z) = q^{3+2\alpha} - zq^{-3-2\alpha} \ , \ A_2^+(z) = q^{\beta+1} - zq^{-\beta-1}.$$

The formula is compatible with Remark 11.10 as $\mathbf{Z}\mathbf{\Psi}_1^{-1} = \Lambda_{1,q^{-4}}\Lambda_{2,q^{-1}}$. Then

$$\chi_q(L(\mathbf{\Psi}_2)) = \mathbf{\Psi}_2 \sum_{\alpha\beta\geq 0, \beta\leq 2\alpha} (A_{1,q^{-4}}A_{1,q^{-8}}\cdots A_{1,q^{-4\alpha}})^{-1}(A_{2,q^{-2}}A_{2,q^{-4}}\cdots A_{2,q^{-2\beta}})^{-1}$$

with the action on the 1-dimensional ℓ-weight space corresponding to each term given by

$$A_1^+(z) = q^{2+2\alpha} - zq^{-2-2\alpha} \ , \ A_2^+(z) = q^{\beta+1} - zq^{-\beta-1}.$$

We get two non twist-equivalent simple representations in the category $\mathcal{O}^\lambda_{\mu,\mathcal{Z}}$ as predicted by Conjecture 12.3. This proves Conjecture 12.3, (A) and (B), for these weights in type B_2.

Remark 12.5. *(i) There are 6 terms in $\chi_q^L(V_2^L(1))$ which is the dimension of the fundamental representation $V_2^L(1)$ of $\mathcal{U}_q({}^L(B_2^{(1)})) = \mathcal{U}_q(A_3^{(2)})$, but the number of vertices in the corresponding crystal of finite type B_2 in only 5. This example also explains the importance of involving the Langlands dual Lie algebra $\mathfrak{g}^L$ in Conjecture 12.3. Indeed, the study of these simple representations in $\mathcal{O}^\lambda_{\mu,\mathcal{Z}}$ would not be compatible with usual q-character of type $C_2^{(1)}$:*

$$Y_{2,1} + Y_{2,q^4}^{-1}Y_{1,q}Y_{1,q^3} + Y_{1,q^5}^{-1}Y_{1,q} + Y_{1,q^3}^{-1}Y_{1,q^5}^{-1}Y_{2,q^2} + Y_{2,q^4}^{-1},$$

or with the monomial crystal :

$$\mathcal{M}(Y_{2,1}) = \{Y_{2,1}, Y_{2,q^2}^{-1}Y_{1,q}^2, Y_{1,q}Y_{1,q^3}^{-1}, Y_{1,q^3}^{-2}Y_{2,q^2}, Y_{2,q^4}^{-1}\}.$$

(ii) The q-character of $V_2^L(1)$ would not be directly relevant either :

$$Z_{2,1} + Z_{2,q^4}^{-1}Z_{1,-q}Z_{1,q} + Z_{1,q}Z_{-q^3}^{-1} + Z_{1,-q}Z_{1,q^3}^{-1} + Z_{2,q^4}Z_{1,q^3}^{-1}Z_{1,-q^3}^{-1} + Z_{2,q^8}^{-1}.$$

(iii) For $\mu = 0$, we have seen the two ℓ-weights can be read in terms of the Langlands dual q-character, involving the monomials of the q-character a Kirillov-Reshetikhin modules of the untwisted quantum affine algebra $\mathcal{U}_q(B_2^{(1)})$:

$$Z_{1,q^2}Z_{1,q^4}^{-1} = Y_{1,q^2}Y_{1,q^4}^{-1} = (Y_{2,q^{-1}}Y_{2,q})A_{2,1}^{-1}A_{2,q^2}^{-1}A_{1,q^2}^{-1},$$

$$Z_{1,1}Z_{1,q^6}^{-1}Z_{2,q^2}^{-1}Z_{2,q^4} = (Y_{2,q^{-1}}Y_{2,q})A_{2,1}^{-1}A_{2,q^2}^{-1}A_{1,q^4}^{-1}.$$

By interpolation, they correspond to the following monomials occurring in the q-characters of the fundamental module $L(Z_{2,1})$ of the twisted quantum affine algebra $\mathcal{U}_q(A_3^{(2)})$:

$$Z_{1,q}Z_{1,-q^3}^{-1} = Z_{2,1}A_{2,q^2}^{-1}A_{1,-q^2}^{-1}\ ,\ Z_{1,-q}Z_{1,q^3}^{-1} = Z_{2,1}A_{2,q^2}^{-1}A_{1,q^2}^{-1}.$$

By folding [He4]*, they correspond to the following monomials occurring in the q-character of the fundamental module $L(Y_{2,1})$ of the simply-laced quantum affine algebra $\mathcal{U}_q(A_3^{(1)})$:*

$$Y_{1,q}Y_{3,q^3}^{-1} = Y_{2,1}A_{2,q}^{-1}A_{3,q^2}^{-1}\ ,\ Y_{3,q}Y_{1,q^3}^{-1} = Y_{2,1}A_{2,q}^{-1}A_{1,q^2}^{-1}.$$

The corresponding eigenvalues of $A_1^+(z)$ and $A_2^+(z)$ are respectively

$$(q^2(1-zq^{-4}), q(1-zq^{-2})) \text{ and } (q^3(1-zq^{-6}), q(1-zq^{-2})),$$

which correspond to the contribution of $\Lambda_{1,q^{-2}}^{-1}\Lambda_{2,q^{-1}}^{-1}$ and $\Lambda_{2,q^{-1}}^{-1}\Lambda_{1,q^{-4}}^{-1}$ (see Remark 10.9).

To complete the picture of fundamental representations in type B_2, let us now set $\lambda = \omega_1^\vee$, $Z_1(z) = 1-z$, $Z_2(z) = 1$. An analog computation gives

$$\chi_q^L(V_1(1)) = Z_{1,1} + Z_{1,q^4}^{-1}Z_{2,q^2} + Z_{2,q^4}^{-1}Z_{1,q^2} + Z_{1,q^6}^{-1}. \tag{12.29}$$

In the same way the following representations descend to the corresponding truncation :
$\mu = \omega_1^\vee$: $\mathbf{\Psi} = (1-z, iq^{-1})$,
$\mu = \lambda - \alpha_1^\vee = \omega_2^\vee - \omega_1^\vee$: $\mathbf{\Psi} = \left(\frac{q}{1-zq^{-4}}, iq^{-1}(1-zq^{-2}))\right)$,
$\mu = \lambda - \alpha_1^\vee - \alpha_2^\vee = \omega_1^\vee - \omega_2^\vee$: $\mathbf{\Psi} = \left(q^{-1}(1-zq^{-2}), \frac{i}{1-zq^{-4}}\right)$,
$\mu = \lambda - 2\alpha_1^\vee - \alpha_2^\vee = -\omega_1^\vee$: $\mathbf{\Psi} = \left(\frac{q^2}{1-zq^{-6}}, iq^{-1}\right)$.

12.6. **Reduction to fundamental representations.** Let us first study the compatibility between fusion products and truncated quantum affine algebras.

Let $\mu_1, \mu_2 \in \Lambda$ and $\lambda_1, \lambda_2 \in \Lambda^+$ so that $\mu_1 \preceq \lambda_1$ and $\mu_2 \preceq \lambda_2$. We consider corresponding set of parameters $\mathcal{Z}_1$, $\mathcal{Z}_2$. The product $\mathcal{Z}_1\mathcal{Z}_2$ is defined component by component.

Proposition 12.6. *If V_1 is in $\mathcal{O}_{\mu_1,\mathcal{Z}_1}^{\lambda_1}$ and V_2 is in $\mathcal{O}_{\mu_2,\mathcal{Z}_2}^{\lambda_2}$ then $V_1 * V_2$ is in $\mathcal{O}_{\mu_1+\mu_2,\mathcal{Z}_1\mathcal{Z}_2}^{\lambda_1+\lambda_2}$.*

Proof. This follows from

$$\Delta_u(\phi_i^\pm(z)) = \phi_i^\pm(z) \otimes \phi_i^\pm(zu).$$

Let us explain it for $\phi_i^+(z)$. Consider elements $A_{i,u}^+(z)$ associated to $\mathcal{Z}_1(z)\mathcal{Z}_2(zu)$. The $\mathcal{A}$-form which defines $V_1 * V_2$ is stable by the coefficients of $A_{i,u}^+(z)$ and

$$\Delta_u(A_{i,u}^+(z)) = A_i^{\mathcal{Z}_1,+}(z) \otimes A_i^{\mathcal{Z}_2,+}(zu).$$

Hence $A_{i,u}^+(z)$ is a polynomial in z on the $\mathcal{A}$-form and $A_i^+(z)$ is polynomial on $V_1 * V_2$. □

Let $\mu \in \Lambda$ and remind the functor $*_{i,a} : \mathcal{O}_\mu \to \mathcal{O}_{\mu+\omega_i^\vee}$. Consider V a representation in $\mathcal{O}_\mu$. Let $\lambda \in \Lambda^+$ so that $\mu \preceq \lambda$. We get the following as for the last Proposition.

Proposition 12.7. *The representation $*_{i,a}(V)$ is in $\mathcal{O}_{\mu+\omega_i^\vee,\mathcal{Z}'}^{\lambda+\omega_i^\vee}$ if and only if V is in $\mathcal{O}_{\mu,\mathcal{Z}}^{\lambda}$. Here $\mathcal{Z}'$ is obtained from $\mathcal{Z}$ by replacing $Z_i(z)$ by $Z_i(z)(1-za)$.*

As a consequence, to prove that the simple representations occurring in Conjecture 12.3 are representations in the category $\mathcal{O}_{\mu,\mathcal{Z}}^{\lambda}$, it suffices to treat the case when V is a fundamental representation, that is when λ is a fundamental coweight.

Consequently, from the examples above, we obtain the following.

Proposition 12.8. *In types A_2, B_2, Conjecture 12.3 (A) is true.*

12.7. **Finite-dimensional representations.** We use Baxter polynomiality of quantum integrable systems to establish the following.

Theorem 12.9. *A simple finite-dimensional representation of a shifted quantum affine algebra $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ is in a category $\mathcal{O}_{\mu,\mathcal{Z}}^{\lambda}$, given by (A) in Conjecture 12.3.*

From Proposition 12.6 and by the classification in Theorem 6.4, it suffices to consider the case of W simple finite-dimensional representation of $\mathcal{U}_q(\hat{\mathfrak{g}})$.

Let w_* be a lower weight vector. Let ω_* be its weight. For $i \in I$, let $T_{i,*}^{\pm}(z) \in \mathbb{C}[[z^{\mp}]]$ be the eigenvalue of $T_i^{\pm}(z)$ on w_*. By Proposition 9.8, $T_{i,*}^{-}(z)/T_{i,\boldsymbol{\Psi}}^{-}(z)$ and $z^{ht_i(\omega(\omega_*)^{-1})}T_{i,\boldsymbol{\Psi}}^{+}(z)/T_{i,*}^{+}(z)$ are polynomial in z of degree $ht_i(\omega(\omega_*)^{-1})$.

Example 12.10. *This can be observed in Example 9.10 with the respective eigenvalues of $T^-(z)$ and $T^+(z)$ on a lowest vector :*

$$(1 - zq^{-1})^2 T_{\boldsymbol{\Psi}}^{-}(z) = T_{\boldsymbol{\Psi}}^{+}(z^{-1}q^2)\ ,\ T_{\boldsymbol{\Psi}}^{+}(z)(1 - z^{-1}q)^{-2} = T_{\boldsymbol{\Psi}}^{-}(z^{-1}q^2).$$

More generally we have the following.

Proposition 12.11. *For ω' a weight of W, on $W_{\omega'}$ the operators*

$$z^{ht_i(\omega'(\omega_*)^{-1})}\frac{T_i^+(z)}{T_{i,*}^+(z)} \text{ and } \frac{T_{i,*}^-(z)}{T_i^-(z)} \in (End(W_{\omega'}))[z]$$

are polynomial in z of degree $ht_i(\omega'(\omega_*)^{-1})$ and coincide up to a constant operator factor.

Proof. It suffices to twist the representation by the morphism σ of [C, Proposition 1.6] as in the proof of Theorem 9.12. Then the statement follows from this Theorem. $\square$

Now we have

$$A_i^{\mathcal{Z},+}(z) = (\overline{\phi}_i^+)^{-1}\overline{Y}_{i,\mathbf{Z}}^+(zq_i^{-1})\frac{T_i^+(z^{-1})}{T_i^+(z^{-1}q_i^2)} = \overline{Y}_{i,\mathbf{Z}}^+(zq_i^{-1})\frac{T_{i,*}^+(z^{-1})}{T_{i,\boldsymbol{\Psi}}^+(z^{-1}q_i^2)}P_i(z),$$

where

$$P_i(z) = (\overline{\phi}_i^+)^{-1}\frac{T_i^+(z^{-1})}{T_{i,*}^+(z^{-1})}\frac{T_{i,\boldsymbol{\Psi}}^+(z^{-1}q_i^2)}{T_i^+(z^{-1}q_i^2)}$$

is a polynomial operator of degree $ht_i(\omega(\omega')^{-1}) + ht_i(\omega'(\omega_*)^{-1}) = ht_i(\omega(\omega_*)^{-1})$. Besides

$$\overline{Y}_{i,\mathbf{Z}}^+(zq_i^{-1})\frac{T_{i,*}^+(z^{-1})}{T_{i,\boldsymbol{\Psi}}^+(z^{-1}q_i^2)} = \overline{Y}_{i,\mathbf{Z}\boldsymbol{\Psi}^{-1}}^+(zq_i^{-1}) \times \frac{T_{i,*}^+(z^{-1})}{T_{i,\boldsymbol{\Psi}}^+(z^{-1})}. \tag{12.30}$$

Note that by Proposition 9.8,

$$T^+_{i,\mathbf{\Psi}}(z^{-1})/T^+_{i,*}(z^{-1}) = \prod_{a\in\mathbb{C}^*}(1-za)^{v_{i,a}}$$

is a polynomial of degree $ht_i(\omega(\omega_*)^{-1})$ where we have denoted $\mathbf{\Psi}^* = \mathbf{\Psi}\prod_{i\in I,a\in\mathbb{C}^*}A_{i,a}^{-v_{i,a}}$ the lowest ℓ-weight of W. We will also denote $u_{i,a}$ the multiplicity of $Y_{i,a}$ in $\mathbf{\Psi}$ for $i\in I$, $a\in\mathbb{C}^*$. Then we set

$$\mathbf{Z} = \prod_{a\in\mathbb{C}^*}\left(\prod_{i\in I,r_i=1}(\mathbf{\Psi}_{i,aq^{-1}}\mathbf{\Psi}_{\bar{i},aq^{1+rh^\vee}})^{u_{i,a}}\right)\left(\prod_{i\in I,r_i=2}(\mathbf{\Psi}_{i,aq^{-2}}\mathbf{\Psi}_{\bar{i},aq^{2+rh^\vee}}\mathbf{\Psi}_{i,a}\mathbf{\Psi}_{\bar{i},aq^{rh^\vee}})^{u_{i,a}}\right)$$
$$\times\left(\prod_{i\in I,r_i=3}(\mathbf{\Psi}_{i,aq^{-3}}\mathbf{\Psi}_{\bar{i},aq^{3+rh^\vee}}\mathbf{\Psi}_{i,aq}\mathbf{\Psi}_{\bar{i},aq^{1+rh^\vee}}\mathbf{\Psi}_{i,aq^{-1}}\mathbf{\Psi}_{\bar{i},aq^{-1+rh^\vee}})^{u_{i,a}}\right),$$

and a corresponding $\mathcal{Z}$, where we use the same notations as in (v) of Remark 12.4. We claim that $\mathbf{\Psi}\preceq_{\mathcal{Z}}\mathbf{Z}$, that $L(\mathbf{\Psi})$ is in the category $\mathcal{O}^\lambda_{\mu,\mathcal{Z}}$ for the corresponding λ and that $M_{\mathbf{\Psi}}$ occurs in the Langlands dual q-character associated to $M_{\mathbf{Z}}$.
For the first point, from (v) of Remark 12.4 and Proposition 12.2, for each $i\in I$, $a\in\mathbb{C}^*$:

$$1\preceq_{\mathcal{Z}}\mathbf{\Psi}_{i,1}\mathbf{\Psi}_{\bar{i},q^{rh^\vee}}.$$

This implies

$$\mathbf{Z}\mathbf{\Psi}^{-1} = \prod_{a\in\mathbb{C}^*}\left(\prod_{i\in I,r_i=1}(\mathbf{\Psi}_{i,aq}\mathbf{\Psi}_{\bar{i},aq^{1+rh^\vee}})^{u_{i,a}}\right)\left(\prod_{i\in I,r_i=2}(\mathbf{\Psi}_{i,aq^{2}}\mathbf{\Psi}_{\bar{i},aq^{2+rh^\vee}}\mathbf{\Psi}_{i,a}\mathbf{\Psi}_{\bar{i},aq^{rh^\vee}})^{u_{i,a}}\right)$$
$$\times\left(\prod_{i\in I,r_i=3}(\mathbf{\Psi}_{i,aq^{3}}\mathbf{\Psi}_{\bar{i},aq^{3+rh^\vee}}\mathbf{\Psi}_{i,aq}\mathbf{\Psi}_{\bar{i},aq^{1+rh^\vee}}\mathbf{\Psi}_{i,aq^{-1}}\mathbf{\Psi}_{\bar{i},aq^{-1+rh^\vee}})^{u_{i,a}}\right)\succeq_{\mathcal{Z}} 1.$$

For the second point, let $\nu_{i,a}\geq 0$ be the power of $\Lambda_{i,a}$ in the factorization of this ℓ-weight. To get $L(\mathbf{\Psi})$ in $\mathcal{O}^\lambda_{\mu,\mathcal{Z}}$, it suffices to prove that

$$\nu_{i,aq_i}\geq v_{i,a} \text{ for any } i\in I,\ a\in\mathbb{C}^*. \tag{12.31}$$

Indeed, we have as in the proof of Proposition 9.6 the eigenvalue of $\overline{Y}^+_{i,\mathbf{Z}\mathbf{\Psi}^{-1}}(zq_i^{-1})$ corresponding to a factor $\Lambda_{i,a}$ is $(1-zq_i^{-1})$. So condition (12.31) implies that the scalar function (12.30) is polynomial. For simply-laced types, we have indeed $\nu_{i,aq}=v_{i,a}$. In general, they may defer. Let $B_{j,b}$ Laurent monomial in the $Z_{i,a}$ so that $\mathbf{\Psi}_{B_{j,b}}=\Lambda_{j,b}$. We see as in the proof of Proposition 12.2 that the power $a_{j,b}$ of $A_{j,b}$ in $Y_{i,a}Y_{\bar{i},aq^{-rh^\vee}}$ and the power $b_{j,b}$ of $B_{j,b}$ in $Z_{i,a}Z_{\bar{i},aq^{-rh^\vee}}$ are related by :
$a_{j,b}=b_{j,b}$ if $r_j=r$,
$a_{j,b}=b_{j,bq^{-1}}+b_{j,bq}$ if $r_j=1$ and $r=2$,
$a_{j,b}=b_{j,bq^{-2}}+b_{j,b}+b_{j,bq^2}$ if $r_j=1$ and $r=3$.
This implies the inequalities (12.31).

The last point follows from the fact that $Z^{-1}_{\bar{i},aq^{-rh^\vee}}$ is a monomial of $V^L_{i,a}$ (see (v) in Remark 12.4). This is analog for $A_i^{\mathcal{Z},-}(z)$, this completes the proof of Theorem 12.9.

Remark 12.12. *Our proof gives one of the possible truncations for a simple finite-dimensional representation as predicted by (A) in Conjecture 12.3.*

Example 12.13. *Let $\mathfrak{g}$ of type B_2 with $r_1 = 2$ and $r_2 = 1$. Let $\mathbf{\Psi} = [-\omega_1]Y_{1,1} = \mathbf{\Psi}_{1,q^{-2}}\mathbf{\Psi}_{1,q^2}^{-1}$ corresponding to a 5-dimensional fundamental representation. We have its lowest-weight*

$$\mathbf{\Psi}^* = [-\omega_1]Y_{1,q^6}^{-1} = [-2\omega_1]\mathbf{\Psi}_{1,q^8}\mathbf{\Psi}_{1,q^4}^{-1} = \mathbf{\Psi}A_{1,q^2}^{-1}A_{2,q^4}^{-1}A_{2,q^2}^{-1}A_{1,q^4}^{-1}.$$

We have also for any $a \in \mathbb{C}^*$:

$$\mathbf{\Psi}_{1,a}\mathbf{\Psi}_{1,aq^6} = \Lambda_{1,aq^2}\Lambda_{2,aq^3}\Lambda_{1,aq^4}.$$

That why we set as is the proof of Theorem 12.9 :

$$\mathbf{Z} = \mathbf{\Psi}_{1,q^{-2}}\mathbf{\Psi}_{1,q^8}\mathbf{\Psi}_{1,1}\mathbf{\Psi}_{1,q^6}$$

polynomial so that

$$\mathbf{\Psi} = \mathbf{Z}\Lambda_{1,q^4}^{-1}\Lambda_{2,q^5}^{-1}\Lambda_{1,q^6}^{-1}\Lambda_{1,q^2}^{-1}\Lambda_{1,q^3}^{-1}\Lambda_{1,q^4}^{-1}.$$

We see that the inequalities (12.31) are satisfied. Hence $L(\mathbf{\Psi})$ is in the category $\mathcal{O}_{\mu,\mathcal{Z}}^{\lambda}$ for a certain $\mathcal{Z}$ compatible with $\mathbf{Z}$. The corresponding Langlands dual standard module is

$$V = V_{1,q^2}^L \otimes V_{1,q^{-8}}^L \otimes V_{1,1}^L \otimes V_{1,q^{-6}}^L.$$

The Langlands dual q-character of the $V_{1,a}^L$ are give by formula (12.29). We obtain that

$$Z_{1,q^2}Z_{1,q^{-2}}^{-1} = Z_{1,q^2}Z_{1,q^{-2}}^{-1}Z_{1,1}Z_{1,1}^{-1}$$

occurs as a monomial in the Langlands dual q-character $\chi_q^L(V)$ as in Conjecture 12.3.

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Université de Paris and Sorbonne Université, CNRS, IMJ-PRG, IUF, F-75006 Paris, France.
Email address: david.hernandez@u-paris.fr