BEYOND KIRILLOV–RESHETIKHIN MODULES

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Abstract. In this survey, we shall be concerned with the category of finite–dimensional representations of the untwisted quantum affine algebra when the quantum parameter q is not a root of unity. We review the foundational results of the subject, including the Drinfeld presentation, the classification of simple modules and q-characters. We then concentrate on particular families of irreducible representations whose structure has recently been understood: Kirillov-Reshetikhin modules, minimal affinizations and beyond.

2000 Mathematics Subject Classification: Primary 17B37, Secondary 81R50, 82B23, 17B67.

Contents

1. Introduction

The representation theory of quantum affine algebras has been an active area of research for the past fifteen or twenty years. As in the case of the affine Kac–Moody algebras, there are two distinct but equally important families of representations that are studied: the positive level representations and the level zero representations. The positive level representations are usually studied via the Chevalley generators and Serre relations (in the quantum case this is known as the Drinfeld–Jimbo presentation) while the level zero representations are studied via the loop realization (Drinfeld realization) of the affine (quantum affine) Kac–Moody algebra.

In this survey, we shall be concerned with a full subcategory of the level zero representations of the quantum affine algebra: namely, the category of finite–dimensional representations and we shall assume moreover that the quantum parameter q is not a root of unity. The non–quantum version of this category had been studied previously in [Cha1, CP1, R] for instance. In those papers, the irreducible finite–dimensional representations of the affine algebra were classified. An explicit description of the irreducible modules were given in terms of the underlying simple Lie algebra thus allowing one to deduce character formulas for the irreducible representations. The results of those papers, after a straightforward reformulation, prove that the irreducible representations are given by an n–tuple of polynomials in an indeterminate u, where n is the rank of the underlying simple Lie algebra. It was also clear that the category of finite–dimensional representations was not semi-simple and that there should be some rich structure theory analogous to the representation theory of algebraic groups in characteristic p. The blocks of the category were determined in [CM1].

After the presentation of quantum affine algebras was given in [Dr2], an identical classification of the simple finite–dimensional modules for quantum affine algebras was given in [CP2, CP3]. In the case of the quantum affine algebra corresponding to $\mathfrak{sl}_2$, the simple modules could be described explicitly in a manner similar to the one for the non–quantum case. Outside this case however, it was known that such a result would not hold in general; for instance in [Dr2], an example was given (in the case of the closely related Yangians) of an irreducible representation of the simple Lie algebra which did not admit the structure of a module for the quantum affine algebra. In [CP3], the structure (regarded as a module for the finite–dimensional simple algebra) of the irreducible "fundamental representations"of the quantum affine algebra was determined. In particular, outside the case of $\mathfrak{sl}_{n+1}$ most of these representations were highly reducible for the simple Lie algebra. It was clear that in the quantum situation that the structure and characters of the irreducible representations was much more complex and thus of intrinsic mathematical interest. Together with this, was the external motivation coming from the work of A.N. Kirillov and N.Reshetikhin, [KR], the work of I. Frenkel and N. Reshetikhin [FiR] and others in mathematical physics. The results of these papers suggested that it would be fruitful to concentrate on understanding particular families of irreducible representations. Thus, the work of [KR] was connected with the irreducible representations of quantum affine algebras corresponding to a multiple of a fundamental representation. These modules are now called the Kirillov–Reshetikhin modules.

The work of [FiR] suggested that there should be some natural minimal family of modules for the quantum affine algebra. This can be also seen in another, purely representation theoretic way: what is the "smallest"representation of the quantum affine algebra corresponding to a given irreducible representation of the simple Lie algebra. This motivated the first author to introduce in [Cha2] the notion of a minimal affinization and these were further studied in [CP4, CP5, CP6]. To do this, one introduces a poset for each dominant integral weight, such that each element of the poset determines a family of irreducible representation of the quantum affine algebra. A minimal affinization is one which corresponds to the minimal element of the poset. Another very general family of representations, the so–called prime representations was introduced in [CP9]. The definition is very natural in the light of the results in [CP2]. It was clear from the results of [CP4, CP5] that the Kirillov–Reshetikhin modules are prime and that they are the minimal affinizations of multiples of the fundamental weights. We shall now see how this and other results motivates the title of our survey.

A very important advance in the theory of finite–dimensional representations was the definition of q–characters introduced in [FR] and studied further in [FM1, FM2]. The q–characters are generalizations of the usual character of a representation of a simple Lie algebra and satisfy all the usual nice properties: they are additive on direct sums and multiplicative on tensor products. To define them, one observes that in the affine case there is an infinite family of commuting elements and the q–characters encode the information of the generalized eigenvalues and eigenspaces for their action. It was shown in those papers that the irreducible representations were determined by their q–character and also that the eigenvalues are given by n–tuples of rational functions in an indeterminate u. However, closed formulas such as an analog of the Weyl–character formula are not known for q–characters. Instead, in [FM1] an algorithm (now called the Frenkel–Mukhin algorithm) was proposed to calculate the q–character. They proved that the algorithm worked for the fundamental representations and it was thought that this procedure might work in general. This is now known to be false through the recent work of [NN4]. However, it was quite reasonable to expect that there were families of representations for which the algorithm worked and this leads to the notion of a regular representation: one where the Frenkel–Mukhin algorithm yields the q–character of the module. It was proved in [FM1] that a representation was regular if it was special, i.e had an unique eigenvalue given by n–tuple of polynomials. From a Lie–theoretic point of view it is more natural to call these minuscule representations as we explain in Section 4. In [Nak4, Nak5] it was proved for simply–laced algebras, by using geometric methods, that the Kirillov–Reshetikhin modules were minuscule. An algebraic proof was given in [H5] for all simple Lie algebras. These results imply the Kirillov–Reshetikhin conjecture, which gives a closed formula for the character of the tensor product of Kirillov–Reshetikhin modules. Later in [H8] it was shown that most minimal affinizations are minuscule.

Another way to study the simple finite–dimensional modules for a quantum affine algebras is via branching rules: namely determining the multiplicity of an irreducible module for the simple Lie algebra in a given module. In the case of $\mathfrak{sl}_2$ and for fundamental representations, this has been discussed earlier in the introduction. In the case of algebras of type C_2, this was done for minimal affinizations in [CP7]. For algebras of type A_n, it is known that the minimal affinization is irreducible for the simple Lie algebra. In the case of the Kirillov–Reshetikhin modules this problem was studied in [Cha3] in the case corresponding to nodes of classical type confirming conjectures of [KR, Kl, HKOTY]. The methods of this paper used the ideas developed in [CP14, CP13] where the notion of a Weyl module was introduced. The Weyl modules are also parametrized by an n–tuple of polynomials but are not necessarily irreducible but do have very nice universal properties. It was conjectured in [CP14, CP13] that the Weyl modules are just a tensor product of fundamental representations and we shall see that this is in fact the case in Section 5.

Relatively little is known about prime representations, in Section 7, we show that most minimal affinizations are prime. One of the main results of [Cha4] gives a necessary condition for a representation to be prime. Additional examples of prime representations are given in [HL]. We also discuss the quasi–minuscule representations, one in which the generalized eigenspaces are of dimension at most one. For some algebras, the Kirillov–Reshetikhin modules and more generally, any minimal affinization is quasi–minuscule. A final family are the small representations closely related to the geometric small property (Borho-MacPherson). We discuss the proof [H7] of a related conjecture [Nak4] implying a description of the singularities in terms of intersection homology of certain projective morphisms of quiver varieties.

The study of Kirillov–Reshetikhin modules has been of immense interest in recent years. Character formulas for these representation have been conjectured from physical considerations. These representations are related to several geometric constructions and to rich combinatorial structures such as crystals [Kas, OS], T–systems [KNS, HKOTY, HKOTT] and their solutions [Nak5, H5] (and more recently cluster algebras [Ke, HL]). In the last section we have a brief overview of these results.

To keep the survey of a manageable length and as a result of our own perspective, we have made many choices and have not elaborated on other important approaches both geometric and combinatorial. In simply-laced cases there are the powerful geometric methods of [Nak1], further studied in [VV]. In particular Nakajima defined an algorithm and proved that it gives the q-character of an arbitrary simple representation of a simply-laced quantum affine algebra [Nak4] (a conjectural algorithm for all cases is defined in [H1]). There is also the connection with the theory of crystal basis and details of this approach can be found in the work of [AK, Kas, OS]. Various important historical references can also be found in [KNT]. We have also restricted ourselves to the case of untwisted affine algebras and when q is not a root of unity. This is primarily because the subject is most well–developed in these cases and several results of this survey have not been established for twisted cases. However there are some results in the twisted case which can be found in [CP11, H9]. The papers [CP12, BK, FM3, Nak4, H2] consider the case when q is a root of unity. Finally, for the quantum toroidal (ie double affine) case see [H10] and references therein.

2. Quantum Affine Algebras: definitions and basic results.

Throughout the paper $\mathbf{C}$ (resp. $\mathbf{Z}$, $\mathbf{Z}_+$, $\mathbf{N}$) denotes the set complex numbers (resp. integers, non–negative integers, positive integers) and $q \in \mathbf{C}$ a fixed non-zero complex number which is not a root of unity.

2.1. Set $I = \{1, \cdots, n\}$ and $\widehat{I} = I \sqcup \{0\}$. Let $A = (a_{ij})_{i,j\in I}$ be an indecomposable Cartan matrix of finite type and let $\hat{A} = (a_{ij})_{i,j\in\hat{I}}$ the corresponding untwisted affine Cartan matrix. Fix a set $\{d_i\}_{i\in\hat{I}}$ of positive integers so that the matrix $\{d_i a_{ij}\}_{i,j\in\widehat{I}}$ is symmetric.

Let $\mathfrak{g}$ and $\widehat{\mathfrak{g}}$ the corresponding finite–dimensional simple and untwisted affine Lie algebra associated to A and $\hat{A}$ respectively. As usual, R denotes the set of roots of $\mathfrak{g}$ with respect to a fixed Cartan subalgebra, and $\{\alpha_i\}_{i\in I}$ (resp. $\{\omega_i\}_{i\in I}$) a set of simple roots (resp. fundamental weights). Let Q (resp. Q^+) and P (resp. P^+) be the $\mathbf{Z}$–span (resp. $\mathbf{Z}_+$–span) of the simple roots and fundamental weights respectively and set $R^+ = R \cap Q^+$. Let θ be the highest root in R^+. Let $\leq$ be the usual partial order on P defined by; for $\lambda, \mu \in P$, we have $\lambda \leq \mu$ iff $\mu - \lambda \in Q^+$.

Let W be the Weyl group of R and for $i \in I$, let $s_i \in W$ be the simple reflection corresponding to α_i and let $\mathbf{l} : W \to \mathbf{Z}_+$ be the length function. The group W acts on the root lattice Q by extending $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$. The affine Weyl group $\hat{W}$ is isomorphic to the semi-direct product $W \ltimes Q$, under the map

$$s_i \to (s_i, 0), \quad i \in I, \quad s_0 \to (s_\theta, \theta),$$

where $s_\theta(\alpha_j) = \alpha_j + a_{0\ j}\theta$. The extended Weyl group $\tilde{W}$ is defined to be the semi-direct product $W \ltimes P$. Regard W and P as subgroups of $\tilde{W}$ via the maps $w \to (w, 0)$ and $\lambda \to t_\lambda = (e, \lambda)$. The affine Weyl group $\hat{W}$ is a normal subgroup of $\tilde{W}$, and the quotient $\mathcal{T} = \tilde{W}/\hat{W}$ is a finite group isomorphic to a subgroup of the group of diagram automorphisms of $\widehat{\mathfrak{g}}$, i.e. the bijections $\tau : \hat{I} \to \hat{I}$ such that $a_{\tau(i)\tau(j)} = a_{ij}$ for all $i, j \in \hat{I}$. Moreover, there is an isomorphism of groups $\tilde{W} \cong \mathcal{T} \ltimes \hat{W}$, where the semi-direct product is defined using the action of $\mathcal{T}$ in $\hat{W}$ given by $\tau.s_i = s_{\tau(i)}\tau$ (see [B]).

2.2. For $i \in \hat{I}$, set $q_i = q^{d_i}$. For $\ell \in \mathbf{Z}$, and $r, p, m \in \mathbf{Z}_+$ with $m \geq p$, set

$$[\ell]_i = \frac{q_i^\ell - q_i^{-\ell}}{q_i - q_i^{-1}}\ , \quad [r]_i! = [r]_i[r-1]_i \cdots [1]_i\ , \quad \begin{bmatrix} m \\ p \end{bmatrix}_i = \frac{[m]_i!}{[m-p]_i![p]_i!}.$$

The quantum affine algebra $\widehat{\mathbf{U}}_q(\mathfrak{g})$ is the associative algebra defined over $\mathbf{C}$ with generators $k_i^{\pm 1}$, $x_i^\pm$ $(i \in \hat{I})$ and relations:

$$k_i k_j = k_j k_i\ , \quad k_i x_j^\pm = q_i^{\pm a_{ij}} x_j^\pm k_i,$$

$$[x_i^+, x_j^-] = \delta_{i,j} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum\nolimits_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_i (x_i^\pm)^{1-a_{ij}-r} x_j^\pm (x_i^\pm)^r = 0 \text{ (for } i \neq j).$$

The assignments,

$$\Delta(x_i^+) = x_i^+ \otimes k_i + 1 \otimes x_i^+, \quad \Delta(x_i^-) = x_i^- \otimes 1 + k_i^{-1} \otimes x_i^-, \quad \Delta(k_i^{\pm 1}) = k_i^{\pm 1} \otimes k_i^{\pm 1},$$
$$S(x_i^+) = -x_i^+ k_i^{-1}, \quad S(x_i^-) = -k_i x_i^-, \quad S(k_i^{\pm 1}) = k_i^{\pm 1},$$
$$\varepsilon(x_i^\pm) = 0, \quad \varepsilon(k_i^{\pm 1}) = 1,$$

for $i \in \hat{I}$, define a Hopf algebra structure on $\widehat{\mathbf{U}}_q(\mathfrak{g})$. The subalgebra $\mathbf{U}_q(\mathfrak{g})$ generated by the elements $x_i^\pm$, k_i, $i \in I$ is a Hopf subalgebra of $\widehat{\mathbf{U}}_q(\mathfrak{g})$ and is isomorphic as a Hopf algebra to the quantized enveloping algebra of $\mathfrak{g}$. If $\theta = \sum_{i \in I} \theta_i \alpha_i$, the element $C = k_0 \prod_{i \in I} k_i^{-\theta_i}$ is central in $\widehat{\mathbf{U}}_q(\mathfrak{g})$.

2.3. For $i \in \hat{I}$ and $m \geq 1$, set $(x_i^\pm)^{(m)} = (x_i^\pm)^m/[m]_i!$ and $(x_i^\pm)^{(0)} = x_i^\pm$. For $i \in \hat{I}$ let T_i be the algebra automorphism of $\widehat{\mathbf{U}}_q(\mathfrak{g})$ defined in [L] by,

$$T_i((x_i^+)^{(m)}) = (-1)^m q^{-m(m-1)} (x_i^-)^{(m)} k_i^m, \quad T_i((x_i^-)^{(m)}) = (-1)^m q^{m(m-1)} k_i^{-m} (x_i^+)^{(m)},$$

$$T_i((x_j^+)^{(m)}) = \sum_{r=0}^{-ma_{ij}} (-1)^r q^{-r} (x_i^+)^{(-ma_{ij}-r)} (x_j^+)^{(m)} (x_i^+)^{(r)} \quad \text{if } i \neq j,$$

$$T_i((x_j^-)^{(m)}) = \sum_{r=0}^{-ma_{ij}} (-1)^r q^{r} (x_i^-)^{(r)} (x_j^-)^{(m)} (x_i^-)^{(-ma_{ij}-r)} \quad \text{if } i \neq j,$$

for all $m \geq 0$. The finite group $\mathcal{T}$ acts as Hopf algebra automorphisms of $\widehat{\mathbf{U}}_q(\mathfrak{g})$ by

$$\tau(x_i^{\pm}) = x_{\tau(i)}^{\pm} \quad \tau(k_i) = k_{\tau(i)}, \quad \text{for all } i \in \hat{I}.$$

Given any $w \in \tilde{W}$, write $w = \tau s_{i_1} \cdots s_{i_p}$ where $i_r \in \hat{I}$ for $1 \leq r \leq p$ and $s_{i_1} \cdots s_{i_p}$ is reduced and set $T_w = \tau T_{i_1} \dots T_{i_m}$. Then, T_w is an automorphism of $\widehat{\mathbf{U}}_q(\mathfrak{g})$ and depends only on w.

2.4. Following [B1], define for $i \in I$, $r \in \mathbf{Z}$, elements $x_{i,r}^{\pm} \in \widehat{\mathbf{U}}_q(\mathfrak{g})$, by

$$x_{i,r}^{\pm} = o(i)^r T_{\omega_i}^{\mp r} x_i^{\pm},$$

where $o : I \to \{\pm 1\}$ is a map such that $o(i) = -o(j)$ whenever $a_{ij} < 0$ (it is clear that there are exactly two possible choices for o). Note that $x_{i,0}^{\pm} = x_i^{\pm}$. It is proved in [B1] that the elements $x_{i,r}^{\pm}$, k_i, C, $i \in I$, $r \in \mathbf{Z}$ generate $\widehat{\mathbf{U}}_q(\mathfrak{g})$. A precise set of defining relations in terms of these generators (now called the Drinfeld generators and relations) is given in [Dr2] and proved in [B1].

For the purposes of this note we identify the following crucial relations. For $i \in I$, $r \in \mathbf{Z}_+$ and $r \neq 0$, set

$$\psi_{i,\pm r} = (q_i - q_i^{-1}) C^{\mp r/2} [x_{i,\pm r}^+, x_{i,0}^-].$$

For all $i, j \in I$ and $r, s \in \mathbf{Z}$, we have

$$[\psi_{i,r}, \psi_{j,s}] \in (C-1)\widehat{\mathbf{U}}_q(\mathfrak{g}).$$

Let $\widehat{\mathbf{U}}_q^{\pm}(\mathfrak{g})$ (resp. $\widehat{\mathbf{U}}_q(\mathfrak{h})$) be the subalgebra generated by the $x_{i,r}^{\pm}$ (resp. the $k_i^{\pm 1}$, $C^{\pm 1}$, $\psi_{i,r}$). We have,

$$\widehat{\mathbf{U}}_q(\mathfrak{g}) = \widehat{\mathbf{U}}_q^-(\mathfrak{g}) \widehat{\mathbf{U}}_q(\mathfrak{h}) \widehat{\mathbf{U}}_q^+(\mathfrak{g}), \tag{2.1}$$

We shall also use:

Lemma. *The assignment $x_{i,r}^{\pm} \to x_{i,-r}^{\mp}$, $\psi_{i,r} \to \psi_{i,-r}$, $q \to q^{-1}$ extends to a $\mathbf{C}$–linear algebra antiautomorphism Ω of $\widehat{\mathbf{U}}_q(\mathfrak{g})$.* □

For each $r \in \mathbf{Z}$ and $i \in I$ let $\mathbf{U}_{i,r}$ be the subalgebra of $\widehat{\mathbf{U}}_q(\mathfrak{g})$ generated by the elements $x_{i,r}^{\pm}$, $k_i^{\pm 1}$ and let $\widehat{\mathbf{U}}_i$ be the subalgebra generated by $\mathbf{U}_{i,r}$, $r \in \mathbf{Z}$ and $C^{\pm 1}$. Note that $\widehat{\mathbf{U}}_i$ is isomorphic to $\widehat{\mathbf{U}}_{q_i}(\mathfrak{sl}_2)$ and that $\mathbf{U}_{i,r}$ is isomorphic to $\mathbf{U}_{q_i}(\mathfrak{sl}_2)$ for $r \in \mathbf{Z}$.

2.5. Following [CP12], define for $i \in I$, $r \in \mathbf{Z}_+$ elements $P_{i,\pm r} \in \widehat{\mathbf{U}_q}(\mathfrak{g})$ by $P_{i,0} = 1$ and

$$P_{i,r} = -\frac{k_i^{-1}}{1 - q_i^{-2r}} \sum_{s=1}^{r} \psi_{i,s} P_{i,r-s}, \qquad P_{i,-r} = \Omega(P_{i,r}), \tag{2.2}$$

and set

$$\Psi_i^{\pm}(u) = k_i^{\pm 1} + \sum_{r \geq 1} \psi_{i,\pm r} u^r, \qquad \mathbf{P}_i^{\pm}(u) = \sum_{r \geq 0} P_{i,\pm r} u^r,$$

where u is an indeterminate. Let $X_i^{\pm}$, $i \in I$, be the subspace of $\widehat{\mathbf{U}}_q(\mathfrak{g})$ spanned by the elements $x_{i,r}^{\pm}$, $r \in \mathbf{Z}$. The following was proved in [CP12].

Proposition. *Let $i \in I$.*

(i) *There is an equality of power series* $\Psi_i^{\pm}(u)\mathbf{P}_i^{\pm}(u) = k_i^{\pm 1}\mathbf{P}_i^{\pm}(q_i^{\mp 2}u)$.

(ii) *The subalgebra of* $\widehat{\mathbf{U}}_q(\mathfrak{g})$ *generated by the elements* $\{P_{i,\pm r} : i \in I, r \in \mathbf{N}\}$ *is a polynomial algebra in these variables.*

(iii) *For* $i \in I$, $r \in \mathbf{Z}_+$, *we have*

$$(x_i^+)^{(r)}(x_{i,1}^-)^{(r)} = (-1)^r q_i^r k_i^r P_{i,r} + X_i^- \widehat{\mathbf{U}}_q(\mathfrak{g}) X_i^+,$$

$$(x_{i,-1}^+)^{(r)}(x_{i,0}^-)^{(r+1)} = (-1)^r q_i^{r+1} k_i^r \sum_{s=1}^{r} x_{i,s+1}^- P_{i,r-s} + \widehat{\mathbf{U}}_q(\mathfrak{g}) X_i^+.$$

□

2.6. In general one does not know explicit formulae for the comultiplication in terms of the generators $x_{i,r}^{\pm}$. However, the next proposition contains partial informations which is sufficient for our purposes. A proof can be found in [B1, BCP, Da].

Proposition. *For* $i \in I$, $r \in \mathbf{Z}$ *we have,*

$$\Delta(x_{i,r}^+) \in \sum_{j\in I} \widehat{\mathbf{U}}_q(\mathfrak{g}) X_j^+ \otimes \widehat{\mathbf{U}}_q(\mathfrak{g}) + \widehat{\mathbf{U}}_q(\mathfrak{g}) \otimes \sum_{j \in I} \widehat{\mathbf{U}}_q(\mathfrak{g}) X_j^+, \tag{2.3}$$

$$\Delta(P_{i,r}) - \sum_{s=0}^{r} P_{i,r-s} \otimes P_s \in \sum_{j\in I} \widehat{\mathbf{U}}_q(\mathfrak{g}) X_j^- \otimes \widehat{\mathbf{U}}_q(\mathfrak{g}) X_j^+. \tag{2.4}$$

□

2.7. We conclude this section by recalling from [CM1] the notion of an ℓ–weight lattice and an ℓ–root lattice, and also the definition of a braid group action on these lattices. These are analogous to the definition of the lattices P and Q and the Weyl group on action on them. The motivation for these ideas will be clear in the next section.

Let u be an indeterminate and let $\mathcal{P}^+$ be the monoid (under coordinate-wise multiplication) of I–tuples of polynomials in u with coefficients in $\mathbf{C}$ and constant term one. Given $i \in I$, $a \in \mathbf{C}^*$, let $\boldsymbol{\pi}_{i,a} \in \mathcal{P}^+$ be defined by requiring the i^{th} coordinate to be $1 - au$ and all other coordinates to be one. Clearly, $\mathcal{P}^+$ is the free abelian monoid generated by the $\boldsymbol{\pi}_{i,a}$, $i \in I$, $a \in \mathbf{C}^*$ and we let $\mathcal{P}$ be the corresponding free abelian group. We call $\mathcal{P}$ the ℓ–weight lattice. Let $\mathrm{wt} : \mathcal{P} \to P$ be the homomorphism of abelian groups given by setting $\mathrm{wt}\,\boldsymbol{\pi}_{i,a} = \omega_i$. If $\boldsymbol{\varpi} = (\boldsymbol{\varpi}_i)_{i\in I} \in \mathcal{P}$, then $\boldsymbol{\varpi}_i \in \mathbf{C}(u)$ and can be written as power series

$$\boldsymbol{\varpi}_i = 1 + \sum_{r\in\mathbf{N}} \boldsymbol{\varpi}_{i,r} u^r, \quad \boldsymbol{\varpi}_{i,r} \in \mathbf{C},$$

and we shall use this fact freely without further comment. Given $\pi \in \mathbf{C}[u]$ with constant term one,set

$$\pi^+ = \pi, \quad \pi^- = u^{\deg\pi}\pi(u^{-1})/(u^{\deg\pi}\pi^+(u^{-1}))|_{u=0},$$

and if $\varpi = \pi/\pi' \in \mathbf{C}(u)$, set $\varpi^{\pm} = \pi^{\pm}/(\pi')^{\pm}$. For $\boldsymbol{\varpi} = (\varpi_1, \cdots, \varpi_n) \in \mathcal{P}$ define elements $\boldsymbol{\varpi}^{\pm} \in \mathcal{P}$ by

$$\boldsymbol{\varpi}^{\pm} = (\varpi_1^{\pm}, \cdots, \varpi_n^{\pm}). \tag{2.5}$$

The element $\boldsymbol{\varpi}^-$ should not be confused with the the elements $(\boldsymbol{\varpi})^{-1}$ coming from the group structure.

2.8. Let B be the braid group associated to W. Thus, B is the group generated by elements $\tilde{T}_i$ $(i \in I)$ and defining relations:

$$\tilde{T}_i\tilde{T}_j = \tilde{T}_j\tilde{T}_i, \text{ if } a_{ij} = 0, \quad \tilde{T}_i\tilde{T}_j\tilde{T}_i = \tilde{T}_j\tilde{T}_i\tilde{T}_j, \text{ if } a_{ij}a_{ji} = 1,$$
$$(\tilde{T}_i\tilde{T}_j)^2 = (\tilde{T}_j\tilde{T}_i)^2, \text{ if } a_{ij}a_{ji} = 2, \quad (\tilde{T}_i\tilde{T}_j)^3 = (\tilde{T}_j\tilde{T}_i)^3, \text{ if } a_{ij}a_{ji} = 3, \quad i, j \in I.$$

The next proposition is a reformulation of [Cha4, Proposition 3.1] and can be easily checked.

Proposition. *There exists a homomorphism of the group B to the automorphism group of $\mathcal{P}$ given by:*

$$(\tilde{T}_i\boldsymbol{\varpi})_i = \frac{1}{\varpi_i(q_i^2u)}, \quad (\tilde{T}_i\boldsymbol{\varpi})_j = \varpi_j, \text{if } a_{ji} = 0, \quad (\tilde{T}_i\boldsymbol{\varpi})_j = \varpi_j(u)\varpi_i(q_iu), \text{ if } a_{ji} = -1,$$
$$(\tilde{T}_i\boldsymbol{\varpi})_j = \varpi_j(u)\varpi_i(q^3u)\varpi_i(qu), \text{ if } a_{ji} = -2,$$
$$(\tilde{T}_i\boldsymbol{\varpi})_j = \varpi_j(u)\varpi_i(q^5u)\varpi_i(q^3u)\varpi_i(qu), \text{ if } a_{ji} = -3,$$

where $\boldsymbol{\varpi} \in \mathcal{P}$ and $i, j \in I$. If $w \in W$ and $s_{i_1} \cdots s_{i_k}$ is a reduced expression of w, the element $\tilde{T}_w(\boldsymbol{\varpi}) = \tilde{T}_{i_1} \cdots \tilde{T}_{i_k}\boldsymbol{\varpi}$, is independent of the reduced expression and we have

$$\mathrm{wt}(\tilde{T}_w(\boldsymbol{\varpi})) = w\,\mathrm{wt}(\boldsymbol{\varpi}).$$

□

2.9. For $i \in I$, set

$$\boldsymbol{\alpha}_{i,a} = (\tilde{T}_i(\boldsymbol{\pi}_{i,a}))^{-1}\boldsymbol{\pi}_{i,a}.$$

This is exactly analogous to defining the root α_i by $\alpha_i = \omega_i - s_i\omega_i$, and so we have $\mathrm{wt}(\boldsymbol{\alpha}_{i,a}) = \alpha_i$. In addition

$$\tilde{T}_j\boldsymbol{\alpha}_{i,a} = \begin{cases} (\boldsymbol{\alpha}_{i,aq_i^2})^{-1}, & j = i, \\ \boldsymbol{\alpha}_{i,a}, & a_{ij} = 0, \\ \boldsymbol{\alpha}_{i,a}\boldsymbol{\alpha}_{j,aq}, & a_i = -1, \\ \boldsymbol{\alpha}_{i,a}\boldsymbol{\alpha}_{j,aq}\boldsymbol{\alpha}_{j,aq^3}, & a_{ij} = -2, \\ \boldsymbol{\alpha}_{i,a}\boldsymbol{\alpha}_{j,aq}\boldsymbol{\alpha}_{j,aq^3}, & a_{ij} = -3. \end{cases} \tag{2.6}$$

Let $\mathcal{Q}$ (resp. $\mathcal{Q}^+$) be the subgroup (resp. monoid) generated by $\boldsymbol{\alpha}_{i,a}$, $i \in I$, $a \in \mathbf{C}^\times$. Set $\mathcal{Q}^- = (\mathcal{Q}^+)^{-1}$. The following is now immediate.

Lemma. *The action of B on $\mathcal{P}$ preserves $\mathcal{Q}$.* □

Let $\preccurlyeq$ be the partial order defined on $\mathcal{P}$ by: $\boldsymbol{\varpi} \preccurlyeq \boldsymbol{\varpi}'$ iff $\boldsymbol{\varpi} \in \boldsymbol{\varpi}'\mathcal{Q}^-$.

Remark. The elements $\boldsymbol{\alpha}_{i,a}$ are essentially the elements $A_{i,a}$ first defined in [FR] from the quantized Cartan matrix and the analog of the partial order $\preccurlyeq$ also appeared first in [FR]. The definitions above were given in [CM1] and turn out to be a natural Lie–theoretic way to introduce these elements and the partial order. A presentation of the group $\mathcal{P}/\mathcal{Q}$ can be found in [CM1].

2.10. In the process of writing this note, we found that the proof of one of the statements of Lemma 2.7 in [CM1] was not complete and we take this opportunity to complete that proof.

Proposition. *Let $w \in W$ and $i \in I$ be such that $w\alpha_i \in R^+$. Then $\tilde{T}_w\boldsymbol{\alpha}_{i,a} \in \mathcal{Q}^+$ for all $a \in \mathbf{C}^\times$.*

Proof. We first prove the Lemma when $\mathfrak{g}$ is of rank two. If $\mathfrak{g}$ is of type A_2, using (2.6) we get

$$\tilde{T}_j\boldsymbol{\alpha}_{i,a} = \boldsymbol{\alpha}_{i,a}\boldsymbol{\alpha}_{j,aq}, \quad \tilde{T}_i\tilde{T}_j\boldsymbol{\alpha}_{i,a} = \boldsymbol{\alpha}_{j,aq}, \quad i \neq j.$$

If $\mathfrak{g}$ is of type B_2 let α_1 be the long root and α_2 the short root, this time we get

$$\tilde{T}_2\boldsymbol{\alpha}_{1,a} = \boldsymbol{\alpha}_{1,a}\boldsymbol{\alpha}_{2,aq}\boldsymbol{\alpha}_{2,aq^3}, \quad \tilde{T}_1\tilde{T}_2\boldsymbol{\alpha}_{1,a} = \boldsymbol{\alpha}_{1,aq^2}\boldsymbol{\alpha}_{2,aq}\boldsymbol{\alpha}_{2,aq^3}, \quad \tilde{T}_2\tilde{T}_1\tilde{T}_2\boldsymbol{\alpha}_{1,a} = \boldsymbol{\alpha}_{1,aq^2},$$
$$\tilde{T}_1\boldsymbol{\alpha}_{2,a} = \boldsymbol{\alpha}_{2,a}\boldsymbol{\alpha}_{1,aq}, \quad \tilde{T}_2\tilde{T}_1\boldsymbol{\alpha}_{2,a} = \boldsymbol{\alpha}_{2,aq^4}\boldsymbol{\alpha}_{1,aq}, \quad \tilde{T}_1\tilde{T}_2\tilde{T}_1\boldsymbol{\alpha}_{2,a} = \boldsymbol{\alpha}_{2,aq^4}.$$

The case of G_2 is similar and we leave the calculation to the reader. This proves the proposition when $\mathfrak{g}$ is of rank two. For the general case, we proceed by induction on the length $\mathbf{l}(w)$ of w. If $\mathbf{l}(w) = 1$ then $w = s_j$ for some $j \neq i$ and (2.6) shows that induction begins. Let $w = s_{i_1}\cdots s_{i_r}$ be a reduced expression for w and assume that $w\alpha_i \in R^+$. If $a_{i_r,i} = 0$ then

$$\tilde{T}_w\boldsymbol{\alpha}_{i,a} = \tilde{T}_{ws_{i_r}}\boldsymbol{\alpha}_{i,a},$$

and since $\mathbf{l}(ws_{i_r}) < \mathbf{l}(w)$ we are done by the induction hypothesis. Otherwise we can write $w = w_1w_2$ where w_2 is in the group generated by s_{i_r}, s_i and $w_2\alpha_i,\ w_1\alpha_i,\ w_1\alpha_{i_r} \in R^+$. By the rank two case, we know that $\tilde{T}_{w_2}\boldsymbol{\alpha}_{i,a}$ is in the monoid generated by $\boldsymbol{\alpha}_{i,c}$, $\boldsymbol{\alpha}_{i_r,d}$ for $c, d \in \mathbf{C}^\times$. Since $w_1\alpha_i,\ w_1\alpha_{i_r} \in R^+$, it follows again by induction that $\tilde{T}_{w_1}\tilde{T}_{w_2}\boldsymbol{\alpha}_{i,a} \in \mathcal{Q}_q^+$ and the proof of the inductive step is complete.

□

Corollary. *If $\boldsymbol{\pi} \in \mathcal{P}^+$ and $w \in W$, then $\tilde{T}_w\boldsymbol{\pi} \preccurlyeq \boldsymbol{\pi}$.*

Proof. It suffices to prove the corollary when $\boldsymbol{\pi} = \boldsymbol{\pi}_{i,a}$ for some $i \in I$, $a \in \mathbf{C}^\times$. We proceed by induction on $\mathbf{l}(w)$. If $w = s_i$ then $\tilde{T}_i(\boldsymbol{\pi}_{i,a}) = \boldsymbol{\pi}_{i,a}\boldsymbol{\alpha}_{i,a}^{-1}$ by definition and if $w = s_j$, then $\tilde{T}_j(\boldsymbol{\pi}_{i,a}) = \boldsymbol{\pi}_{i,a}$. If $\mathbf{l}(w) > 1$ write $w = w's_{i_r}$ for some w' with $\mathbf{l}(w') < \mathbf{l}(w)$. If $i_r \neq i$, then $\tilde{T}_w\boldsymbol{\pi}_{i,a} = \tilde{T}_{w'}\boldsymbol{\pi}_{i,a}$ and the result follows by induction. If $i_r = i$ then $w'\alpha_i \in R^+$ and we have

$$\tilde{T}_w\boldsymbol{\pi}_{i,a} = \tilde{T}_{w'}\boldsymbol{\pi}_{i,a}(\tilde{T}_{w'}\boldsymbol{\alpha}_{i,a})^{-1},$$

and the result follows from the inductive hypothesis and the proposition. □

3. The category $\widehat{\mathcal{F}}_q(\mathfrak{g})$: simple modules and Weyl modules

Let $\widehat{\mathcal{F}}_q(\mathfrak{g})$ be the category whose objects are finite–dimensional $\widehat{\mathbf{U}}_q(\mathfrak{g})$-modules V satisfying,

$$V = \oplus_{\mu\in P}V_\mu, \quad V_\mu = \{v \in V : k_i^{\pm 1}v = q_i^{\pm\mu(i)}v, \quad i \in \hat{I}\}, \quad \mu = \sum_{i\in I}\mu(i)\omega_i,$$

and where the morphisms between two objects are maps of $\widehat{\mathbf{U}}_q(\mathfrak{g})$–modules. Since $\widehat{\mathbf{U}}_q(\mathfrak{g})$ is a Hopf algebra the category $\widehat{\mathcal{F}}_q(\mathfrak{g})$ is closed under taking tensor products and duals and we let $\widehat{\operatorname{Rep}}(\mathfrak{g})$ be the corresponding Grothendieck ring. This category is far from semi-simple and a parametrization of the blocks of the category can be found in [CM1].

In this section we shall see that given $\boldsymbol{\pi} \in \mathcal{P}^+$ we can associate to it canonically two modules: $V(\boldsymbol{\pi})$ which is simple and $W(\boldsymbol{\pi})$ which has nice universal properties and see that $\mathcal{P}^+$ parametrizes the simple objects in $\widehat{\mathcal{F}}_q(\mathfrak{g})$. These results can be found in [CP2, CP3, CP14, CP13]. We include a sketch of a proof in some cases for the readers convenience and for motivating some of the later results.

We shall also begin the discussion the q–character of objects in $\widehat{\mathcal{F}}_q(\mathfrak{g})$. These were originally defined and studied in a slightly different formulation in [FR]. For our purposes, we define it as follows. It is not hard to see [CP2], that the element $C-1$ of $\widehat{\mathbf{U}}_q(\mathfrak{g})$ acts trivially on an object of $\widehat{\mathbf{U}}_q(\mathfrak{g})$ and hence the elements $k_i^{\pm 1}$, $P_{i,r}$, $i \in I$, $r \in \mathbf{Z}$ act as a family of commuting operators on V. Hence we can write V as a direct of generalized eigenspaces for their action, i.e we have

$$V = \oplus_{\mathbf{d}} V_{\mathbf{d}}, \quad \mathbf{d} = (d_{i,r})_{i \in I, r \in \mathbf{Z}}, \quad d_{i,r} \in \mathbf{C}, \quad ,$$

where

$$V_{\mathbf{d}} = \{v \in V : k_i^{\pm 1} v = q^{\pm d_{i,0}} v, \quad (P_{i,r} - d_{i,r})^{N_{i,r}} v = 0, \text{ for some } N_{i,r} \in \mathbf{N}\}.$$

The formal sum

$$\mathrm{ch}_q(V) = \sum_{\mathbf{d}} \dim V_{\mathbf{d}} \mathbf{d}, \tag{3.1}$$

is called the q–character of V.

3.1. We shall assume that the reader is aware that the category $\mathcal{F}_q(\mathfrak{g})$ of (type 1) finite–dimensional representations of $\mathbf{U}_q(\mathfrak{g})$ is essentially the same as the corresponding category for the simple Lie algebra as long as q is not a root of unity. Thus, the category is semi-simple, the irreducible representations are parametrized by dominant integral weights and given $\lambda = \sum_{i \in I} \lambda(i)\omega_i \in P^+$, we let $V(\lambda)$ be the irreducible finite–dimensional module generated by an element v_λ, with defining relations:

$$k_i^{\pm 1} v_\lambda = q^{\pm \lambda(i)} v_\lambda, \quad x_i^+ v_\lambda = 0, \quad (x_i^-)^{\lambda(i)+1} v_\lambda = 0, \quad i \in I.$$

In particular any finite–dimensional representation V of $\mathbf{U}_q(\mathfrak{g})$ can be written as

$$V = \oplus_{\mu \in P} V_\mu, \quad V_\mu = \{v \in V : k_i^{\pm 1} v = q^{\pm \mu(i)} v\}$$

and we let

$$\mathrm{ch}(V) = \sum_{\mu \in P} \dim V_\mu e(\mu),$$

be the element of the group ring $\mathbf{Z}[P]$. Set

$$\mathrm{wt}(V) = \{\mu \in P : V_\mu \neq 0\}.$$

Any object of $\mathcal{F}_q(\mathfrak{g})$ is completely determined, up to isomorphism, by its character. Details of all these facts can be found in any of the standard books on quantum groups (for example [CP7]). Since any object V of $\widehat{\mathcal{F}}_q(\mathfrak{g})$ can also be regarded as an object in $\mathcal{F}_q(\mathfrak{g})$ the set $\mathrm{wt}(V)$ is defined and we see immediately, that $\mathrm{ch}_q(V)$ is much finer than the character $\mathrm{ch}(V)$.

3.2. We introduce the notion of a highest weight module adapted to the triangular decomposition given in (2.1).

Definition. We say that a $\widehat{\mathbf{U}}_q(\mathfrak{g})$-module V is ℓ-highest weight with highest weight vector v and highest weight $\mathbf{d} = \{d_{i,r} \in \mathbf{C} : i \in I,\ r \in \mathbf{Z}\}$ if $V = \widehat{\mathbf{U}}_q(\mathfrak{g})v$ and

$$x_{i,r}^{+}v = 0, \quad k_i^{\pm 1}v = q_i^{\pm d_{i,0}}v, \quad P_{i,r}v = d_{i,r}v, \qquad r \in \mathbf{Z}, r \neq 0.$$

We begin with the following result which is an immediate consequence of Proposition 2.6.

Lemma. *Let V_p be ℓ–highest weights $\mathbf{d}^p$ and highest weight vectors v_p, $p = 1, 2$. The $\widehat{\mathbf{U}}_q(\mathfrak{g})$ submodule of $V_1 \otimes V_2$ generated by $v_1 \otimes v_2$ is also ℓ–highest weight, with highest weight vector $v_1 \otimes v_2$ and ℓ–highest weight $\mathbf{d}$ given by*

$$d_{i,\pm r} = \sum_{s=0}^{r} d_{i,\pm s}^{1} d_{i,\pm(r-s)}^{2}, \quad i \in I, r \in \mathbf{Z}_{+}.$$

More generally, if $V_p \in \widehat{\mathcal{F}}_q(\mathfrak{g})$ then the q-character of $V_1 \otimes V_2$ is the product of the q–characters of V_1 and V_2. □

3.3. The next result gives a necessary condition for an ℓ-highest weight module to be finite–dimensional.

Lemma. *Let V be an ℓ–highest weight module with highest weight $\{d_{i,r} : i \in I,\ r \in \mathbf{Z}\}$ and highest weight vector v. Then $\dim(V) < \infty$ only if for all $i \in I$, we have:*

$$d_{i,0} = S_i \in \mathbf{Z}_{+}, \qquad d_{i,r} = 0, \quad |r| > S_i, \qquad d_{i,\pm S_i} \neq 0, \tag{3.2}$$

$$1 + \sum_{r \geq 1} d_{i,-r}u^r = u^{S_i} + d_{i,S_i}^{-1} \sum_{r \geq 1} d_{i,r}u^{S_i - r}. \tag{3.3}$$

In other words, if we set $\pi_i = \sum_{r \geq 0} d_{i,r}u^r$ and $\boldsymbol{\pi} = (\pi_1, \cdots, \pi_n)$, then $\boldsymbol{\pi} \in \mathcal{P}^{+}$ and the highest weight of V is given by $\boldsymbol{\pi}$, in the sense that

$$\sum_{r \geq 0} d_{i,\pm r}u^r = \boldsymbol{\pi}^{\pm}(u), \quad S_i = \deg \pi_i.$$

Proof. For $i \in I$, $r \in \mathbf{Z}$, regard V as a module for the subalgebra $\mathbf{U}_{i,r}$ which we recall, is isomorphic to $\mathbf{U}_{q^i}(\mathfrak{sl}_2)$. Since V is finite–dimensional there exists $S_i \in \mathbf{Z}_{+}$ minimal such that $(x_{i,r}^{-})^{S_i+1}v = 0$. It is now immediate from the representation theory of $\mathbf{U}_q(\mathfrak{sl}_2)$ that $S_i + 1 = d_{i,0} + 1$ and hence $d_{i,0} = S_i$. To prove that $d_{i,r} = 0$ if $|r| > S_i$ we use Proposition 2.5(iii) by noting that the second term on the right hand side is zero on v and the left hand side is zero on v if $r > S_i$ by the preceding discussion. Finally (3.3) is obtained by using Proposition 2.5(i). □

3.4. We now turn our attention to the converse problem. Given $\boldsymbol{\pi} \in \mathcal{P}^{+}$, the Weyl module $W(\boldsymbol{\pi})$ is defined in [CP14] as the $\widehat{\mathbf{U}}_q(\mathfrak{g})$–module generated by an element $w_{\boldsymbol{\pi}}$ with relations:

$$x_{i,r}^{+}w_{\boldsymbol{\pi}} = 0, \ k_i^{\pm 1}w_{\boldsymbol{\pi}} = q^{\pm \deg \pi_i}w_{\boldsymbol{\pi}}, \qquad \mathbf{P}_i^{\pm}(u)w_{\boldsymbol{\pi}} = \pi_i^{\pm}(u)w_{\boldsymbol{\pi}}, \quad (x_{i,0}^{-})^{\deg(\pi_i)+1}w_{\boldsymbol{\pi}} = 0,$$

for all $i \in I$ and $r \in \mathbf{Z}$. The following theorem was proved in [CP14], although part (iii) of the theorem was proved much earlier in [CP2, CP3]. Note that part (ii) shows that Weyl modules are universal ℓ–highest weight modules in $\widehat{\mathcal{F}}_q(\mathfrak{g})$.

Theorem 1. (i) For $\boldsymbol{\pi} \in \mathcal{P}^+$ we have $W(\boldsymbol{\pi}) \in \operatorname{Ob} \mathcal{F}_q(L(\mathfrak{g}))$. Moreover,

$$\dim(W(\boldsymbol{\pi}))_{\operatorname{wt} \boldsymbol{\pi}} = 1, \quad \operatorname{wt}(W(\boldsymbol{\pi})) \subset \operatorname{wt} \boldsymbol{\pi} - Q^+.$$

In particular, $W(\boldsymbol{\pi})$ is indecomposable and has a unique irreducible quotient $V(\boldsymbol{\pi})$.

(ii) Any ℓ–highest weight module in $\widehat{\mathcal{F}}_q(\mathfrak{g})$ is a quotient of $W(\boldsymbol{\pi})$ for some $\boldsymbol{\pi} \in \mathcal{P}^+$.

(iii) If $V \in \operatorname{Ob} \widehat{\mathcal{F}}_q(\mathfrak{g}))$ and V is simple, then $V \cong V(\boldsymbol{\pi})$ for some $\boldsymbol{\pi} \in \mathcal{P}^+$.

□

Remark. Lemma 3.2 and the preceding theorem makes clear the motivation for defining the monoid $\mathcal{P}^+$, it is the analog of the fact that the monoid P^+ parametrizes the irreducible finite–dimensional representations of $\mathfrak{g}$.

Definition. For $i \in I$, $a \in \mathbf{C}^\times$, the module $V(\boldsymbol{\pi}_{i,a})$ is called the i^{th}–fundamental module with parameter a.

The following is a consequence of the Theorem and Lemma 3.2 and justifies the definition of the fundamental module. It is analogous to the result that any object in $\mathcal{F}_q(\mathfrak{g})$ occurs in the tensor product of the fundamental representations $V(\omega_i)$, $i \in I$.

Corollary. *Let $\boldsymbol{\pi} = \prod_{i \in I} \prod_{s=1}^{\deg \pi_i} \boldsymbol{\pi}_{i,b_{i,s}} \in \mathcal{P}^+$. The module $V(\boldsymbol{\pi})$ is a subquotient of the tensor product (in arbitrary order) of the modules $V(\boldsymbol{\pi}_{i,b_{i,s}})$ $i \in I$, $b_{i,s} \in \mathbf{C}^\times$, $1 \le s \le \deg \pi_i$.*

4. The category $\widehat{\mathcal{F}}_q(\mathfrak{sl}_2)$ and q–characters as elements of $\mathbf{Z}[\mathcal{P}]$

In this section, we shall see that the formal sum

$$\operatorname{ch}_q(V) = \sum_{\mathbf{d}} \dim V_{\mathbf{d}} \mathbf{d}, \tag{4.1}$$

can be regarded as an element of the integral group ring of $\mathcal{P}$ and explain how this is related to the original formulation of q–characters in [FR]. This requires an understanding of the simple objects of $\widehat{\mathcal{F}}_q(\mathfrak{sl}_2)$ and we also discuss the q–characters of the irreducible modules and the Weyl modules in the $\mathfrak{sl}_2$–case.

4.1. It is convenient at this point to introduce some definitions. Given $V \in \operatorname{Ob} \widehat{\mathcal{F}_q}(\mathfrak{g})$ and $\boldsymbol{\varpi} \in \mathcal{P}$ recall the elements $\boldsymbol{\varpi}^\pm = (\varpi_1^\pm, \cdots \varpi_n^\pm)$ defined in (2.5) and set

$$V_{\boldsymbol{\varpi}} = V_{\{\varpi_{i,r}^\pm : i \in I, r \in \mathbf{Z}\}}, \quad \varpi_i^\pm(u) = \sum_{r \ge 0} \varpi_{i,r}^\pm u^r,$$

Note that since $\dim W(\boldsymbol{\pi})_{\operatorname{wt} \boldsymbol{\pi}} = 1 = \dim V(\boldsymbol{\pi})_{\operatorname{wt} \boldsymbol{\pi}}$, it follows also that $\dim W(\boldsymbol{\pi})_{\boldsymbol{\pi}} = 1 = \dim V(\boldsymbol{\pi})_{\boldsymbol{\pi}}$.

Definition. We say that $V \in \operatorname{Ob}\widehat{\mathcal{F}_q}(\mathfrak{g})$ is *minuscule* if there exists exactly one element $\boldsymbol{\pi} \in \mathcal{P}^+$ such that $\dim V_{\boldsymbol{\pi}} \neq 0$. We say that $V \in \operatorname{Ob}\widehat{\mathcal{F}_q}(\mathfrak{g})$ is *quasi–minuscule* if for all $\boldsymbol{\pi} \in \mathcal{P}$ we have $\dim V_{\boldsymbol{\pi}} \leq 1$. Finally we say that $V \in \operatorname{Ob}\widehat{\mathcal{F}_q}(\mathfrak{g})$ is *prime* iff V cannot be written as a tensor product of nontrivial objects of $\widehat{\mathcal{F}_q}(\mathfrak{g})$.

By Theorem 1, it is clear that given an $V \in \operatorname{Ob}\widehat{\mathcal{F}_q}(\mathfrak{g})$ there must exist $\boldsymbol{\pi} \in \mathcal{P}^+$ such that $V_{\boldsymbol{\pi}} \neq 0$. If V is minuscule it follows that any irreducible constituent of V must be isomorphic to $V(\boldsymbol{\pi})$ and in particular, if $\dim V_{\boldsymbol{\pi}} = 1$, then $V \cong V(\boldsymbol{\pi})$. In the literature, so far simple minuscule modules are called special [Nak4] and quasi–minuscule modules are called thin [H7]. We believe that our notation is more consistent with the representation theory of simple Lie algebras, where a minuscule representation is one with a unique dominant integral weight and a quasi–minuscule is one where all weight spaces are of dimension at most one. In fact it is quite easy to see that if $V \in \operatorname{Ob}\widehat{\mathcal{F}}_q(\mathfrak{g})$ is minuscule (resp. quasi–minuscule) as an object of $\mathcal{F}_q(\mathfrak{g})$ then it is minuscule (resp. quasi-minuscule) as an object of $\widehat{\mathcal{F}}_q(\mathfrak{g})$. The converse is far from true as will be clear from the examples of minuscule and quasi-minuscule representations given in the rest of the paper. Finally, note that if $V \in \operatorname{Ob}\widehat{\mathcal{F}}_q(\mathfrak{g})$ is irreducible when regarded as an object of $\mathcal{F}_q(\mathfrak{g})$ then V is prime. The modules $V(\boldsymbol{\pi}_{i,a})$ for $i \in I$ and $a \in \mathbf{C}^\times$ are clearly prime but as we shall see in general not irreducible as an object of $\mathcal{F}_q(\mathfrak{g})$.

4.2. As in the case of the representation theory of simple Lie algebras it is crucial to first understand the case when $\mathfrak{g}$ is of type $\mathfrak{s}l_2$. In this case an element of $\mathcal{P}^+$ is just a single polynomial π in u with constant term one. Our goal is to describe the structure of $V(\pi)$ and $W(\pi)$.

Given $a \in \mathbf{C}^\times$ and $m \in \mathbf{Z}_+$, set

$$\pi_a^m(u) = \prod_{j=1}^{m}(1 - q^{m-2j+1}au), \quad m \geq 1, \quad \pi_a^0 = 1.$$

It is proved in [CP2] that $V(\pi_a^m)$ is irreducible when regarded as module for $\mathbf{U}_q(\mathfrak{sl}_2)$ and we have

$$\dim V(\pi_a^m) = m+1, \quad \operatorname{wt}(V(\pi_a^m)) = \{m-2j : 0 \leq j \leq m\}.$$

Moreover, if $v \in V(\pi_a^m)_{m-2j}$, we have an equality of power series [CM1],

$$P_1^{\pm} v = (\pi_{aq^{-j}}^{m-j})^{\pm}((\pi_{aq^{m-j+2}}^{j})^{\pm})^{-1} v = \pi_a^m(\boldsymbol{\alpha}_{aq^{m-1}} \cdots \boldsymbol{\alpha}_{aq^{m-2j+1}})^{-1} v, \tag{4.2}$$

where $P_1^{\pm}$ are the elements defined in (2.2). The first equality in the preceding formula appears in a different form in [CP2], while the second appears in [FR] using the elements $\psi_{i,\pm r}$ and $A_{i,a}^{-1}$.

In particular we see that $V(\pi_a^m)$ is minuscule, quasi-minuscule and prime. Moreover we have,

$$\mathbf{d} = \{d_r\}_{r \in \mathbf{Z}}, \; V(\pi_a^m)_{\mathbf{d}} \neq 0, \quad \Longrightarrow \sum_{r \geq 1} d_{\pm r} u^r = \varpi^{\pm}(u), \quad \text{for some } \; \varpi \in \mathcal{P}.$$

Suppose now that $\pi \in \mathcal{P}^+$. It is easy to see that there exists:

(i) a unique partition of $\deg \pi = (m_1, m_2, \cdots, m_s)$ $(m_1 \geq m_2 \geq \cdots \geq m_s \geq 1)$,

(ii) and unique elements $a_1, \cdots, a_s \in \mathbf{C}^\times$ satisfying

$$a_k/a_p \notin \{q^{\pm(m_k+m_p)}, q^{\pm(m_k+m_p-2)}, \cdots, q^{\pm(m_k-m_p+2)}\}, \quad p > k,$$

such that $\pi = \prod_{k=1}^s \pi_{m_k}^{a_k}$. We call this the q–factorization of π and the elements $\pi_{m_k}^{a_k}$ the q–factors of π. The following theorem was proved in [CP2].

Theorem 2. Let $\boldsymbol{\pi} \in \mathcal{P}^+$ and assume that $\boldsymbol{\pi} = \prod_{k=1}^s \pi_{a_k}^{m_k}$ is a q–factorization of π. Then

$$V(\boldsymbol{\pi}) \cong V(\pi_{a_1}^{m_1}) \otimes \cdots \otimes V(\pi_{a_s}^{m_s}).$$

Corollary. *Let $\boldsymbol{\pi} \in \mathcal{P}^+$ and assume that $\boldsymbol{\pi} = \prod_{k=1}^s \pi_{a_k}^{m_k}$ is a q–factorization of π. Then, $V(\boldsymbol{\pi})$ is minuscule iff for $1 \le j \ne k \le m$, we have that,*

$$a_k \notin \{a_j q^{m_j - m_k - 2}, \cdots, a_j q^{-m_j - m_k + 2}\},$$

and it is quasi-minuscule iff $V(\boldsymbol{\pi})$ is minuscule and for $j \ne k$,

$$a_k \ne a_j q^{m_j - m_k}.$$

Finally the irreducible prime objects in $\widehat{\mathcal{F}}_q(\mathfrak{g})$ are the $V(\pi_a^m)$, $m \in \mathbf{Z}_+$, $a \in \mathbf{C}^\times$.

Proof. The first statement is proved in [FR, Lemma 4] while the last is a direct consequence of Theorem 2. For the second, suppose that $a_k q^{m_k} \in \{a_j q^{m_j}, \cdots, a_j q^{-m_j+2}\}$, say $a_k q^{m_k - 1} = a_j q^{m_j - 1 - 2r}$ where $r \le m_j - 1$. So $\boldsymbol{\pi}\boldsymbol{\alpha}^{-1}_{a_j q^{m_j}} \boldsymbol{\alpha}^{-1}_{a_j q^{m_j - 2}} \cdots \boldsymbol{\alpha}^{-1}_{a_j q^{m_j - 2r}}$ has multiplicity at least 2 in $\mathrm{ch}_q(V(\boldsymbol{\pi}))$ and so $V(\boldsymbol{\pi})$ is not quasi-minuscule.

Otherwise for $j \ne k$, we have $\{a_k q^{m_k - 1}, \cdots a_k q^{1 - m_k}\} \cap \{a_j q^{m_j - 1}, \cdots, a_j q^{1 - m_j}\} = \emptyset$ and the quasi-minuscule property follows. □

4.3. For general $\mathfrak{g}$, we have the following consequence of Theorem 2 which first appeared in [FR].

Proposition. *Let $\mathfrak{g}$ be simple and $V \in \widehat{\mathcal{F}}_q(\mathfrak{g})$. Then $V = \oplus_{\varpi \in \mathcal{P}} V_\varpi$. In particular,* $\mathrm{ch_q}(V) \in \mathbf{Z}[\mathcal{P}]$ *and we have a ring homomorphism* $\mathrm{ch_q} : \widehat{Rep}(\mathfrak{g}) \to \mathbf{Z}[\mathcal{P}]$.

Proof. In the case when $\mathfrak{g}$ is $\mathfrak{sl}_2$ and V is irreducible the Proposition is immediate from Theorem 2. If V is reducible then the result follows again since the eigenspaces for the action of $P_{1,r}$ on V are just the sum of the eigenspaces coming from a Jordan–Holder series for V. For arbitrary $\mathfrak{g}$, the result follows by regarding V as a module for $\widehat{\mathbf{U}}_i$, $i \in I$. The fact that ch_q is a ring homomorphism is immediate from Lemma 3.2. □

Remark. We now explain the connection with original formulation of [FR]. Let $\mathcal{Y}$ be the polynomial ring over the integers in the infinitely many variables $Y_{i,a}^{\pm 1}$, $i \in I$, $a \in \mathbf{C}^\times$ The assignment

$$\boldsymbol{\pi} = \prod_{i \in I, a \in \mathbf{C}^*} \boldsymbol{\pi}_{i,a}^{r_{i,a}} \mapsto m = \prod_{i \in I, a \in \mathbf{C}^*} Y_{i,a}^{r_{i,a}},$$

is clearly an isomorphism of rings and hence $\mathrm{ch}_q(V)$ can also be regarded as an element of $\mathcal{Y}$, For some applications the monomial notation is more convenient, and in the following we shall use both notations without comment. Any object V in $\widehat{\mathcal{F}}_q(\mathfrak{g})$ is obviously a finite–dimensional module for $\mathbf{U}_q(\mathfrak{g})$ and it follows from the fact that $\mathrm{ch}_q(V) \in \mathbf{Z}[\mathcal{P}]$ that $\mathrm{ch}(V) = \mathrm{wt}(\mathrm{ch}_q(V))$, or equivalently [FR], in the monomial notation one replaces $Y_{i,a}$ by $y_i = e(\omega_i)$ to get the usual character.

4.4. We go back to the case $\mathfrak{g}$ of type $\mathfrak{sl}_2$ and turn to understanding the Weyl modules. It was proved in [CP13] that if $\pi = \prod_{r=1}^{s} \pi_{a_r}^1$, and $a_r/a_k \neq q^{-2}$ if $k > r$, then

$$W(\pi) \cong V(\pi_{a_1}^1) \otimes \cdots \otimes V(\pi_{a_r}^1). \tag{4.3}$$

We now compute the q–character of $W(\pi)$. Define a map

$$\mathcal{P}^+(\pi) = \{\pi' \in \mathcal{P}^+ : \pi(\pi')^{-1} \in \mathcal{P}^+\} \to \mathcal{P}, \quad \pi' \mapsto \pi'(u)\pi'(q^2u)(\pi(q^2u))^{-1}.$$

We claim that the map is injective. For, if

$$\pi'(u)\pi'(q^2u) = \eta'(u)\eta'(q^2u),$$

then choose $a \in \mathbf{C}^\times$ so that $(1 - au)$ divides the left hand side and $(1 - aq^k u)$ does not divide it for any $k < 0$. It is immediate that $(1 - au)$ divides $\pi'(u)$ and similarly must divide $\eta'(u)$. An obvious induction on $\deg \pi'$ now proves that $\pi' = \eta'$. Given, $\pi \in \mathcal{P}^+$ and $\pi' \in \mathcal{P}^+(\pi)$, write

$$\pi = \prod_{j=1}^{k}(1 - a_j u)^{r_j}, \quad \pi' = \prod_{j=1}^{k}(1 - a_j u)^{S_j},$$

where $a_j \in \mathbf{C}^\times$, $1 \le j \le k$ are distinct, and set

$$d_\pi(\pi') = \binom{r_1}{S_1}\binom{r_2}{S_2}\cdots\binom{r_k}{S_k}.$$

Using (4.2), (4.3) and the multiplicative property of ch_q, we see that

$$\{\varpi \in \mathcal{P} : W(\pi)_\varpi \neq 0\} = \{\pi'(u)\pi'(q^2u)(\pi(q^2u))^{-1} : \pi' \in \mathcal{P}^+(\pi)\},$$

and we have proved the following:

Proposition. *Let $\pi \in \mathcal{P}^+$. We have*

$$ch_q(W(\pi)) = \sum_{\pi'|\pi} d_\pi(\pi')\pi'(u)\pi'(q^2u)(\pi(q^2u))^{-1}.$$

In particular, $W(\pi)$ is quasi–minuscule iff π has distinct roots.

5. Irreducibility of Tensor products and q–characters of Weyl modules

In this section, we give a partial analog of Theorem 2 for general $\mathfrak{g}$. Thus we give a sufficient condition [Cha4] for a tensor product of two irreducible objects of $\widehat{\mathcal{F}}_q(\mathfrak{g})$ to be irreducible and hence also a necessary condition for a representation to be prime. We shall then see that this result along with results in [Kas, Nak1, BN] can be used to substantially strengthen Corollary 3.4: namely that any ℓ–highest weight module is actually a quotient of an ordered tensor product of fundamental modules. This is essentially equivalent (by the universal property of Weyl modules) to proving that for all $\boldsymbol{\pi} \in \mathcal{P}^+$ the module $W(\boldsymbol{\pi})$ is isomorphic to an ordered tensor product of fundamental modules. Finally, we discuss q–characters of the fundamental representations and hence also the q–characters of Weyl modules.

5.1. We shall say that two polynomials π and π' with constant term one are in general position if the q–factorization of $\pi\pi'$ is just the product of the q–factorization of π and π'. We say that π is in general position with respect to π' if for every q–factor π_a^m of π and π_b^r of π' we have

$$a \neq bq^{-(m+r-2p)}, \quad 0 \leq p < \min(m,r).$$

The following was proved in [Cha4]. It was motivated by the following classical result: if V is any finite–dimensional $\mathfrak{g}$–module and $w \in W$, then $\dim V_\mu = \dim V_{w\mu}$ for all $\mu \in P$. In the case of the quantum affine algebra it was natural to ask if something similar was true for $\mathrm{wt}_\ell(V)$. This motivated the definition of the braid group action on $\mathcal{P}$ (in the quantum case, it is quite natural to replace the Weyl group by the braid group). Part (i) of the theorem shows that the analog of the classical result works for the highest weight $\boldsymbol{\pi}$; the discussion in the $\mathfrak{s}l_2$ case shows that this is false for an arbitrary weight. However, even this partial information is enough to find conditions for a tensor product of irreducible representations to be irreducible.

Theorem 3. Let $\boldsymbol{\pi}, \boldsymbol{\pi}' \in \mathcal{P}^+$. For all $w \in W$ we have

$$\dim V(\boldsymbol{\pi})_{T_w\boldsymbol{\pi}} = \dim V(\boldsymbol{\pi})_{\mathrm{wt}(T_w\boldsymbol{\pi})} = 1. \tag{5.1}$$

Further, $V(\boldsymbol{\pi}) \otimes V(\boldsymbol{\pi}')$ is an ℓ–highest weight module (resp. irreducible) if given a reduced expression $w_0 = s_{i_1} \cdots s_{i_N}$ of the longest element in W, we have

- (i) the elements π_{i_N} is in general position with respect to π'_{i_N} (resp. π_{i_N} and π'_{i_N} are in general position),
- (ii) For $2 \leq j \leq N$, the elements $(\tilde{T}_{i_j} \cdots \tilde{T}_{i_N}\boldsymbol{\pi})_{i_{j-1}}$ is in general position with respect to $\pi'_{i_{j-1}}$ (resp. $(\tilde{T}_{i_j} \cdots \tilde{T}_{i_N}\boldsymbol{\pi})_{i_{j-1}}$ and $\pi'_{i_{j-1}}$ are in general position).

Proof. The idea of the proof is as follows. In the case when $\mathfrak{g}$ is of type $\mathfrak{sl}_2$, this was done in [CP2]. Then one observes that for all $w \in W$, the weight space $\dim V(\boldsymbol{\pi})_{w\,\mathrm{wt}\,\boldsymbol{\pi}} = 1$ and hence is an eigenspace for the action of the elements $P_{i,r}$, $i \in I$, $r \in \mathbf{Z}_+$. The corresponding eigenvalue is $\tilde{T}_w(\boldsymbol{\pi})$. Given a reduced expression $s_{i_1} \cdots s_{i_N}$ of w_0, set $v_N = v_{\boldsymbol{\pi}}$ and fix non–zero elements $v_j \in V(\boldsymbol{\pi})_{s_{i_{j+1}} \cdots s_{i_N}\,\mathrm{wt}\,\boldsymbol{\pi}}$ for $1 \leq j \leq N-1$. It is easy to see that $x_{j,r}^+ v_j = 0$ for all $r \in \mathbf{Z}_+$. We proceed by induction to show that $v_j \otimes v_{\boldsymbol{\pi}'} \in \widehat{\mathbf{U}}_q(\mathfrak{g})(v_{\boldsymbol{\pi}} \otimes v_{\boldsymbol{\pi}'})$. The inductive step follows from the fact that the element $v_j \otimes v_{\boldsymbol{\pi}'}$ is an ℓ–highest weight vector for the subalgebra $\widehat{\mathbf{U}}_i$ (which is isomorphic to $\widehat{\mathbf{U}}_{q^i}(\mathfrak{sl}_2)$) and by using condition (ii) in the statement of the theorem. □

Let $\boldsymbol{\pi} \in \mathcal{P}^+$ and write $\boldsymbol{\pi} = \prod_{i\in I}\prod_{p=1}^{S_i} \boldsymbol{\pi}_{i,a_{i,p}}$ for some $a_{i,p} \in \mathbf{C}^\times$. Fix an ordering $\leq$ of the set $\{a_{i,p} : i \in I, 1 \leq p \leq S_i\}$ so that the polynomial $(1 - a_{i,p}u)$ is in general position with respect to $(1 - a_{j,r}u)$ if $a_{i,p} < a_{j,r}$.

Corollary. *The ordered tensor product $\otimes_{i\in I} \otimes_{p=1}^{S_i} V(\boldsymbol{\pi}_{i,a_{i,p}})$ is ℓ–highest weight and hence a quotient of $W(\boldsymbol{\pi})$. In particular if $a_{i,p} = 1$ for all $i \in I$ and $1 \leq p \leq s$ the tensor product is irreducible.*

The corollary had been proved in [AK] when $\mathfrak{g}$ is of type A_n or C_n, a geometric proof was given for the simply–laced algebras in [VV] and a complete proof was given in [Kas] by using crystal bases. In [Nak1], Nakajima introduced the notion of standard modules for $\widehat{\mathbf{U}}_q(\mathfrak{g})$

through the geometry of quiver varieties and the results of [VV] show in the simply-laced case, that an algebraic definition of the standard module is just the tensor product given in the corollary.

5.2. We can now state the following result which can be viewed as giving a presentation of the standard modules.

Theorem 4. Let $\boldsymbol{\pi} \in \mathcal{P}^+$ and write $\boldsymbol{\pi} = \prod_{i\in I}\prod_{p=1}^{S_i} \boldsymbol{\pi}_{i,a_{i,p}}$ for some $a_{i,p} \in \mathbf{Z}_+$. Fix an ordering $\leq$ of the set $\{a_{i,p} : i \in I, 1 \leq p \leq S_i\}$ so that the polynomial $(1 - a_{i,p}u)$ is in general position with respect to $(1 - a_{j,r})$ if $a_{i,p} < a_{j,r}$. The ordered tensor product $\otimes_{i\in I} \otimes_{p=1}^{S_i} V(\boldsymbol{\pi}_{i,a_{i,p}})$ is isomorphic to $W(\boldsymbol{\pi})$. $\square$

This theorem was conjectured in [CP14]. A proof was given in the case of $\mathfrak{sl}_2$ in [CP13]. For arbitrary simple Lie algebras this follows easily from the results of [Nak1, BN] and a proof can be found in [CM1]. Another approach which is more algebraic and similar to the proofs given for the $\mathfrak{sl}_2$ case can be found for $\mathfrak{s}l_{n+1}$ in [CL] and a proof using Demazure modules can be found in [FL] in the general simply–laced case.

5.3. Given $V \in \operatorname{Ob} \widehat{\mathcal{F}}_q(\mathfrak{g})$, set

$$\operatorname{wt}_\ell(V) = \{\boldsymbol{\varpi} \in \mathcal{P}^+ : V_{\boldsymbol{\varpi}} \neq 0\}.$$

Theorem 5. (i) For $i \in I$ and $a \in \mathbf{C}^\times$, we have $\operatorname{wt}_\ell V(\boldsymbol{\pi}_{i,a}) \subset \boldsymbol{\pi}_{i,a}\mathcal{Q}^-$.
(ii) For $\boldsymbol{\pi} \in \mathcal{P}^+$, we have $\operatorname{wt}_\ell(W(\boldsymbol{\pi})) \subset \boldsymbol{\pi}\mathcal{Q}^-$.
(iii) The representations $V(\boldsymbol{\pi}_{i,a})$, $i \in I$, $a \in \mathbf{C}^\times$ are minuscule.
(iv) For algebras of type A_n, B_n and C_n the representations $V(\boldsymbol{\pi}_{i,a})$ are quasi-minuscule.

The following corollary of the theorem is immediate from the universal property of Weyl modules.

Corollary. *Let $V \in \widehat{\mathcal{F}}_q(\mathfrak{g})$ be an ℓ–highest weight module with ℓ–highest weight $\boldsymbol{\pi}$. Then*

$$ch_q(V) = \sum_{\boldsymbol{\varpi}\in\boldsymbol{\pi}\mathcal{Q}^-} \dim V_{\boldsymbol{\varpi}}\boldsymbol{\varpi}.$$

The first part of the theorem was proved in [FM1] and using the Corollary to Theorem 1 they deduced immediately that if $V(\boldsymbol{\pi})$ is irreducible, then $\operatorname{ch}_q(V(\boldsymbol{\pi})) = \sum_{\boldsymbol{\varpi}\in\boldsymbol{\pi}\mathcal{Q}^-} \dim V(\boldsymbol{\pi})_{\boldsymbol{\varpi}}\boldsymbol{\varpi}$ (a stronger general condition on terms of $\operatorname{ch}_q(V(\boldsymbol{\pi}))$ is proved in [H3]). In fact, a stronger result which implies part (i) is proved in [H6, Theorem 5.21]. In the case of classical Lie algebras, another proof of part (i) and a Weyl–character type formula for the q–character of the fundamental representations is given in [CM1].

Part (ii) of the theorem is now immediate from from Theorem 4. Part (iii) of the theorem was proved in [FM1]. Closed formulae for the characters of fundamental representations for the classical Lie algebras can be found in [FM1, KS, CM2]. The q–character of fundamental representations of exceptional types were computed in [Nak6, H1, H4]. Part (iv) can be established directly from the closed formulae in [KS, CM2], and was first observed (and proved by a different method) in [H4]. The quasi–minuscule fundamental representations for algebras of type D_n can be found in [CM2] while the case of F_4 and G_2 can be found in [H1, H4].

5.4. We now give a proof of parts (i), (iii) and (iv) of the Theorem for algebras of type A_n since it follows easily from the results discussed so far. The elements $\omega_i \in P^+$, $i \in I$ are minuscule i.e there do no exist dominant integral weights $\mu \in P^+$ such that $\omega_i - \mu \in Q^+ \setminus \{0\}$ and $V(\omega_i)$ is a minuscule representation of $\mathbf{U}_q(\mathfrak{g})$. Since $\mathrm{wt}\, V(\boldsymbol{\pi}_{i,a}) \subset \omega_i - Q^+$, by Theorem 1, it follows that $V(\boldsymbol{\pi}_{i,a}) \cong V(\omega_i)$ as $\mathbf{U}_q(\mathfrak{g})$–modules. Hence

$$V(\boldsymbol{\pi}_{i,a})_\mu \neq 0 \iff \mu = w\omega_i, \quad \dim V(\boldsymbol{\pi}_{i,a})_{w\omega_i} = 1, \quad w \in W.$$

Using Theorem 3, we get

$$\dim V(\boldsymbol{\pi}_{i,a})_{w\omega_i} = \dim V(\boldsymbol{\pi}_{i,a})_{\tilde{T}_w \boldsymbol{\pi}_{i,a}},$$

which proves (iv) . By Corollary 2.10, we see that $\tilde{T}_w \boldsymbol{\pi}_{i,a} \in \boldsymbol{\pi}_{i,a}\mathcal{Q}^-$. Let $W_i = \{w \in W : w\omega_i = \omega_i\}$, and let W^i be a set of coset of representatives for W/W_i. Then, we have by Theorem 3 that

$$\mathrm{ch}_q V(\boldsymbol{\pi}_{i,a}) = \sum_{w \in W^i} \tilde{T}_w \boldsymbol{\pi}_{i,a},$$

and the proof of parts (i) and (iv) of the theorem is complete.

5.5. Let us consider the case of $V(\boldsymbol{\pi}_{1,a})$ for A_n. Here, we have

$$\mathrm{wt}_\ell\, V(\boldsymbol{\pi}_{1,a}) = \{(\boldsymbol{\pi}_{i,aq^{i+1}})^{-1} \boldsymbol{\pi}_{i+1,aq^i} : 0 \leq i \leq n\},$$

where we understand that $\boldsymbol{\pi}_{n+1,aq^n} = 1$. The following is analogous to Proposition 4.4 and is proved in a similar fashion.

Proposition. *Suppose that $\mathfrak{g}$ is of type A_n and that $\boldsymbol{\pi} = (\pi, 1, \cdots, 1) \in \mathcal{P}^+$. Then*

$$ch_q(W(\boldsymbol{\pi})) = \sum_{\boldsymbol{\pi}' \in \mathcal{P}^+(\boldsymbol{\pi})} d_{\boldsymbol{\pi}}(\boldsymbol{\pi}')\boldsymbol{\pi}',$$

where

$$\mathcal{P}^+(\boldsymbol{\pi}) = \left\{ (\frac{\pi_1(u)}{\pi_2(q^2u)}, \frac{\pi_2(qu)}{\pi_3(q^3u)}, \cdots, \frac{\pi_n(q^{n-1}u)}{\pi_{n+1}(q^{n+1}u)}) : \pi = \prod_{s=1}^{n+1} \pi_j,\ \pi_j \in \mathbf{C}[u] \right\},$$

and

$$d_{\boldsymbol{\pi}}(\boldsymbol{\pi}') = \prod_{s=1}^{k} \prod_{i=1}^{n+1} \binom{r_s - m_{s,1} - m_{s,2} - \cdots - m_{s,i-1}}{m_{s,i}},$$

where $\pi = \prod_{s=1}^k (1 - a_k u)^{r_k}$, $a_k \neq a_m$ if $k \neq m$ and $\pi_j = \prod_{s=1}^k (1 - a_k u)^{m_{s,j}}$, $1 \leq j \leq n+1$. In particular, $W(\boldsymbol{\pi})$ is quasi–minuscule iff the roots of π are distinct. □

5.6. Suppose now that $\mathfrak{g}$ is a simple Lie algebra and assume that $i \in I$ is such that the coefficient of α_i in the highest root is one. Then it is known that the corresponding $\mathbf{U}_q(\mathfrak{g})$–representation $V(\omega_i)$ is quasi–minuscule and also that ([CP7] for instance) $V(\boldsymbol{\pi}_{i,a}) \cong V(\omega_i)$. In particular, $V(\boldsymbol{\pi}_{i,a})$ is a quasi–minuscule representation of $\widehat{\mathbf{U}}_q(\mathfrak{g})$. The converse is also true for type A_n as we see from the preceding discussion : if a fundamental representation is quasi-minuscule then it is simple as a $\mathbf{U}_q(\mathfrak{g})$-module. For D_n from [CM2] and more generally it has been observed by Nakajima that the converse is also true for all simply–laced algebras. In the non–simply laced case however, the converse need not be true, for instance by Theorem 5 (iv), all the fundamental representations of B_n are quasi–minuscule, but $V(\boldsymbol{\pi}_{i,a})$ is not irreducible as a $\mathbf{U}_q(\mathfrak{g})$–module if $i \neq 1, n$, see [CP7] for instance. Examples for G_2 can be found in [H1].

5.7. As a consequence of Theorem 5(i), more precisely its Corollary 5.3, and by using the grading of q-characters by the usual weight lattice of $\mathfrak{g}$, the q-characters of simple representations are clearly $\mathbf{Z}$-linearly independent (as for usual characters of simple representations of $\mathfrak{g}$). Then it follows [FR] that $\mathrm{ch}_q : \widehat{Rep}(\mathfrak{g}) \to \mathbf{Z}[\mathcal{P}]$ is an injective ring homomorphism.

The image of ch_q has been characterized in [FM1] as the intersection of the kernel of screening operators (this is analogous to the invariance for the Weyl group action of usual characters). Besides an element of $\mathrm{Im}(\mathrm{ch}_q)$ is characterized [FR, FM1] by the multiplicity of the dominant terms $\boldsymbol{\pi} \in \mathcal{P}^+$. In other words, we have $\mathrm{ch}_q(V) = \mathrm{ch}_q(W)$ iff $\dim V_{\boldsymbol{\pi}} = \dim W_{\boldsymbol{\pi}}$ for all $\boldsymbol{\pi} \in \mathcal{P}^+$.

6. Affinizations of λ and the poset $\mathbf{D}_\lambda$

One tool that has been used very effectively for example in [Nak6, H1] in the computation of q–characters of certain families of modules is the Frenkel–Mukhin algorithm [FM1] and we shall describe this in another section. The braid group action on $\mathcal{P}$ and Theorem 3 can be used to compute q–characters although this approach has not as yet been fully explored outside the fundamental modules. As explained in the introduction, in the ADE-case an algorithm is given in [Nak4] which gives the q-character of an arbitrary simple finite dimensional representation. In practice however, it is difficult to write closed formulae for the q-character and the first thing is to identify suitable families of modules for which these methods can be made to work and possibly lead to closed formulae. The fundamental modules and the Weyl modules introduced in the previous section are examples of such families and in this section we shall identify some other natural families of modules which could provide further examples. We shall also be interested in branching rules, i.e the decomposition of an object of $\widehat{\mathcal{F}}_q(\mathfrak{g})$ as a direct of sum of simple objects of $\mathcal{F}_q(\mathfrak{g})$.

6.1. Since any $V \in \widehat{\mathcal{F}}_q(\mathfrak{g})$ is completely reducible as a $\mathbf{U}_q(\mathfrak{g})$–module, we have

$$V \cong_{\mathbf{U}_q(\mathfrak{g})} \bigoplus_{\mu \in P^+} V(\mu)^{\oplus m_\mu(V)}, \quad m_\mu(V) = \dim \mathrm{Hom}_{\mathbf{U}_q(\mathfrak{g})}(V(\mu), V).$$

Set,

$$\mathrm{ch}_{\mathfrak{g}}(V) = \sum_{\mu \in P^+} m_\mu(V) \mathrm{ch}(V(\mu)). \tag{6.1}$$

Clearly knowing either ch or ch_q implies that one knows $\mathrm{ch}_{\mathfrak{g}}$ in principle. In practice however it is very hard to find from this information, a closed formula for $\mathrm{ch}_{\mathfrak{g}}$. This is seen in other situations: for instance in the case of objects of $\mathcal{F}_q(\mathfrak{g})$, the character $\mathrm{ch}(V(\lambda))$ is given by the Weyl character formula and since ch is multiplicative, one knows the character of the tensor product of $V(\lambda) \otimes V(\mu)$. But the understanding of $\dim \mathrm{Hom}_{\mathbf{U}_q(\mathfrak{g})}(V(\nu), V(\lambda) \otimes V(\mu))$ is a hard problem. In this section, we shall see some examples of families of modules where $\mathrm{ch}_{\mathfrak{g}}$ is known.

6.2. We shall need two automorphisms introduced in [CP2, Cha2] of $\widehat{\mathbf{U}}_q(\mathfrak{g})$. Given $a \in \mathbf{C}^\times$, there exists an automorphism $\tau_a : \widehat{\mathbf{U}}_q(\mathfrak{g}) \to \widehat{\mathbf{U}}_q(\mathfrak{g})$ which is defined on generators by

$$\tau_a(x_{i,r}^{\pm}) = a^{\pm r} x_{i,r}^{\pm}, \quad \tau_a(\psi_{i,\pm m}) = a^{\pm r}\psi_{i,\pm m}, \quad \tau_a(k_i^{\pm 1}) = k_i^{\pm 1},$$

for all $i \in I$, $r \in \mathbf{Z}$, $m \in \mathbf{Z}_+$. Let $\sigma : \widehat{\mathbf{U}}_q(\mathfrak{g}) \to \widehat{\mathbf{U}}_q(\mathfrak{g})$ be the involution satisfying,

$$\sigma(x_{i,r}^{\pm}) = x_{i,-r}^{\mp}\ ,\ \sigma(\phi_{i,\pm m}^{\pm}) = \phi_{i,\mp m}^{\mp}\ ,\ \sigma(k_i) = k_i^{-1},$$

for $i \in I$, $r \in \mathbf{Z}$, $m \in \mathbf{Z}_+$. Given $V \in \operatorname{Ob}\widehat{\mathcal{F}}_q(\mathfrak{g})$, let $\tau_a^*(V)$ and $\sigma^*(V)$ be the object obtained by pulling V back via the automorphism τ_a and σ respectively. Define corresponding automorphisms $\tau_a : \mathbf{Z}[\mathcal{P}] \to \mathbf{Z}[\mathcal{P}]$ sending $\boldsymbol{\pi} \to \boldsymbol{\pi}_a$ (resp. $\sigma : \mathbf{Z}[\mathcal{P}] \to \mathbf{Z}[\mathcal{P}]$ sending $\boldsymbol{\pi} \to \boldsymbol{\pi}_\sigma$) by:

$$\tau_a(\boldsymbol{\pi}_{i,b}) = \boldsymbol{\pi}_{i,ab}, \quad i \in I, \quad b \in \mathbf{C}^\times,$$
$$\sigma(\boldsymbol{\pi}_{i,b}) = \boldsymbol{\pi}_{-w_\circ(i),q_i^2 b^{-1}}, \quad i \in I, \quad b \in \mathbf{C}^\times,$$

where we recall that $w_\circ$ is the longest element of the Weyl group and that $-w_\circ$ induces an involution of the Dynkin diagram of $\mathfrak{g}$. The following can be found in [Cha2] and [CP5, Proposition 5.1]

Proposition. *Let $\boldsymbol{\pi} \in \mathcal{P}^+$ and $a \in \mathbf{C}^\times$. Then*

$$\tau_b^* V(\boldsymbol{\pi}) \cong V(\boldsymbol{\pi}_b), \quad \sigma^*(V(\boldsymbol{\pi})) \cong \tau_h^* V(\boldsymbol{\pi}_\sigma),$$

for some $h \in \mathbf{C}^\times$ which is independent of $\boldsymbol{\pi}$.

In fact, h is the dual Coxeter number of $\mathfrak{g}$. The statements of the following corollary were first proved in [FR] and [H8] respectively and are consequence of the fact that for $V \in \widehat{\mathcal{F}}_q(\mathfrak{g})$

$$\dim(\sigma^* V)_{\sigma\boldsymbol{\varpi}} = \dim V_{\boldsymbol{\varpi}} = \dim(\tau_a^* V)_{\tau_a \boldsymbol{\varpi}}.$$

Corollary. *Let $\boldsymbol{\pi} \in \mathcal{P}^+$. Then,*

$$ch_q(\tau_a^* V(\boldsymbol{\pi})) = \tau_a\, ch_q(V(\boldsymbol{\pi})), \quad ch_{\mathrm{q}}(\sigma^* \mathrm{V}(\boldsymbol{\pi})) = \sigma(ch_{\mathrm{q}}(\mathrm{V}(\boldsymbol{\pi})).$$

6.3. Define an equivalence relation $\sim$ on objects of $\widehat{\mathcal{F}_q}(\mathfrak{g})$ by: $V \sim V'$ iff V and V' are isomorphic as $\mathbf{U}_q(\mathfrak{g})$–modules and let $[V]$ denote the equivalence class of V. Note that by Proposition 6.2, we have

$$[V] = [\tau_a^*(V)], \quad a \in \mathbf{C}^\times.$$

It is shown in [Cha2] and can be verified easily, that one has a partial order on the set of equivalence classes given by: $[V] \geq [V']$ iff for all $\nu \in P^+$, either

- $m_\nu(V) \geq m_\nu(V')$, or
- there exists $\mu \in P^+$ with $\mu \geq \nu$ such that $m_\mu(V) > m_\mu(V')$.

For $\lambda \in P^+$, set

$$\mathbf{D}_\lambda = \{[V(\boldsymbol{\pi})] : \operatorname{wt} \boldsymbol{\pi} = \lambda\}.$$

The following is proved in [Cha2], we include a proof here, since it follows easily from the results of the previous sections.

Proposition. *For all $\lambda \in P^+$ the set $\mathbf{D}_\lambda$ is a finite poset.*

Proof. Suppose that $\boldsymbol{\pi} \in \mathcal{P}^+$ and $\operatorname{wt}\boldsymbol{\pi} = \lambda$. Then $V(\boldsymbol{\pi})$ is a quotient of $W(\boldsymbol{\pi})$. Since $W(\boldsymbol{\pi})$ is a finite–dimensional $\mathbf{U}_q(\mathfrak{g})$–module it follows that

$$\mathrm{ch}_{\mathfrak{g}} W(\boldsymbol{\pi}) = \sum_{\nu \in P^+} m_\nu(W(\boldsymbol{\pi}))V(\nu),$$

and hence we have

$$\mathrm{ch}_{\mathfrak{g}} V(\boldsymbol{\pi}) = \sum_{\nu \in P^+} m_\nu(V(\boldsymbol{\pi}))V(\nu), \quad m_\nu(V(\boldsymbol{\pi})) \le n_\nu(V(\boldsymbol{\pi})).$$

The proposition follows since there are only finitely many $\nu \in P^+$ with $m_\nu(W(\boldsymbol{\pi})) \neq 0$. □

6.4. Using Theorem 2, it is easy to describe the poset $\mathbf{D}_\lambda$ in the case when $\mathfrak{g}$ is of type $\mathfrak{sl}_2$. Suppose that $\lambda = s\omega_1$ and let π, π' be such that $\operatorname{wt}\pi = \lambda = \operatorname{wt}\pi'$. Then, it is trivially seen that

$$V(\pi) \cong_{\mathbf{U}_q(\mathfrak{sl}_2)} V(\pi') \iff \pi = \prod_{k=1}^K \pi_{a_k}^{m_k}, \quad \pi' = \prod_{k=1}^K \pi_{b_k}^{m_k},$$

where $\pi = \prod_{k=1}^K \pi_{a_k}^{m_k}$ (resp. $\pi' = \prod_{k=1}^K \pi_{b_k}^{m_k}$), is the q-factorization of π (resp. π'). In other words, the elements of $\mathbf{D}_\lambda$ are just elements of $\mathrm{Par}(s)$, the set partitions of s. Consider the reverse lexicographic ordering on $\mathrm{Par}(s)$, this is the order where $\{s\}$ is the smallest partition and $\{1, \cdots, 1\}$ the maximal element. It is now an elementary exercise in the representation theory of $\mathfrak{sl}_2$ to prove the following.

Lemma. *Assume that $\mathfrak{g}$ is of type $\mathfrak{sl}_2$ and that $\lambda = s\omega_1$. Then $\mathbf{D}_\lambda$ is isomorphic as a poset to the set $\mathrm{Par}(s)$ of partitions of s equipped with the reverse lexicographic order.*

6.5. In the case of an arbitrary simple Lie algebra, very little is known about the poset $\mathbf{D}_\lambda$ except for the minimal and maximal elements of the poset. We first show that the maximal element of $\mathbf{D}_\lambda$ is just the equivalence class of $W(\boldsymbol{\pi})$ for $\boldsymbol{\pi} \in \mathcal{P}^+$ with $\operatorname{wt}\boldsymbol{\pi} = \lambda$. For this, it suffices by Theorem 1 to prove that there exists $\boldsymbol{\pi}_0 \in \mathcal{P}^+$ with $\operatorname{wt}(\boldsymbol{\pi}_0) = \lambda$ such that $W(\boldsymbol{\pi}_0)$ is irreducible. Write $\lambda = \sum_{i \in I} m_i\omega_i$ and set $\boldsymbol{\pi}_0 = \prod_{i \in I}(\boldsymbol{\pi}_{i,1})^{m_i}$. By Corollary 5.1 it follows that $\otimes_{i \in I} V(\boldsymbol{\pi}_{i,1})^{\otimes m_i}$ is irreducible and by Theorem 4 we see that it is isomorphic to $W(\boldsymbol{\pi}_0)$ and we have proved:

Lemma. *For $\lambda \in P^+$, there exists $\boldsymbol{\pi} \in \mathcal{P}^+$ with $\operatorname{wt}\boldsymbol{\pi} = \lambda$ such that $[W(\boldsymbol{\pi})]$ is the unique maximal element of $\mathbf{D}_\lambda$.*

6.6. The minimal elements of the poset $\mathbf{D}_\lambda$ were studied in [Cha2, CP4, CP5, CP6]. Notice first that if $\lambda = \omega_i$ and $a \in \mathbf{C}^\times$, then $\mathbf{D}_{\omega_i} = \{[V(\boldsymbol{\pi}_{i,a})]\}$, and in particular any affinization of ω_i is minimal. For general λ, the picture is more complicated. Assume that the nodes of the Dynkin diagram are numbered as in [B]. For $\lambda \in P^+$, set

$$\operatorname{supp}\lambda = \{i \in I : \lambda(i) > 0\},$$

and let $i_{\min\lambda}$ be the minimal element of $\operatorname{supp}\lambda$.

6.7. If $\mathfrak{g}$ is not of type D or E, then there is a unique minimal element in $\mathbf{D}_\lambda$. Moreover there exist two elements $\boldsymbol{\pi}_{\min\lambda}[k] \in \mathcal{P}^+$, $k = 1, 2$ such that:
• if $|\operatorname{supp}_\lambda| > 1$, there does not exist $a \in \mathbf{C}^\times$ such that $\boldsymbol{\pi}_{\min\lambda}[1] = \tau_a(\boldsymbol{\pi}_{\min\lambda}[2])$,
• if $\boldsymbol{\pi} \in \mathcal{P}^+$ is such that $V(\boldsymbol{\pi}) \sim V(\boldsymbol{\pi}_{\min\lambda}[k])$ for some $k \in \{1,2\}$, then there exists $a \in \mathbf{C}^\times$ such that $\boldsymbol{\pi} = \tau_a(\boldsymbol{\pi}_{\min\lambda}[k])$.

The elements are defined as follows: If $i \notin \operatorname{supp}\lambda$ the i^{th} coordinate of $\boldsymbol{\pi}_{\min\lambda}[k]$ is one while if $i \in \operatorname{supp}\lambda$, we let

$$(\boldsymbol{\pi}_{\min\lambda}[1])_i = \pi_{c_i(\lambda)}^{\lambda(i)}, \quad c_i(\lambda) = q^{-\sum_{i_{\min}\lambda \le j \le i}(d_j\lambda(j) + d_{j+1}\lambda(j+1) + d_{j+1} - a_{j+1,j} - 1)}, \tag{6.2}$$

$$(\boldsymbol{\pi}_{\min\lambda}[2])_i = \pi_{c'_i(\lambda)}^{\lambda(i)}, \quad c'_i(\lambda) = q^{\sum_{i_{\min}\lambda \le j \le i} d_j\lambda(j) + d_{j+1}\lambda(j+1) + d_j - a_{j,j+1} - 1}. \tag{6.3}$$

6.8. We turn now to the case of D or E where the number of minimal elements in $\mathbf{D}_\lambda$ can depend on $\lambda \in P^+$. Let i_0 be the trivalent node of the Dynkin diagram and let I_j, $1 \le j \le 3$ be the disjoint connected components of $I\backslash\{i_0\}$. For $1 \le j \le 3$ and $\lambda \in P^+$, set $\lambda_j = \sum_{i \in I_j} \lambda(j)\omega_j$ and for $\boldsymbol{\pi} = (\pi_1, \cdots, \pi_n) \in \mathcal{P}^+$ let $\boldsymbol{\pi}_j$ be defined by setting the i^{th} co-ordinate to be 1 if $i \notin I_j$ and to be π_i if $i \in I_j$. Clearly

$$\lambda = \lambda(i_0)\omega_{i_0} + \lambda_1 + \lambda_2 + \lambda_3, \quad \boldsymbol{\pi} = \boldsymbol{\pi}_{i_0}^{\lambda(i_0)} \boldsymbol{\pi}_1 \boldsymbol{\pi}_2 \boldsymbol{\pi}_3.$$

Suppose first that $\lambda \in P^+$ is such that

$$\operatorname{supp}_\lambda \subset I_{j_1} \cup \{i_0\} \cup I_{j_2}, \tag{6.4}$$

for some $j_1, j_2 \in \{1,2,3\}$. Then the situation is identical to the one discussed in Section 6.7.

Suppose next that

$$i_0 \in \operatorname{supp}_\lambda \quad \text{and} \quad I_j \cap \operatorname{supp}_\lambda \neq \emptyset, \text{ for all } \ 1 \le j \le 3. \tag{6.5}$$

Then $\mathbf{D}_\lambda$ has three minimal elements which are obtained as follows. We have $[V(\boldsymbol{\pi})]$ is a minimal element of $\mathbf{D}_\lambda$ iff:
• $[V(\boldsymbol{\pi}_j \boldsymbol{\pi}_{i_0})]$ is a minimal element of $\mathbf{D}_{\lambda_j + \lambda(i_0)\omega_{i_0}}$ for all $1 \le j \le 3$,
• there exists a permutation σ of $\{1,2,3\}$ such that the elements $[V(\boldsymbol{\pi}_{\sigma(1)}\boldsymbol{\pi}_{i_0}\boldsymbol{\pi}_{\sigma(2)})]$ and $[V(\boldsymbol{\pi}_{\sigma(1)}\boldsymbol{\pi}_{i_0}\boldsymbol{\pi}_{\sigma(3)})]$ are minimal elements of $\mathbf{D}_{\lambda_{\sigma(1)} + \lambda(i_0)\omega_{i_0} + \lambda_{\sigma(2)}}$ and $\mathbf{D}_{\lambda_{\sigma(1)} + \lambda(i_0)\omega_{i_0} + \lambda_{\sigma(3))}}$ respectively.
Since $\lambda_j + \lambda(i_0)\omega_{i_0}$ and $\lambda_j + \lambda(i_0)\omega_{i_0} + \lambda_k$ for $1 \le j \neq k \le 3$ satisfy the condition in (6.4) one can write down the elements $\boldsymbol{\pi}$ explicitly using (6.2) and (6.3) and we find that there are three minimal elements.

The remaining case when $i_0 \notin \operatorname{supp}_\lambda$ has been studied when $\mathfrak{g}$ is of type D_4 in [CP6]. Here, one finds that the number of minimal elements in $\mathbf{D}_\lambda$ "increases"with λ. Virtually nothing is known in this case beyond D_4.

6.9. Consider the special case $\lambda = m\omega_i$. The preceding results show that the poset $\mathbf{D}_{m\omega_i}$ has a unique minimal element and that $\boldsymbol{\pi}_{\min m\omega_i} = (\pi_1, \cdots, \pi_n)$ where

$$\pi_j(u) = \begin{cases} 1, & \text{if } j \neq i, \\ \prod_{k=1}^m (1 - q_i^{m-2k+1}u) & \text{if } j = i. \end{cases}$$

Definition. The modules $V(\tau_a^*(\boldsymbol{\pi}_{\min m\omega_i}))$, $a \in \mathbf{C}^\times$ are called the Kirillov–Reshetikhin modules.

We shall discuss these further in the last section.

6.10. We now discuss the $\mathrm{ch}_{\mathfrak{g}}$–characters of minimal affinizations. In the case when $\mathfrak{g}$ is of type $\mathfrak{sl}_{n+1}$, one can prove [CP8] by using the evaluation homomorphism defined by Drinfeld and Jimbo that

$$V(\boldsymbol{\pi}_{\min\lambda}[k]) \cong_{\mathfrak{sl}_{n+1}} V(\lambda), \quad \lambda \in P^+. \tag{6.6}$$

In the case when $\mathfrak{g}$ is of type C_2, it was proved in [Cha2] that

$$V(\boldsymbol{\pi}_{\min\lambda}[k]) \cong_{\mathfrak{g}} \bigoplus_{r=0}^{s} V(\lambda - 2r\lambda(2)\omega_2),$$

where s is the integer part of $\lambda(2)/2$. For other Lie algebras, virtually nothing is known about the $\mathrm{ch}_{\mathfrak{g}}$ –character of the minimal affinizations in general.

In the case when $\lambda = m\omega_i$, and i is of classical type, i.e, α_i occurs in the highest root θ with multiplicity at most two, the $\mathrm{ch}_{\mathfrak{g}}$ –character is studied in [Cha3] and the results confirm the conjectures in [HKOTY, HKOTT, Kl, KR]. As an example we give the $\mathfrak{g}$–structure of in the case when $i = 2$ for the algebras of type B_n: here we have that

$$V(\boldsymbol{\pi}_{\min m\omega_2}) \cong_{\mathfrak{g}} \bigoplus_{r=0}^{m} V(r\omega_2).$$

Graded versions and classical analogs of these results have been studied in [CM3, CM4]. Other approaches can also be found in [CG]. If $i \in I$ is not classical, then the corresponding Lie algebra is exceptional and such a decomposition for $\mathrm{ch}_{\mathfrak{g}}(V(\boldsymbol{\pi}_{\min\omega_i}))$ is not known. In the case of E_8 for instance, even conjectural decomposition formulas are not known when i is the trivalent node and $m > 1$. However, character formulas in a different form have been proved (see the last section).

7. Properties of minimal affinizations

In this section, we review results which give sufficient conditions for minimal affinizations to be prime, minuscule or quasi–minuscule.

7.1. We can prove the following.

Proposition. *Let $\lambda \in P^+$ and suppose that $a_{ij} \in \{0, -1\}$ for all $i, j \in \operatorname{supp}\lambda$ and $i \neq j$. Then $V(\boldsymbol{\pi}_{\min\lambda})$ is prime.*

Proof. Notice first that under the hypothesis on λ, the elements $\boldsymbol{\pi}_{\min\lambda}[k]$, $k = 1, 2$ are given explicitly in (6.2),(6.3). If $\mathfrak{g}$ is of type A_n, then one knows from (6.6) that $V(\boldsymbol{\pi}_{\min\lambda}[k]) \cong V(\lambda)$ as $\mathbf{U}_q(\mathfrak{g})$–modules. It is an elementary exercise to see that $V(\lambda)$ can never be written as a tensor product of two non–trivial representations of $\mathbf{U}_q(\mathfrak{g})$ and hence the proposition follows in this case. For the general case, suppose that there exists $\boldsymbol{\pi}, \boldsymbol{\pi}' \in \mathcal{P}^+$, such that $V(\boldsymbol{\pi}_{\min\lambda}[k]) \cong V(\boldsymbol{\pi}) \otimes V(\boldsymbol{\pi}')$. Let I_0 be the connected component of I containing $\operatorname{supp}\lambda$. The subalgebra

$\widehat{\mathbf{U}}_0$ generated by $\widehat{\mathbf{U}}_i$, $i \in I_0$ is isomorphic to $\widehat{\mathbf{U}}_q(\mathfrak{s}l_{r+1})$, where $r = |I_0|$. Using the formulae for comultiplication given in Lemma 2.6, one sees that

$$\widehat{\mathbf{U}_0}(v_{\boldsymbol{\pi}_{\min\lambda[k]}}) \cong \widehat{\mathbf{U}}_0 v_{\boldsymbol{\pi}} \otimes \widehat{\mathbf{U}}_0 v_{\boldsymbol{\pi}'},$$

which means that $\widehat{\mathbf{U}}_0 v_{\boldsymbol{\pi}_{\min\lambda[k]}}$ is not a minimal affinization for λ, where we regard λ as an element of P_0^+ (the weight lattice corresponding to $\mathfrak{sl}_{r+1}$). But again, an inspection of (6.2), (6.3) shows that this is a contradiction. □

Remark. More generally, one can prove that any minimal affinization is prime and a proof will appear elsewhere. The converse statement however is not true, examples of prime representations which are not minimal can be found for $\mathfrak{s}l_3$ in [CP9].

7.2. We explain two essential tools that are needed to continue our study. The first is an algorithm defined in [FM1] which can be used to compute the q-character of minuscule representations.

Given $\boldsymbol{\pi} = (\pi_1, \cdots, \pi_n) \in \mathcal{P}^+$, let $\mathcal{Q}_{\boldsymbol{\pi}}^-$ be the submonoid of $\mathcal{Q}^-$ generated by the elements

$$\{(\boldsymbol{\alpha}_{i,aq^m})^{-1} : i \in I,\ m \in \mathbf{Z},\ a \in \mathbf{C}^\times,\ \prod_{i=1}^n \pi_i(a^{-1}) = 0\}.$$

The set $\boldsymbol{\pi}\mathcal{Q}_{\boldsymbol{\pi}}^-$ is countable and we fix an enumeration of this set $\{\boldsymbol{\varpi}_r\}_{r\geq 0}$ so that,
- $\boldsymbol{\varpi}_0 = \boldsymbol{\pi}$,
- $r \geq r'$ implies $\boldsymbol{\varpi}_r \preccurlyeq \boldsymbol{\varpi}_{r'}$.

For $i \in I$, define $\mathbf{p}_i : \mathcal{P} \to \mathbf{Z}[\mathcal{P}]$ by: for $\boldsymbol{\varpi} = (\varpi_1, \cdots, \varpi_n)$, we have

$$\mathbf{p}_i(\boldsymbol{\varpi}) = \begin{cases} 0 & \varpi_i \notin \mathbf{C}[u], \\ (\varpi_1, \cdots, \varpi_{i-1}, 1, \varpi_{i+1}, \cdots, \varpi_n)\mathrm{ch}_q^i V(\varpi_i) & \varpi_i \in \mathbf{C}[u], \end{cases}$$

where ch_q^i is the q–character of the module $V(\varpi_i)$ for $\widehat{\mathbf{U}}_i$, expect that in $(\varpi_i)^{-1}\mathrm{ch}_q^i$ we use the $\boldsymbol{\alpha}_{i,a}^{-1}$ instead of the roots $\boldsymbol{\alpha}_a^{-1}$ of $\widehat{\mathbf{U}}_i$. If $\boldsymbol{\varpi}' \in \mathcal{P}$ we let $\mathbf{p}_i(\boldsymbol{\varpi})_{\boldsymbol{\varpi}'}$ be the coefficient of $\boldsymbol{\varpi}'$ in $\mathbf{p}_i(\boldsymbol{\varpi})$. Note that $\mathbf{p}_i(\boldsymbol{\varpi})_{\boldsymbol{\varpi}'} \in \mathbf{Z}_+$.

For $i \in I$ and $r \in \mathbf{Z}_+$ define integers $s(\boldsymbol{\varpi}_r)$, $s_i(\boldsymbol{\varpi}_r)$, inductively and simultaneously by:

$$s(\boldsymbol{\varpi}_0) = 1, \quad s_i(\boldsymbol{\varpi}_0) = 0,$$
$$s_i(\boldsymbol{\varpi}_r) = \sum_{r'<r} (s(\boldsymbol{\varpi}_{r'}) - s_i(\boldsymbol{\varpi}_{r'}))[\mathbf{p}_i(\boldsymbol{\varpi}_{r'})]_{\boldsymbol{\varpi}_r}\ , \quad r \geq 1,$$
$$s(\boldsymbol{\varpi}_r) = \mathrm{Max}_{j\in I}(s_j(\boldsymbol{\varpi}_r)), \quad r \geq 1$$

Finally, set $FM(\boldsymbol{\pi}) = \sum_{r\geq 0} s(\boldsymbol{\varpi}_r)\boldsymbol{\varpi}_r$.

Definition. The Frenkel-Mukhin algorithm is said to be well-defined if $FM(\boldsymbol{\pi}) \in \mathbf{Z}[\mathcal{P}]$, i.e $s(\boldsymbol{\varpi}_r) = 0$ for all but finitely many r. We say that $V(\boldsymbol{\pi})$ is regular if $\mathrm{ch}_q(V(\boldsymbol{\pi})) = FM(\boldsymbol{\pi})$.

Theorem 6. [FM1] A minuscule simple module is regular.

In general a simple representation is not regular (examples were first considered in [NN4] for type C_3). Note that if $V(\boldsymbol{\pi})$ is regular, then a representation theoretical interpretation of the integers $s(\boldsymbol{\pi}_r) - s_i(\boldsymbol{\pi}_r)$ can be found in [H7].

In [H1], a $F(\boldsymbol{\pi}) \in \mathbf{Z}[\mathcal{P}]$ has been constructed for any dominant $\boldsymbol{\pi}$. If $V(\boldsymbol{\varpi})$ is minuscule, then $F(\boldsymbol{\varpi}) = FM(\boldsymbol{\varpi}) = \mathrm{ch}_q(V(\boldsymbol{\varpi}))$. For general $\boldsymbol{\varpi}$, $F(\boldsymbol{\varpi}) \in \mathrm{Im}(\mathrm{ch}_q)$, but may have negative coefficients. $FM(\boldsymbol{\varpi})$ has nonnegative coefficients, but may not belong to the image of ch_q.

7.3. The Frenkel-Mukhin algorithm produces elements of $\mathcal{P}$ which could occur in the set $\mathrm{wt}_\ell V(\boldsymbol{\pi})$. We now give a result which gives a sufficient condition [H7] for an element of $\mathcal{P}$ to not be in the set $\mathrm{wt}_\ell V(\boldsymbol{\pi})$. This elimination theorem is useful to prove that a representation is minuscule. Another application to the smallness conjecture is given later.

Define a morphism of monoids $\mathrm{ht} : \mathcal{Q}^\pm \to \mathbf{Z}_+$ by extending

$$\mathrm{ht}(\boldsymbol{\alpha}_{i,a})^{\pm 1} = 1, \quad i \in I, a \in \mathbf{C}^\times.$$

Let $\mathcal{P}_i^+ = \{\boldsymbol{\varpi} \in \mathcal{P} : \pi_i \in \mathbf{C}[u]\}$. The following is proved in [H7].

Theorem 7. *Let $\boldsymbol{\pi} \in \mathcal{P}^+$ and assume that $\boldsymbol{\varpi} \preccurlyeq \boldsymbol{\pi}$ satisfies the following conditions for some $i \in I$.*

(i) There exists a unique element $\tilde{\boldsymbol{\varpi}} \in \mathrm{wt}_\ell(V(\boldsymbol{\pi})) \cap \mathcal{P}_i^+ \cap \boldsymbol{\varpi}\mathcal{Q}^-$ such that $\tilde{\boldsymbol{\varpi}} \neq \boldsymbol{\varpi}$, and its multiplicity is 1 ,
(ii) $x_{i,r}^+(V(\boldsymbol{\pi})_{\tilde{\boldsymbol{\varpi}}}) = \{0\}$ for all $r \in \mathbf{Z}$,
(iii) $\mathbf{p}_i(\tilde{\boldsymbol{\varpi}})_{\boldsymbol{\varpi}} = 0$,
(iv) if $\boldsymbol{\varpi}'$ is such that $V(\boldsymbol{\pi})_{\boldsymbol{\varpi}'} \cap \widehat{\mathbf{U}}_i V_{\tilde{\boldsymbol{\varpi}}} \neq 0$ and $\boldsymbol{\varpi}'_i \in \mathbf{C}[u]$, then $\mathrm{ht}(\boldsymbol{\varpi}'\boldsymbol{\pi}^{-1}) \geq \mathrm{ht}(\boldsymbol{\varpi}\boldsymbol{\pi}^{-1})$,
(v) for all $j \neq i$, we have

$$\{\boldsymbol{\varpi}' \in \mathrm{wt}_\ell(V(\boldsymbol{\pi})) : \mathrm{ht}(\boldsymbol{\varpi}'\boldsymbol{\pi}^{-1}) < \mathrm{ht}(\boldsymbol{\varpi}\boldsymbol{\pi}^{-1})\} \cap \boldsymbol{\varpi}\mathcal{Q}_j = \emptyset,$$

where $\mathcal{Q}_j$ is the subgroup of $\mathcal{Q}$ generated by the elements $\boldsymbol{\alpha}_{j,a}$, $a \in \mathbf{C}^\times$.

Then $V(\boldsymbol{\pi})_{\boldsymbol{\varpi}} = 0$.

7.4. We now consider the problem of giving explicit formulae for the q–character of a minimal affinization. It is useful to see the introduction of [H8] for references on previous results in this direction; in particular in type A the results can be found in [Che1, NT], in the case of Yangians and in [FM2] for quantum affine algebras. In the special case of the Kirillov-Reshetikhin, the results can be extracted from [Nak5, H5] and we refer to the next section, in which they are discussed in greater detail, for references.

The following result is proved in [H8].

Theorem 8. *Let $\lambda \in P^+$ and assume that $\boldsymbol{\pi}_{\min\lambda} \in \{\boldsymbol{\pi}_{\min\lambda}[k] : k = 1, 2\}$.*

(i) If $\mathfrak{g}$ is of type A_n, B_n or G_2, then $V(\boldsymbol{\pi}_{\min\lambda})$ and $\sigma^* V(\boldsymbol{\pi}_{\min\lambda})$ are minuscule.
(ii) if $\mathfrak{g}$ is of type C_n and $\lambda(n) = 0$, then $V(\boldsymbol{\pi}_{\min\lambda})$ is minuscule if $\boldsymbol{\pi}_{\min\lambda}$ satisfies (6.2) and $\sigma^* V(\boldsymbol{\pi}_{\min\lambda})$ is minuscule if $\boldsymbol{\pi}_{\min\lambda}$ satisfies (6.3). An analogous result holds if $\mathfrak{g}$ is of type F_4 if we assume that $\lambda(4) = 0$.
(iii) If $\mathfrak{g}$ is of type D_n and $\lambda(n-1) = \lambda(n)$, then $V(\boldsymbol{\pi}_{\min\lambda})$ is minuscule if $\boldsymbol{\pi}_{\min\lambda}$ satifies (6.2) and $\sigma^* V(\boldsymbol{\pi}_{\min\lambda})$ is minuscule if $\boldsymbol{\pi}_{\min\lambda}$ satifies (6.3). □

The main points of the proof are a generalization of the methods of [H5]. There are two main steps: the first one is to compute the terms which occur "at the top" of the q-characters, that is to say the first few terms. This is done by Theorem 7 to show that some terms cannot occur in the q-character. Of the terms that occur, only the highest weight is in $\mathcal{P}^+$. Then most have the following right-negative property :

Definition. [FM1] A non trivial $\varpi = \prod_{i\in I, a\in\mathbf{C}^*} \boldsymbol{\omega}^{u_{i,a}(\varpi)}$ is said to be right-negative if for all $a \in \mathbf{C}^*, j \in I$ we have $u_{j,aq^{L_a}}(\varpi) \neq 0 \Rightarrow u_{j,aq^{L_a}}(m) < 0$ where

$$L_a = \max\{l \in \mathbf{Z}/\exists i \in I, u_{i,aq^L}(\varpi) \neq 0\}.$$

The second step is to use the information of the top of the q-character, the right-negative property, to prove that all other elements in $\mathrm{wt}_\ell(V(\boldsymbol{\pi}))$ also have the right negative property. This implies immediately, that they are not elements of $\mathcal{P}^+$. For this step, we need a representation theoretical induction inside the module and a crucial ingredient is the structure of Weyl-module in the sl_2-case.

Although, the theorem is not the best possible in the case of algebras of type C_n, D_n and F_4, it is not true that all minimal affinizations are minuscule. For instance, when $\mathfrak{g}$ of type C_3, and $\lambda = 2\omega_2 + \omega_3$, then one can see that a corresponding minimal affinization is not minuscule although it satisfies (6.2). Other counter-examples may be found in [H8].

7.5. Recall the classical result that the Jacobi–Trudi determinant gives the character of the irreducible representation $V(\lambda)$ of $\mathfrak{sl}_{n+1}$. A generalization of the Jacobi–Trudi determinant to other classical Lie algebras in terms of tableaux can be found in [KOS] for type B, and [NN1, NN2, NN3] for general classical type. In [NN1, Conjecture 2.2] Nakai and Nakanishi conjectured that when $\mathfrak{g}$ is of classical type, the Jacobi-Trudi type determinant gives the q-character of an irreducible representation of the corresponding quantum affine algebra. The following is proved in [H8] and confirms their conjecture when $\mathfrak{g}$ is of type A or B.

Theorem 9. *Assume that $\mathfrak{g}$ of type A or B and let $\lambda \in P^+$ be such that $\lambda(n)$ is even if $\mathfrak{g}$ is of type B_n. The q-character of the minimal affinization of λ is given by the corresponding Jacobi-Trudi determinant. In particular, the corresponding minimal affinizations are quasi-minuscule.* □

If $\mathfrak{g}$ of type C_4, for instance the minimal affinization of $2\omega_3$ is not quasi–minuscule. As discussed earlier in the paper, for D_n, there exist fundamental representations which are not quasi–minuscule.

7.6. As an illustration, we give the q-characters for type B_n predicted by the conjecture of [NN2].

Recall that a partition $\lambda = (\lambda_1, \lambda_2, \cdots)$ is a sequence of weakly decreasing non-negative integers with finitely many non-zero terms. The conjugate partition is denoted by $\lambda' = (\lambda'_1, \lambda'_2, \cdots)$. Given two partitions λ and μ, we say that $\mu \subset \lambda$ if $\lambda_i \geq \mu_i$ for all $i \geq 1$ and the corresponding skew diagram denoted λ/μ is

$$\lambda/\mu = \{(i,j) \in \mathbf{N}\times\mathbf{N} : \mu_i + 1 \leq j \leq \lambda_i\} = \{(i,j) \in \mathbf{N}\times\mathbf{N} : \mu'_j + 1 \leq i \leq \lambda'_j\},$$

and let $d(\lambda/\mu)$ be the length of the longest column of λ/μ.

Assume now that that $d(\lambda/\mu) \leq n$ and also that λ/μ is connected (i.e. $\mu_i + 1 \leq \lambda_{i+1}$ if $\lambda_{i+1} \neq 0$).

Let $\mathbf{B} = \{1, \cdots, n, 0, \overline{n}, \cdots, \overline{1}\}$. We give the ordering $\prec$ on the set $\mathbf{B}$ by

$$1 \prec 2 \prec \cdots \prec n \prec 0 \prec \overline{n} \prec \cdots \prec \overline{2} \prec \overline{1}.$$

As it is a total ordering, we can define the corresponding maps succ and prec. For $a \in \mathbf{C}^*$, let (we use the notations of [FR] explained in Section 4.3)

$$\boxed{i}_a = Y^{-1}_{i-1,aq^{2i}} Y_{i,aq^{2(i-1)}} \;,\; \boxed{\bar{i}}_a = Y_{i-1,aq^{4n-2i-2}} Y^{-1}_{i,aq^{4n-2i}} \text{ for } 2 \leq i \leq n-1,$$

$$\boxed{n}_a = Y^{-1}_{n-1,aq^{2n}} Y_{n,aq^{2n-1}} Y_{n,aq^{2n-3}} \;,\; \boxed{\overline{n}}_a = Y_{n-1,aq^{2n-2}} Y^{-1}_{n,aq^{2n+1}} Y^{-1}_{n,aq^{2n-1}},$$

$$\boxed{0}_a = Y^{-1}_{n,aq^{2n+1}} Y_{n,aq^{2n-3}},$$

(we denote $Y_{0,a} = Y_{n+1,a} = 1$). For $T = (T_{i,j})_{(i,j)\in\lambda/\mu}$ a tableaux of shape λ/μ with coefficients in $\mathbf{B}$, let

$$m_{T,a} = \prod_{(i,j)\in\lambda/\mu} \boxed{T_{i,j}}_{aq^{4(j-i)}} \in \mathcal{Y}.$$

Let $\mathrm{Tab}(B_n, \lambda/\mu)$ be the set of tableaux of shape λ/μ with coefficients in $\mathbf{B}$ satisfying :

$$[T_{i,j} \preceq T_{i,j+1} \text{ and } (T_{i,j}, T_{i,j+1}) \neq (0,0)] \text{ and } [T_{i,j} \prec T_{i+1,j} \text{ or } (T_{i,j}, T_{i+1,j}) = (0,0)].$$

The tableaux expression of the Jacobi-Trudi determinant [KOS, NN1] is :

$$\chi_{\lambda/\mu,a} = \sum_{T \in \mathrm{Tab}(B_n,\lambda/\mu)} m_{T,a} \in \mathcal{Y}.$$

Note that we get minimal affinizations of Theorem 9 for $\mu = 0$. The highest weight is $\sum_j \sum_{\{i|\lambda_i=j\}} (3 - r_j)\omega_j$. The proof of the conjecture for more general representations will appear elsewhere.

8. Kirillov-Reshetikhin modules

We conclude this paper with a discussion of the Kirillov–Reshetikhin (KR) modules. These were first introduced in [KR] and since then have been widely studied. They have a number of interesting properties, some of which we have already seen. They are the minimal affinizations $W^{(i)}_{m,a} = \tau_a^* V(\boldsymbol{\pi}_{\min m\omega_i})$ of $m\omega_i$, $m \in \mathbf{Z}_+$, $i \in I$, $a \in \mathbf{C}^\times$ of section 6.9 and are prime by Proposition 7.1.

We have also seen in Theorem 8 that in some cases they are minuscule and we have seen in section 6.10 that closed formulas are known for $\mathrm{ch}_{\mathfrak{g}}$ when i is of classical type. Some of these results can be improved for these modules as we shall see below. In particular it is one of the first infinite family of simple finite dimensional representations of $\widehat{\mathbf{U}}_q(\mathfrak{g})$ where explicit uniform character formulas can be given for all types : this important property of KR modules is the KR conjecture proved by Nakajima (ADE case) and the second author (general case) and discussed in this section.

Another important property that is expected is that the modules $V(\boldsymbol{\pi}_{\min m\omega_i})$ have a crystal basis (for a choice of the spectral parameter) and moreover that a module $V(\boldsymbol{\pi})$ has a crystal basis if and only if it is a tensor product of modules of the form $V(\boldsymbol{\pi}_{\min m\omega_i})$. Another important motivation for the study of the Kirillov–Reshetikhin modules is their connections with solvable lattice models [HKOTY, HKOTT]. There is extensive literature on the subject and we just give a quick overview in this section, with pointers to the appropriate references.

8.1. By Proposition 6.2 we see that $W_{m,a}^{(i)} \cong_{\mathfrak{g}} W_{m,1}^{(i)}$ and hence $\mathrm{ch}W_{m,a}^{(i)} = \mathrm{ch}W_{m,1}^{(i)}$. The first part of the Kirillov-Reshetikhin conjecture gives a closed formula for the character ch of an arbitrary tensor product of the modules $W_{m,1}^{(i)}$.

For a sequence $\nu = (\nu_k^{(i)})_{i\in I, k>0}$ of non–negative integers, such that all but finitely many $\nu_k^{(i)}$ are zero we set :

$$\mathcal{F}(\nu) = \sum_{N=(N_k^{(i)})} \prod_{i\in I,k>0} \binom{P_k^{(i)}(\nu,N)+N_k^{(i)}}{N_k^{(i)}} e(-kN_k^{(i)}\alpha_i)$$

where

$$P_k^{(i)}(\nu,N) = \sum_{l=1\ldots\infty} \nu_l^{(i)}\mathrm{min}(k,l) - \sum_{j\in I,l>0} N_l^{(j)} r_i C_{i,j}\mathrm{min}(k/r_j, l/r_i),$$

$$\binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(a-b+1)\Gamma(b+1)},$$

and Γ is the usual gamma function.

Theorem 10 (The KR conjecture)**.** For a sequence $\nu = (\nu_k^{(i)})_{i\in I,k>0}$ such that for all but finitely many $\nu_k^{(i)}$ are zero, we have :

$$\prod_{i\in I,k\geq 1} (\mathrm{ch}(W_{k,1}^{(i)}))^{\nu_k^{(i)}} \prod_{\alpha\in\Delta_+} (1-e(-\alpha)) = \mathcal{F}(\nu). \tag{8.1}$$

□

The following theorems 12, 11, 13 (that imply Theorem 10 by [HKOTY, HKOTT, KNT]) are due to Nakajima [Nak4, Nak5] for simply–laced algebras (the proof uses geometric methods) and in full generality to the second author [H5] (the general proof uses purely algebraic different methods described in section 7.4).

In both proofs the crucial step is the following :

Theorem 11. The modules $W_{k,a}^{(i)}$ are minuscule.

Recently it was proved combinatorially in [DK] that (8.1) can be rewritten in a different form with positive coefficients.

8.2. We now discuss the relationship of Theorem 10 to T and Q systems. The T–systems were originally introduced in [KNS] as functional relations. They can also be viewed as a system of induction relations on the characters of the Kirillov–Reshetikhin modules and to do this, we introduce the following representations. For $i \in I$, $k \geq 1$, $a \in \mathbf{C}^*$ define the $\widehat{\mathbf{U}}_q(\mathfrak{g})$-module $S_{k,a}^{(i)}$ by :

$$S_{k,a}^{(i)} = \begin{cases} (\bigotimes_{j/a_{j,i}=-1} W_{k,aq_i}^{(j)}) \otimes (\bigotimes_{j/a_{j,i}\leq -2} W_{d_i k,aq}^{(j)}) & \text{if } d_i \geq 2, \\ (\bigotimes_{j/a_{i,j}=-1} W_{k,aq}^{(j)}) \otimes (\bigotimes_{j/a_{i,j}=-2} W_{r,aq}^{(j)} \otimes W_{r,aq^3}^{(j)}) & \text{if } d_i = 1,\ \mathfrak{g} \neq G_2,\ k = 2r, \\ (\bigotimes_{j/a_{i,j}=-1} W_{k,aq}^{(j)}) \otimes (\bigotimes_{j/a_{i,j}=-2} W_{r+1,aq}^{(j)} \otimes W_{r,aq^3}^{(j)}) & \text{if } d_i = 1,\ \mathfrak{g} \neq G_2,\ k = 2r+1, \end{cases}$$

and for $r_i = 1$ and $\mathfrak{g}$ of type G_2 ($j \neq i$ is the other node) :

$$S_{k,a}^{(i)} = \begin{cases} W_{r,aq}^{(j)} \otimes W_{r,aq^3}^{(j)} \otimes W_{r,aq^5}^{(j)} & \text{if } k = 3r, \\ W_{r+1,aq}^{(j)} \otimes W_{r,aq^3}^{(j)} \otimes W_{r,aq^5}^{(j)} & \text{if } k = 3r+1, \\ W_{r+1,aq}^{(j)} \otimes W_{r+1,aq^3}^{(j)} \otimes W_{r,aq^5}^{(j)} & \text{if } k = 3r+2. \end{cases}$$

The $S_{k,a}^{(i)}$ are well-defined as the modules in the definition commute for $\otimes$. Moreover $S_{k,a}^{(i)}$ is minuscule and so is simple. We denote by $[V]$ the image in $\widehat{\mathrm{Rep}}(\mathfrak{g})$ of a module V.

Theorem 12 (The T-system)**.** Let $a \in \mathbf{C}^*$, $k \geq 1$, $i \in I$. We have in $\widehat{\mathrm{Rep}}(\mathfrak{g})$:

$$[W_{k,a}^{(i)}][W_{k,aq_i^2}^{(i)}] = [W_{k+1,a}^{(i)}][W_{k-1,aq_i^2}^{(i)}] + [S_{k,a}^{(i)}].$$

The T-system holds in the Grothendieck ring, but it can also be written as an exact sequence of representations. The T-system implies the Q-system which is just the corresponding statement in $\mathrm{Rep}(\mathfrak{g})$.

For example in the case of sl_2, the T-system is just the following relation which can be easily checked by using the explicit formulas given above :

$$[W_{k,a}][W_{k,aq^2}] = [W_{k+1,a}][W_{k-1,aq^2}] + 1.$$

The Q-system is

$$Q_k^2 = Q_{k+1}Q_{k-1} + 1$$

where $Q_k = \mathrm{ch}(W_{k,a})$ does not depend on the spectral parameter a. This just an elementary relation between the characters $\mathrm{ch}(Q_k) = e(k\omega) + e((k-2)\omega) + \cdots + e(-k\omega)$.

8.3. A convergence property of the q-characters of KR modules holds :

Theorem 13. The normalized q-character of $W_{k,a}^{(i)}$ considered as a polynomial in $\boldsymbol{\alpha}_{j,b}^{-1}$ has a limit as a formal power series :

$$\exists \lim_{k\to\infty} \frac{\mathrm{ch}_q(W_{k,aq_i^{-2k}}^{(i)})}{\boldsymbol{\pi}_{k,aq_i^{-2k}}^{(i)}} \in \mathbf{Z}[[\boldsymbol{\alpha}_{j,aq^m}^{-1}]]_{j\in I, m\in\mathbf{Z}},$$

where $\boldsymbol{\pi}_{k,aq_i^{-2k}}^{(i)}$ is the highest term of $W_{k,aq_i^{-2k}}^{(i)}$.

As a direct consequence, a convergence property holds for the characters of KR module : $\mathcal{Q}_k^{(i)} = e(-k\omega_i)\mathrm{ch}(W_{k,a}^{(i)})$ considered as a polynomial in $e(-\alpha_j)$ has a limit as a formal power series :

$$\exists \lim_{k\to\infty} \mathcal{Q}_k^{(i)} \in \mathbf{Z}[[e(-\alpha_j)]]_{j\in I}.$$

With the Q-system, these representation theoretical results imply by combinatorial arguments [HKOTY, HKOTT, KNT] explicit character in Theorem 10. [DK] deals with the problem of rewriting (8.1) into a different expression.

8.4. It is expected that KR modules (for a special choice of the spectral parameter) have a crystal basis. This is known for fundamental representations [Kas] (see [HN] and references therein for explicit descriptions). As an application of the branching rules of KR modules discussed in Section 6.10 (the branching rules in [H9] for twisted cases), the conjecture about crystal basis has been proved for classical types (see [OS] and references therein).

8.5. Let us now go to the question of q-characters of KR modules. The fact that they are minuscule implies that they are regular and that their q–character can in principle be calculated by using the Frenkel-Mukhin algorithm. In classical types, there are explicit formulas in [KOS, KNH] (the formulas for fundamental representations are given in [KS, CM2]) which follow from the minuscule property. But explicit formulas for their q-character are not known in other cases. It would be interesting as well, to give analogs of fermionic formulas for their q-characters.

8.6. In simply-laced cases, Nakajima [Nak4] defined t-analogs of q-characters (see [H1] for non simply laced cases based on a different proof of the existence). Nakajima's construction of q, t-characters is closely related to the geometry of quiver varieties. The geometric small property (Borho-MacPherson) of projective morphisms implies a description of their singularities in terms of intersection homology. This notion for certain resolutions of quiver varieties [Nak4] (analogs of the Springer resolution) can be translated in terms of q, t-characters. Then by using a modification of the proof of Theorem in [Nak4], it is proved in [H7] that we have the following purely representation theoretical characterization of small modules involving q-characters without q, t-characters. We will use it as a definition :

Theorem 14. Let $\boldsymbol{\pi} \in \mathcal{P}^+$. $V(\boldsymbol{\pi})$ is small if and only if for all $\boldsymbol{\pi}' \in \mathcal{P}^+$ satisfying $\boldsymbol{\pi}' \preceq \boldsymbol{\pi}$, $V(\boldsymbol{\pi}')$ is minuscule.

Note that a small module is necessarily minuscule. From the geometric point of view it is important to determine which modules are small. In particular, Nakajima [Nak4, Conjecture 10.4] raised the problem of characterizing the Drinfeld polynomials of small standard modules corresponding to KR modules. The main result of [H7] is an explicit answer to this question (Theorem 15). First let us note in general the standard modules corresponding to KR modules are not necessarily small : for type A_3, $V(\boldsymbol{\pi}_{2,1}\boldsymbol{\pi}_{2,q^2}\boldsymbol{\pi}_{2,q^4})$ is not small since $V(\boldsymbol{\pi}_{1,q}\boldsymbol{\pi}_{3,q}\boldsymbol{\pi}_{2,q^4})$ is not minuscule and $\boldsymbol{\pi}_{1,q}\boldsymbol{\pi}_{3,q}\boldsymbol{\pi}_{2,q^4} \preceq \boldsymbol{\pi}_{2,1}\boldsymbol{\pi}_{2,q^2}\boldsymbol{\pi}_{2,q^4}$.

8.7. Let us give a characterization of small KR modules.

Definition. A node $i \in \{1, \cdots, n\}$ is said to be extremal (resp. trivalent) if there is a unique $j \in I$ (resp. three distinct $j, k, l \in I$) such that $a_{i,j} < 0$ (resp. $a_{i,j} < 0$, $a_{i,k} < 0$ and $a_{i,l} < 0$).

For $i \in I$, we denote by k_i the minimal $k \geq 1$ such that there are distinct $i = i_1, \cdots, i_k \in I$ satisfying $a_{i_j, i_{j+1}} < 0$ and i_k is trivalent. If there does not exist such k, set $k_i = +\infty$.

For example for $\mathfrak{g}$ of type A, we have $k_i = +\infty$ for all $i \in I$.

Theorem 15. [Smallness problem][H7] Let $k \geq 0, i \in I, a \in \mathbf{C}^*$. Then $W_{k,a}^{(i)}$ is small if and only if $k \leq 2$ or (i is extremal and $k \leq k_i + 1$).

In particular for $\mathfrak{g} = sl_2$ or $\mathfrak{g} = sl_3$, all KR modules are small (it proves the corresponding [Nak4, Conjecture 10.4]). In general it gives an explicit criterion so that the geometric smallness holds. A few words about the proof : the "only if" part is proved by writing down explicitly an element $\boldsymbol{\pi} \in \mathcal{P}^+$ so that $\boldsymbol{\pi} \preccurlyeq \tau_a \boldsymbol{\pi}_{\min k\omega_i}$ which proves that the module is not minuscule. For the "if" part, all dominant monomials lower than $\tau_a \boldsymbol{\pi}_{\min k\omega_i}$ are computed, and then it is proved by using the elimination theorem that they correspond to minuscule representations.

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