MONOMIALS OF q AND q, t-CHARACTERS FOR NON SIMPLY-LACED QUANTUM AFFINIZATIONS

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Abstract. Nakajima [N2, N3] introduced the morphism of q, t-characters for finite dimensional representation of simply-laced quantum affine algebras : it is a t-deformation of the Frenkel-Reshetikhin's morphism of q-characters (sum of monomials in infinite variables). In [H2] we generalized the construction of q, t-characters for non simply-laced quantum affine algebras. First in this paper we prove a conjecture of [H2] : the monomials of q and q, t-characters of standard representations are the same in non simply-laced cases (the simply-laced cases were treated in [N3]) and the coefficients are non negative. In particular those q, t-characters can be considered as t-deformations of q-characters. In the proof we show that for quantum affine algebras of type A, B, C and quantum toroidal algebras of type $A^{(1)}$ the l-weight spaces of fundamental representations are of dimension 1. Eventually we show and use a generalization of a result of [FR3, FM, N1] : for general quantum affinizations we prove that the l-weights of a l-highest weight simple module are lower than the highest l-weight in the sense of monomials.

Contents

1. Introduction

V.G. Drinfel'd [D] and M. Jimbo [J] associated, independently, to any symmetrizable Kac-Moody algebra $\mathfrak{g}$ and any complex number $q \in \mathbb{C}^*$ a Hopf algebra $\mathcal{U}_q(\mathfrak{g})$ called quantum Kac-Moody algebra.

In this paper we suppose that $q \in \mathbb{C}^*$ is not a root of unity. In the case of a semi-simple Lie algebra $\mathfrak{g}$ of rank n (ie. with a finite Cartan matrix), the structure of the Grothendieck ring $\mathrm{Rep}(\mathcal{U}_q(\mathfrak{g}))$ of finite dimensional representations of the quantum finite algebra $\mathcal{U}_q(\mathfrak{g})$ is well understood. It is analogous to the classical case $q = 1$.

For the general case of Kac-Moody algebras the picture is less clear. The representation theory of the quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ is of particular interest (see [CP1, CP2]). In this case there is a crucial property of $\mathcal{U}_q(\hat{\mathfrak{g}})$: it has two realizations, the usual Drinfel'd-Jimbo realization and a new realization (see [D2, Be]) as a quantum affinization of a quantum finite algebra $\mathcal{U}_q(\mathfrak{g})$.

To study the finite dimensional representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ Frenkel and Reshetikhin [FR3] introduced q-characters which encode the (pseudo)-eigenvalues of some commuting elements $\phi^{\pm}_{i,\pm m}$ $(m \geq 0)$ of the Cartan subalgebra $\mathcal{U}_q(\hat{\mathfrak{h}}) \subset \mathcal{U}_q(\hat{\mathfrak{g}})$ (see also [K]) : for V a finite dimensional representations we have :

$$V = \bigoplus_{\gamma \in \mathbb{C}^{I\times\mathbb{Z}}} V_\gamma$$

where for $\gamma = (\gamma^{(\pm)}_{i,\pm m})_{i\in I, m\geq 0}$, V_γ is a simultaneous generalized eigenspace (l-weight space):

$$V_\gamma = \{x \in V / \exists p \in \mathbb{N}, \forall i \in I, \forall m \geq 0, (\phi^{(\pm)}_{i,\pm m} - \gamma^{(\pm)}_{i,\pm m})^p.x = 0\}$$

The morphism of q-characters is an injective ring homomorphism:

$$\chi_q : \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) \to \mathcal{Y} = \mathbb{Z}[Y^{\pm}_{i,a}]_{i\in I, a\in\mathbb{C}^*}$$

where $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ is the Grothendieck ring of finite dimensional (type 1)-representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ and $I = \{1, ..., n\}$, and :

$$\chi_q(V) = \sum_{\gamma\in\mathbb{C}^{I\times\mathbb{Z}}} \dim(V_\gamma) m_\gamma$$

where $m_\gamma \in \mathcal{Y}$ depends of γ. In particular $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ is commutative and isomorphic to $\mathbb{Z}[X_{i,a}]_{i\in I, a\in\mathbb{C}^*}$.

In the finite simply laced-case (type ADE) Nakajima [N2, N3] introduced t-analogs of q-characters. The motivations are the study of filtrations induced on representations by (pseudo)-Jordan decompositions, the study of the decomposition in irreducible modules of tensorial products and the study of cohomologies of certain quiver varieties. The morphism of q,t-characters is a $\mathbb{Z}$-linear map :

$$\chi_{q,t} : \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) \to \mathcal{Y}_t = \mathbb{Z}[Y^{\pm}_{i,a}, t^{\pm}]_{i\in I, a\in\mathbb{C}^*}$$

which is a deformation of χ_q and multiplicative in a certain sense. A combinatorial axiomatic definition of q,t-characters is given. But the existence is non-trivial and is proved with the geometric theory of quiver varieties which holds only in the simply laced case.

In [H2] we defined and constructed q,t-characters for a finite (non necessarily simply-laced) Cartan matrix C with a new approach motivated by the non-commutative structure of $\mathcal{U}_q(\hat{\mathfrak{h}}) \subset \mathcal{U}_q(\hat{\mathfrak{g}})$, the study of screening currents of [FR2] and of deformed screening operators $S_{i,t}$ of [H1]. It generalizes the construction of Nakajima to non-simply laced cases.

The quantum affinization process (that Drinfel'd [D2] described for constructing the second realization of a quantum affine algebra) can be extended to all symmetrizable quantum Kac-Moody algebras $\mathcal{U}_q(\mathfrak{g})$ (see [Jin, N1, H4]). One obtains a new class of algebras called quantum affinizations : the quantum affinization of $\mathcal{U}_q(\mathfrak{g})$ is denoted by $\mathcal{U}_q(\hat{\mathfrak{g}})$. It has a triangular decomposition [H4]. For example the quantum affinization of a quantum affine algebra is called a quantum toroidal algebra. The quantum affine algebras are the simplest examples and are very special because they are also quantum Kac-Moody algebras. In the following, general quantum affinization means with an invertible quantum Cartan matrix (it includes most interesting cases like affine and toroidal quantum affine algebras, see section 2.2). In [H4] we developed the representation theory of general quantum affinizations and constructed a generalization of the q-characters morphism which appears to be a powerful tool for this investigation. In particular we proved that the new Drinfel'd coproduct leads to the construction of a fusion product on the Grothendieck group.

The results of this paper can be divided in three parts :

1) First we prove that for general quantum affinizations, the l-weights $m' \in \mathcal{Y}$ of a simple module of l-highest weight $m \in \mathcal{Y}$ are lower than m in the sense of monomials (theorem 3.2) : it means that $m'm^{-1}$ is a product of certain $A^{-1}_{i,l} \in \mathcal{Y}$. For C finite, this result was conjectured and partly proved in [FR3] and proved in [N1] (ADE-case) and [FM] (finite case). In the general case no universal $\mathcal{R}$-matrix has been defined : so we propose a new proof based on the study of $\mathcal{U}_q(\hat{sl_2})$-Weyl modules introduced in [CP3]. This first result is used in the proof of the following points :

2) We prove a conjecture of [H2] : let $\mathcal{U}_q(\hat{\mathfrak{g}})$ be a quantum affine algebra (C finite) and M be a standard module of $\mathcal{U}_q(\hat{\mathfrak{g}})$ (tensorial product of fundamental representations). We prove that the coefficients of $\chi_{q,t}(M)$ are in $\mathbb{N}[t^{\pm}]$ and that the monomials of $\chi_{q,t}(M)$ are the monomials of $\chi_q(M)$ (theorem 7.5) (the case ADE follows from the work of Nakajima [N3]; in this paper the non-simply laced case is treated.) In particular the q,t-characters for quantum affine algebras have a finite number of monomials : this result shows that the q,t-characters of [H2] can be considered as t-deformations of q-characters for all quantum affine algebras. In particular it is an argument for the existence of a geometric model behind the q,t-characters in non simply-laced cases (in the simply laced-case the standard module can be realized in the K-theory of quivers varieties).

3) In the proof of the conjecture we study combinatorial properties of q-characters : we prove that for quantum affinizations of type $A, A^{(1)}, B, C$ the l-weight spaces of fundamental representations are of dimension 1 (theorem 3.5). Note that this property is not true in general, for example for type D.

Our proof is based on an investigation of the classical algorithm (see [FM, H3]) which gives q-characters. The proof is direct without explicit computation. Note that for type A, B, C the result could follow from explicit computation of the specialized $\mathcal{R}$-matrix, as explained in [FR3]. The result should produce the formulas of [FR1]; however with this method it would not be easy to decide if the coefficients are 1 (for example it is not the case for type D_4). Moreover it allows us to extend the proof to quantum toroidal algebras of type $A^{(1)}$.

This paper is organized as follows : in section 2 we give reminders on representations of quantum affinizations and their q-characters. In section 3 we state and prove theorem 3.2 (the l-weights of a l-highest weight simple module are lower than the highest l-weight in the sense of monomials) and state theorem 3.5 (on q-characters of fundamental representations) and give technical complements. The proof of theorem 3.5 is based on a case by case investigation explained in sections 4, 5, 6. In section 7 we give reminders on q,t-characters and we prove theorem 7.5 (on coefficients of q,t-characters of standard monomials). For the theorem 7.5 type F_4, our proof is based on results obtained by a computer program written with Travis Schedler, and the results are given in the appendix (section 8).

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2. Reminder

2.1. **Representations of quantum affinizations.** Let $C = (C_{i,j})_{1\leq i,j\leq n}$ be a symmetrizable (non necessarily finite) Cartan matrix and $I = \{1,...,n\}$. Let $D = \text{diag}(r_1,...,r_n)$ such that $B = DC$ is symmetric. We consider $(\mathfrak{h}, \Pi, \Pi^\vee)$ a realization of C, the weight lattice $P \subset \mathfrak{h}^*$, the roots $\alpha_1,...,\alpha_n \in P$, the set of dominant weights $P^+ \subset P$, the relation $\leq$ on P, the map $\nu : \mathfrak{h}^* \to \mathfrak{h}$ (see [H4]).

Let $q \in \mathbb{C}^*$ not a root of unity. Let $\mathcal{U}_q(\mathfrak{g})$ be the quantum Kac-Moody algebra of Cartan matrix C. Let $\mathcal{U}_q(\hat{\mathfrak{g}}) \supset \mathcal{U}_q(\mathfrak{g})$ be the quantum affinization of $\mathcal{U}_q(\mathfrak{g})$, with generators $x_{i,r}^{\pm}, k_h, c^{\pm\frac{1}{2}}, \phi_{i,\pm m}^{(\pm)}$, where $i \in I, r \in \mathbb{Z}, m \geq 0, h \in \mathfrak{h}$ (see for example [H4]). A $\mathcal{U}_q(\hat{\mathfrak{g}})$-module is said to be integrable if it is integrable as a $\mathcal{U}_q(\mathfrak{g})$-module.

Denote by P_l the set of l-weights, that is to say of couple (λ, Ψ) such that $\lambda \in \mathfrak{h}^*$, $\Psi = (\Psi_{i,\pm m}^{\pm})_{i\in I, m\geq 0}$, $\Psi_{i,\pm m}^{\pm} \in \mathbb{C}$, $\Psi_{i,0}^{\pm} = q_i^{\pm\lambda(\alpha_i^\vee)}$. Note that if C is finite, λ is uniquely determined by Ψ.

A $\mathcal{U}_q(\hat{\mathfrak{g}})$-module V is said to be of l-highest weight $(\lambda, \Psi) \in P_l$ if there is $v \in V$ such that ($i \in I, r \in \mathbb{Z}, m \geq 0, h \in \mathfrak{h}$):

$$x_{i,r}^{+}.v = 0 \ , \ V = \mathcal{U}_q(\hat{\mathfrak{g}}).v \ , \ \phi_{i,\pm m}^{\pm}.v = \Psi_{i,\pm m}^{\pm} v \ , \ k_h.v = q^{\lambda(h)}.v$$

For $(\lambda, \Psi) \in P_l$, let $L(\lambda, \Psi)$ be the simple $\mathcal{U}_q(\hat{\mathfrak{g}})$-module of l-highest weight (λ, Ψ) (see [H4]).

Let P_l^+ be the set of dominant l-weights, that is to say the set of $(\lambda, \Psi) \in P_l$ such that there exist (Drinfel'd)-polynomials $P_i(z) \in \mathbb{C}[z]$ ($i \in I$) of constant term 1 such that in $\mathbb{C}[[z]]$ (resp. in $\mathbb{C}[[z^{-1}]]$):

$$\sum_{m\geq 0} \Psi_{i,\pm m}^{\pm} z^{\pm m} = q_i^{\deg(P_i)} \frac{P_i(zq_i^{-1})}{P_i(zq_i)}$$

Theorem 2.1. *For $(\lambda,\Psi)\in P_l$, $L(\lambda,\Psi)$ is integrable if and only $(\lambda,\Psi)\in P_l^+$.*

If $\mathfrak{g}$ is finite (case of a quantum affine algebra) it is a result of Chari-Pressley in [CP1, CP2]. Moreover in this case the integrable $L(\lambda,\Psi)$ are finite dimensional. If C is simply-laced the result is proved by Nakajima in [N1]. If C is of type $A_n^{(1)}$ (toroidal $s\hat{l}_n$-case) the result is proved by Miki in [M1]. In general the result is proved in [H4].

Denote by $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ the Grothendieck group of decomposable integrable representations of type 1 (see [H4]). The operators $k_h,\phi^{(\pm)}_{i,\pm m}\in\mathcal{U}_q(\hat{\mathfrak{g}})$ $(h\in\mathfrak{h},i\in I,m\in\mathbb{Z})$ commute on $V\in\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$. So we have a l-weight space decomposition:

$$V=\bigoplus_{(\lambda,\gamma)\in P_l}V_{\lambda,\gamma}$$

$$V_{\lambda,\gamma}=\{x\in V_\lambda/\exists p\in\mathbb{N},\forall i\in I,\forall m\geq 0,(\phi^{(\pm)}_{i,\pm m}-\gamma^{(\pm)}_{i,\pm m})^p.x=0\}\subset V_\lambda=\{v\in V/\forall h\in\mathfrak{h},k_h.v=q^{\lambda(h)}v\}$$

Let $QP_l^+\subset P_l$ be the set of $(\mu,\gamma)\in P_l$ such that there exist polynomials $Q_i(z),R_i(z)\in\mathbb{C}[z]$ $(i\in I)$ of constant term 1 such that in $\mathbb{C}[[z]]$ (resp. in $\mathbb{C}[[z^{-1}]]$):

$$\sum_{m\geq 0}\gamma^{\pm}_{i,\pm m}z^{\pm m}=q_i^{\deg(Q_i)-\deg(R_i)}\frac{Q_i(zq_i^{-1})R_i(zq_i)}{Q_i(zq_i)R_i(zq_i^{-1})}$$

In particular $P_l^+\subset QP_l^+$.

Proposition 2.2. *Let V be a module in Rep$(\mathcal{U}_q(\hat{\mathfrak{g}}))$ and $(\mu,\gamma)\in P_l$. If dim$(V_{\mu,\gamma})>0$ then $(\mu,\gamma)\in QP_l^+$.*

The result is proved in [FR3] for C finite. The generalization is straightforward (see [H4]).

2.2. **q-characters.** Let z be an indeterminate. We denote $z_i=z^{r_i}$ and for $l\in\mathbb{Z}$, $[l]_z=\frac{z^l-z^{-l}}{z-z^{-1}}\in\mathbb{Z}[z^{\pm}]$. Let $C(z)$ be the quantized Cartan matrix defined by $(i\neq j\in I)$:

$$C_{i,i}(z)=z_i+z_i^{-1}\ ,\ C_{i,j}(z)=[C_{i,j}]_z$$

In the rest of this paper we suppose that $C(z)$ is invertible, that is to say $\det(C(z))\neq 0$. We have seen in lemma 6.9 of [H3] that the condition $(C_{i,j}<-1\Rightarrow -C_{j,i}\leq r_i)$ implies that $\det(C(z))\neq 0$. In particular finite and affine Cartan matrices (where we impose $r_1=r_2=2$ for $A_1^{(1)}$) satisfy this condition.

Consider formal variables $Y_{i,a}^{\pm}$ $(i\in I,a\in\mathbb{C}^*)$ and k_ω $(\omega\in\mathfrak{h})$ $(k_0=1)$. Let A be the commutative group of monomials of the form $m=\prod_{i\in I,a\in\mathbb{C}^*}Y_{i,a}^{u_{i,a}(m)}k_{\omega(m)}$ where a finite number of $u_{i,a}(m)\in\mathbb{Z}$ are non zero, $\omega(m)\in\mathfrak{h}$ (the coweight of m), and such that for $i\in I:\alpha_i(\omega(m))=r_i\sum_{a\in\mathbb{C}^*}u_{i,a}(m)$.

For $(\mu,\Gamma)\in QP_l^+$ we define $Y_{\mu,\Gamma}\in A$ by:

$$Y_{\mu,\Gamma}=k_{\nu(\mu)}\prod_{i\in I,a\in\mathbb{C}^*}Y_{i,a}^{\beta_{i,a}-\gamma_{i,a}}$$

where $\beta_{i,a},\gamma_{i,a}\in\mathbb{Z}$ are defined by $Q_i(u)=\prod_{a\in\mathbb{C}^*}(1-ua)^{\beta_{i,a}}$, $R_i(u)=\prod_{a\in\mathbb{C}^*}(1-ua)^{\gamma_{i,a}}$.

For $(\mu,\Gamma)\in QP_L^+$ and V a $\mathcal{U}_q(\hat{\mathfrak{g}})$-module, we denote $V_m=V_{\mu,\Gamma}$ where $m=Y_{\mu,\Gamma}$.

For $\chi\in A^{\mathbb{Z}}$ we say $\chi\in\mathcal{Y}$ if for $\lambda\in\mathfrak{h}$, there is a finite number of monomials of χ such that $\omega(m)=\lambda$ and there is a finite number of element $\lambda_1,...,\lambda_s\in\mathfrak{h}^*$ such that the coweights of monomials of χ are in $\bigcup_{j=1...s}\nu(D(\lambda_j))$ (where $D(\lambda_j)=\{\mu\in\mathfrak{h}^*/\mu\leq\lambda_j\}$). In particular $\mathcal{Y}$ has a structure of $\mathfrak{h}$-graded $\mathbb{Z}$-algebra.

Definition 2.3. *For $V\in$ Rep$(\mathcal{U}_q(\hat{\mathfrak{g}}))$ a representation, the q-character of V is:*

$$\chi_q(V)=\sum_{(\mu,\Gamma)\in QP_l^+}dim(V_{\mu,\Gamma})Y_{\mu,\Gamma}\in\mathcal{Y}$$

If C is finite the construction is given in [FR3] and it is proved that $\chi_q : \text{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) \rightarrow \mathcal{Y}$ is an injective ring homomorphism (with the ring structure on $\text{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ deduced from the Hopf algebra structure of $\mathcal{U}_q(\hat{\mathfrak{g}})$).

For general C, χ_q is defined in [H4]. A priori there is no ring structure on $\mathcal{Y}$ that comes from a tensor product, but we proved [H4]:

Theorem 2.4. *The image $Im(\chi_q) \subset \mathcal{Y}$ is a subalgebra of $\mathcal{Y}$.*

Let $*$ be the induced commutative product on $\text{Im}(\chi_q) \subset \mathcal{Y}$. Using a deformation of the new Drinfel'd coproduct we proved in [H4] :

Theorem 2.5. *For $(\lambda, \Psi), (\lambda', \Psi') \in P_l^+$ we have :*

$$L(\lambda, \Psi) * L(\lambda', \Psi') = L(\lambda + \lambda', \Psi\Psi') + \sum_{(\mu,\Phi)\in P_l^+/\mu<\lambda+\lambda'} Q_{\lambda,\Psi,\lambda',\Psi'}(\mu, \Phi)L(\mu, \Phi)$$

where the integers $Q_{\lambda,\Psi,\lambda',\Psi'}(\mu, \Phi) \geq 0$.

2.3. **Notations and technical tools.** For $i \in I$ and $a \in \mathbb{C}^*$ we set:

$$A_{i,a} = k_{r_i\alpha_i^\vee} Y_{i,aq_i^{-1}} Y_{i,aq_i} \prod_{j/C_{j,i}<0} \prod_{r=C_{j,i}+1,C_{j,i}+3,...,-C_{j,i}-1} Y_{j,aq^r}^{-1} \in A$$

The $A_{i,l}$ are algebraically independent (see [H2]). Let $\mathcal{A} = \mathbb{Z}[A_{i,a}^{-1}]_{i\in I,a\in\mathbb{C}^*} \subset \mathcal{Y}$.

For a product $M \in A$ such that $\omega(M) \in \mathbb{Z}\alpha_1 \oplus ... \oplus \mathbb{Z}\alpha_n$, denote $\omega(M) = -v_1(M)\alpha_1 - ... - v_n(M)\alpha_n$ and $v(M) = v_1(M) + ... + v_n(M)$. v defines a $\mathbb{N}$-gradation on $\mathcal{A}$.

Definition 2.6. *For $m, m' \in A$, we say that $m \geq m'$ if $m'm^{-1} \in \mathcal{A}$.*

For $m \in A$ and $J \subset I$, denote $u_J(m) = \sum_{j\in J,a\in\mathbb{C}^*} u_{j,a}(m)$, $m^{(J)} = k_{\omega(m)} \prod_{j\in J,a\in\mathbb{C}^*} Y_{j,a}^{u_{j,a}(m)}$ and ($j \in I$):

$$u_j^\pm(m) = \pm \sum_{l\in\mathbb{Z}/\pm u_{j,l}(m)>0} u_{j,l}(m) \ , \ u_J^\pm(m) = \sum_{j\in J} u_j^\pm(m)$$

For $J \subset I$, denote $B_J \subset A$ the set of J-dominant monomials (ie $\forall j \in J, l \in \mathbb{Z}$, $u_{j,l}(m) \geq 0$) and $B = B_I$. Note that for $(\lambda, \Psi) \in QP_l^+$: $((\lambda, \Psi) \in P_l^+ \Leftrightarrow Y_{\lambda,\Psi} \in B)$.

For $m \in B$ denote $V_m = L(\lambda, \Psi) \in \text{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ where $(\lambda, \Psi) \in P_l^+$ is given by $Y_{\lambda,\Psi} = m$. In particular for $i \in I, a \in \mathbb{C}^*$ denote $V_i(a) = V_{k_{\nu(\Lambda_i)}Y_{i,a}}$ and $X_{i,a} = \chi_q(V_{i,a})$. The simple modules $V_i(a)$ are called fundamental representations.

Denote $M_m = \prod_{i\in I,a\in\mathbb{C}^*} V_{i,a}^{*u_{i,a}(m)} \in \text{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$. We have $\chi_q(M_m) = \prod_{i\in I,a\in\mathbb{C}^*} X_{i,a}^{u_{i,a}(m)}$.

For $J \subset I$ we denote by $\mathfrak{g}_J$ the Kac-Moody algebra of Cartan matrix $(C_{i,j})_{i,j\in J}$ and by χ_q^J the morphism of q-characters for $\mathcal{U}_q(\hat{\mathfrak{g}}_J) \subset \mathcal{U}_q(\hat{\mathfrak{g}})$. Let us recall the definition of the morphism τ_J (section 3.3 in [FM] for finite case and [H4] for general case) :

We suppose that $\mathfrak{g}_J$ is finite. Let $\mathfrak{h}_J^\perp = \{\omega \in \mathfrak{h}/\forall i \in J, \alpha_i(\omega) = 0\}$ and $\mathfrak{h}_J = \bigoplus_{i\in J} \mathbb{Q}\Lambda_i^\vee$. Consider formal variables k'_ω ($\omega \in \mathfrak{h}_J$), k_ω ($\omega \in \mathfrak{h}_J^\perp$), $Y_{i,a}^\pm$ ($i \in J, a \in \mathbb{C}^*$), $Z_{j,c}$ ($j \in I - J$, $c \in \mathbb{C}^*$). Let $A^{(J)}$ be the commutative group of monomials :

$$m = k'_{\omega'(m)} k_{\omega(m)} \prod_{i\in J,a\in\mathbb{C}^*} Y_{i,a}^{u_{i,a}(m)} \prod_{j\in I-J,c\in\mathbb{C}^*} Z_{j,c}^{z_{j,c}(m)}$$

where a finite number of $u_{i,a}(m), z_{j,c}(m), r(m) \in \mathbb{Z}$ are non zero, $\omega(m) \in \mathfrak{h}_J^\perp$ and such that for $i \in J$, $\alpha_i(\omega'(m)) = r_i u_i(m) = r_i \sum_{a\in\mathbb{C}^*} u_{i,a}(m)$.

Let $\tau_J : A \to A^{(J)}$ be the group morphism defined formally by ($j\in I$, $a\in\mathbb{C}^*$, $\lambda\in\mathfrak{h}$):

$$\tau_J(Y_{j,a}) = Y_{j,a}^{\epsilon_{j,J}} \prod_{k\in I-J}\prod_{r\in\mathbb{Z}} Z_{k,aq^r}^{p_{j,k}(r)} \ ,\ \tau_J(k_\lambda) = k'_{\sum\limits_{i\in J}\alpha_i(\lambda)\Lambda_i^\vee} k_{\lambda-\sum\limits_{i\in J}\alpha_i(\lambda)\Lambda_i^\vee}$$

where $j\in J \Leftrightarrow \epsilon_{j,J}=1$ and $j\notin J\Leftrightarrow \epsilon_{j,J}=0$. The $p_{i,j}(r)\in\mathbb{Z}$ are defined in the following way : we write $\tilde{C}(z) = \frac{\tilde{C}'(z)}{d(z)}$ where $d(z), \tilde{C}'_{i,j}(z)\in\mathbb{Z}[z^{\pm}]$ and $(D(z)\tilde{C}'(z))_{i,j} = \sum\limits_{r\in\mathbb{Z}} p_{i,j}(r)z^r$.

It is proved in [FM] (finite case) and in [H4] (the proof is given for the $\tau_{\{i\}}$ ($i\in I$), but the proof for τ_J ($J\subset I$, $\mathfrak{g}_J$ finite) is the same) :

Lemma 2.7. *Consider V a module in $Rep(\mathcal{U}_q(\hat{\mathfrak{g}}))$ and a decomposition $\tau_J(\chi_q(V)) = \sum\limits_k P_k Q_k$ where $P_k\in\mathbb{Z}[Y_{i,a}^{\pm}, k'_h]_{i\in J, a\in\mathbb{C}^*, h\in\mathfrak{h}_J}$, Q_k is a monomial in $\mathbb{Z}[Z_{j,c}^{\pm}, k_h]_{j\in I-J, c\in\mathbb{C}^*, h\in\mathfrak{h}_J^\perp}$ and all monomials Q_k are distinct. Then the restriction of V to $\mathcal{U}_q(\hat{\mathfrak{g}}_J)$ is isomorphic to $\bigoplus\limits_k V_k$ where V_k is a $\mathcal{U}_q(\hat{\mathfrak{g}}_J)$-module and $\chi_q^J(V_k) = P_k$.*

2.4. **Classical algorithm.** Consider $\mathfrak{K} = \bigcap\limits_{i\in I}\mathfrak{K}_i \subset \mathcal{Y}$ where $\mathfrak{K}_i = \mathrm{Ker}(S_i)\subset\mathcal{Y}$ is the kernel of the screening operator S_i (see [H4]).

Theorem 2.8. *We have $\mathfrak{K} = Im(\chi_q)$ and it is a subalgebra of $\mathcal{Y}$.*

The result in proved in [FM] for C finite and in [H4] in general. Note that for $m\in B_i$, there is a unique $F_i(m)\in\mathfrak{K}_i$ such that m is the unique i-dominant monomial of $F_i(m)$ (see [H2]).

In [H2] a classical algorithm (and also a t-deformation of it) is proposed : if it is well-defined, it gives for $m\in B$ a $F(m)\in\mathfrak{K}$ such that m is the unique dominant monomial of $F(m)$. Such an algorithm was first used in [FM] for finite case (see also [H3]). Note that if $F(m)$ exists, it is unique (see [H2]). Let us describe this algorithm : first for $m\in B$ we have to define the set D_m :

Definition 2.9. *For $m\in B$, we say that $m'\in D_m$ if and only if there is a finite sequence ($m_0 = m, m_1, ..., m_R = m'$), such that for all $1\le r\le R$, there is $j\in I$ such that $m_{r-1}\in B_j$ and m_r is a monomial of $F_j(m_{r-1})$.*

In particular the set D_m is countable (see [H2]) and $m'\in D_m \Rightarrow m'\le m$. Denote $D_m = \{m_0 = m, m_1, m_2, ...\}$ such that $m_r\le m_{r'}\Rightarrow r\ge r'$.

For $r, r'\ge 0$ and $j\in I$ denote $[F_j(m_{r'})]_{m_r}\in\mathbb{Z}$ the coefficient of m_r in $F_j(m_{r'})$.

We call classical algorithm the following inductive definition of the sequences $(s(m_r))_{r\ge0}\in\mathbb{Z}^{\mathbb{N}}$, $(s_j(m_r))_{r\ge 0}\in\mathbb{Z}^{\mathbb{N}}$ ($j\in I$) : $s(m_0)=1$, $s_j(m_0)=0$ and for $r\ge 1, j\in I$:

$$s_j(m_r) = \sum_{r'<r}(s(m_{r'}) - s_j(m_{r'}))[F_j(m_{r'})]_{m_r}$$

$$m_r\notin B_j \Rightarrow s(m_r) = s_j(m_r)\ ,\ m_r\in B\Rightarrow s(m_r) = 0$$

It follows from theorem 2.8 that the classical algorithm is weel-defined and for all $m\in B$, $F(m)\in\mathfrak{K}$ exists (see section 5.5.4 in [H4]).

3. Monomials of q-characters

In this section we state the two main results on q-characters of this paper : theorems 3.2 and 3.5.

3.1. **First result.** In this section we prove that for m' a l-weight of V_m we have $m'\le m$ (theorem 3.2). This result is proved in [FR3, FM] for C finite. In the general case a universal $\mathcal{R}$-matrix has not been defined so we propose a new proof based on the Weyl modules introduced in [CP3].

Definition 3.1. *For $m\in B$, denote $L(m) = \chi_q(V_m)$ and by $D(m)$ the set of monomials of $L(m)$.*

The partial order on monomials is set in definition 2.6.

Theorem 3.2. *For $m \in B$ and $m' \in D(m)$, we have $m' \leq m$.*

In this section 3 we prove this theorem. First let us show some lemmas which will be useful :

Lemma 3.3. *Let V be a $\mathcal{U}_q(\hat{\mathfrak{g}})$-module and $W \subset V$ a $\mathcal{U}_q(\hat{\mathfrak{h}})$-submodule of V. Then for $i \in I$, $W_i' = \sum\limits_{r\in\mathbb{Z}} x_{i,r}^-.W$ is a $\mathcal{U}_q(\hat{\mathfrak{h}})$-submodule of V.*

Proof: For $w \in W$, $j \in J$, $m, r \in \mathbb{Z}$ $(m \neq 0)$, $h \in \mathfrak{h}$ we have :

$$h_{j,m}.(x_{i,r}^-.w) = x_{i,r}^-.(h_{j,m}.w) - \frac{1}{m}[mB_{i,j}]_q x_{i,m+r}^-.w \in W_i'$$

$$k_h.(x_{i,r}^-.w) = x_{i,r}^-.(q^{\alpha_i(h)}k_h.w) \in W_i'$$

□

Note that q-character of an (integrable) $\mathcal{U}_q(\hat{\mathfrak{h}})$-module is well-defined (see section 5.4 of [H4]).

Lemma 3.4. *Suppose that $\mathfrak{g} = sl_2$ and let L be a finite dimensional $\mathcal{U}_q(\hat{\mathfrak{g}})$-module ($\Lambda^\vee$ is the fundamental coweight).*

(i) If L is of l-highest weight M then $L_{m'} \neq \{0\}$ implies $m' \leq M$.

(ii) For $p \in \mathbb{Z}$, let $L_p = \sum\limits_{\lambda\in P^*/\lambda(\Lambda^\vee)\geq p} L_\lambda$ and $L_p' = \sum\limits_{r\in\mathbb{Z}} x_r^-.L_p$. Then L_p, L_p' are $\mathcal{U}_q(\hat{\mathfrak{h}})$-submodule of L and $(L_p')_{m'} \neq 0 \Rightarrow \exists m, m' \leq m$ and $(L_p)_m \neq \{0\}$.

Proof: (i) Consider the Weyl module $W_q(M)$ of l-highest weight M defined in [CP3] : $W_q(M)$ is the universal finite dimensional module of l-highest weight M such that all finite dimensional module of highest l-weight M is a quotient of $W_q(M)$. In particular L is a quotient of $W_q(M)$. So it suffices to study $W_q(M)$. For $\mathcal{U}_q(\hat{sl_2})$, the Weyl modules are explicitly described in [CP4] : in particular the dimension of $W_q(M)$ is 2^m where $m = u(M) = \sum\limits_{i\in I, a\in\mathbb{C}^*} u_{i,a}(M)$. But (see [VV, AK, FM]) there is a standard module (tensorial product of fundamental representation) of highest l-weight M. The dimension of such a standard module is 2^m and it is a quotient of $W_q(M)$. So $W_q(M)$ is isomorphic to a standard module. The q-character of a standard module is known (see section 2.3), in particular for a l-weight m' of $W_q(M)$ we have $m' \leq M$.

(ii) L_p is a $\mathcal{U}_q(\hat{\mathfrak{h}})$-submodule of L because the action of $\mathcal{U}_q(\hat{\mathfrak{h}})$ does not change the weight, so it follows from lemma 3.3 that L_p' is a $\mathcal{U}_q(\hat{\mathfrak{h}})$-submodule of L. Let us prove the second point by induction on $\dim(L_p)$: if $L_p = \{0\}$ we have $L_p' = \{0\}$. In general let v be a l-highest weight vector of L_p (there is at least one, see the proof of proposition 5.2 in [H4]) and denote by M his l-weight. Consider $V = \mathcal{U}_q(\hat{\mathfrak{g}}).v$. It is a l-highest weight module and so it follows from (i) that $V_m \neq \{0\} \Rightarrow m \leq M$. We can use the induction hypothesis with $L^{(1)} = L/V$ and we get the result because $\chi_q(L) = \chi_q(V) + \chi_q(L^{(1)})$. □

End of the proof of theorem 3.2 :

We prove the result by induction on $v(m'm^{-1}) \geq 0$. For $v(m'm^{-1}) = 0$ we have $m' = m$. In general suppose that the result is known for $v(m'm^{-1}) \leq p$ and let $W = \sum\limits_{m'/v(m'm^{-1})\leq p} (V_m)_{m'}$. Note that W is a $\mathcal{U}_q(\hat{\mathfrak{h}})$-submodule of V_m. It follows from the triangular decomposition of $\mathcal{U}_q(\hat{\mathfrak{g}})$ (see [H4]) that :

$$\bigoplus_{m'/v(m'm^{-1})=p+1} (V_m)_{m'} \subset \sum_{i\in I} W_i' \text{ where } W_i' = \sum_{r\in\mathbb{Z}} x_{i,r}^-.W$$

For $i \in I$, W_i' is a $\mathcal{U}_q(\hat{\mathfrak{h}})$-submodule of V_m (lemma 3.3). In particular $W_i' = \bigoplus\limits_{m'}(W_i' \cap (V_m)_m') = \bigoplus\limits_{m'}(W_i)_{m'}$ and it suffices to show that for $i \in I$, $(W_i')_{m'} \neq \{0\} \Rightarrow m' \leq m$.

Consider the decomposition of lemma 2.7 with $J = \{i\}$: $V = \bigoplus_k V_k$. We have $W = \bigoplus_k (V_k \cap W)$ and so $W_i' = \bigoplus_k (V_k \cap W_i')$ (because V_k is a sub $\mathcal{U}_q(\hat{\mathfrak{g}}_i)$-module of V_m). So we can use the (ii) of lemma 3.4 to the $\mathcal{U}_{q_i}(\hat{sl}_2)$-module V_k with p_k such that $(V_k)_{p_k} = V_k \cap W$. We get that for m a monomial of $\chi_q^i(W_i')$ there are m' a monomial $\chi_q^i(W)$ and $m'' \in \mathbb{Z}[Y_{i,a}^{-1} Y_{i,aq_i^2}^{-1} (k_{2r_i}^{(i)})^{-1}]_{a \in \mathbb{C}^*}$ such that $m = m'm''$ (the $k_r^{(i)}$ are the k_h' for $\mathcal{U}_{q_i}(\hat{sl}_2)$, see [H4]). It follows from the lemma 5.9 of [H4] (see also [FM]) that $\tau_i(A_{i,aq_i}) = Y_{i,a} Y_{i,aq_i^2} k_{2r_i}^{(i)}$. So for M a monomial of $\chi_q(W_i')$ there is M' a monomial of $\chi_q(W)$ such that $M \leq M'$. □

3.2. Second result.

Theorem 3.5. *Let $\mathfrak{g}$ be of type A_n $(n \geq 1)$, $A_l^{(1)}$ $(l \geq 2)$, B_n $(n \geq 2)$ or C_n $(n \geq 2)$. Let $i \in I, a \in \mathbb{C}^*$. Then for $m \in D(Y_{i,a})$, for all $j \in I, l \in \mathbb{Z}$, $u_{j,l}(m) \leq 1$. In particular all coefficients of $L(Y_{i,a})$ are equal to* 1 *and all l-weight space of $V_i(a)$ are of dimension 1.*

The last part of the result for type A_n is established in [N4].

Note that for type D_n the statement is false : for example for the type D_4, the monomial $Y_{2,2}Y_{2,4}^{-1}$ has a coefficient 2 in $\chi_q(V_2(q^0))$ (see the figure 1 in [N2]). For type F_4 it is also false (see section 8).

Let us explain the main points of the proof : it is based on the study of the classical algorithm in a case by case investigation : for type A_n a proof is given in [H2] and recalled in section 4. The result for type $A_l^{(1)}$ is proved in section 4, the result for type B_n is proved in section 5, the result for type C_n in section 6. In each case we suppose the existence of a $m \in D(Y_{i,a})$, such that there is $j \in I, l \in \mathbb{Z}$, $u_{j,l}(m) \geq 2$. The classical algorithm starts from the highest weight monomial. In our proof we look at a monomial m with $u_{j,l}(m) \geq 2$ and show that inductively that it implies a condition on some monomials of higher weight. In particular it leads to a contradiction on the highest weight monomial.

Note that for type A_n, B_n, C_n the result could follow from explicit computation of $\chi_q(V_i(a))$. We would have to compute the specialized $\mathcal{R}$-matrix, as explained in [FR3]. The result should produce the formulas of [FR1]. However with this method it would not easy to decide if the coefficients are 1 (for example it is not the case for type D_4). In this paper the proof is direct without explicit computation. In particular it allows us to extend the proof to $A_l^{(1)}$.

3.3. Notations.

In the following (sections 3, 4, 5, 6) we can forget the terms k_λ because we work in a set $D(m)$ or D_m : indeed m' such that $m' \leq m$ is uniquely determined by m and the $v_{i,l}(m'm^{-1})$.

For $J \subset I$, $j \in J, a \in \mathbb{C}^*$ consider $A_{j,a}^{J,\pm} = (A_{j,a}^{\pm})^{(J)}$. Define $\mu_J^I : \mathbb{Z}[A_{j,a}^{J,\pm}]_{j \in J, a \in \mathbb{C}^*} \to \mathbb{Z}[A_{j,a}^{\pm}]_{j \in J, a \in \mathbb{C}^*}$ the ring morphism such that $\mu_J^I(A_{j,a}^{J,\pm}) = A_{j,a}^{\pm}$. For $m \in B_J$, denote $L^J(m^{(J)})$ defined for $\mathfrak{g}_J$ ($\mathfrak{g}_J$ is the Kac-Moody algebra of Cartan matrix $(C_{i_1,i_2})_{i_1,i_2 \in J}$). Define :

$$L_J(m) = m^{(I-J)} \mu_J^I((m^{(J)})^{-1} L^J(m^{(J)}))$$

Definition 3.6. *For $J \subset I$ and $m \in B_J$, denote by $D_J(m)$ the set of monomials of $L_J(m)$.*

For $J = \{i\}$ and $m \in B_i$, an explicit description of $D_i(m)$ is given in [FR3] : a $\sigma \subset \mathbb{Z}$ is called a 2-segment if σ is of the form $\sigma = \{l, l+2, ..., l+2k\}$ where $l \in \mathbb{Z}, k \geq 0$. Two 2-segment are said to be in special position if their union is a 2-segment that properly contains each of them. All finite subset of $\mathbb{Z}$ with multiplicity $(l, u_l)_{l \in \mathbb{Z}}$ $(u_l \geq 0)$ can be broken in a unique way into a union of 2-segments which are not in pairwise special position. For $m \in B_i$ and $r \in \{1, ..., 2r_i\}$, consider $(\sigma_j^{(r)})_j$ the decomposition of the $(l, u_{r+2r_i l}(m))_{l \in \mathbb{Z}}$ as above. Let $m^{(i)} = \prod_{r=1,...,2r_i} \prod_j m_{r,j}$ where $m_{r,j} = \prod_{l \in \sigma_j^{(r)}} Y_{i,r+2r_i l}$, and we have :

$$D_i(m) = m^{(I-\{i\})} \prod_{r=1,...,2r_i} \prod_j D_i(m_{r,j})$$

where for $m = \prod_{k=1...r} Y_{i,l+2r_ik}$:

$$D_i(m) = \{mA^{-1}_{i,l+2r_ik+r_i}, mA^{-1}_{i,l+2r_ik+r_i}A^{-1}_{i,l+2r_i(k-1)+r_i}, ..., mA^{-1}_{i,l+2r_ik+r_i}A^{-1}_{i,l+2r_i(k-1)+r_i}...A^{-1}_{i,l+r_i}\}$$

In particular :

Lemma 3.7. *For $m \in B_i$ such that $\forall l \in \mathbb{Z}, u_{i,l}(m) \leq 1$, we have $F_i(m) = L_i(m)$.*

Definition 3.8. *Let $J \subset I$ and $i \in I, a \in \mathbb{C}^*$. For $m, m' \in D(Y_{i,a})$, we denote :*

$m \rightarrow_J m'$ (or $m' \leftarrow_J m$) if $m \in B_J$ and $m' \in D_J(m)$.

$m \rightharpoonup_J m'$ (or $m' \leftharpoondown_J m$) if $v(m'Y_{i,a}^{-1}) \geq v(mY_{i,a}^{-1})$ and $\exists m'' \in D(Y_{i,a})$ such that $m'' \rightarrow_J m$ and $m'' \rightarrow_J m'$.

In particular $m \rightarrow_J m'$ implies $m \rightharpoonup_J m'$. For $J = \{j\}$ (resp. $J = I$) we denote $\rightarrow_j, \rightharpoonup_J$ (resp. $\rightarrow, \rightharpoonup$).

3.4. **Technical complements.**

Proposition 3.9. *For $m \in B$ and $J \subset I$ such that $\mathfrak{g}_J$ is finite, there is a unique decomposition:*

$$L(m) = \sum_{m' \in B_J \cap D(m)} \lambda_J(m')L_J(m')$$

where $\lambda_J(m') \geq 0$.

Proof: Consider the decomposition of lemma 2.7 with J : $V = \bigoplus_k V_k$. We can decompose each V_k in a sum of simple $\mathcal{U}_q(\hat{\mathfrak{g}}_J)$-modules in the Grothendieck group : $\chi_q^J(V_k) = \sum_{k'} \lambda_{k,k'} L^J(m_{k,k'})$ where $m_{k.k'} \in B_J$ and $\lambda_{k,k'} \geq 0$. In particular $\tau_J^{-1}(P_kQ_k) = \sum_{k'} \lambda_{k,k'} L_J(\tau_J^{-1}(m_{k,k'}Q_k))$ (consequence of lemma 5.9 of [H4]). For the uniqueness the $L_J(m')$ ($m' \in B_J$) are linearly independent. □

We say that a monomial $m \in B$ is right (resp. left) negative if : for $b \in \mathbb{C}^*$ such that ($\exists j \in I$, $u_{j,b}(m') \neq 0$ and $\forall k \in I, l > 0$ (resp. $l < 0$), $u_{k,bq^l}(m') = 0$), we have $\forall k \in I$, $u_{k,b}(m') \leq 0$ (see [FM]). A product of right (resp. left) negative monomials is right (resp. left) negative.

Corollary 3.10. *For $i \in I, a \in \mathbb{C}^*$ and $m' \in D(Y_{i,a})$, we have :*

1) for $J \subset I$ such that $\mathfrak{g}_J$ is finite, there is $m'' \rightarrow_J m'$.

2) there is a finite sequence $Y_{i,a} = m_0 > m_1 > m_2 > ... > M_R = m'$ such that for all $1 \leq r \leq R$, $\exists j_r \in I$, $m_{r-1} \rightarrow_{j_r} m_r$.

3) if $m' \neq Y_{i,a}$, then m' is right negative

4) for $b \in \mathbb{Z}$ and $j \in I$, we have $u_{j,b}(m') \neq 0 \Rightarrow b \in aq^{\mathbb{Z}}$.

Note that the (1) will be used intensively in the following. For C finite those results are proved in [FM].

Proof: 1) Consequence of proposition 3.9.

2) We use (1,3) inductively.

3) For $m' \in D_{Y_{i,a}} - \{Y_{i,a}\}$, we have $m' < Y_{i,a}$ (theorem 3.2) and m' is right or left negative, so not dominant. So as in [FM] m' is right negative.

4) As for $m \in B_j \cap \mathbb{Z}[Y_{i,aq^m}]_{m \in \mathbb{Z}}$ implies $L_j(m) \in \mathbb{Z}[Y_{i,aq^m}]_{m \in \mathbb{Z}}$ (see section 3.3), we have $M \in B \cap \mathbb{Z}[Y_{i,aq^m}]_{m \in \mathbb{Z}}$ implies $D_m \subset \mathbb{Z}[Y_{i,aq^m}]_{m \in \mathbb{Z}}$ (see also [FM]). □

As a right negative monomial is not dominant, we have :

Corollary 3.11. *For $i \in I, a \in \mathbb{C}^*$, $L(Y_{i,a}) = F(Y_{i,a})$ has a unique dominant monomial $Y_{i,a}$.*

For $c \in \mathbb{C}^*$, let $\beta_c : \mathcal{Y} \rightarrow \mathcal{Y}$ be the ring morphism such that $\beta_c(Y_{i,a}) = Y_{i,ac}$.

Proposition 3.12. *For $a,b\in\mathbb{C}^*$, $L(Y_{i,a})=\beta_{ab^{-1}}(L(Y_{i,b}))$.*

Proof: For $c\in\mathbb{C}^*$, we have $\beta_c(\mathfrak{K})=\mathfrak{K}$ (see [FM, H3]). □

It suffices to study (see (4) of corollary 3.10 and [H3]) :

$$\chi_q:\mathbb{Z}[X_{i,q^l}]_{i\in I,l\in\mathbb{Z}}\to\mathbb{Z}[Y_{i,q^l}]_{i\in I,l\in\mathbb{Z}}$$

In the following we denote Rep $=\mathbb{Z}[X_{i,q^l}]_{i\in I,l\in\mathbb{Z}}$, $X_{i,l}=X_{i,q^l}$, $\mathcal{Y}=\mathbb{Z}[Y_{i,q^l}^{\pm}]_{i\in I,l\geq 0}$, $Y_{i,l}^{\pm}=Y_{i,q^l}^{\pm}$. A Rep-monomials is a product of the $X_{i,l}$.

Lemma 3.13. *For $m\in B$, we have :*

$$D(m)\subset\prod_{j\in I,l\in\mathbb{Z}}D(Y_{j,l})^{u_{j,l}(m)}$$

Proof: $\prod_{j\in I,l\in\mathbb{Z}}D(Y_{j,l})^{u_{j,l}(m)}$ is the set of monomials of $\chi_q(M_m)$. Then see theorem 2.5. □

Lemma 3.14. *Let $m_1,m_2\in B_i$ such that $\forall l\in\mathbb{Z}$, $u_{i,l}(m_1)\leq 1$ and $u_{i,l}(m_2)\leq 1$. Then $D_i(m_1)=D_i(m_1)$ (resp. $D_i(m_2)=D_i(m_2)$) and $D_i(m_1)\cap D_i(m_2)=\emptyset\Leftrightarrow m_1\neq m_2$.*

Proof: Let us write $m_1^{(i)}=\prod_{r=1,...,2r_i}\prod_j m^{(r)}_{\sigma_j^{(r)}}$ as in section 3.3 . Denote $\overline{\sigma_j^{(r)}}=\sigma_j^{(r)}\cup\{\max(\sigma_j^{(r)})+2r_i\}$. It follows from the hypothesis of the lemma that $(j,r)\neq(j',r')\Rightarrow\overline{\sigma_j^{(r)}}\cap\overline{\sigma_{j'}^{(r')}}=\emptyset$. Moreover for $m'\in D_i(m_{\sigma_j^{(r)}})$, we have $u_{i,r+2lr_i}(m')\neq 0\Rightarrow\exists j,l\in\overline{\sigma_j^{(r)}}$. In particular the given of m' suffices to determine the $\overline{\sigma_j^{(r)}}$: for example we can find the set $\mathcal{M}=\{\max(\overline{\sigma_j^{(r)}})/j,r\}$ and $\mathcal{M}'=\{\min(\overline{\sigma_j^{(r)}})/j,r\}$ in the following way :

if $u_{i,l}(m')=1$ and $u_{i,l+2r_i}(m')=0$ and $u_{i,l+4r_i}(m')\geq 0$, then $l+2r_i\in\mathcal{M}$

if $u_{i,l}(m')=-1$ and $u_{i,l+2r_i}(m')\geq 0$, then $l\in\mathcal{M}$

if $u_{i,l}(m')=-1$ and $u_{i,l-2r_i}(m')=0$ and $u_{i,l-4r_i}(m')\leq 0$, then $l-2r_i\in\mathcal{M}'$

if $u_{i,l}(m')=1$ and $u_{i,l-2r_i}(m')\leq 0$, then $l\in\mathcal{M}'$

So if $m'\in D_i(m_1)\cap D_i(m_2)$, we have the same decomposition for m_1 and m_2, that is to say $m_1=m_2$. □

Lemma 3.15. *Let $m\in B$, $m'\in D(m)\cap B_j$. We suppose that for all $m''\in D(m)$ such that $v(m''m^{-1})<v(m'm^{-1})$, all $i\in I,l\in\mathbb{Z}$ we have $u_{i,l}(m'')\leq 1$. Then $D_j(m')\subset D(m)$.*

Proof: Let $p=v(m'm^{-1})$. Consider the decomposition of proposition 3.9 with $J=\{j\}$: $L(m)=\sum_{M\in B_j\cap D(m)}\lambda_j(M)L_j(M)$. It follows from the hypothesis and from the lemma 3.7 that for $v(Mm^{-1})<p$, $m'\notin D_j(M)$. So $\lambda_j(m')>0$, and $D_j(m')\subset D(m)$. □

Proposition 3.16. *Let $i\in I$ such that all $m\in D(Y_{i,L})$ satisfies : for $j\in I$, if $m\in B_j$ then $\forall l\in\mathbb{Z},u_{j,l}(m)\leq 1$. Then all coefficients of $L(Y_{i,L})$ are equal to 1.*

Proof: We can compute the coefficients of $L(Y_{i,L})=F(Y_{i,L})$ thanks to the classical algorithm (see section 3.3) : let us show by induction on $v(mY_{i,L}^{-1})$ that the coefficients of m is equal to 1. For a monomial $m<Y_{i,L}$, there is $j\in I$ such that $m\notin B_j$. There is $M\to_j m$. It follows from the lemma 3.14 that M is entirely determined by m. So the coefficient of m is the coefficient of M in $L(Y_{i,l})$ multiplied with the coefficient of m in $L_j(M)=F_j(M)$, that is to say 1 (section 3.3). □

4. Type A, $A^{(1)}$

Proposition 4.1. *The property of theorem 3.5 is true for $\mathfrak{g}$ of type A_n $(n\geq 1)$ or $A_l^{(1)}$ $(l\geq 2)$.*

Technical consequences of this result which will be used in the following are discussed in section 4.3.

4.1. **Type** A. Let $n \geq 1$ and $\mathfrak{g}$ of type A_n. For $i \in \{2, ..., n-1\}$, $l \in \mathbb{Z}$:

$$A_{i,l} = Y_{i,l+1} Y_{i,l-1} Y_{i+1,l}^{-1} Y_{i-1,l}^{-1}$$

$$A_{1,l} = Y_{1,l+1} Y_{1,l-1} Y_{2,l}^{-1} \text{ , } A_{n,l} = Y_{n,l+1} Y_{n,l-1} Y_{n-1,l}^{-1}$$

In particular for all $i \in I, l \in \mathbb{Z}$, $u(A_{i,l}^{-1}) \leq 0$. So $m \leq m' \Rightarrow u(m) \leq u(m')$.

We can suppose $Y_{i,L} = Y_{i,0}$ (proposition 3.12).

Lemma 4.2. *For $m \in B$ and $m' \in D(m)$ we have $u(m) \geq v_n(m'm^{-1})$.*

Proof: For all $i \in I$, we have $\omega_i + \omega_{n+1-i} \in \alpha_n + \sum_{j \leq n-1} \mathbb{Z}\alpha_j$ (see [Bo]).

Consider $m' \in D(m)$. It follows from the lemma 6.8 of [FM] that $\omega(m) \geq \omega(m') \geq -\sum_{i \in I} u_i(m)\omega_{n+1-i}$, and so $-\omega(m'm^{-1}) \leq \sum_{i \in I} u_i(m)(\omega_i + \omega_{n+1-i})$. So $v_n(m'm^{-1}) \leq \sum_{i \in I} u_i(m) = u(m)$. □

Lemma 4.3. *For $j \in I$, if $m \in B_j \cap D(Y_{i,0})$ then $u_j(m) \leq 1$.*

Proof: Suppose there is $j \in I$ and $m_1 \in B_j \cap D(Y_{i,0})$ such that $u_j(m_1) \geq 2$. Let $J_1 = \{k \in I / k < j\}$, $J_2 = \{k \in I / k > j\}$ and $J = J_1 \cup J_2$. Let $m_2 \rightarrow_J m_1$ and $v = v_{j-1}(m_1 m_2^{-1}) + v_{j+1}(m_1 m_2^{-1})$. It follows from lemma 4.2 (for $\mathfrak{g}_{J_1}$ and $\mathfrak{g}_{J_2}$) that $u_{J_1}(m_2) + u_{J_2}(m_2) \geq v$. Moreover we have $u_j(m_2) = u_j(m_1) - v \geq 2 - v$. So $u(m_2) = u_{J_1}(m_2) + u_j(m_2) + u_{J_2}(m_2) \geq 2$, contradiction because $m_2 \leq Y_{i,0}$. □

The proposition 4.1 for type A_n follows from proposition 3.16 and lemma 4.3.

4.2. **Type** $A_l^{(1)}$. Let $l \geq 2$ and $\mathfrak{g}$ of type $A_l^{(1)}$. For $i \in I$, $l \in \mathbb{Z}$ (where $Y_{-1,L} = Y_{n,L}$, $Y_{n+1,L} = Y_{0,L}$):

$$A_{i,l} = Y_{i,l+1} Y_{i,l-1} Y_{i+1,l}^{-1} Y_{i-1,l}^{-1}$$

In particular for all $i \in I, l \in \mathbb{Z}$, $u(A_{i,l}^{-1}) \leq 0$. So $m \leq m' \Rightarrow u(m) \leq u(m')$.

We have an analog of lemma 4.3 by putting in the proof $J = I - \{j\}$ instead of $J_1 \cup J_2$. In particular we get proposition 4.1 for type $A_l^{(1)}$.

4.3. **Consequences.** In this section $\mathfrak{g}$ is general and consider $J \subset I$ such that $\mathfrak{g}_J$ is of type A_m ($m \leq n$). We prove technical results which will be useful in the following. Let $i \in I, a \in \mathbb{C}^*$.

Lemma 4.4. *Let $m \in B_J$, $j \in J$ and $m' \in B_j$ such that $m' \in D_J(m)$. We have $u_J(m) \geq u_j(m')$.*

Proof: It follows from lemma 3.13 that we can write :

$$m' = m^{(I-J)} \prod_{k \in J, l \in \mathbb{Z}} m'_{k,l,1} ... m'_{k,l,u_{k,l}(m)}$$

where $m'_{k,l,1} \in D_J(Y_{k,l})$. For $\alpha \in J \times \mathbb{Z} \times \mathbb{N}$, it follows from lemma 4.3 that $u_{j,L}^+(m'_\alpha) \geq 1 \Rightarrow (m'_\alpha)^{(j)} = Y_{j,L}$. So there are $\alpha_1, ..., \alpha_{u_j(m')}$ such that the $(m'_{\alpha_p})^{(j)} = Y_{j,l_p}$ and $m'_{\alpha_1} ... m'_{\alpha_{u_j(m')}} = (m')^{(j)}$. So $u_j(m') \leq \sum_{k \in J, l \in \mathbb{Z}} u_{k,l}(m) = u_J(m)$. □

Lemma 4.5. *Let $M \in B_J$ such that $u_J(M) \geq 2$. The following properties are equivalent :*

(i) there are $j \in J$, $l \in \mathbb{Z}$, $M_1 \in D_J(M) \cap B_j$ such that $u_{j,l}(M_1) \geq 2$

(ii) there are $M' \in D_J(M) \cap B_J$, $i_1, i_2 \in J$, $l_1, l_2 \in \mathbb{Z}$ such that $i_2 - i_1 \geq |l_1 - l_2|$ and $(i_2 - i_1) - (l_2 - l_1)$ is even and $u_{i_1,l_1}(M') \geq 1$, $u_{i_2,l_2}(M') \geq 1$.

Moreover one can choose M' such that $M_1 \in D_J(M')$.

Proof: We can suppose that $\mathfrak{g} = \mathfrak{g}_J$ is of type A_n. For $K \subset I$, in this proof the notation $\rightarrow_K, \rightharpoonup_K$ is defined as in definition 3.8 by putting $D(M)$ instead of $D(Y_{i,a})$.

Let us show that $(ii) \Rightarrow (i)$: if $i_2 = i_1$ we have $u_{i_1,l_1}(M') \geq 2$. If $i_2 - i_1 > 0$, suppose that (i) is not true. In this situation we can use the lemma 3.15. Consider the integers:

$$K = \frac{(i_2 - i_1) + (l_2 - l_1)}{2} \ , \ K' = \frac{(i_2 - i_1) + (l_1 - l_2)}{2}$$

We have $K, K' \geq 0$. Denote $i = i_1 + K = i_2 - K'$, $l = l_1 + K = l_2 + K'$ and consider :

$$V = A^{-1}_{i_1,l_1+1} A^{-1}_{i_1+1,l_1+2} ... A^{-1}_{i_1+(K-1),l_1+K} A^{-1}_{i_2,l_2+1} A^{-1}_{i_2-1,l_2+2} ... A^{-1}_{i_2-(K'-1),l_2+K'}$$

There is $M_1 \in D(M')$ such that $M_1 \leq M'V$ and $v_i(M_1(M')^{-1}) = 0$ (lemma 3.15). In particular $M_1 \in B_i$ and $u_{i,l}(M_1) \geq 2$, contradiction.

Let us show that $(i) \Rightarrow (ii)$: it follows from lemma 3.13 and proposition 4.1 that we can suppose that $u(M) = 2$. Denote $M = Y_{i_1,l_1} Y_{i_2,l_2}$, and let us show the result by induction on n. For $n = 1$ we have $M_1 = M$ and (ii) is clear. In general let $M_1 \in D(M) \cap B_j$ such that $M_1^{(j)} = Y^2_{j,l}$. We can suppose that $v(M_1 M^{-1})$ is minimal. If M_1 is dominant, we put $M_1 = M'$. Otherwise consider $J' = \{1, ..., n-1\}$ if $j \leq n-1$, and $J' = \{2, ..., n\}$ if $j = n$ (we suppose that $j \leq n-1$, the case $j = n$ can be treated in the same way). Let $M_2 \rightarrow_{J'} M_1$. The induction with $\mathfrak{g}_{J'}$ of type A_{n-1} gives that $M_2^{(J')} = Y_{i_1,l_1} Y_{i_2,l_2}$ where $i_2 - i_1 \geq |l_1 - l_2|$ and $(i_2 - i_1) - (l_2 - l_1)$ is even. We have $u(M_2) \leq 2$ and so $u_n(M_2) = u(M_2) - u_{J'}(M_2) \leq 0$. If $M_2^{(n)} = 1$, we put $M_2 = M'$. Otherwise it follows from the lemma 4.4 that we are in one the following cases α, β, γ:

α) if $M_2^{(n)} = Y^{-1}_{n,K_1} Y^{-1}_{n,K_2}$, we have :

$$M_2 \leftarrow_n M_3 = Y_{i_1,l_1} Y_{i_2,l_2} Y^{-1}_{n-1,K_1-1} Y^{-1}_{n-1,K_2-1} Y_{n,K_1-2} Y_{n,K_2-2}$$

If $Y_{i_2,l_2} \neq Y_{n-1,K_1-1}$ and $Y_{i_2,l_2} \neq Y_{n-1,K_2-1}$, there is $M_4 = M_3(M_1 M_2^{-1}) \in D(M_3)$ (lemma 3.15) such that $M_4^{(j)} = Y^2_{j,l}$ and $v(M_4 M^{-1}) < v(M_1 M^{-1})$, contradiction. So for example we have $M_3 = Y_{i_1,l_1} Y^{-1}_{n-1,K_2-1} Y_{n,K_1-2} Y_{n,K_2-2}$ and $i_2 = n-1$, $l_2 = K_1 - 1$. We have :

$$M_3 \leftarrow_{\{i_1+1,...,n-1\}} M_4 = Y_{i_1,l_1} Y^{-1}_{i_1,K_2-1-(n-i_1-1)} Y_{i_1+1,K_2-2-(n-i_1-1)} Y_{n,K_1-2}$$

If $Y_{i_1,l_1} Y^{-1}_{i_1,K_2-1-(n-i_1-1)} \neq 1$, there is :

$$M_4 \leftarrow_{\{1,...,i_1\}} M_5 = Y_{1,K_3} Y_{i_1,l_1} Y^{-1}_{i_1+1,K_2-2-(n-i_1-1)} Y_{i_1+1,K_2-2-(n-i_1-1)} Y_{n,K_1-2}$$

and $u(M_5) = 3$, impossible. So $l_1 = K_2 - 1 - (n - i_1 - 1)$ and $M_4 = Y_{i_1',l_1'} Y_{i_2',l_2'}$ where $i_1' = i_1 + 1$, $i_2' = n$, $l_1' = K_2 - 2 - (n - i_1 - 1)$, $l_2' = K_1 - 2$. Let $M' = M_4$. We have $i_2' - i_1' = i_2 - i_1 \geq |l_2 - l_1| = |l_2' - l_1'|$ and $(i_2' - i_1') - (l_2' - l_1') = (i_2 - i_1) - (l_2 - l_1)$ is even.

β) if $M_2^{(n)} = Y_{n,K_1} Y^{-1}_{n,K_2}$, we have :

$$M_2 \leftarrow_n M_3 = Y_{i_1,l_1} Y_{i_2,l_2} Y^{-1}_{n-1,K_2-1} Y_{n,K_1-2} Y_{n,K_2-2}$$

and $u(M_3) = 3$, impossible.

γ) if $M_2^{(n)} = Y^{-1}_{n,K_1}$, we have :

$$M_2 \leftarrow_n M_3 = Y_{i_1,l_1} Y_{i_2,l_2} Y^{-1}_{n-1,K_1-1} Y_{n,K_1-2}$$

If $Y_{n-1,K_1-1} \neq Y_{i_1,l_1}$ and $Y_{n-1,K_1-1} \neq Y_{i_2,l_2}$, there is $M_4 = M_3(M_1 M_2^{-1}) \in D(M_3)$ (lemma 3.15) such that $M_4^{(j)} = Y^2_{j,l}$ and $v(M_4 M^{-1}) < v(M_1 M^{-1})$, contradiction. So for example we have $M_3 = Y_{i_1,l_1} Y_{n,K_1-2}$ and $i_2 = n-1$, $l_2 = K_1 - 1$. Let $M' = M_3$ and we have $n - i_1 = i_2 - i_1 + 1 \geq |l_2 - l_1| + 1 \geq |(K_1 - 2) - l_1 + 1| + 1 \geq |(K_1 - 2) - l_1|$ and $n - i_1 - ((K_1 - 2) - l_1) = (i_2 - i_1) - (l_2 - l_1)$ is even.

For the last point, the arguments of this proof can be used for any $M' \in B$ such that $M_1 \in D_J(M')$. $\square$

5. Type B

5.1. **Statement.** In this section $\mathfrak{g}$ is of type B_n $(n\geq 2)$. For $i\in\{2,...,n-2\}$, $l\in\mathbb{Z}$:

$$A_{i,l}=Y_{i,l+2}Y_{i,l-2}Y_{i+1,l}^{-1}Y_{i-1,l}^{-1}\ ,\ A_{1,l}=Y_{1,l+2}Y_{1,l-2}Y_{2,l}^{-1}\ ,\ A_{n,l}=Y_{n,l+1}Y_{n,l-1}Y_{n-1,l}^{-1}$$

$$A_{n-1,l}=Y_{n-1,l+2}Y_{n-1,l-2}Y_{n-2,l}^{-1}Y_{n,l-1}^{-1}Y_{n,l+1}^{-1}$$

In this section we prove:

Proposition 5.1. *The property of theorem 3.5 is true for $\mathfrak{g}$ of type B_n $(n\geq 2)$.*

Denote $J=\{1,...,n-1\}$. We can suppose $Y_{i,L}=Y_{i,0}$ (proposition 3.12).

As $u(A_{n-1,l}^{-1})>0$, the $m'\leq m$ does not imply $u(m')\leq u(m)$.

5.2. **Proof of the proposition 5.1.** Suppose that there is $m\in D(Y_{i,0})$ such that there is $j\in J,l\in\mathbb{Z}$, $u_{j,l}(m)\geq 2$, and let m such that $v(mY_{i,0}^{-1})$ is minimal with this property.

Lemma 5.2. *There is $M\in D(Y_{i,0})$ such that $v(MY_{i,0}^{-1})<v(mY_{i,0}^{-1})$ and $\exists l'\in\mathbb{Z}$, $u_{n,l'}(M)\geq 2$.*

Proof: Suppose that M does not exists. Let $m_1\rightarrow_J m$. It follows from lemma 4.5 that $m_1=m_1'Y_{i_1,l_1}Y_{i_2,l_2}$ where $m_1'\in B_J$, $2(i_2-i_1)\geq|l_1-l_2|$, $(i_2-i_1)-(l_2-l_1)/2$ is even. Let $m_2\rightharpoonup_n m_1$ such that $v(m_1m_2^{-1})=1$ (m_2 exists because it follows from the hypothesis and lemma 3.15 that $M_2\rightarrow_n m_2$ implies $D_n(M_2)\subset D(Y_{i,0})$). We have $m_2=m_2'Y_{i_1,l_1}Y_{i_2,l_2}Y_{n-1,L}^{-1}Y_{n,L-1}$ where $m_2'\in B_J$ and $u_{n,L-1}(m_2')\geq 0$. If $Y_{i_1,l_1}Y_{i_2,l_2}Y_{n-1,L}^{-1}\notin B$, there is $m_2\rightharpoonup_J M_2=m_2(mm_1^{-1})$ (lemma 3.15) such that $u_{j,l}(M_2)\geq 2$ and $v(M_2Y_{i,0}^{-1})<v(mY_{i,0}^{-1})$, contradiction. So for example $Y_{i_2,l_2}=Y_{n-1,L}$, and $m_2\in B_J$. m_2 is not dominant (because we would have $u(m_2)\geq 2$), so there is $m_3\rightharpoonup_n m_2$ such that $v(m_2m_3^{-1})=1$ (same argument as above for the existence of m_3). We have $m_3=m_3'Y_{i_1,l_1}Y_{n-1,L'}^{-1}Y_{n,L-1}Y_{n,L'-1}$ where $m_3'\in B_J$ and $u_{n,L-1}(m_3')\geq 0$, $u_{n,L'-1}(m_3')\geq 0$.

if we can choose $L'\neq L+2$, the same argument gives $Y_{i_1,l_1}=Y_{n-1,L'}$ because :

$$m_4=Y_{i_1,l_1}Y_{i_2,l_2}Y_{n-1,L'}^{-1}Y_{n,L-1'}m_4'\rightharpoonup_n m_1$$

where $m_4'\in B_J$. So we have $i_1=i_2=n-1$, so $l_1=l_2$ and $L'=L$, ie $u_{n,L}(m_3)\geq 2$, contradiction.

if we can not choose $L'\neq L+2$, we can not use the same argument (because : $1\notin L_n(Y_{n,L-1}Y_{n,L+1})$). We have $(k\geq 1)$:

$$m_3\leftharpoondown_n m_5=m_5'Y_{i_1,l_1}Y_{n-1,L+2}^{-1}Y_{n-1,L+4}^{-1}...Y_{n-1,L-1+2k+1}^{-1}Y_{n,L-1}Y_{n,L+1}...Y_{n,L-1+2k}$$

where $m_5'\in B$. Suppose that $m_5'Y_{i_1,l_1}Y_{n-1,L+4}^{-1}\notin B$. Then we have :

$$m_5\leftharpoondown_{n-1} m_6=m_5'Y_{i_1,l_1}Y_{n-1,L}Y_{n-1,L+2}^{-1}Y_{n-1,L+6}^{-1}...Y_{n-1,L-1+2k+1}^{-1}Y_{n,L-1}Y_{n,L+5}...Y_{n,L-1+2k}$$

But $u_{n-1,L}(m_6A_{n,L}^{-1})=2$, contradiction. In the same way we prove by induction that $m_5'Y_{i_1,l_1}Y_{n-1,L+4}^{-1}Y_{n-1,L+8}^{-1}...\in B$ and so :

$$m_5=m_5''Y_{i_1,l_1}Y_{n-1,L+2}^{-1}Y_{n-1,L+6}^{-1}...Y_{n-1,L+2+4K'}^{-1}Y_{n,L-1}Y_{n,L+1}...Y_{n,L-1+2k}$$

where $m_5''\in B$. Suppose that $m_5\notin B$. As m_5 is right negative, we have $L+2+4K'=L+2k$ and so $k=1+2K'$. Consider $m_7\rightarrow_J m_5$. We get that m_7 is dominant, and so $m_7=Y_{i,0}$. Let $K''=u_-(m_5)\leq K'$. We have $1=u(m_7)\geq u_J(m_7)+u_n(m_7)\geq K''+(k+1-2K'')=1+k-K''\geq k-K'\geq 1+K'$. So $K'=0$ and $k=1$. In particular $m_5=m_5''Y_{i_1,l_1}Y_{n-1,L+2}^{-1}Y_{n,L-1}Y_{n,L+1}$ and so we have $i_1=n-1-j$, $l_1=L+2-2j$ where $j\geq 0$ (otherwise we would have $u(m_7)\geq 2$). It implies $i_2-i_1-(l_2-l_1)/2=j-(j-1)=1$ not even, contradiction. So $m_5\in B$ and $m_5=Y_{i,0}$. But $u(m_5)\geq u_n(m_5)\geq 2$, contradiction. □

Lemma 5.3. *Let $j\in I$ and $m\in D(Y_{i,0})\cap B_j$ such that $u_j(m)=2$. For $L,L'\in\mathbb{Z}$ such that $m^{(j)}=Y_{j,L}Y_{j,L'}$, we have $L\neq L'$.*

Proof: It follows from lemma 5.2 that we can suppose that $j = n$ and that for $v(m'Y_{i,0}^{-1}) \leq v(mY_{i,0}^{-1})$, for all $j \in J, l \in \mathbb{Z}$, $u_{j,l}(m') \leq 1$. Let M such that $\exists l \in \mathbb{Z}$, $u_{n,l}(M) \geq 2$ and suppose that $v(MY_{i,0}^{-1})$ is minimal with this property. Let L be maximal such that $u_{n,L-1}(M) \geq 2$. First it follows from lemma 4.5 that $M \in B_n$. We have $M \to_n M' = MA_{n,L}^{-1}$ and the coefficient of M' in $L_n(M)$ is at least 2. Suppose that there is $j \in J$ such that $M' \notin B_j$. Let $M'' \to_j M'$. It follows from lemma 3.14 that M'' is uniquely determined by M', and that the coefficient of M' in $F_j(M')$ is 1. But the coefficient of M'' in $L(Y_{i,0})$ is 1, so it follows from the proposition 3.9 that the coefficient of M' is 1, contradiction. So $M' \in B_J$. So $M = Y_{n-1,L}^{-1}\tilde{M}$ where $\tilde{M} \in B$ and $u_{n,L-1}(\tilde{M}) \geq 2$. As $u_n(M) \geq 2$, $M \notin B$. So there is $M_0 \to_J M$. We have :

$$M_0 = \tilde{M}_0 Y_{n,L-1} Y_{n,L-3}^{-1}$$

where $\tilde{M}_0 \in B$ and $u_J(\tilde{M}_0) \geq 1$. But M_0 is not right negative, so M_0 is dominant (corollary 3.10). But $u(M_0) \geq u_J(\tilde{M}_0) + u_{n,L-1}(M_0) \geq 2$, contradiction. □

So the proposition 5.1 follows from proposition 3.16 and lemma 5.3.

5.3. **Complement : degree of monomials.** The aim of this section is to prove that the degrees are bounded (it is a complement independent of the proof of theorem 3.5):

Proposition 5.4. *For $j \in I$ and $m \in B_j \cap D(Y_{i,0})$, then $u_j(m) \leq 2$.*

Note that it follows from proposition 5.1 that we can use the lemma 3.15.

For $m \in A$ denote $w(m) = (u_J^+(m), u_J^-(m), u_n^+(m), u_n^-(m))$.

Suppose that there is $j \in J$ and $m_0 \in D(Y_{i,0}) \cap B_j$ such that $u_j(m_0) \geq 3$. It follows from lemma 4.4 that there is $m \to_J m_0$ such that $u_J(m) \geq 3$. Suppose that $v(mY_{i,0}^{-1})$ is minimal for this property.

Lemma 5.5. *There is $M \in D(Y_{i,0}) \cap B_n$ such that $M > m$ and $u_n(M) \geq 3$.*

Proof: We have $m \in B_J$ and $u_J(m) = 3 :$ $m = Y_{i_1,l_1}Y_{i_2,l_2}Y_{i_3,l_3}m'$ where $(m')^{(J)} = 1$. There is

$$m \leftharpoondown_n m_1 = Y_{i_1,l_1}Y_{i_2,l_2}Y_{i_3,l_3}Y_{n-1,L}^{-1}m_1'Y_{n,L-1}$$

where $(m_1')^{(J)} = 1$ and $u_{n,L-1}(m_1') \geq 0$. If $Y_{i_1,l_1}Y_{i_2,l_2}Y_{i_3,l_3}Y_{n-1,L}^{-1} \notin B$, there is $M_1 \to_J m_1$ such that $u_J(M_1) \geq 3$, contradiction. So for example $m_1 = Y_{i_1,l_1}Y_{i_2,l_2}m_1'Y_{n,l_3-1}$. There is

$$m_1 \leftharpoondown_n m_2 = Y_{i_1,l_1}Y_{i_2,l_2}Y_{n-1,L'}^{-1}m_2'Y_{n,l_3-1}Y_{n,L'-1}$$

where $(m_2')^{(J)} = 1$ and $u_{n,l_3-1}(m_2') \geq 0$, $u_{n,L'-1}(m_2') \geq 0$. It follows from lemma 5.3 that $l_3 \neq L'$. If $L' = l_3 + 1$, we have $m_2'' \to_J m_2$ where m_2'' is dominant and $u(m_2'') \geq u_J(m_2'') \geq 2$, contradiction. So we see as above that for example $m_2 = Y_{i_1,l_1}m_2'Y_{n,l_3-1}Y_{n,l_2-1}$. In the same way

$$m_2 \leftharpoondown_n m_3 = m_3'Y_{i_1,l_1}Y_{n-1,L''}^{-1}Y_{n,l_3-1}Y_{n,l_2-1}Y_{n,L''-1}$$

where $(m_3')^{(J)} = 1$ and $u_{n,l_3-1}(m_3') \geq 0$, $u_{n,l_2-1}(m_3') \geq 0$, $u_{n,l_1-1}(m_3') \geq 0$. We can conclude with lemma 4.4. □

For $m \in A$, denote $w(m) = (u_J^+(m), u_J^-(m), u_n^+(m), u_n^-(m))$.

End of the proof of proposition 5.4 :

Suppose that there is $M \in D(Y_{i,0}) \cap B_n$ such that $u_n(M) \geq 3$. Suppose that $v(MY_{i,0}^{-1})$ is minimal for this property. It follows from lemma 5.5 that for $M' \in D(Y_{i,0})$, $M' \geq M$ for $j \in J, l \in \mathbb{Z}$, $u_{j,l}(M') \leq 2$.

We have $M \in B_{\{1,...,n-2\}}$ (if not we would have $M' \to_{\{1,...,n-2\}} M$ with $M' > M$ and $u_n(M') \geq 3$).

If $u_n(M) = 3 :$ there is $M \leftarrow_J M_1$. If $v_{n-1}(M_1M^{-1}) = 1$, we have $w(M_1) = (a, 0, 3, 2)$ or $(a, 0, 2, 1)$ or $(a, 0, 1, 0)$ where $a = 1$ or 2. For the first two cases we have $u_n^+(M_1) + u_n^-(M_1) \geq 3$, so there is $M_1' \to_n M_1$ such that $u_n(M_1) \geq 3$, contradiction. For the last case M_1 is dominant with $u(M_1) \geq 2$, contradiction. So $v_{n-1}(M_1M^{-1}) = 2$ and $w(m_1) = (2, 0, 3, 4)$ or $(2, 0, 2, 3)$ or $(2, 0, 1, 2)$ or $(2, 0, 0, 1)$. As above we have $w(M_1) = (2, 0, 0, 1)$. Let $M_2 \to_n M_1$. If $w(M_2) = (1, 0, 1, 0)$, M_2 is dominant with

$u(M_2) \geq 2$, contradiction. So $w(M_2) = (2,1,1,0)$. Let $M_3 \rightarrow_J M_2$. If $w(M_3) = (2,0,1,2)$, there is $M_3' > M_3$ such that $u_n(M_3') \geq 3$, contradiction. So $w(M_3) = (2,0,0,1)$. We continue and we get an infinite sequence such that $w(M_{2k}) = (2,1,1,0)$ and $w(M_{2k+1}) = (2,0,0,1)$. Contradiction because the sequence $v(M_k Y_{i,0}^{-1}) \geq 0$ decreases strictly.

If $u_n(M) = 4$: there is $M \leftarrow_J M_1$. If $v_{n-1}(M_1 M^{-1}) = 1$, we have $w(M_1) = (a,0,4,2)$ or $(a,0,3,1)$ or $(a,0,2,0)$ where $a = 1$ or 2. We see as above that it is impossible. So $v_{n-1}(M_1 M^{-1}) = 2$ and $w(m_1) = (2,0,4,4)$ or $(2,0,3,3)$ or $(2,0,2,2)$ or $(2,0,1,1)$. As above we have $w(M_1) = (2,0,1,1)$. Let $M_2 \rightarrow_n M_1$. If $w(M_2) = (1,0,2,0)$, M_2 is dominant with $u(M_2) \geq 2$, contradiction. So $w(M_2) = (2,1,2,0)$. Let $M_3 \rightarrow_J M_2$. If $w(M_3) = (2,0,2,2)$, there is $M_3' > M_3$ such that $u_n(M_3') \geq 3$, contradiction. So $w(M_3) = (2,0,1,1)$. We continue and we get an infinite sequence such that $w(M_{2k}) = (2,1,2,0)$ and $w(M_{2k+1}) = (2,0,1,1)$. Contradiction because the sequence $v(M_k Y_{i,0}^{-1}) \geq 0$ decreases strictly. □

6. Type C

6.1. **Statement.** Let $\mathfrak{g}$ be of type C_n $(n \geq 2)$. For $i \in \{2,...,n-1\}$, $l \in \mathbb{Z}$:

$$A_{i,l} = Y_{i,l+1} Y_{i,l-1} Y_{i-1,l}^{-1} Y_{i+1,l}^{-1}$$

$$A_{1,l} = Y_{1,l+1} Y_{1,l-1} Y_{2,l}^{-1} \text{ , } A_{n,l} = Y_{n,l+2} Y_{n,l-2} Y_{n-1,l+1}^{-1} Y_{n-1,l-1}^{-1}$$

In particular for all $i \in I, l \in \mathbb{Z}$, $u(A_{i,l}^{-1}) \leq 0$. So $m \leq m' \Rightarrow u(m) \leq u(m')$.

In this section we prove :

Proposition 6.1. *The property of theorem 3.5 is true for $\mathfrak{g}$ of type C_n $(n \geq 2)$.*

Denote $J = \{1,...,n-1\} \subset I$.

6.2. **Proof of proposition 6.1.** We can suppose $Y_{i,L} = Y_{i,0}$ (proposition 3.12).

Lemma 6.2. *(i) For $m \in B_n \cap D(Y_{i,0})$, we have $u_n(m) \leq 1$.*

(ii) For $j \leq n-1$ and $m \in B_j \cap D(Y_{i,0})$, we have $u_j(m) \leq 2$.

(iii) Let $j \leq n-1$ and $m \in D(Y_{i,0}) \cap B_j$ such that $u_j(m) = 2$. For $L, L' \in \mathbb{Z}$ such that $m^{(j)} = Y_{j,L} Y_{j,L'}$, we have $L \neq L'$.

Proof: (i) suppose that there is $m_1 \in B_n \cap D(Y_{i,0})$ such that $u_n(m_1) \geq 2$. Let $m_2 \rightarrow_J m_1$. We have $u_n(m_2) \geq 2 - v_{n-1}(m_2 m_1^{-1})$. But $u_J(m_2) \geq v_{n-1}(m_2 m_1^{-1})$ (lemma 4.2) and so $u(m_2) = u_J(m_2) + u_n(m_2) \geq 2$. As $Y_{i,0} \geq m_2$ it is impossible.

(ii) suppose that there is $j \leq n-1$ and $m_1 \in B_j \cap D(Y_{i,0})$ such that $u_j(m_1) \geq 3$. Let $m_2 \rightarrow_J m_1$. Then we have $u_J(m_2) \geq 3$ (lemma 4.4) and so $u(m_2) \geq 3 + u_n(m_1) \geq 2$ (it follows from (i) that $u_n(m_1) \geq -1$). Contradiction.

(iii) let $j \neq n$ and $m_1 \in D(Y_{i,0}) \cap B_j$ such that $m_1^{(j)} = Y_{j,L}^2$. We can suppose that $v(m_1 m^{-1})$ is minimal. Let $m_2 \rightarrow_J m_1$. It follows from lemma 4.5 for $\mathfrak{g}_J$ of type A_{n-1} that $m_2^{(J)} = Y_{i_1,L_1} Y_{i_2,L_2}$ with $i_2 - i_1 \geq |L_1 - L_2|$ and $(i_2 - i_1) - (L_2 - L_1)$ is even. As $u(m_2) \leq 1$ and $u_n(m_2) \geq -1$, we have $u_n(m_2) = -1$ and $u_J(m_2) = 2$. In particular $m_2 = Y_{i_1,L_1} Y_{i_2,L_2} Y_{n,K}^{-1}$. There is $m_2 \leftarrow_n m_3 = Y_{i_1,L_1} Y_{i_2,L_2} Y_{n-1,K-1}^{-1} Y_{n-1,K-3}^{-1} Y_{n,K-4}$. We are in one the following cases $\alpha, \beta, \gamma, \delta$:

α) $Y_{i_1,L_1} Y_{i_2,L_2} Y_{n-1,K-1}^{-1} Y_{n-1,K-3}^{-1} = 1$: impossible because $i_1 = i_2 \Rightarrow L_1 = L_2$.

β) $Y_{i_1,L_1} Y_{i_2,L_2} Y_{n-1,K-1}^{-1} Y_{n-1,K-3}^{-1} = Y_{i_1,L_1} Y_{n-1,K-1}^{-1} \neq 1$ (or in the same way $Y_{i_2,L_2} Y_{n-1,K-1}^{-1} \neq 1$). There is $m_3 \leftarrow_{n-1} m_4 = m_3 A_{n-1,K-2}$. In particular $m_4^{(n)} = Y_{n,K-4} Y_{n,K-2}^{-1}$, contradiction with (i).

γ) $Y_{i_1,L_1} Y_{i_2,L_2} Y_{n-1,K-1}^{-1} Y_{n-1,K-3}^{-1} = Y_{i_1,L_1} Y_{n-1,K-3}^{-1} \neq 1$ (or in the same way $Y_{i_2,L_2} Y_{n-1,K-3}^{-1} \neq 1$). In particular $i_2 = n-1$, $L_2 = K-1$ and $m_3 = Y_{i_1,L_1} Y_{n-1,K-3}^{-1} Y_{n,K-4}$. Let $J' = \{i_1+1,...,n-1\}$ (J' can be empty) and $m_3 \leftarrow_{J'} m_4 = Y_{i_1,L_1}Y^{-1}_{i_1,K-2-n+i_1}Y_{i_1+1,K-3-n+i_1}$. If $Y_{i_1,L_1}Y^{-1}_{i_1,K-2-n+i_1} \neq 1$, let $m_5 \rightarrow_{i_1} m_4$. If $i_1 = 1$, we have $u(m_5) = 2$, impossible. If $i_1 \geq 2$, we have $m_5 = Y^{-1}_{i_1-1,K-3-n+i_1}Y_{i_1,L_1}Y_{i_1,K-4-n+i_1}$. Let $J'' = \{1, ..., i_1 - 1\}$ and $m_5 \leftarrow_{J''} m_6 = Y_{1,K'}Y_{i_1,L_1}$. We have $u(m_6) = 2$, impossible. So $Y_{i_1,L_1} = Y_{i_1,K-2-n+i_1}$, that is to say $L_1 = K - 2 - n + i_1$. So $L_2 - L_1 = n - i_1 + 1 = i_2 - i_1 + 2 > i_2 - i_1$, contradiction.

δ) $\{(i_1, L_1), (i_2, L_2)\} \cap \{(n-1, K-1), (n-1, K-3)\}$ is empty : there is $m_3 \rightarrow_J m_4 = m_3(m_1m_2^{-1})$ such that $m_4^{(j)} = Y^2_{j,l}$ (lemma 3.15) and $v(m_4m^{-1}) < v(m_1m^{-1})$, contradiction. □

The proposition 6.1 follows from proposition 3.16 and lemma 6.2.

7. Application to q, t-characters

In this section we state and prove the main result of this paper on q,t-characters (theorem 7.5).

7.1. **Reminder on q, t-characters** [N2, N3, H1, H2, H3]**.** We define the product $*_t$ on $A \times (\mathcal{A} \otimes \mathbb{Z}[t\pm])$ such that : for $(m, v), (m', v') \in A \times \mathcal{A}$ (m, m', v, v' monomials):

$$(m, v) *_t (m', v') = t^{D((m,v),(m',v'))}(mm', vv')$$

where :

$$D((m, v), (m', v')) = \sum_{i\in I,l\in\mathbb{Z}} 2u_{i,l+r_i}(m)v_{i,l}(v') + 2v_{i,l+r_i}(v)u_{i,l}(m') + v_{i,l+r_i}(v)u_{i,l}(v') + u_{i,l+r_i}(v)v_{i,l}(v')$$

(see [N3] for the ADE-case and [H2, H3] for other cases).

Let $\mathcal{Y}_t = \mathcal{Y} \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm}]$. One can define $\mathfrak{K}_{i,t}, \mathfrak{K}_t \subset \mathcal{Y}_t$ with deformed screening operators (see [H1, H3]).

Definition 7.1. *We say that a $\mathbb{Z}$-linear map $\chi_{q,t} : Rep \rightarrow \mathcal{Y}_t$ is a morphism of q, t-characters if :*

1) For M a Rep-monomial define $m = \prod_{i\in I,l\in\mathbb{Z}} (Y_{i,l})^{x_{i,l}(M)} \in B$. We have :

$$\chi_{q,t}(M) = m + \sum_{m'<m} a_{m'}(t)m' \text{ (where } a_{m'}(t) \in \mathbb{Z}[t^{\pm}])$$

2) The image of $\chi_{q,t}$ is contained in $\mathfrak{K}_t$.

3) Let M_1, M_2 be Rep-monomials. If $max\{l/\sum_{i\in I} x_{i,l}(M_1) > 0\} \leq min\{l/\sum_{i\in I} x_{i,l}(M_2) > 0\}$ then :

$$(M_1M_2, (M_1M_2)^{-1}\chi_{q,t}(M_1M_2)) = (M_1, M_1^{-1}\chi_{q,t}(M_1)) *_t (M_2, M_2^{-1}\chi_{q,t}(M_2))$$

Those properties are generalizations of the axioms that Nakajima [N3] defined for the ADE-case.

Theorem 7.2. *(*[N3, H2, H3]*) For C such that $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$, there is a unique morphism of q, t-characters.*

This result (among others) was proved by Nakajima [N3] for C of type ADE. For C finite it is proved in [H2], and for C such that $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$ in [H3] (it includes quantum affine and toroidal algebras except $A_1^{(1)}, A_2^{(2)}$). The existence of $\chi_{q,t}$ for symmetric toroidal type is also mentioned in [N5].

In [H2] we defined a t-deformed algorithm : for $m \in B$, if it is well-defined it gives an element $F_t(m) \in \mathfrak{K}_t$ such that m is the unique dominant monomial of $F_t(m)$ (an algorithm was also used by Nakajima in the ADE-case in [N2]). If we set $t = 1$ we get the classical algorithm. It follows from theorem 7.2 that the t-deformed algorithm is well defined if $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$. We proved in [H2] that if the t-deformed algorithm is well-defined, for $i \in I, j \in I, l \in \mathbb{Z}$: $F_t(Y_{i,l})F_t(Y_{j,l}) = F_t(Y_{j,l})F_t(Y_{i,l})$.

Note that $\chi_{q,t}$ is injective and we have (see [H2]):

$$(1) \quad \chi_{q,t}(\prod_{i\in I,l\in\mathbb{Z}} X^{x_{i,l}}_{i,l}) = \overrightarrow{\prod_{l\in\mathbb{Z}}}\prod_{i\in I} F_t(Y_{i,l})^{x_{i,l}} = ...(\prod_{i\in I} F_t(Y_{i,l-1})^{x_{i,l-1}})(\prod_{i\in I} F_t(Y_{i,l})^{x_{i,l}})(\prod_{i\in I} F_t(Y_{i,l+1})^{x_{i,l+1}})...$$

7.2. **Technical complement.**

Proposition 7.3. *(i) Let $m \in B_j$ such that for all $l \in \mathbb{Z}$, $u_{j,l}(m) \leq 1$. Then $F_{i,t}(m) = F_i(m) = L_i(m)$ and all coefficients are equal to 1.*

(ii) Let $i \in I$ such that all $m \in D(Y_{i,L})$ satisfies : for $j \in I$, if $m \in B_j$ then $\forall l \in \mathbb{Z}, u_{j,l}(m) \leq 1$. Then $F_t(Y_{i,L}) = F(Y_{i,L}) = L(Y_{i,L}) \in \mathcal{Y}_t$ is in $\mathfrak{K}_t$ and all coefficients are equal to 1.

Proof: (i) Direct consequence of the lemma 4.13 of [H2].

(ii) Let j be in I and consider the decomposition of proposition 3.9 :

$$L(Y_{i,L}) = \sum_{m' \in B_j \cap D(Y_{i,L})} \lambda_j(m') L_j(m')$$

But it follows from (i) that $m' \in B_j \cap D(Y_{i,L})$ implies that $L_j(m') = F_{j,t}(m')$. And so:

$$L(Y_{i,l}) = \sum_{m' \in B_j \cap D(Y_{i,L})} \lambda_j(m') F_{j,t}(m') \in \mathfrak{K}_{j,t}$$

So $L(Y_{i,L}) \in \mathfrak{K}_t$ and $F_t(Y_{i,L}) = L(Y_{i,L}) = F(Y_{i,l})$. □

7.3. **New results for q,t-characters.** It follows also from theorem 3.5 and proposition 7.3 :

Proposition 7.4. *Let $\mathfrak{g}$ be of type A_n $(n \geq 1)$, $A_l^{(1)}$ $(l \geq 2)$, B_n $(n \geq 2)$ or C_n $(n \geq 2)$. For $i \in I, a \in \mathbb{C}^*$, we have $\chi_{q,t}(V_i(a)) = \chi_q(V_i(a))$ and all coefficients are equal to 1.*

We prove a conjecture of [H2]:

Theorem 7.5. *Let $\mathcal{U}_q(\hat{\mathfrak{g}})$ be a quantum affine algebra (C finite) and M be a standard module of $\mathcal{U}_q(\hat{\mathfrak{g}})$. The coefficients of $\chi_{q,t}(M)$ are in $\mathbb{N}[t^{\pm}]$ and the monomials of $\chi_{q,t}(M)$ are the monomials of $\chi_q(M)$.*

In particular the q,t-characters for quantum affine algebras have a finite number of monomials and this result shows that the q,t-characters of [H2] can be considered as a t-deformation of q-characters for all quantum affine algebras. In particular it is an argument for the existence of a geometric model behind the q,t-characters in non simply-laced cases.

Proof: If follows from formula (1) in section 7.1 that it suffices to look at the $F_t(Y_{i,l})$. We do it with a case by case investigation :

the case ADE follows from the work of Nakajima [N3]

the case BC follows from theorem 3.5 and proposition 7.4 (ii)

the case G_2 follows from an explicit computation in [H2]

the case F_4 follows from an explicit computation on computer (see section 8). □

8. Appendix : explicit computations on computer for type F_4

The proof of theorem 7.5 for type F_4 is based on an explicit computation on computer. A computer program written in C with Travis Schedler computes explicitly the q,t-characters of fundamental representations.

For type F_4 there are 4 fundamental representations (see [Bo] for the numbers on the Dynkin diagram) : $\dim(V_1(a)) = 26$ (26 monomials), $\dim(V_1(a)) = 299$ (283 monomials), $\dim(V_1(a)) = 1703$ (1532 monomials), $\dim(V_1(a)) = 53$ (53 monomials). We checked that the coefficients are in $\mathbb{N}[t^{\pm}]$. We give an explicit list of terms of fundamental representations of type F_4 whose coefficient is not 1 (the complete list of monomials can be found on http://www.dma.ens.fr/∼dhernand/f4monomials.pdf). We can see that the coefficient are all $(t + t^{-1}) \in \mathbb{N}[t^{\pm}]$. They appear only in fundamental representations 2 and 3 :

Fundamental representation 2 :

Monomial 70: $(t^{-1}+t)\ Y_{1,10}Y_{2,7}Y_{2,9}^{-1}Y_{2,11}^{-1}Y_{4,6}$
Monomial 87: $(t^{-1}+t)\ Y_{1,12}^{-1}Y_{2,7}Y_{2,9}^{-1}Y_{4,6}$
Monomial 89: $(t^{-1}+t)\ Y_{1,10}Y_{2,7}Y_{2,9}^{-1}Y_{2,11}^{-1}Y_{3,8}Y_{4,10}^{-1}$
Monomial 105: $(t^{-1}+t)\ Y_{1,12}^{-1}Y_{2,7}Y_{2,9}^{-1}Y_{3,8}Y_{4,10}^{-1}$
Monomial 109: $(t^{-1}+t)\ Y_{1,10}Y_{2,7}Y_{3,12}^{-1}$
Monomial 120: $(t^{-1}+t)\ Y_{1,8}Y_{1,10}Y_{2,9}^{-1}Y_{3,8}Y_{3,12}^{-1}$
Monomial 124: $(t^{-1}+t)\ Y_{1,12}^{-1}Y_{2,7}Y_{2,11}Y_{3,12}^{-1}$
Monomial 142: $(t^{-1}+t)\ Y_{2,7}Y_{2,13}^{-1}$
Monomial 143: $(t^{-1}+t)\ Y_{1,8}Y_{1,12}^{-1}Y_{2,9}^{-1}Y_{2,11}Y_{3,8}Y_{3,12}^{-1}$
Monomial 151: $(t^{-1}+t)\ Y_{1,10}^{-1}Y_{1,12}^{-1}Y_{2,11}Y_{3,8}Y_{3,12}^{-1}$
Monomial 155: $(t^{-1}+t)\ Y_{1,8}Y_{2,9}^{-1}Y_{2,13}^{-1}Y_{3,8}$
Monomial 168: $(t^{-1}+t)\ Y_{1,10}^{-1}Y_{2,13}^{-1}Y_{3,8}$
Monomial 173: $(t^{-1}+t)\ Y_{1,8}Y_{2,11}Y_{2,13}^{-1}Y_{3,12}^{-1}Y_{4,10}$
Monomial 188: $(t^{-1}+t)\ Y_{1,10}^{-1}Y_{2,9}Y_{2,11}Y_{2,13}^{-1}Y_{3,12}^{-1}Y_{4,10}$
Monomial 193: $(t^{-1}+t)\ Y_{1,8}Y_{2,11}Y_{2,13}^{-1}Y_{4,14}^{-1}$
Monomial 206: $(t^{-1}+t)\ Y_{1,10}^{-1}Y_{2,9}Y_{2,11}Y_{2,13}^{-1}Y_{4,14}^{-1}$

Fundamental representation 3 :

Monomial 64: $(t^{-1}+t)\ Y_{1,3}Y_{1,9}Y_{2,6}Y_{2,8}^{-1}Y_{2,10}^{-1}Y_{4,5}$
Monomial 90: $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,9}Y_{2,4}Y_{2,6}Y_{2,8}^{-1}Y_{2,10}^{-1}Y_{4,5}$
Monomial 91: $(t^{-1}+t)\ Y_{3,5}Y_{3,9}^{-1}Y_{4,5}$
Monomial 93: $(t^{-1}+t)\ Y_{1,3}Y_{1,11}^{-1}Y_{2,6}Y_{2,8}^{-1}Y_{4,5}$
Monomial 96: $(t^{-1}+t)\ Y_{1,3}Y_{1,9}Y_{2,6}Y_{2,8}^{-1}Y_{2,10}^{-1}Y_{3,7}Y_{4,9}^{-1}$
Monomial 117: $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,11}^{-1}Y_{2,4}Y_{2,6}Y_{2,8}^{-1}Y_{4,5}$
Monomial 125: $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,9}Y_{2,4}Y_{2,6}Y_{2,8}^{-1}Y_{2,10}^{-1}Y_{3,7}Y_{4,9}^{-1}$
Monomial 126: $(t^{-1}+t)\ Y_{3,5}Y_{3,7}Y_{3,9}^{-1}Y_{4,9}^{-1}$
Monomial 128: $(t^{-1}+t)\ Y_{1,3}Y_{1,11}^{-1}Y_{2,6}Y_{2,8}^{-1}Y_{3,7}Y_{4,9}^{-1}$
Monomial 138: $(t^{-1}+t)\ Y_{1,3}Y_{1,9}Y_{2,6}Y_{3,11}^{-1}$
Monomial 152: $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,11}^{-1}Y_{2,4}Y_{2,6}Y_{2,8}^{-1}Y_{3,7}Y_{4,9}^{-1}$
Monomial 159: $(t^{-1}+t)\ Y_{2,8}Y_{2,10}Y_{3,5}Y_{3,9}^{-1}Y_{3,11}^{-1}$
Monomial 162: $(t^{-1}+t)\ Y_{1,3}Y_{1,7}Y_{1,9}Y_{2,8}^{-1}Y_{3,7}Y_{3,11}^{-1}$
Monomial 165: $(t^{-1}+t)\ Y_{1,3}Y_{1,11}^{-1}Y_{2,6}Y_{2,10}Y_{3,11}^{-1}$
Monomial 166: $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,9}Y_{2,4}Y_{2,6}Y_{3,11}^{-1}$
Monomial 194: $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,11}^{-1}Y_{2,4}Y_{2,6}Y_{2,10}Y_{3,11}^{-1}$
Monomial 208: $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,7}Y_{1,9}Y_{2,4}Y_{2,8}^{-1}Y_{3,7}Y_{3,11}^{-1}$
Monomial 209: $(t^{-1}+t)\ Y_{1,11}Y_{2,8}Y_{2,12}^{-1}Y_{3,5}Y_{3,9}^{-1}$
Monomial 220: $(t^{-1}+t)\ Y_{1,3}Y_{2,6}Y_{2,12}^{-1}$
Monomial 221: $(t^{-1}+t)\ Y_{1,3}Y_{1,7}Y_{1,11}^{-1}Y_{2,8}^{-1}Y_{2,10}Y_{3,7}Y_{3,11}^{-1}$
Monomial 237: $(t^{-1}+t)\ Y_{1,9}Y_{1,11}Y_{2,10}^{-1}Y_{2,12}^{-1}Y_{3,5}$
Monomial 238: $(t^{-1}+t)\ Y_{1,7}Y_{1,9}Y_{2,6}^{-1}Y_{2,8}^{-1}Y_{3,5}Y_{3,7}Y_{3,11}^{-1}$
Monomial 251: $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{2,4}Y_{2,6}Y_{2,12}^{-1}$
Monomial 252: $(t^{-1}+t)\ Y_{1,3}Y_{1,9}^{-1}Y_{1,11}^{-1}Y_{2,10}Y_{3,7}Y_{3,11}^{-1}$
Monomial 253: $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,7}Y_{1,11}^{-1}Y_{2,4}Y_{2,8}^{-1}Y_{2,10}Y_{3,7}Y_{3,11}^{-1}$
Monomial 257: $(t^{-1}+t)\ Y_{1,3}Y_{1,7}Y_{2,8}^{-1}Y_{2,12}^{-1}Y_{3,7}$
Monomial 281: $(t^{-1}+t)\ Y_{1,13}^{-1}Y_{2,8}Y_{3,5}Y_{3,9}^{-1}$
Monomial 289: $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,9}^{-1}Y_{1,11}^{-1}Y_{2,4}Y_{2,10}Y_{3,7}Y_{3,11}^{-1}$
Monomial 294: $(t^{-1}+t)\ Y_{1,7}Y_{1,9}Y_{3,7}Y_{3,9}^{-1}Y_{3,11}^{-1}Y_{4,7}$

Monomial 296: $(t^{-1}+t)\ Y_{1,9}Y_{1,11}Y_{2,6}Y_{2,8}Y_{2,10}^{-1}Y_{2,12}^{-1}Y_{3,9}^{-1}Y_{4,7}$
Monomial 298: $(t^{-1}+t)\ Y_{1,3}Y_{1,9}^{-1}Y_{2,12}^{-1}Y_{3,7}$
Monomial 300: $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,7}Y_{2,4}Y_{2,8}^{-1}Y_{2,12}^{-1}Y_{3,7}$
Monomial 303: $(t^{-1}+t)\ Y_{1,7}Y_{1,11}^{-1}Y_{2,6}^{-1}Y_{2,8}^{-1}Y_{2,10}Y_{3,5}Y_{3,7}Y_{3,11}^{-1}$
Monomial 320: $(t^{-1}+t)\ Y_{1,9}Y_{1,13}^{-1}Y_{2,10}^{-1}Y_{3,5}$
Monomial 332: $(t^{-1}+t)\ Y_{1,3}Y_{1,7}Y_{2,10}Y_{2,12}^{-1}Y_{3,11}^{-1}Y_{4,9}$
Monomial 351: $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,9}^{-1}Y_{2,4}Y_{2,12}^{-1}Y_{3,7}$
Monomial 353: $(t^{-1}+t)\ Y_{1,7}Y_{2,6}^{-1}Y_{2,8}^{-1}Y_{2,12}^{-1}Y_{3,5}Y_{3,7}$
Monomial 359: $(t^{-1}+t)\ Y_{1,9}^{-1}Y_{1,11}^{-1}Y_{2,6}^{-1}Y_{2,10}Y_{3,5}Y_{3,7}Y_{3,11}^{-1}$
Monomial 361: $(t^{-1}+t)\ Y_{1,7}Y_{1,11}^{-1}Y_{2,10}Y_{3,7}Y_{3,9}^{-1}Y_{3,11}^{-1}Y_{4,7}$
Monomial 362: $(t^{-1}+t)\ Y_{1,11}^{-1}Y_{1,13}^{-1}Y_{3,5}$
Monomial 368: $(t^{-1}+t)\ Y_{1,7}Y_{1,9}Y_{3,7}Y_{3,11}^{-1}Y_{4,11}^{-1}$
Monomial 370: $(t^{-1}+t)\ Y_{1,9}Y_{1,11}Y_{2,6}Y_{2,8}Y_{2,10}^{-1}Y_{2,12}^{-1}Y_{4,11}^{-1}$
Monomial 382: $(t^{-1}+t)\ Y_{1,3}Y_{1,9}^{-1}Y_{2,8}Y_{2,10}Y_{2,12}^{-1}Y_{3,11}^{-1}Y_{4,9}$
Monomial 384: $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,7}Y_{2,4}Y_{2,10}Y_{2,12}^{-1}Y_{3,11}^{-1}Y_{4,9}$
Monomial 394: $(t^{-1}+t)\ Y_{1,9}Y_{1,13}^{-1}Y_{2,6}Y_{2,8}Y_{2,10}^{-1}Y_{3,9}^{-1}Y_{4,7}$
Monomial 399: $(t^{-1}+t)\ Y_{1,3}Y_{1,7}Y_{2,10}Y_{2,12}^{-1}Y_{4,13}^{-1}$
Monomial 414: $(t^{-1}+t)\ Y_{1,9}^{-1}Y_{2,6}^{-1}Y_{2,12}^{-1}Y_{3,5}Y_{3,7}$
Monomial 422: $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,9}^{-1}Y_{2,4}Y_{2,8}Y_{2,10}Y_{2,12}^{-1}Y_{3,11}^{-1}Y_{4,9}$
Monomial 428: $(t^{-1}+t)\ Y_{1,9}^{-1}Y_{1,11}^{-1}Y_{2,8}Y_{2,10}Y_{3,7}Y_{3,9}^{-1}Y_{3,11}^{-1}Y_{4,7}$
Monomial 431: $(t^{-1}+t)\ Y_{1,7}Y_{1,11}^{-1}Y_{2,10}Y_{3,7}Y_{3,11}^{-1}Y_{4,11}^{-1}$
Monomial 432: $(t^{-1}+t)\ Y_{1,11}^{-1}Y_{1,13}^{-1}Y_{2,6}Y_{2,8}Y_{3,9}^{-1}Y_{4,7}$
Monomial 436: $(t^{-1}+t)\ Y_{1,7}Y_{2,6}^{-1}Y_{2,10}Y_{2,12}^{-1}Y_{3,5}Y_{3,11}^{-1}Y_{4,9}$
Monomial 438: $(t^{-1}+t)\ Y_{1,7}Y_{2,12}^{-1}Y_{3,7}Y_{3,9}^{-1}Y_{4,7}$
Monomial 461: $(t^{-1}+t)\ Y_{1,3}Y_{1,9}^{-1}Y_{2,8}Y_{2,10}Y_{2,12}^{-1}Y_{4,13}^{-1}$
Monomial 463: $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,7}Y_{2,4}Y_{2,10}Y_{2,12}^{-1}Y_{4,13}^{-1}$
Monomial 469: $(t^{-1}+t)\ Y_{1,9}Y_{1,13}^{-1}Y_{2,6}Y_{2,8}Y_{2,10}^{-1}Y_{4,11}^{-1}$
Monomial 495: $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,9}^{-1}Y_{2,4}Y_{2,8}Y_{2,10}Y_{2,12}^{-1}Y_{4,13}^{-1}$
Monomial 498: $(t^{-1}+t)\ Y_{1,9}Y_{1,11}^{-1}Y_{1,13}^{-1}Y_{2,6}Y_{2,10}^{-1}Y_{4,7}$
Monomial 500: $(t^{-1}+t)\ Y_{1,9}^{-1}Y_{2,8}Y_{2,12}^{-1}Y_{3,7}Y_{3,9}^{-1}Y_{4,7}$
Monomial 502: $(t^{-1}+t)\ Y_{1,7}Y_{2,8}Y_{2,10}Y_{2,12}^{-1}Y_{3,9}^{-1}Y_{3,11}^{-1}Y_{4,7}Y_{4,9}$
Monomial 505: $(t^{-1}+t)\ Y_{1,9}^{-1}Y_{2,6}^{-1}Y_{2,8}Y_{2,10}Y_{2,12}^{-1}Y_{3,5}Y_{3,11}^{-1}Y_{4,9}$
Monomial 512: $(t^{-1}+t)\ Y_{1,9}^{-1}Y_{1,11}^{-1}Y_{2,8}Y_{2,10}Y_{3,7}Y_{3,11}^{-1}Y_{4,11}^{-1}$
Monomial 514: $(t^{-1}+t)\ Y_{1,11}^{-1}Y_{1,13}^{-1}Y_{2,6}Y_{2,8}Y_{4,11}^{-1}$
Monomial 526: $(t^{-1}+t)\ Y_{1,7}Y_{2,6}^{-1}Y_{2,10}Y_{2,12}^{-1}Y_{3,5}Y_{4,13}^{-1}$
Monomial 528: $(t^{-1}+t)\ Y_{1,7}Y_{2,12}^{-1}Y_{3,7}Y_{4,11}^{-1}$
Monomial 564: $(t^{-1}+t)\ Y_{2,10}^{-1}Y_{2,12}^{-1}Y_{3,7}Y_{4,7}$
Monomial 575: $(t^{-1}+t)\ Y_{1,7}Y_{1,9}Y_{1,11}^{-1}Y_{1,13}^{-1}Y_{2,8}^{-1}Y_{2,10}^{-1}Y_{3,7}Y_{4,7}$
Monomial 577: $(t^{-1}+t)\ Y_{1,9}^{-1}Y_{2,8}^{2}Y_{2,10}Y_{2,12}^{-1}Y_{3,9}^{-1}Y_{3,11}^{-1}Y_{4,7}Y_{4,9}$
Monomial 581: $(t^{-1}+t)\ Y_{1,9}Y_{1,11}^{-1}Y_{1,13}^{-1}Y_{2,6}Y_{2,10}^{-1}Y_{3,9}Y_{4,11}^{-1}$
Monomial 583: $(t^{-1}+t)\ Y_{1,9}^{-1}Y_{2,8}Y_{2,12}^{-1}Y_{3,7}Y_{4,11}^{-1}$
Monomial 586: $(t^{-1}+t)\ Y_{1,7}Y_{2,8}Y_{2,10}Y_{2,12}^{-1}Y_{3,9}^{-1}Y_{4,7}Y_{4,13}^{-1}$
Monomial 591: $(t^{-1}+t)\ Y_{1,9}^{-1}Y_{2,6}^{-1}Y_{2,8}Y_{2,10}Y_{2,12}^{-1}Y_{3,5}Y_{4,13}^{-1}$
Monomial 622: $(t^{-1}+t)\ Y_{1,7}Y_{2,8}Y_{2,10}Y_{2,12}^{-1}Y_{3,11}^{-1}Y_{4,9}Y_{4,11}^{-1}$
Monomial 648: $(t^{-1}+t)\ Y_{2,10}^{-1}Y_{2,12}^{-1}Y_{3,7}Y_{3,9}Y_{4,11}^{-1}$
Monomial 656: $(t^{-1}+t)\ Y_{2,8}Y_{2,12}^{-1}Y_{3,11}^{-1}Y_{4,7}Y_{4,9}$

Monomial 657: $(t^{-1}+t)\ Y_{1,7}Y_{1,9}Y_{1,11}^{-1}Y_{1,13}^{-1}Y_{2,8}^{-1}Y_{2,10}^{-1}Y_{3,7}Y_{3,9}Y_{4,11}^{-1}$
Monomial 666: $(t^{-1}+t)\ Y_{1,7}Y_{1,9}Y_{1,11}^{-1}Y_{1,13}^{-1}Y_{3,11}^{-1}Y_{4,7}Y_{4,9}$
Monomial 669: $(t^{-1}+t)\ Y_{1,7}Y_{2,8}Y_{2,10}Y_{2,12}^{-1}Y_{4,11}^{-1}Y_{4,13}^{-1}$
Monomial 672: $(t^{-1}+t)\ Y_{1,9}^{-1}Y_{2,8}^{2}Y_{2,10}Y_{2,12}^{-1}Y_{3,9}^{-1}Y_{4,7}Y_{4,13}^{-1}$
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Monomial 696: $(t^{-1}+t)\ Y_{1,9}^{-1}Y_{2,8}^{2}Y_{2,10}Y_{2,12}^{-1}Y_{3,11}^{-1}Y_{4,9}Y_{4,11}^{-1}$
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Monomial 764: $(t^{-1}+t)\ Y_{2,8}Y_{2,12}^{-1}Y_{4,7}Y_{4,13}^{-1}$
Monomial 765: $(t^{-1}+t)\ Y_{1,7}Y_{1,9}Y_{1,11}^{-1}Y_{1,13}^{-1}Y_{4,7}Y_{4,13}^{-1}$
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Monomial 770: $(t^{-1}+t)\ Y_{1,9}Y_{1,11}^{-1}Y_{2,6}Y_{2,14}^{-1}$
Monomial 771: $(t^{-1}+t)\ Y_{1,7}Y_{1,9}Y_{1,11}^{-1}Y_{1,13}^{-1}Y_{2,8}^{-1}Y_{2,12}Y_{3,7}Y_{3,13}^{-1}$
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Monomial 998: $(t^{-1}+t)\ Y_{1,13}^{-1}Y_{2,8}Y_{2,10}^{-1}Y_{2,14}Y_{3,15}^{-1}Y_{4,7}$
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Monomial 1017: $(t^{-1}+t)\ Y_{1,7}Y_{1,9}Y_{2,12}^{-1}Y_{2,14}^{-1}Y_{4,9}$
Monomial 1026: $(t^{-1}+t)\ Y_{1,13}^{-1}Y_{2,8}Y_{3,13}^{-1}Y_{4,9}$
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Monomial 1075: $(t^{-1}+t)\ Y_{1,13}^{-1}Y_{1,15}Y_{2,8}Y_{2,10}^{-1}Y_{2,16}^{-1}Y_{4,7}$
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Monomial 1156: $(t^{-1}+t)\ Y_{1,7}Y_{1,11}^{-1}Y_{2,10}Y_{2,12}^{-1}Y_{2,14}^{-1}Y_{3,11}Y_{4,13}^{-1}$
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Monomial 1204: $(t^{-1}+t)\ Y_{1,9}^{-1}Y_{1,11}^{-1}Y_{2,8}Y_{2,10}Y_{2,12}^{-1}Y_{2,14}^{-1}Y_{3,11}Y_{4,13}^{-1}$
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Monomial 1277: $(t^{-1}+t)\ Y_{1,15}Y_{2,8}Y_{2,14}^{-1}Y_{2,16}^{-1}$
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Monomial 1452: $(t^{-1}+t)\ Y_{1,11}^{-1}Y_{1,17}^{-1}Y_{2,10}Y_{2,12}Y_{2,14}^{-1}Y_{4,15}^{-1}$

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