

EXTREMAL MONOMIAL PROPERTY OF q -CHARACTERS AND POLYNOMIALITY OF THE X -SERIES

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To Vajjayanthi Chari on her birthday

ABSTRACT. The character of every irreducible finite-dimensional representation of a simple Lie algebra has the highest weight property. The invariance of the character under the action of the Weyl group W implies that there is a similar “extremal weight property” for every weight obtained by applying an element of W to the highest weight. In this paper we conjecture an analogous “extremal monomial property” of the q -characters of simple finite-dimensional modules over the quantum affine algebras, using the braid group action on q -characters defined by Chari. In the case of the identity element of W , this is the highest monomial property of q -characters proved in [FM]. Here we prove it for simple reflections. Somewhat surprisingly, the extremal monomial property for each $w \in W$ turns out to be equivalent to polynomiality of the “ X -series” corresponding to w , which we introduce in this paper. We show that these X -series are equal to certain limits of the generalized Baxter operators for all $w \in W$. Thus, we find a new bridge between q -characters and the spectra of XXZ-type quantum integrable models associated to quantum affine algebras. This leads us to conjecture polynomiality of all generalized Baxter operators, extending the results of [FH1].

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1. INTRODUCTION

The theory of q -characters of finite-dimensional representations of quantum affine algebras introduced in [FR] is a powerful tool to investigate the structure of these representations and the corresponding category, as well as the spectra of the Hamiltonians in the quantum integrable models of XXZ type associated to these representations. However, questions such as explicit formulas for the q -characters of arbitrary simple modules still remain open. In this paper we introduce a new property of q -characters that we believe can be useful for tackling these questions.

Let $U_q(\widehat{\mathfrak{g}})$ be the quantum affine algebra associated to a finite-dimensional simple Lie algebra $\mathfrak{g}$. Recall that the q -character $\chi_q(V)$ of a finite-dimensional representation V of $U_q(\widehat{\mathfrak{g}})$ is defined in [FR] as an element of the ring

$$\mathcal{Y} := \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times},$$

where $I = \{1, \dots, n\}$ and n is the rank of $\mathfrak{g}$. The variables $Y_{i,a}$ are the analogues of the fundamental weights ω_i of $\mathfrak{g}$. There are also Laurent monomials $A_{i,a}$, $i \in I, a \in \mathbb{C}^\times$, which are the analogues of the simple roots α_i , $i \in I$, of $\mathfrak{g}$ in the following sense: Let P be the lattice of integral weights of $\mathfrak{g}$ and $\mathbb{Z}[P]$ its group ring. Consider the homomorphism $\varpi : \mathcal{Y} \rightarrow \mathbb{Z}[P]$ given by $\varpi(Y_{i,a}) = \omega_i$ for all $i \in I, a \in \mathbb{C}^\times$. Then $\varpi(A_{i,a}) = \alpha_i$.

Furthermore, for any finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module V , consider its restriction $\bar{V}$ to $U_q(\mathfrak{g}) \subset U_q(\widehat{\mathfrak{g}})$. Then the ordinary character $\chi(\bar{V})$ of $\bar{V}$, viewed as an element of $\mathbb{Z}[P]$, is equal to $\varpi(\chi_q(V))$, so the q -character of V is a refinement of its ordinary character.

The ordinary character $\chi(U)$ of any simple finite-dimensional $U_q(\mathfrak{g})$ -module U (which coincides with the character of the corresponding $\mathfrak{g}$ -module) has the familiar *highest weight property*: there is a dominant integral weight $\lambda \in P^+ \subset P$, such that all other weights $\mu \in P$ in $\chi(U)$ have the form $\mu = \lambda - \sum_{i \in I} a_i \alpha_i$, where $a_i \in \mathbb{Z}_{\geq 0}$, $i \in I$.

The q -characters of simple $U_q(\widehat{\mathfrak{g}})$ -modules have a similar property, which was conjectured in [FR] and proved in [FM]. Namely, if L is a simple $U_q(\widehat{\mathfrak{g}})$ -module, then there is a monomial

m in the variables $Y_{i,a}$ (raised to non-negative powers only), called the *highest monomial*, such that the q -character $\chi_q(L)$ of L satisfies

$$(1.1) \quad \chi_q(L) \in m \cdot \mathbb{Z}[A_{i,a}^{-1}]_{i \in I, a \in \mathbb{C}^\times}.$$

The highest monomial m corresponds to the ordinary weight $\lambda := \varpi(m)$. Note that the construction of simple $U_q(\widehat{\mathfrak{g}})$ -modules given in [Dr, CP] immediately implies that the λ -weight subspace in $\overline{L}$ is one-dimensional and so m is the only monomial in $\chi_q(L)$ with the ordinary weight λ (i.e. $\varpi(m) = \lambda$). Now, the statement (1.1) (proved in [FM]) means that every other monomial in $\chi_q(L)$ has the form $m \prod_{k=1, \dots, p} A_{i_k, a_k}^{-1}$ for some $i_k \in I, a_k \in \mathbb{C}^\times$. This statement has important consequences, such as an algorithm [FM] for computing the q -characters of the fundamental representations of $U_q(\widehat{\mathfrak{g}})$.

Next, recall that the Weyl group W of $\mathfrak{g}$ acts on P . The W -invariance of the character $\chi(U)$ of any simple finite-dimensional $U_q(\mathfrak{g})$ -module U (which is equal to the character of the corresponding simple $\mathfrak{g}$ -module) implies that $\chi(U)$ has the following “extremal weight property” for each element $w \in W$: all weights $\mu \in P$ appearing in $\chi(U)$ have the form

$$\mu = w(\lambda) - \sum_{i \in I} a_i w(\alpha_i),$$

where $a_i \in \mathbb{Z}_{\geq 0}, i \in I$. Here $w(\lambda), w \in W$, is the *extremal weight* of U corresponding to w . Just like the highest weight λ of U , which corresponds to $w = e$, all extremal weights have multiplicity one.

In this paper we describe a generalization of the extremal weight property for simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -modules.

To explain it, we use the braid group action on $\mathcal{Y}$, which was introduced by Chari in [C]. This action is an analogue of the W -action on $\mathbb{Z}[P]$. In fact, it is intertwined with the W -action on $\mathbb{Z}[P]$ under the homomorphism ϖ . Given $w \in W$, let $w = s_{i_1} \dots s_{i_k}$ be its reduced decomposition into simple reflections $s_i, i \in I$. Then $T_w := T_{i_1} \dots T_{i_k}$, where $T_i, i \in I$, are the generators of the Chari braid group action, is a well-defined automorphism of $\mathcal{Y}$ assigned to w . Moreover, T_w preserves the group of monomials in $\mathcal{Y}$.

Chari proved in [C] that for every $w \in W$, the monomial $T_w(m)$, where m denotes the highest monomial of a simple $U_q(\widehat{\mathfrak{g}})$ -module L , appears in $\chi_q(L)$ and is in fact the unique monomial m_w in $\chi_q(L)$ with the ordinary weight λ (i.e. $\varpi(m_w) = w(\lambda)$).

Let $A_{i,a}^w := T_w(A_{i,a}), i \in I$ (these are the q -character analogues of $w(\alpha_i)$). In this paper we conjecture the following *extremal monomial property* of the q -character of every simple module $L(m)$ for each $w \in W$:

$$(1.2) \quad \chi_q(L) \in T_w(m) \cdot \mathbb{Z}[(A_{i,a}^w)^{-1}]_{i \in I, a \in \mathbb{C}^\times}.$$

This is the statement of Conjecture 4.4, one of the two main conjectures of the present paper.

For $w = e$ (the identity element of W), formula (1.2) becomes formula (1.1), which has been proved in [FM]. Its validity implies formula (1.2) in the case when w is the longest element of W , as we show in Theorem 4.6. And in Theorem 6.13 below, we will prove (1.2) when w is a simple reflection in the Weyl group.

We will also prove a weaker property for all $w \in W$ (in which we allow negative powers of $A_{i,a}^w$ to appear):

$$(1.3) \quad \chi_q(L) \in T_w(m) \cdot \mathbb{Z}[(A_{i,a}^w)^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times}.$$

This is Theorem 4.10.

Somewhat surprisingly, it turns out that Conjecture 4.4 (expressed by formula (1.2)) is equivalent to Conjecture 6.9 (which is our second main conjecture) about certain explicit formal Taylor power series $X_{w(\omega_i)}(z)$ in the variable z which are defined in terms of the Drinfeld-Cartan generators of $U_q(\widehat{\mathfrak{g}})$. We call them the *X-series*. Namely, it is the statement that all eigenvalues of the (suitably renormalized) *X-series* on any simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module (which could in principle be infinite formal power series in z) are in fact polynomials in z .

In the case $w = e$, Conjecture 6.9 was proved in [FH1], and in the case when w is the longest element of W it was proved in [Z] (this provides an alternative proof of Conjecture 4.4 in this case). In Section 6 below, we will prove Conjecture 6.9 (and hence Conjecture (1.2)) when w is a simple reflection $s_i \in W, i \in I$.

What motivated us to study the *X-series* in the first place is the fact that they appear as certain limits of the *generalized Baxter operators* which we defined in our recent paper [FH4]. In that paper, we introduced a family of simple modules $L(\Psi_{w(\omega_i),a}), w \in W, i \in I$, in the category $\mathcal{O}$ of modules over the Borel subalgebra of $U_q(\widehat{\mathfrak{g}})$ defined in [HJ]. The generalized Baxter operators are the transfer-matrices associated to these modules (up to some normalization factors). In this paper, it will be more convenient to consider a “dual picture”; namely, we introduce the modules $L'(\Omega(\Psi_{w(\omega_i),a})), w \in W, i \in I$, in the “dual category” $\mathcal{O}^*$ (here Ω is an automorphism on ℓ -weights, see Sections 2.5 and 7.5 for details). The corresponding transfer-matrices, which we denote by $t_{w(\omega_i),a}(z, u)$, form the family of the “dual” generalized Baxter operators.

In our earlier work [FH1] we considered the case $w = e$, which corresponds to the transfer-matrices associated to the modules $L'(\Psi_{\omega_i,a}), i \in I, a \in \mathbb{C}^\times$, in $\mathcal{O}^*$ (they are called the *prefundamental representations*). We proved that all eigenvalues of these transfer-matrices on any simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module L are *polynomials*, up to an overall scalar factor depending only on L . This crucial property has been used in [FH1] to describe explicitly the spectra of the Hamiltonians of the quantum integrable models of XXZ type associated to $U_q(\widehat{\mathfrak{g}})$ (following a conjecture from [FR]). In the special case $\mathfrak{g} = \mathfrak{sl}_2$, this reproduces the celebrated results of Baxter on the spectra of the Heisenberg XXZ model, which corresponds to $U_q(\widehat{\mathfrak{sl}_2})$ (see the Introduction of [FH1] for details).

Motivated by this result, in this paper we conjecture that a similar polynomiality property holds for all generalized Baxter operators $t_{w(\omega_i),a}(z, u)$ (and not just the ones corresponding to $w = e$, which was the case considered in [FH1]); that is, the eigenvalues of $t_{w(\omega_i),a}(z, u)$ on simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -modules are *polynomials* (up to a scalar factor), for all $w \in W$ and $i \in I$ (Conjecture 7.19). Using the dual extended *TQ*-relations of Section 7.5, we then obtain many alternative descriptions of the spectra of the Hamiltonians of the XXZ-type model associated to $U_q(\widehat{\mathfrak{g}})$ in terms of these polynomials, one for every $w \in W$.

Next, we observe that the X -series $X_{w(\omega_i)}(z)$, $w \in W, i \in I$, can be identified with certain limits of the corresponding generalized Baxter operators $t_{w(\omega_i),a}(z, u)$ (after replacing z by za). Therefore polynomiality of the eigenvalues of the Baxter operators implies the same property for the eigenvalues of the X -series – and this is precisely the statement of our Conjecture 6.9!

Thus, Conjecture 6.9 may be viewed as a special (degenerate) case of the polynomiality conjecture for the generalized Baxter operators. As we explained above, Conjecture 6.9 has a special feature: it is equivalent to the extremal monomial property of the q -characters, which is Conjecture 4.4 discussed above. This, of course, leads us to a natural question: Is there an analogue of Conjecture 4.4 for the generalized Baxter operators?

It is also worth adding that the results and conjectures of this paper are strongly inspired by the extended TQ - and QQ -relations we introduced in [FH4] as well as other tools from quantum integrable models (of XXZ type) associated to quantum affine algebras. These relations also inspired our construction in [FH3] of the Weyl group action on an extension of the ring $\mathcal{Y}$ of q -characters. We proved in [FH4] that the subring of q -characters in $\mathcal{Y}$ is precisely the ring of invariants of $\mathcal{Y}$ under this Weyl group action. However, the image of a monomial in $\mathcal{Y}$ under this action is in general an infinite linear combination of monomials, and therefore the Weyl group does not act by permuting monomials of a given q -character (in contrast to the Weyl group action on the ordinary characters). That's why Conjecture 1.2 does not follow directly from this Weyl group action.

The paper is organized as follows. In Section 2 we recall various definitions and results regarding quantum affine algebras and their representations. We also discuss representations of the Borel subalgebra of a quantum affine algebra from the category $\mathcal{O}$ and the dual category $\mathcal{O}^*$. In Section 3 we recall the braid group action defined by Chari [C] and its generalization which we defined in [FH4]. In Section 4 we introduce twisted root monomials parametrized by elements of the Weyl group, state the extremal monomial property of q -characters (Conjecture 4.4), and prove its weaker version (Theorem 4.10). In Section 5 we introduce formal power series $X_{w(\omega_i)}(z)$, $w \in W, i \in I$, of the Drinfeld-Cartan generators of $U_q(\widehat{\mathfrak{g}})$, which we call the X -series. We also introduce a pairing which we prove to be invariant with respect to an action of the Weyl group W (Proposition 5.6). These results enable us to relate Conjecture 4.4 to the polynomiality of the eigenvalues of the X -series (Conjecture 6.9) in Section 6. Namely, we derive from the W -invariance of our pairing some identities between the eigenvalues of the X -series on a simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module (Corollary 6.2). We then prove that up to a normalization factor (depending only on the module) these eigenvalues are expansions of rational functions (Theorem 6.6). This result is equivalent to Theorem 4.10. Moreover, we conjecture that in fact these rational functions are polynomials (Conjecture 6.9) and show that this property is equivalent to Conjecture 4.4. Finally, we use some additional results to prove Conjecture 6.9 (and hence Conjecture 4.4) for the simple reflections from the Weyl group (Theorem 6.12). In Section 7, we recall the definition of the transfer-matrices, introduce the generalized Baxter operators, and the TQ -relations from [FH4], as well as their duals. We conjecture that polynomiality of the eigenvalues of the X -series (Conjecture 6.9) extends to the (suitably renormalized) generalized Baxter operators.

Acknowledgments. It is a pleasure to dedicate this paper to Vyjayanthi Chari who has made fundamental contributions to representation theory of quantum affine algebras. In particular, the braid group action on the q -characters that she introduced in [C] is one of the key ingredients of the present paper.

2. BACKGROUND ON QUANTUM AFFINE ALGEBRAS

In this section, we recall some definitions and results on quantum affine algebras and modules over them that we will need in the paper. See [CP] and the surveys [CH, L] for more details. We also recall some results on modules from the category $\mathcal{O}$ of the Borel subalgebra of a quantum affine algebra (see [HJ, FH1] for more details) and the corresponding quantum integrable models.

2.1. Simple Lie algebras and Weyl groups. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank n and $\mathfrak{h}$ its Cartan subalgebra. Using the conventions of [Kac], we denote by $C = (C_{i,j})_{i,j \in I}$ the corresponding Cartan matrix, where $I = \{1, \dots, n\}$ and by $h^\vee$ (resp. $r^\vee$) the dual Coxeter number (resp. the lacing number) of $\mathfrak{g}$.

Let $\{\alpha_i\}_{i \in I}$, $\{\alpha_i^\vee\}_{i \in I}$, $\{\omega_i\}_{i \in I}$, $\{\omega_i^\vee\}_{i \in I}$ be the sets of simple roots, simple coroots, fundamental weights, and fundamental coweights. Let

$$Q := \bigoplus_{i \in I} \mathbb{Z}\alpha_i, \quad Q^+ := \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i, \quad P := \bigoplus_{i \in I} \mathbb{Z}\omega_i, \quad P^\vee := \bigoplus_{i \in I} \mathbb{Z}\omega_i^\vee,$$

and let $P_{\mathbb{Q}} := P \otimes \mathbb{Q}$.

The set of roots is denoted by $\Delta \subset Q$ and $\Delta_+ = (\Delta \cap Q^+)$ is the set of positive roots. Let $D = \text{diag}(d_1, \dots, d_n)$ be the unique diagonal matrix such that $B = DC$ is symmetric and d_i 's are relatively prime positive integers.

We have a partial ordering on $P_{\mathbb{Q}}$ defined by the rule $\omega \leq \omega'$ if and only if $\omega' - \omega \in Q^+$.

There is a unique such form κ_0 [Kac] with the property that the dual form $(\kappa_0|_{\mathfrak{h}})^{-1}$ on $\mathfrak{h}^*$ to the restriction $\kappa_0|_{\mathfrak{h}}$ of κ_0 to $\mathfrak{h} \subset \mathfrak{g}$ satisfies $(\alpha_i, \alpha_i) = 2d_i$. We denote the restriction of $(\kappa_0|_{\mathfrak{h}})^{-1}$ to $P \subset \mathfrak{h}^*$ by $(\cdot, \cdot)$. We have for $i, j \in I$,

$$(\alpha_i, \alpha_j) = d_i C_{i,j} \quad \text{and} \quad (\alpha_i, \omega_j) = d_j \delta_{i,j}.$$

The bases $(\alpha_i)_{i \in I}$, $(\omega_i/d_i)_{i \in I}$ are dual to each other with respect to this form.

Let W be the Weyl group of $\mathfrak{g}$. It is generated by the simple reflections $s_i, i \in I$. For $w \in W$, denote by $l(w)$ its length. The formula

$$s_k(\omega_j) = \omega_j - \delta_{k,j}\alpha_j, \quad j, k \in I$$

defines an action of the group W on P . The form $(\cdot, \cdot)$ is W -invariant, that is for any $\lambda, \mu \in P$, we have

$$(2.4) \quad (w\lambda, w\mu) = (\lambda, \mu) \text{ for any } w \in W.$$

The following lemma is well-known (see [Hu, Lemma 1.6, Corollary 1.7, Proposition 1.15]).

Lemma 2.1. *Let $w \in W$ and $i \in I$. We have the following.*

- (1) $w(\alpha_i) \in \Delta_+$ if and only if $l(ws_i) = l(w) + 1$.
- (2) $w(\omega_i) = \omega_i$ if and only if w is product of $s_j, j \neq i$.

(3) *The W -orbits of the fundamental weights are disjoint.*

Denote by w_0 the longest element of the Weyl group. We have the bar involution on the set I defined by the formula $w_0(\alpha_i) = -\alpha_{\bar{i}}, i \in I$.

2.2. Quantum affine algebras and their Borel subalgebras. Denote by $\widehat{\mathfrak{g}}$ the untwisted affine Kac-Moody Lie algebra associated to $\mathfrak{g}$. Let $(C_{i,j})_{0 \leq i,j \leq n}$ be the corresponding indecomposable Cartan matrix. We will use the Kac labels $[\text{Kac}]$ denoted by $a_0, \dots, a_n$. We have $a_0 = 1$ and we set $\alpha_0 = -(a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n)$.

Once and for all, fix a non-zero complex number q that is not a root of unity and choose $h \in \mathbb{C}$ such that $q = e^h$ and define $q^r \in \mathbb{C}^\times$ for any $r \in \mathbb{Q}$ as e^{hr} . Since q is not a root of unity, the map $\mathbb{Q} \rightarrow \mathbb{C}^\times$ sending $r \mapsto q^r$ is injective. Let $q^\mathbb{Q} := \{e^{rh}, r \in \mathbb{Q}\}$ and $\mathfrak{t}^\times := \text{Maps}(I, q^\mathbb{Q}) = (q^\mathbb{Q})^I$ seen as a group with pointwise multiplication. We have a group isomorphism $^- : P_\mathbb{Q} \longrightarrow \mathfrak{t}^\times$ by setting

$$\overline{r\omega_i}(j) := q_i^{r\delta_{i,j}} = e^{rd_i h \delta_{i,j}}, \quad i, j \in I, r \in \mathbb{Q}.$$

The partial ordering on P induces a partial ordering on $\mathfrak{t}^\times$.

We will use the standard symbols for the q -integers

$$[m]_z = \frac{z^m - z^{-m}}{z - z^{-1}}, \quad [m]_z! = \prod_{j=1}^m [j]_z, \quad \begin{bmatrix} s \\ r \end{bmatrix}_z = \frac{[s]_z!}{[r]_z! [s-r]_z!}.$$

We set $q_i = q^{d_i}$.

The quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ of level 0 is the $\mathbb{C}$ -algebra with the generators $e_i, f_i, k_i^{\pm 1}$ ($0 \leq i \leq n$) and the following relations:

$$\begin{aligned} k_i k_j &= k_j k_i, \quad k_0^{a_0} k_1^{a_1} \dots k_n^{a_n} = 1, & k_i e_j k_i^{-1} &= q_i^{C_{i,j}} e_j, \quad k_i f_j k_i^{-1} = q_i^{-C_{i,j}} f_j, \\ [e_i, f_j] &= \delta_{i,j} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}, \\ \sum_{r=0}^{1-C_{i,j}} (-1)^r e_i^{(1-C_{i,j}-r)} e_j e_i^{(r)} &= 0 \quad (i \neq j), & \sum_{r=0}^{1-C_{i,j}} (-1)^r f_i^{(1-C_{i,j}-r)} f_j f_i^{(r)} &= 0 \quad (i \neq j). \end{aligned}$$

Here $0 \leq i, j \leq n$ and $x_i^{(r)} = x_i^r / [r]_{q_i}!$ ($x_i = e_i, f_i$). The algebra $U_q(\widehat{\mathfrak{g}})$ has a Hopf algebra structure with

$$\Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i, \quad \Delta(k_i) = k_i \otimes k_i \text{ for } 0 \leq i \leq n.$$

The algebra $U_q(\widehat{\mathfrak{g}})$ admits also a Drinfeld presentation with generators

$$(2.5) \quad x_{i,r}^\pm \quad (i \in I, r \in \mathbb{Z}), \quad \phi_{i,\pm m}^\pm \quad (i \in I, m \in \mathbb{Z}), \quad k_i^{\pm 1} \quad (i \in I).$$

We will use the generating series ($i \in I$)

$$(2.6) \quad \phi_i^\pm(z) = \sum_{m \in \mathbb{Z}} \phi_{i,m}^\pm z^m = k_i^{\pm 1} \exp \left(\pm (q_i - q_i^{-1}) \sum_{r > 0} h_{i,\pm r} z^{\pm r} \right).$$

The algebra $U_q(\widehat{\mathfrak{g}})$ has a $\mathbb{Z}$ -grading defined by the formulas $\deg(e_i) = \deg(f_i) = \deg(k_i^{\pm 1}) = 0$ for $i \in I$ and $\deg(e_0) = -\deg(f_0) = 1$.

For any $a \in \mathbb{C}^\times$, there is an automorphism $\tau_a : U_q(\widehat{\mathfrak{g}}) \rightarrow U_q(\widehat{\mathfrak{g}})$ such that $\tau_a(g) = a^m g$ for any element g of degree $m \in \mathbb{Z}$.

The *Borel subalgebra* $U_q(\widehat{\mathfrak{b}})$ is, by definition, the subalgebra of $U_q(\widehat{\mathfrak{g}})$ generated by e_i and $k_i^{\pm 1}$ with $0 \leq i \leq n$. It is a Hopf subalgebra of $U_q(\widehat{\mathfrak{g}})$. For $a \in \mathbb{C}^\times$, the subalgebra $U_q(\widehat{\mathfrak{b}})$ is stable under τ_a . We will denote its restriction to $U_q(\widehat{\mathfrak{b}})$ by the same symbol.

2.3. Category $\mathcal{O}$ of modules over the Borel subalgebra. For a $U_q(\widehat{\mathfrak{b}})$ -module V and $\omega \in \mathfrak{t}^\times$, we set the weight subspace of weight ω :

$$(2.7) \quad V_\omega := \{v \in V \mid k_i v = \omega(i)v \ (\forall i \in I)\}.$$

We say that V is Cartan-diagonalizable if $V = \bigoplus_{\omega \in \mathfrak{t}^\times} V_\omega$.

Definition 2.2. A series $\Psi = (\Psi_{i,m})_{i \in I, m \geq 0}$ of complex numbers such that $\Psi_{i,0} \in q^{\mathbb{Q}}$ for all $i \in I$ is called an ℓ -weight.

We denote by $\mathfrak{t}_\ell^\times$ the set of ℓ -weights.

We will identify the collection $(\Psi_{i,m})_{m \geq 0}$ with its generating series

$$\Psi = (\Psi_i(z))_{i \in I}, \quad \Psi_i(z) := \sum_{m \geq 0} \Psi_{i,m} z^m.$$

For example, for $i \in I$ and $a \in \mathbb{C}^\times$ we have the prefundamental ℓ -weight $\Psi_{i,a}$ defined as

$$(2.8) \quad (\Psi_{i,a})_j(z) = \begin{cases} (1 - za)^{\pm 1} & (j = i), \\ 1 & (j \neq i). \end{cases}$$

Since each $\Psi_i(z)$ is an invertible formal power series, $\mathfrak{t}_\ell^\times$ has a natural group structure. We also have a surjective homomorphism of groups $\varpi : \mathfrak{t}_\ell^\times \rightarrow \mathfrak{t}^\times \simeq P_{\mathbb{Q}}$ defined by the assignment $(\Psi_i(z))_{i \in I} \mapsto (\Psi_i(0))_{i \in I}$.

Definition 2.3. A $U_q(\widehat{\mathfrak{b}})$ -module V is called a *highest ℓ -weight module* with the highest ℓ -weight $\Psi \in \mathfrak{t}_\ell^\times$ if there is $v \in V$ such that $V = U_q(\widehat{\mathfrak{b}})v$ and the following equations hold:

$$e_i v = 0 \quad (i \in I), \quad \phi_{i,m}^+ v = \Psi_{i,m} v \quad (i \in I, m \geq 0).$$

The highest ℓ -weight $\Psi \in \mathfrak{t}_\ell^\times$ is uniquely determined by V . The vector v is said to be a highest ℓ -weight vector of V .

Lemma 2.4. [HJ, Section 3.2] For any $\Psi \in \mathfrak{t}_\ell^\times$, there exists a simple highest ℓ -weight module $L(\Psi)$ over $U_q(\widehat{\mathfrak{b}})$ with highest ℓ -weight Ψ . This module is unique up to isomorphism.

Definition 2.5. [HJ] For $i \in I$ and $a \in \mathbb{C}^\times$, let $L_{i,a}^\pm = L(\Psi_{i,a}^{\pm 1})$. We call $L_{i,a}^+$ (resp. $L_{i,a}^-$) a positive (resp. negative) prefundamental representation.

For $\lambda \in P_{\mathbb{Q}}$, we define the 1-dimensional representation $[\lambda] = L(\Psi_\lambda)$ with $\Psi_\lambda := (q_i^{\lambda(\alpha_i^\vee)})_{i \in I}$.

For $\lambda \in \mathfrak{t}^\times$, let $D(\lambda) = \{\omega \in \mathfrak{t}^\times \mid \omega \leq \lambda\}$. Consider the category $\mathcal{O}$ introduced in [HJ].

Definition 2.6. A $U_q(\widehat{\mathfrak{b}})$ -module V is said to be in category $\mathcal{O}$ if

- (i) V is Cartan-diagonalizable;
- (ii) for all $\omega \in \mathfrak{t}^\times$ we have $\dim(V_\omega) < \infty$;
- (iii) there exist a finite number of elements $\lambda_1, \dots, \lambda_s \in \mathfrak{t}^\times$ such that the weights of V are in $\bigcup_{j=1, \dots, s} D(\lambda_j)$.

It is easy to check that the category $\mathcal{O}$ is a monoidal category.

Let $\mathfrak{r}$ be the subgroup of $\mathfrak{t}_\ell^\times$ consisting of Ψ such that $\Psi_i(z)$ is an expansion in positive powers of z of a rational function in z for any $i \in I$.

Theorem 2.7. [HJ] Let $\Psi \in \mathfrak{t}_\ell^\times$. The simple module $L(\Psi)$ is an object of $\mathcal{O}$ if and only if $\Psi \in \mathfrak{r}$.

Let $K_0(\mathcal{O})$ be the (completed) Grothendieck ring of the category $\mathcal{O}$ as in [FH4, Definition 2.15] (see also [HL, Section 3.2]).

2.4. Monomials and finite-dimensional modules. Following [FR], we define the ring of Laurent polynomials $\mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times}$ in the indeterminates $\{Y_{i,a}\}_{i \in I, a \in \mathbb{C}^\times}$, containing the multiplicative group $\mathcal{M}$ of Laurent monomials in $\mathcal{Y}$.

For $i \in I, a \in \mathbb{C}^\times$, this group contains, in particular, the simple root monomials $A_{i,a} \in \mathcal{M}$ defined by the formula

$$(2.9) \quad A_{i,a} := Y_{i,aq_i^{-1}} Y_{i,aq_i} \left(\prod_{\{j \in I | C_{j,i} = -1\}} Y_{j,a} \prod_{\{j \in I | C_{j,i} = -2\}} Y_{j,aq^{-1}} Y_{j,aq} \prod_{\{j \in I | C_{j,i} = -3\}} Y_{j,aq^{-2}} Y_{j,a} Y_{j,aq^2} \right)^{-1}.$$

We have an injective group homomorphism $\mathcal{M} \rightarrow \mathfrak{r}$ by

$$Y_{i,a} \mapsto [\omega_i] \Psi_{i,aq_i^{-1}} \Psi_{i,aq_i}^{-1}.$$

Note that $\varpi(Y_{i,a}) = \omega_i$ and $\varpi(A_{i,a}) = \alpha_i$.

For any $m \in \mathcal{M}$, we will denote by $L(m)$ the corresponding simple $U_q(\widehat{\mathfrak{b}})$ -module, where we identify the monomial $m \in \mathcal{M}$ with its image in $\mathfrak{r}$.

Let $\mathcal{C}$ be the category of (type 1) finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$. A monomial $m \in \mathcal{M}$ is said to be *dominant* if the powers of the variables $Y_{j,b}$, $j \in I, b \in \mathbb{C}^\times$, occurring in this monomial are all positive. We will denote by $\mathcal{M}^+$ the set of dominant monomials.

Part (1) of the following theorem was proved in [HJ] using the result of part (2), which was established in [CP] following [Dr].

Theorem 2.8. (1) For $m \in \mathcal{M}$, the simple $U_q(\widehat{\mathfrak{b}})$ -module $L(m)$ is finite-dimensional if and only if $m \in \mathcal{M}^+$. In this case, the action of $U_q(\widehat{\mathfrak{b}})$ on $L(m)$ can be uniquely extended to an action of $U_q(\widehat{\mathfrak{g}})$.

(2) Every simple module in the category $\mathcal{C}$ is of the form $L(m), m \in \mathcal{M}^+$.

In particular, for $i \in I, a \in \mathbb{C}^\times$, we have the fundamental representation $L(Y_{i,a})$ of $U_q(\widehat{\mathfrak{g}})$.

2.5. The dual category $\mathcal{O}^*$. For V a Cartan-diagonalizable $U_q(\widehat{\mathfrak{b}})$ -module, we define a structure of $U_q(\widehat{\mathfrak{b}})$ -module on its graded dual $V^* = \bigoplus_{\beta \in \mathfrak{t}^\times} V_\beta^*$ by

$$(xu)(v) = u(S^{-1}(x)v) \quad (u \in V^*, v \in V, x \in U_q(\widehat{\mathfrak{b}})).$$

Definition 2.9. Let $\mathcal{O}^*$ be the category of Cartan-diagonalizable $U_q(\widehat{\mathfrak{b}})$ -modules V such that V^* is in category $\mathcal{O}$.

A $U_q(\widehat{\mathfrak{b}})$ -module V is said to be of lowest ℓ -weight $\Psi \in \mathfrak{t}_\ell^\times$ if there is $v \in V$ such that $V = U_q(\widehat{\mathfrak{b}})v$ and the following hold:

$$U_q(\widehat{\mathfrak{b}})^-v = \mathbb{C}v, \quad \phi_{i,m}^+ v = \Psi_{i,m} v \quad (i \in I, m \geq 0).$$

For $\Psi \in \mathfrak{t}_\ell^\times$, we have the simple $U_q(\widehat{\mathfrak{b}})$ -module $L'(\Psi)$ of lowest ℓ -weight Ψ . The notions of characters and q -characters of the objects of the category $\mathcal{O}^*$ are defined in the same way as for the objects of the category $\mathcal{O}$.

Proposition 2.10. [HJ] For $\Psi \in \mathfrak{t}_\ell^\times$ we have $(L'(\Psi))^* \simeq L(\Psi^{-1})$.

In Section 7 we will discuss the prefundamental representations $R_{i,a}^\pm$ in $\mathcal{O}^*$ defined by the property $(R_{i,a}^\pm)^* \simeq L_{i,a}^\mp$.

Remark 2.11. If V is any finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module, then its double dual $(V^*)^*$ is isomorphic to V up to a twist by $\tau_{q^{-2r^\vee h^\vee}} : \tau_{q^{-2r^\vee h^\vee}}(V^*)^* \simeq V$. But for a general (simple) module V in $\mathcal{O}^*$ or $\mathcal{O}$, it is not clear how to express $(V^*)^*$ in terms of V . This is closely related to the fact that the left and right dual modules are not isomorphic to each other (not even up to a twist by $\tau_{q^{-2r^\vee h^\vee}}$ in general).

2.6. q -characters. For any module V in the category $\mathcal{O}$, we define the character of V to be the following element of the group algebra of $P_\mathbb{Q}$ generated by the $[\omega] = \delta_{\omega, \cdot}$ for $\omega \in P_\mathbb{Q}$.

$$(2.10) \quad \chi(V) = \sum_{\omega \in \mathfrak{t}^\times} \dim(V_\omega) [\omega].$$

We recall the notion of q -character of a finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module.

For any $U_q(\widehat{\mathfrak{b}})$ -module V and $\Psi \in \mathfrak{t}_\ell^\times$, the ℓ -weight subspace of V of ℓ -weight Ψ is

$$(2.11) \quad V_\Psi = \{v \in V \mid \exists p \geq 0, \forall i \in I, \forall m \geq 0, (\phi_{i,m}^+ - \Psi_{i,m})^p v = 0\}.$$

We will identify $\mathcal{M}$ and $\mathcal{Y}$ with their images in $\mathfrak{t}$ and $\mathcal{E}_\ell$, respectively, under these homomorphisms.

It is proved in [FR] that for any finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module V , all ℓ -weights appearing in V are in the image of $\mathcal{M}$ in $\mathfrak{t}$. Hence, one can define the q -character of V

$$(2.12) \quad \chi_q(V) = \sum_{\Psi \in \mathfrak{t}} \dim(V_\Psi) [\Psi]$$

as an element of $\mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times}$.

Theorem 2.12. [FR, FM]

- (1) χ_q defines an injective ring morphism $K_0(\mathbb{C}) \rightarrow \mathcal{Y}$.
- (3) For any simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module $L(m)$, we have

$$\chi_q(L(m)) \in m\mathbb{Z}[A_{i,a}^{-1}]_{i \in I, a \in \mathbb{C}^\times}.$$

Let $\varphi : \mathcal{M} \rightarrow P$ be the restriction of the homomorphism ϖ to $\mathcal{M}$. For example, for $i \in I, a \in \mathbb{C}^\times$, we have $\varphi(Y_{i,a}) = \omega_i$ and $\varphi(A_{i,a}) = \alpha_i$. If $m \in \mathcal{M}^+$, then $\varphi(m) \in P^+$.

For any finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module V , the ordinary character of its restriction to $U_q(\mathfrak{g})$ is equal to $\varphi(\chi_q(V))$.

3. CHARI'S BRAID GROUP ACTION AND ITS EXTENSION

In this section, we recall the action of the braid group on $\mathcal{Y}$ introduced by Chari in [C] and its extension to $\mathcal{Y}'$ which we introduced in [FH4, Section 3.2].

3.1. Extremal monomials. Consider the simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module $L(m)$, $m \in \mathcal{M}^+$, and set $\lambda = \varphi(m) \in P^+$. The ordinary character of the restriction of $L(m)$ to $U_q(\mathfrak{g})$ is invariant under the action of the Weyl group W . For any $w \in W$, the dimension of the weight subspace of $L(m)$ of weight $w(\omega)$ is 1, and it is spanned by an extremal weight vector (in the sense of [Kas]). We will denote this vector by v_w . (In particular, if $w = e$, the identity element of the Weyl group W , the vector v_e is the highest weight vector of $L(m)$.) Hence v_w is an ℓ -weight vector.

V. Chari introduced in [C] a braid group action on $\mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times}$. Namely, she defined operators $T_i, i \in I$, on $\mathcal{Y}$ by formulas

$$(3.13) \quad T_i(Y_{i,a}) = Y_{i,a} A_{i,aq_i}^{-1} \text{ and } T_i(Y_{j,a}) = Y_{j,a} \text{ if } j \neq i$$

for all $i, j \in I$ and $a \in \mathbb{C}^\times$. It was established in [C] (a closely related result was proved earlier in [BP]) that the T_i 's satisfy the relations of the braid group corresponding to $\mathfrak{g}$. Therefore for any $w \in W$, we have a well-defined operator T_w acting on $\mathcal{Y}$ obtained by considering a reduced decomposition $w = s_{i_1} \dots s_{i_k}$ and setting

$$T_w := T_{i_1} T_{i_2} \dots T_{i_k}.$$

Theorem 3.1. [C, CM] *For each $w \in W$, the monomial corresponding to the ℓ -weight of $v_w \in L(m)$ is $T_w(m)$. In particular, the multiplicity of $T_w(m)$ in $\chi_q(L(m))$ is equal to 1.*

We call $T_w(m)$ the *extremal monomial* of $L(m)$ corresponding to w .

We showed in [FH4, Section 2.6] that the following notation is well-defined (that is, depends only on $w(\omega_i) \in P$, and not separately on w and ω_i):

$$Y_{w(\omega_i),a} := T_w(Y_{i,a}).$$

For example, $Y_{\omega_i,a} = Y_{i,a}$, $Y_{s_i(\omega_i),a} = Y_{i,a} A_{i,aq_i}^{-1}$.

3.2. Extension of Chari's action. In [FH4], we extended Chari's braid group action to the following extension $\mathcal{Y}'$ of the ring $\mathcal{Y}$:

$$(3.14) \quad \mathcal{Y}' := \mathbb{Z}[\Psi_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times} \otimes_{\mathbb{Z}} \mathbb{Z}(P) = \mathbb{Z}[\Psi_{i,a}^{\pm 1}, y_j^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times; j \in I} \supset \mathcal{Y},$$

where

$$(3.15) \quad Y_{i,a} = [\omega_i] \Psi_{i,aq_i^{-1}} \Psi_{i,aq_i}^{-1}.$$

Recall the elements $\tilde{\Psi}_{i,a} \in \mathcal{Y}'$, $i \in I$, which we introduced in [FH2]:

$$(3.16) \quad \tilde{\Psi}_{i,a} := \Psi_{i,a}^{-1} \left(\prod_{j \in I, C_{i,j} = -1} \Psi_{j,aq_i} \right) \left(\prod_{j \in I, C_{i,j} = -2} \Psi_{j,a} \Psi_{j,aq^2} \right) \left(\prod_{j \in I, C_{i,j} = -3} \Psi_{j,aq^{-1}} \Psi_{j,aq} \Psi_{j,aq^3} \right).$$

Define a ring automorphism σ of $\mathcal{Y}'$ by the formula

$$(3.17) \quad \sigma(\Psi_{i,a}) := \Psi_{i,a^{-1}}^{-1}, \quad i \in I, \quad a \in \mathbb{C}^\times; \quad \sigma([\omega]) = [\omega], \quad \omega \in P.$$

Define a ring automorphism $T'_i : \mathcal{Y}' \rightarrow \mathcal{Y}'$, $i \in I$, by the formulas

$$(3.18) \quad T'_i(\Psi_{i,a}) = \sigma(\tilde{\Psi}_{i,a^{-1}q_i^{-2}}^{-1}),$$

$$(3.19) \quad T'_i(\Psi_{j,a}) = \Psi_{j,a}, \quad j \neq i; \quad T'_i[\omega] = [s_i(\omega)], \quad \omega \in P.$$

Theorem 3.2. [FH4] (1) *The operators T'_i , $i \in I$, generate an action of the braid group associated to $\mathfrak{g}$ on $\mathcal{Y}'$.*

(2) *The restriction of T'_i to $\mathcal{Y} \subset \mathcal{Y}'$ is equal to the Chari operator T_i .*

Variations of the braid group action are also considered in [GHL, W].

4. EXTREMAL MONOMIAL PROPERTY OF q -CHARACTERS

In this section, we introduce w -twisted root monomials, where w is an arbitrary element of the Weyl group of $\mathfrak{g}$. We then conjecture a new property of q -characters of simple finite-dimensional $U_q(\hat{\mathfrak{g}})$ -modules which we call the *extremal monomial property* corresponding to w (Conjecture 4.4). We will prove a weaker property, stated in Theorem 4.10. In the case of the identity element of W , Conjecture 4.4 has been established in [FM]. For the longest element of W , we have two proofs: the first is given in the proof of Theorem 4.6 and the second is obtained by combining Theorem 6.11 and the proof of Conjecture 6.8 obtained in [Z]. In the case of simple reflections, we will prove Conjecture 6.9 (and hence Conjecture 4.4 which is equivalent to it by Theorem (6.11)) in Section 6 (Theorem 6.12).

4.1. Ordinary characters. The character $\chi(V_i)$ of a fundamental representation V_i of $\mathfrak{g}$ (or, equivalently, of $U_q(\mathfrak{g})$) is invariant for the action of the Weyl group W . Consequently

$$\chi(V_i) \in [w(\omega_i)] + \sum_{\lambda \prec_w w(\omega_i)} \mathbb{Z}[\lambda],$$

where the partial ordering $\prec_w$ is defined by

$$\lambda \preceq_w \mu \Leftrightarrow \mu - \lambda \in \sum_{j \in I} \mathbb{Z}[w(\alpha_j)] \Leftrightarrow w^{-1}(\lambda) \preceq w^{-1}(\mu).$$

The Weyl group symmetry of q -character established in [FH3] does not give a direct analogue of this statement because the image of a monomial in $\mathcal{Y}$ is not a monomial $\mathcal{Y}$. Instead, we find a new *extremal monomial property* of q -characters.

4.2. Root monomials. Let us introduce the w -twisted root monomials for $i \in I$, $a \in \mathbb{C}^\times$ and $w \in W$:

$$A_{i,a}^w = T_w(A_{i,a}) \text{ for } i \in I \text{ and } a \in \mathbb{C}^\times.$$

The corresponding weight is

$$\varpi(A_{i,a}^w) = w(\alpha_i).$$

Lemma 4.1. *For any $w \in W$, T_w induces a ring automorphism of $\mathbb{Z}[A_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times}$.*

Proof. It suffices to establish the statement for the $T_i, i \in I$. It is established in [CM] (by direct computation) that $\mathbb{Z}[A_{j,a}^{\pm 1}]_{j \in I, a \in \mathbb{C}^\times}$ is preserved by the T_i :

$$\begin{aligned} T_i(A_{i,a}) &= A_{i,aq_i^2}^{-1} \\ T_i(A_{j,a}) &= A_{j,a} \text{ if } C_{i,j} = 0, \\ T_i(A_{j,a}) &= A_{j,a} A_{i,aq_i} \text{ if } C_{i,j} = -1, \\ T_i(A_{j,a}) &= A_{j,a} A_{i,a} A_{i,aq_i^2} \text{ if } C_{i,j} = -2, \\ T_i(A_{j,a}) &= A_{j,a} A_{i,aq_i^{-1}} A_{i,aq_i} A_{i,aq_i^3} \text{ if } C_{i,j} = -3. \end{aligned}$$

Now we have for $i, j \in I$:

$$\begin{aligned} A_{i,a} &= T_i(A_{i,aq_i^{-2}}^{-1}) \\ A_{j,a} &= T_i(A_{j,a}) \text{ if } C_{i,j} = 0, \\ A_{j,a} &= T_i(A_{j,a} A_{i,aq_i^{-1}}) \text{ if } C_{i,j} = -1, \\ A_{j,a} &= T_i(A_{j,a} A_{i,a} A_{i,aq_i^{-2}}) \text{ if } C_{i,j} = -2, \\ A_{j,a} &= T_i(A_{j,a} A_{i,aq_i} A_{i,aq_i^{-1}} A_{i,aq_i^{-3}}) \text{ if } C_{i,j} = -3. \end{aligned}$$

Hence T_i induces an automorphism. □

4.3. Extremal monomial property defined.

Proposition 4.2. *There is a well-defined partial ordering $\preceq_w$ on $\mathcal{M}$ defined by the property*

$$m \preceq_w m' \Leftrightarrow m'm^{-1} \in \mathbb{Z}[(A_{j,b}^w)^{-1}]_{j \in I, b \in \mathbb{C}^\times} \Leftrightarrow T_w^{-1}(m) \preceq T_w^{-1}(m').$$

Proof. This partial ordering is well-defined since the elements $A_{j,b}^w$ are algebraically independent (for $w = e$, this is known from [FR], and for general $w \in W$ the result follows from Lemma 4.1). $\square$

Remark 4.3. *For $w = e$, this is sometimes called the Nakajima partial ordering.*

The following is the first main conjecture of this paper, which expresses the *extremal monomial property* of the q -characters of simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -modules. It is a generalization to an arbitrary element $w \in W$ of the highest monomial property of the q -characters, conjectured in [FR] and proved in [FM], which corresponds to the identity element $w = e$ of the Weyl group (see Theorem 2.12, (ii)).

Conjecture 4.4. *Let $L(m)$ be the simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module with the highest monomial m and $w \in W$. Then*

$$(4.20) \quad \chi_q(L(m)) \in T_w(m) \mathbb{Z}[(A_{j,b}^w)^{-1}]_{j \in I, b \in \mathbb{C}^\times}.$$

Equivalently,

$$\chi_q(L(m)) = T_w(m) + \sum_{m' \prec_w T_w(m)} c_{m'} m', \quad c_{m'} \in \mathbb{Z}_{\geq 0}.$$

Remark 4.5. *Conjecture 4.4 is equivalent to the following statement: for all $i \in I$, $a \in \mathbb{C}^\times$, and $w \in W$,*

$$\chi_q(L(Y_{i,a})) \in Y_{w_0(\omega_i), a} \mathbb{Z}[(A_{j,b}^w)^{-1}]_{j \in I, b \in \mathbb{C}^\times}.$$

Theorem 4.6. *Conjecture 4.4 holds when w is the identity element and the longest element of the Weyl group.*

Proof. In the case $w = e$, Conjecture 4.4 has been proved in [FM]. Applying to this result the involution studied in [He2, Section 4.2], we obtain

$$\chi_q(L(Y_{i,a})) \in Y_{w_0(\omega_i), a} \mathbb{Z}[A_{j,b}]_{j \in I, b \in \mathbb{C}^\times}.$$

This implies the statement of Conjecture 4.4 for $w = w_0$ (the longest element of the Weyl group) because one can readily see that $(A_{j,b}^{w_0})^{-1} = A_{\bar{j}, bq^h \vee_r \vee}^{-1}$, where the bar involution of I is defined by the formula $\alpha_{\bar{i}} = -w_0(\alpha_i)$. In fact, $T_{w_0}(Y_{j,b}) = Y_{\bar{j}, bq^h \vee_r \vee}^{-1}$ by [C, FM]. $\square$

The partial ordering $\preceq_w$ induces a partial ordering on simple modules, which we denote by the same symbol:

$$L(m) \preceq_w L(m') \Leftrightarrow m \preceq_w m'.$$

Then the statement of the Conjecture 4.4 implies that for simple modules $L(m)$ and $L(m')$, we have the following relation in the Grothendieck ring of finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -modules:

$$[L(m) \otimes L(m')] = [L(mm')] + \sum_{L \prec_w L(mm')} k_L [L], \quad k_L \in \mathbb{Z}_{\geq 0}$$

because $T_w(mm') = T_w(m)T_w(m')$ occurs with multiplicity 1 in $\chi_q(L(mm'))$.

Remark 4.7. *It would be interesting to find out whether the partial orderings $\preceq_w$ on simple modules are related to the partial orderings introduced in [KKOP].*

Example 4.8. (i) *Let $\mathfrak{g} = \mathfrak{sl}_3$ and $w = s_1$. Then*

$$Y_{w(\omega_1),a} = Y_{1,aq^2}^{-1} Y_{2,aq}, \quad Y_{w(\omega_2),a} = Y_{2,a}, \quad A_{1,a}^w = A_{1,aq^2}^{-1}, \quad A_{2,a}^w = A_{2,a} A_{1,aq},$$

$$\chi_q(L(Y_{1,a})) = Y_{w(\omega_1),a} (1 + (A_{1,aq^{-1}}^w)^{-1} + (A_{1,aq}^w)^{-1} (A_{2,aq^2}^w)^{-1}),$$

$$\chi_q(L(Y_{2,a})) = Y_{w(\omega_2),a} (1 + (A_{2,aq}^w)^{-1} + (A_{2,aq}^w)^{-1} (A_{1,a}^w)^{-1}).$$

(ii) *Let $\mathfrak{g} = B_2$ ($d_1 = 2, d_2 = 1$) and $w = s_1 s_2$. Then*

$$Y_{w(\omega_1),a} = Y_{1,aq^4}^{-1} Y_{2,aq} Y_{2,aq^3}, \quad Y_{w(\omega_2),a} = Y_{1,aq^5}^{-1} Y_{2,aq^4},$$

$$A_{1,a}^w = A_{1,aq^2} A_{2,aq^2} A_{2,a}, \quad A_{2,a}^w = A_{2,aq^2}^{-1} A_{1,aq^4}^{-1},$$

$$\begin{aligned} \chi_q(L(Y_{1,a})) &= Y_{w(\omega_1),a} (1 + (A_{1,aq^2}^w A_{2,a}^w)^{-1} + (A_{1,aq^2}^w A_{2,a}^w A_{1,a}^w A_{2,aq^{-2}}^w)^{-1} + (A_{1,aq^2}^w)^{-1} \\ &\quad + (A_{2,aq^{-2}}^w A_{1,aq^{-2}}^w A_{2,aq^{-4}}^w)^{-1}), \end{aligned}$$

$$\chi_q(L(Y_{2,a})) = Y_{w(\omega_2),a} (1 + (A_{1,aq^3}^w A_{2,aq}^w)^{-1} + (A_{2,aq^{-1}}^w A_{1,aq^{-1}}^w A_{2,aq^{-3}}^w)^{-1} + (A_{2,aq^{-1}}^w)^{-1}).$$

Remark 4.9. *The example (i) shows that the extremal monomial property does not follow from a reduction to the $\mathfrak{sl}_2$ -case (contrary to the case $w = e$, as shown in the proof of the highest monomial property in [FM]). Indeed, the factors $(A_{1,aq^{-1}}^w)^{-1}$ and $(A_{1,aq}^w)^{-1}$ occur with different spectral parameters therein.*

The following theorem is a weaker version of Conjecture 4.4; namely, in Conjecture 4.4 we have polynomials in $(A_{j,b}^w)^{-1}$ (see formula (4.20)), whereas in this theorem we have polynomials in $(A_{j,b}^w)^{\pm 1}$ (see formula (4.21)).

Theorem 4.10. *Let $L(m)$ be the simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module with the highest monomial m and $w \in W$. Then*

$$(4.21) \quad \chi_q(L(m)) \in T_w(m) \mathbb{Z}[(A_{j,b}^w)^{\pm 1}]_{j \in I, b \in \mathbb{C}^\times} \subset \mathbb{Z}[(Y_{w(\omega_j),b})^{\pm 1}]_{j \in I, b \in \mathbb{C}^\times}.$$

Proof. It suffices to establish the result for $m = Y_{i,a}$, $i \in I$ and $a \in \mathbb{C}^\times$. By definition, $Y_{w(\omega_i),a}$ occurs in $\chi_q(L(Y_{i,a}))$ with multiplicity 1. This implies that

$$\chi_q(L(Y_{i,a})) \in Y_{w(\omega_i),a} \mathbb{Z}[(A_{j,b})^{\pm 1}]_{j \in I, b \in \mathbb{C}^\times}.$$

But, by Lemma 4.1, we have

$$\mathbb{Z}[(A_{j,b})^{\pm 1}]_{j \in I, b \in \mathbb{C}^\times} = T_w(\mathbb{Z}[(A_{j,b})^{\pm 1}]_{j \in I, b \in \mathbb{C}^\times}) = \mathbb{Z}[(A_{j,b}^w)^{\pm 1}]_{j \in I, b \in \mathbb{C}^\times}.$$

□

5. AN INVARIANT PAIRING

In this section, we develop technical tools to establish Conjecture 4.4 for simple reflections, and potentially for all elements of the Weyl group. We will introduce the X -series which are certain formal Taylor power series $X_{w(\omega_i)}(z)$, $w \in W, i \in I$, of Drinfeld-Cartan generators in $U_q(\widehat{\mathfrak{g}})$. We also introduce a pairing that we prove to be invariant with respect to the braid group operators $T_w, w \in W$ (Proposition 5.6). We then relate the ℓ -weights $\Psi_{w(\omega_i),a}$ to the X -series $X_{w(\omega_i)}(za)$. (Proposition 5.10).

5.1. Fundamental and prefundamental series. In [FH1], we defined the *prefundamental series* $X_i(z)$, $i \in I$, by the formulas

$$(5.22) \quad X_i(z) = \exp \left(\sum_{m>0} z^m \frac{\tilde{h}_{i,-m}}{[d_i]_q [m]_{q_i}} \right),$$

where

$$(5.23) \quad \tilde{h}_{i,-m} = \sum_{j \in I} [d_j]_q \tilde{C}_{j,i}(q^m) h_{j,-m} \text{ and } [d_i]_q h_{i,-m} = \sum_{j \in I} C_{j,i}(q^m) \tilde{h}_{j,-m}.$$

Remark 5.1. In [FH1], the series $X_i(z)$ was denoted by $T_i(z)$.

The series $X_i(z)$ are formal Taylor power series in z which are related to the *fundamental series* $Y_i(z)$ defined in [FR] as follows:

$$Y_i(z) = \bar{k}_i^{-1} \frac{X_i(zq_i^{-1})}{X_i(zq_i)} = \bar{k}_i^{-1} \exp \left(\sum_{m>0} z^m (q^{-1} - q) \tilde{h}_{i,-m} \right),$$

where the elements $\tilde{h}_{i,n}$ are defined by formula (5.23), and the elements $\bar{k}_i$ are defined in the adjoint version of quantum affine algebra by the relation:

$$\prod_{j \in I} \bar{k}_j^{C_{j,i}} = k_i.$$

The relation between $X_i(z)$, $Y_i(z)$, and the eigenvalues of transfer-matrices will be discussed below (this will also justify the terminology). Note that $X_i(z)$ commutes with the $x_{j,r}^\pm$ for $j \neq i$ and $r \in \mathbb{Z}$ (since $[\tilde{h}_{i,-m}, x_{j,r}^\pm] = 0$ for $i \neq j$).

Example 5.2. In the case $\mathfrak{g} = sl_2$, we have $X_1(z) = \exp \left(\sum_{m>0} z^m \frac{h_{1,-m}}{[m]_q (q^m + q^{-m})} \right)$.

Since the series $Y_i(az)$, $i \in I$, $a \in \mathbb{C}^\times$, are algebraically independent in $U_q(\widehat{\mathfrak{h}})[[z]]$, there is a unique injective group homomorphism

$$(5.24) \quad \mathcal{J} : \mathcal{M} \rightarrow (U_q(\widehat{\mathfrak{h}})[[z]])^\times$$

such that $\mathcal{J}(Y_{i,a}) = Y_i(za)$.

Recall from [FR] that

$$(5.25) \quad \phi_i^-(z^{-1}) = Y_i(zq_i) Y_i(zq_i^{-1}) \prod_{j, C_{j,i}=-1} Y_j(z) \prod_{j, C_{j,i}=-2} Y_j(zq^{-1}) Y_j(zq) \prod_{j, C_{j,i}=-3} Y_j(zq^{-2}) Y_j(z) Y_j(zq^2).$$

Comparing with formula (2.9), we obtain that $\mathcal{J}(A_{i,a}) = \phi_i^-(z^{-1}a^{-1})$.

5.2. Definition of the pairing. Given $m \in \mathcal{M}$ and a formal power series $h(z)$ in $U_q(\widehat{\mathfrak{h}})[[z]]$, we define a formal power series

$$\langle h(z), m \rangle \in \mathbb{C}[[z]],$$

as the eigenvalue of $h(z)$ on the ℓ -weight space associated to the monomial m . This defines a pairing

$$\langle \cdot, \cdot \rangle : U_q(\widehat{\mathfrak{h}})[[z]] \times \mathcal{M} \rightarrow \mathbb{C}[[z]].$$

It is related to the inner product $(\cdot, \cdot)$ on the lattice P of weights of $\mathfrak{g}$ introduced in Section 2.1 as follows:

$$(5.26) \quad \langle k_j, m \rangle = q^{(\alpha_j, \varpi(m))}, \quad j \in I.$$

This pairing satisfies the following property: for $m_1, m_2 \in \mathcal{M}$, $i \in I$, $m \neq 0$, we have

$$(5.27) \quad \begin{aligned} \langle h_{i,m}, m_1 m_2 \rangle &= \langle h_{i,m}, m_1 \rangle + \langle h_{i,m}, m_2 \rangle, \quad \langle k_i, m_1 m_2 \rangle = \langle k_i, m_1 \rangle \langle k_i, m_2 \rangle, \\ \langle X_i(z), m_1 m_2 \rangle &= \langle X_i(z), m_1 \rangle \langle X_i(z), m_2 \rangle, \quad \langle Y_i(z), m_1 m_2 \rangle = \langle Y_i(z), m_1 \rangle \langle Y_i(z), m_2 \rangle. \end{aligned}$$

By definition, for $i, j \in I$ and $m > 0$, we have

$$\langle \widetilde{h}_{i,-m}, Y_{j,a} \rangle = [d_i]_q [m]_{q_i} a^{-m} \frac{\widetilde{C}_{i,j}(q^m)}{m},$$

and therefore

$$\langle h_{i,-m}, Y_{j,a} \rangle = \delta_{i,j} a^{-m} \frac{[m]_{q_i}}{m}, \quad \langle \widetilde{h}_{i,-m}, A_{j,a} \rangle = \delta_{i,j} a^{-m} [d_i]_q \frac{[m]_{q_i}}{m}.$$

According to the results of [FH1], for $i, j \in I$, $a \in \mathbb{C}^\times$, we have

$$(5.28) \quad \langle X_i(z), Y_{j,a} \rangle = \exp \left(\sum_{r>0} z^r a^{-r} \frac{\widetilde{C}_{i,j}(q^r)}{r} \right),$$

$$(5.29) \quad \langle X_i(z), A_{j,a} \rangle = (1 - za^{-1})^{-\delta_{i,j}}.$$

In particular, if $m = \prod_{j \in I, b \in \mathbb{C}^\times} Y_{j,b}^{u_{j,b}(m)}$, where $u_{j,b}(m) \in \mathbb{Z}_{\geq 0}$, then

$$(5.30) \quad \langle X_i(z), m \rangle = \prod_{j \in I, b \in \mathbb{C}^\times} \exp \left(u_{j,b}(m) \sum_{r>0} z^r b^{-r} \frac{\widetilde{C}_{i,j}(q^r)}{r} \right).$$

Now consider the monomials appearing in the q -character of the simple $U_q(\widehat{\mathfrak{g}})$ -module $L(m)$. According to [FM], they all have the form

$$M = m \cdot A_{i_1, a_1}^{-1} A_{i_2, a_2}^{-1} \cdots A_{i_N, a_N}^{-1},$$

where $i_1, \dots, i_N \in I$ and $a_1, \dots, a_N \in \mathbb{C}^\times$. From the above formulas it follows that

$$\langle X_i(z), M \rangle = \langle X_i(z), m \rangle \cdot \prod_{1 \leq k \leq N, i_k = i} (1 - za_k^{-1}).$$

5.3. W -invariance of the pairing. Let us introduce the following series for $i \in I$:

$$U_i(z) = \exp \left(\sum_{m>0} z^{-m} \frac{h_{i,-m}}{[m]_{q_i}} \right)$$

so that we have

$$\phi_i^-(z) = k_i^{-1} \frac{U_i(zq_i)}{U_i(zq_i^{-1})}.$$

Remark 5.3. A series $\mathcal{P}_i(z)$ of Cartan-Drinfeld elements is defined in [CP, Lemma 12.2.7] in a different way, but it satisfies $\phi_i^-(z) = k_i^{-1} \mathcal{P}_i^-(q_i^{-2}z) / \mathcal{P}_i^-(z)$ and $\mathcal{P}_i(0) = 1$. And so $\mathcal{P}_i^-(z) = (U_i(zq_i))^{-1}$.

Then we have

$$U_i(z) = \prod_{j \in I} \exp \left(\sum_{m>0} z^{-m} C_{j,i}(q^m) \frac{[d_j]_q [m]_{q_j}}{[d_i]_q [m]_{q_i}} \frac{\tilde{h}_{j,-m}}{[d_j]_q [m]_{q_j}} \right).$$

But

$$C_{j,i}(q^m) \frac{[d_j]_q [m]_{q_j}}{[d_i]_q [m]_{q_i}} = C_{i,j}(q^m) \frac{[d_i]_{q^m} [d_j]_q [m]_{q_j}}{[d_j]_{q^m} [d_i]_q [m]_{q_i}} = C_{i,j}(q^m).$$

Hence

$$(5.31) \quad U_i(z^{-1}) = X_i(zq_i) X_i(zq_i^{-1}) \prod_{j, C_{i,j}=-1} X_j(z) \prod_{j, C_{i,j}=-2} X_j(zq^{-1}) X_j(zq) \prod_{j, C_{i,j}=-3} X_j(zq^{-2}) X_j(z) X_j(zq^2).$$

Remark 5.4. Note that in comparison to (5.25), the coefficients $C_{i,j}$ appear, not $C_{j,i}$. However, this is not the expression of the root monomials $A_{i,a}$ for the Langlands dual Lie algebra. Rather, it can be identified with the expression of the variables $\Lambda_{i,a}$ in [He3, Section 9.2].

Example 5.5. In the case $\mathfrak{g} = \mathfrak{sl}_2$, we have $U_1(z) = \exp \left(\sum_{m>0} z^{-m} \frac{h_{1,-m}}{[m]_q} \right)$.

Now we define an automorphism T_i of $U_q(\widehat{\mathfrak{h}})$, and hence of $U_q(\widehat{\mathfrak{h}})[[z^{\pm 1}]]$, by setting

$$T_i(\tilde{h}_{j,m}) = \tilde{h}_{j,m} - \delta_{i,j} q_i^{-m} [d_i]_q h_{i,m} \text{ and } T_i(\tilde{k}_j) = \tilde{k}_j k_i^{-\delta_{i,j}}$$

for $i, j \in I$ and $m \in \mathbb{Z} \setminus \{0\}$, so that

$$(5.32) \quad T_i(Y_j(z)) = Y_j(z) (\phi_i^-(z^{-1} q_i^{-1}))^{-\delta_{i,j}}.$$

This implies that the homomorphism $\mathcal{J}$ given by formula (5.24) is W -equivariant. It also follows that

$$T_i(X_j(z)) = X_j(z) U_i(z^{-1} q_i^{-1})^{-\delta_{i,j}}.$$

The results cited above imply that the operators $T_i, i \in I$, satisfy the braid relations, and therefore we can define operators T_w for $w \in W$ by using a reduced decomposition of w .

Proposition 5.6. *The pairing $\langle \cdot, \cdot \rangle$ is W -invariant in the following sense:*

$$\langle T_w(h(z)), T_w(m) \rangle = \langle h(z), m \rangle$$

for all $w \in W$, $h(z) \in U_q(\widehat{\mathfrak{h}})[[z]]$ and $m \in \mathcal{M}$.

Proof. If $h(z) = k_i^{\pm 1}$, then W -invariance follows from formula (5.26) and W -invariance of $(\cdot, \cdot)$. So it suffices to check that for $i, j, k \in I$ and $a \in \mathbb{C}^\times$, we have

$$\langle T_k(\tilde{h}_{i,-m}), T_k(Y_{j,a}) \rangle = \langle \tilde{h}_{i,-m}, Y_{j,a} \rangle.$$

The left hand side is equal to the right hand side $\langle \tilde{h}_{i,-m}, Y_{j,a} \rangle$ plus the sum of the term

$$\langle \tilde{h}_{i,-m}, A_{j,aq_j}^{-\delta_{k,j}} \rangle = -\delta_{k=j=i}(aq_i)^{-m}[d_i]_q \frac{[m]_{q_i}}{m}$$

and the term

$$-\delta_{i,k} q_i^m [d_i]_q \langle \tilde{h}_{i,-m}, Y_{j,a} A_{j,aq_j}^{-\delta_{k,j}} \rangle = \delta_{k=j=i} q_i^m [d_i]_q (aq_i^2)^{-m} \frac{[m]_{q_i}}{m}.$$

This gives us the desired result because the sum of these last two terms is zero. $\square$

For $w \in W$ and $i \in I$, we introduce the following formal Taylor power series in z :

$$X_{w(\omega_i)}(z) := T_w(X_i(z)).$$

By Lemma 2.1, $T_w(X_i(z))$ depends only on $w(\omega_i) \in P$ (and not separately on $w \in W$ and $i \in I$), so this notation is justified. We will call these series the X -series.

Example 5.7. (i) For $\mathfrak{g} = B_2$, we have

$$X_{s_2(\omega_1)}(z) = X_1(z) \text{ and } X_{s_2(\omega_2)}(z) = X_2^{-1}(zq^2)X_1(zq^2)X_1(z).$$

(ii) More generally, for a simple reflection s_i , one has

$$X_{s_i(\omega_j)}(z) = X_i(z) \text{ if } j \neq i,$$

$$\begin{aligned} X_{s_i(\omega_i)}(z) &= X_i(z)U_i(z^{-1}q_i^{-1})^{-1} = (X_i(zq_i^2))^{-1} \left(\prod_{j \in I, C_{i,j}=-1} X_j(zq_i) \right) \left(\prod_{j \in I, C_{i,j}=-2} X_j(zq^2)X_j(z) \right) \\ &\quad \times \left(\prod_{j \in I, C_{i,j}=-3} X_j(zq^3)X_j(zq)X_j(zq^{-1}) \right). \end{aligned}$$

Proposition 5.8. For $i, j \in I$, $a \in \mathbb{C}^\times$ and $w \in W$, we have

$$\langle X_{w(\omega_i)}(z), (A_{k,a}^w)^{-1} \rangle = (1 - za^{-1})^{\delta_{i,k}}.$$

Moreover, we have

$$X_{w(\omega_i)}(z) = \prod_{j \in I, a \in \mathbb{C}^\times} X_j(za)^{m_{j,a}^i}$$

where the $m_{j,a}^i \in \mathbb{Z}$ are defined by

$$A_{i,1} = \prod_{j \in I, a \in \mathbb{C}^\times} (A_{j,a^{-1}}^w)^{m_{i,a}^j}.$$

Proof. By W -invariance of the pairing and formula (5.29), we have

$$(X_{w(\omega_i)}(z), (A_{k,a}^w)^{-1}) = \langle X_i(z), A_{k,a}^{-1} \rangle = (1 - za^{-1})^{\delta_{i,k}}.$$

Next, we write

$$X_{w(\omega_i)}(z) = \prod_{j \in I, a \in \mathbb{C}^\times} X_j(za)^{m_{j,a}^i}$$

where $m_{j,a}^i \in \mathbb{Z}$. Let us write for $k \in I$

$$A_{k,1}^{-1} = \prod_{l \in I, d \in \mathbb{C}^\times} (A_{l,d-1}^w)^{-w_{l,d}^k},$$

where $w_{l,d}^k \in \mathbb{Z}$. Then

$$\langle X_{w(\omega_i)}(z), A_{k,1}^{-1} \rangle = \prod_{d \in \mathbb{C}^\times} (1 - zd^{-1})^{w_{i,d}^k} = \prod_{a \in \mathbb{C}^\times} (1 - za^{-1})^{m_{k,a}^i}.$$

This implies that $w_{i,d}^k = m_{k,d}^i$. □

We also define

$$Y_{w(\omega_i)}(z) := T_w(Y_i(z)) = T_w(\bar{k}_i^{-1}) \frac{X_{w(\omega_i)}(zq_i^{-1})}{X_{w(\omega_i)}(zq_i)} = T_w(\bar{k}_i^{-1}) \prod_{j \in I, a \in \mathbb{C}^\times} \frac{(X_j(zaq_i^{-1}))^{m_{j,a}^i}}{(X_j(zaq_i))^{m_{j,a}^i}}.$$

Recall the injective group homomorphism $\mathcal{J}$ given by formula (5.24). The next lemma follows from the W -equivariance of $\mathcal{J}$ and formulas (3.13), (5.32).

Lemma 5.9. *For any $i \in I$, we have*

$$(5.33) \quad \mathcal{J}(Y_{w(\omega_i),a}) = Y_{w(\omega_i)}(za).$$

Next, we compute the series $X_{w(\omega_i)}(z)$. In order to do this, note that $X_{w(\omega_i)}(z)$ is characterized by the formula

$$Y_{w(\omega_i)}(z) = T_w(\bar{k}_i^{-1}) X_{w(\omega_i)}(zq_i^{-1}) / X_{w(\omega_i)}(zq_i),$$

which is analogous to the formula

$$\sigma(Y_{w(\omega_i),a^{-1}}) = [w(\omega_i)] \Psi_{w(\omega_i),aq_i^{-1}} / \Psi_{w(\omega_i),aq_i},$$

where σ (as defined by formula 3.17) satisfies $\sigma(Y_{j,b}) = Y_{j,b^{-1}}$, (see [FH4, Proposition 4.5]). The ℓ -weights $\Psi_{w(\omega_i),a}$ were defined in [FH4] (this definition will be recalled in Section 7.3).

Comparing these formulas and using Lemma 5.9, we obtain the following result. Let $\mathcal{M}'$ be the group of monomials in $\mathcal{Y}' = \mathbb{Z}[\Psi_{i,a}]_{i \in I, a \in \mathbb{C}^\times}$.

Proposition 5.10. (1) *The homomorphism $\mathcal{J}$ extends uniquely to a homomorphism*

$$(5.34) \quad \mathcal{J}' : \mathcal{M}' \rightarrow (U_q(\widehat{\mathfrak{h}})[[z]])^\times$$

such that

$$(5.35) \quad \mathcal{J}'(\Psi_{i,a}) = X_i(za^{-1}) \quad i \in I, a \in \mathbb{C}^\times.$$

(2) *For $i \in I$, $a \in \mathbb{C}^\times$ and $w \in W$, we have*

$$\mathcal{J}'(\Psi_{w(\omega_i),a}) = X_{w(\omega_i)}(za^{-1}).$$

This means that the powers of $X_j(zb)$ in the factorization of $X_{w(\omega_i)}(z)$ are the same as the powers of the $\Psi_{j,b-1}$ in the factorization of $\Psi_{w(\omega_i),1}$. The latter are given in Section 7.3. This implies the following.

Proposition 5.11. *The series $X_{w(\omega_i)}(z)$ are obtained from the factorization of $Y_{w(\omega_i),1}$ in the variables $Y_{k,a}^{\pm 1}$ by replacing*

- $Y_{k,a}$ by $X_k(za)$ if $d_i = d_k$,
- $Y_{k,a}$ by $X_k(zaq)X_k(zaq^{-1})$ if $C_{i,k} = -2$,
- $Y_{k,a}$ by $X_k(zaq^{-2})X_k(za)X_k(zaq^2)$ if $C_{i,k} = -3$,
- $W_{k,a}$ by $X_k(za)$ if $C_{k,i} < -1$, where $W_{i,a}$ is given by (7.46).

6. EXTREMAL MONOMIAL PROPERTY AND POLYNOMIALITY

In this section, we relate the extremal monomial property of q -characters, Conjecture 4.4 (resp. a weaker property, Theorem 4.10) to the polynomiality (resp. rationality) of the eigenvalues of the renormalized X -series $X_{w(\omega_i)}^N(z)$ acting on simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -modules. We first derive certain identities between the eigenvalues of the renormalized X -series on a simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module (Corollary 6.2). Then we show that our Theorem 4.10 implies rationality of the eigenvalues of $X_{w(\omega_i)}^N(z)$ on these modules (Theorem 6.4), from which we derive that the operator $X_{w(\omega_i)}^N(z)$ itself is an expansion of a rational function in z (Theorem 6.6). We conjecture that it is in fact a polynomial in z (Conjecture 6.8). In [FH1] we proved this polynomiality for $w = e$, and in [Z] it was proved for $w = w_0$. Here, we prove polynomiality of the eigenvalues of $X_{w(\omega_i)}^N(z)$ when w is a simple reflection (Theorem 6.12). This implies Conjecture 4.4 for simple reflections.

In the next section (Section 7), we will introduce the generalized Baxter operators $t_{w(\omega_i)}(z, u)$ (they are the dual versions of the Baxter operators we had previously introduced in [FH4]) and show that the X -series $X_{w(\omega_i)}(z)$ are certain limits of these operators. Therefore we obtain a link between polynomiality of the eigenvalues of $X_{w(\omega_i)}^N(z)$ and polynomiality of the eigenvalues of these Baxter operators (properly renormalized). This has important consequences for the quantum integrable models of XXZ type associated to $U_q(\widehat{\mathfrak{g}})$; namely, it allows one to give different descriptions (one for each $w \in W$) of the spectra of the commuting Hamiltonians in these models in terms of the polynomial eigenvalues of the renormalized Baxter operators $t_{w(\omega_i)}(z, u)$. For $w = e$ this description has been proved in [FH1] (see Section 7.5), and for general $w \in W$ this is our Conjecture 7.19.

6.1. Rationality of $X_{w(\omega_i)}^N(z)$. Let $L(m)$ be the simple $U_q(\widehat{\mathfrak{g}})$ -module with the highest monomial m . Recall that $\langle X_{w(\omega_i)}(z), T_w(m) \rangle$ is the eigenvalue of $X_{w(\omega_i)}(z)$ on the ℓ -weight corresponding to the monomial $T_w(m)$. From the W -invariance of the pairing $\langle \cdot, \cdot \rangle$ (see Proposition 5.6), we obtain the following key result.

Theorem 6.1. *For any $i \in I$, the pairing $\langle X_{w(\omega_i)}(z), T_w(m) \rangle$ is independent of $w \in W$.*

This implies the following statement.

Corollary 6.2. *The eigenvalue of $X_{w(\omega_i)}(z)$ on the ℓ -weight corresponding to the monomial $T_w(m)$ is independent of $w \in W$.*

Let us denote this eigenvalue by $f_{i,m}(z)$. Thus, we have

$$\langle X_{w(\omega_i)}(z), T_w(m) \rangle = f_{i,m}(z), \quad \forall w \in W.$$

In [FH1], we computed it in the case $w = e$, and the result is the RHS of formula (5.30):

$$(6.36) \quad f_{i,m}(z) = \prod_{j \in I, b \in \mathbb{C}^\times} \exp \left(u_{j,b}(m) \sum_{r>0} (zb)^r \frac{\tilde{C}_{i,j}(q^r)}{r} \right),$$

where the numbers $u_{j,b}(m) \in \mathbb{Z}_{\geq 0}$ are the powers appearing in the formula

$$m = \prod_{j \in I, b \in \mathbb{C}^\times} Y_{j,b}^{u_{j,b}(m)}.$$

Now let us define the *renormalized X -series* acting on the simple module $L(m)$ by the formula

$$(6.37) \quad X_{w(\omega_i)}^N(z) := (f_{i,m}(z))^{-1} X_{w(\omega_i)}(z).$$

Thus, $X_{w(\omega_i)}^N(z)$ is a formal Taylor power series in z of the form

$$X_{w(\omega_i)}^N(z) = \sum_{k \geq 0} X_{w(\omega_i),k}^N z^k$$

whose coefficients $X_{w(\omega_i),m}^N$ are elements of the commutative algebra $U_q(\hat{\mathfrak{h}})$. One can derive from this that any finite-dimensional $U_q(\hat{\mathfrak{g}})$ -module $L(m)$ has a basis (over $\mathbb{C}$) in which the matrix of the formal power series $X_{w(\omega_i)}^N(z)$ is upper-triangular, with the diagonal entries being the eigenvalues of the operator $X_{w(\omega_i),m}^N$ acting on $L(m)$.

Example 6.3. *Here are two examples of such matrices:*

$$\begin{pmatrix} z + z^2 & z^3 \\ 0 & z + z^2 \end{pmatrix}, \quad \begin{pmatrix} z + z^2 & \sum_{k \geq 0} \frac{z^k}{k!} \\ 0 & z + z^2 \end{pmatrix}$$

In both cases, the single eigenvalue is a polynomial in z (namely, $z + z^2$). While in the first case, the matrix itself is also a polynomial in z , in the second case it is not (not even an expansion of a rational function in z ; in fact, on the intersection of the first row and second column we have the expansion of $\exp(z)$, which is not an expansion of a rational function).

Here are two more examples:

$$\begin{pmatrix} z + z^2 & \sum_{k \geq 0} z^k & 0 & 0 \\ 0 & z + z^2 & 0 & 0 \\ 0 & 0 & \sum_{k \geq 0} k z^k & z^5 \\ 0 & 0 & 0 & \sum_{k \geq 0} k z^k \end{pmatrix}, \quad \begin{pmatrix} z + z^2 & \sum_{k \geq 0} z^k & 0 & 0 \\ 0 & z + z^2 & 0 & 0 \\ 0 & 0 & \sum_{k \geq 0} k z^k & \sum_{k \geq 0} k! z^k \\ 0 & 0 & 0 & \sum_{k \geq 0} k z^k \end{pmatrix}$$

In both cases, the matrix has two eigenvalues, which are expansions of two rational functions (namely, $z + z^2$ and $z/(1 - z)$). But while in the first case all entries of the matrix are

expansions of rational functions, in the second case this is not so (indeed, the entry on the intersection of the 3rd row and the 4th column is not an expansion of a rational function).

If all entries of the matrix of a given operator in its Jordan form, in the sense discussed above, are polynomials (resp. expansions of rational functions) in z , we will say that this operator is a polynomial (resp. an expansion of a rational function) in z .

As the next theorem shows, the eigenvalues of the renormalized operator $X_{w(\omega_i)}^N(z)$ on any simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module are expansions of rational functions in z .

Theorem 6.4. *Every eigenvalue of the renormalized operator $X_{w(\omega_i)}^N(z)$ acting on any simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module is an expansion of a rational function in z .*

Proof. This follows immediately from Proposition 5.8 and Theorem 4.10. $\square$

Remark 6.5. *It follows from the results of [FH1] that the degree of the rational function expressing the eigenvalue of $X_{w(\omega_i)}^N(z)$ on the ℓ -weight space corresponding to a given monomial m' appearing in the q -character $\chi_q(L(m))$ equals the multiplicity of $w(\alpha_i)$ in the decomposition of the ordinary weight of the monomial $T_w(m)(m')^{-1} \in \mathcal{M}$.*

More precisely, this is established in [FH1] for $w = e$. For a general $w \in W$, by Proposition 5.6, we have $\langle X_{w(\omega_i)}(z), m' \rangle = \langle X_i(z), T_w^{-1}(m') \rangle$ and so

$$(6.38) \quad \langle X_{w(\omega_i)}^N(z), m' \rangle = \langle X_i^N(z), T_w^{-1}(m') \rangle = \langle X_i(z), m^{-1}T_w^{-1}(m') \rangle.$$

Here $m^{-1}T_w^{-1}(m') \in \mathbb{Z}[A_{j,b}^{\pm 1}]_{j \in I, b \in \mathbb{C}^}$ and therefore it follows from (5.28) that the degree of the rational function $\langle X_i(z), m^{-1}T_w^{-1}(m') \rangle$ is the multiplicity of α_i in the ordinary weight of $mT_w^{-1}((m')^{-1})$, which is equal to the weight written above.*

In the next theorem, we claim the rationality of the operator $X_{w(\omega_i)}^N(z)$ itself. Hence it is a stronger statement than Theorem 4.10.

Theorem 6.6. *The renormalized X -series $X_{w(\omega_i)}^N(z)$ acting on any simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module is an expansion of a rational function in z .*

Proof. By [FH1, Theorem 5.17], $X_{\omega_i}^N(z)$ is a polynomial in z .¹ Hence the statement of the theorem is true for $w = e$. Now, for a general $w \in W$, recall that we have established in Proposition 5.8 that $X_{w(\omega_i)}(z)$ is a Laurent monomial in the $X_j(zb) = X_{\omega_j}(zb)$, $j \in I, b \in \mathbb{C}^\times$. By the above result, the matrix of $X_j(zb)$ on $L(m)$ is equal to its eigenvalue on the weight subspace of $L(m)$ associated to the weight $w(\varpi(m))$ times a rational matrix-valued function in z . This implies that $X_{w(\omega_i)}(z)$ is equal to its eigenvalue on the (one-dimensional) weight subspace of $L(m)$ associated to the weight $w(\varpi(m))$ times a rational matrix-valued function in z . But as we have shown above, the latter eigenvalue is equal to $f_{i,m}(z)$. Therefore the matrix of $X_{w(\omega_i)}^N(z)$, which is given by formula (6.37), acting on $L(m)$, is an expansion of a rational function in z . $\square$

Remark 6.7. *We note that rationality of closely related power series, which can be expressed as certain products of shifts of the X -series $X_i(z)$ acting on finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -modules (not necessarily those of highest weight) was recently established in [GTL].*

¹Note that the sentence “By Proposition 5.5, Corollary 5.10 implies:” before Theorem 5.17 in [FH1] should read “By Proposition 5.5, Theorem 5.9 implies:”.

6.2. Polynomiality of $X_{w(\omega_i)}^N(z)$. Here is the second main conjecture of this paper.

Conjecture 6.8. *The renormalized operator $X_{w(\omega_i)}^N(z)$ acting on any simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module $L(m)$ is a polynomial in z .*

The following conjecture is a weaker version of Conjecture 6.8, as it states polynomiality of the *eigenvalues* of $X_{w(\omega_i)}^N(z)$.

Conjecture 6.9. *Every eigenvalue of the renormalized X -series $X_{w(\omega_i)}^N(z)$ acting on any simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module $L(m)$ is a polynomial in z .*

Remark 6.10. *Note that although this statement of Conjecture 6.9 in the case $w = e$ was proved in [FH1], equation (6.38) by itself does not imply that $\langle X_{w(\omega_i)}^N(z), m' \rangle$ is a polynomial because for a general monomial m' occurring in $\chi_q(L(m))$, the monomial $T_w^{-1}(m')$ does not necessarily occur in $\chi_q(L(m))$.*

Now we relate these conjectures to Conjecture 4.4.

Theorem 6.11. *Conjecture 4.4 is equivalent to Conjecture 6.9, which in turn follows from Conjecture 6.8.*

More precisely, the statement of Conjecture 4.4 for the simple module $L(m)$ and $w \in W$ is equivalent to the statement of Conjecture 6.9 for $X_{w(\omega_i)}^N(z), i \in I$, and $L(m)$.

Proof. As in the proof of Theorem 6.4, Proposition 5.8 proves the equivalence of Conjectures 4.4 and 6.9.

Conjecture 4.4 claims polynomiality of the eigenvalues of $X_{w(\omega_i)}^N(z)$, whereas Conjecture 6.8 claims polynomiality of the operator $X_{w(\omega_i)}^N(z)$ itself. Hence Conjecture 4.4 follows from Conjecture 6.8. $\square$

For $w = e$, Conjectures 4.4 and 6.8 were established in [FH1]. For $w = w_0$, the longest element of the Weyl group, Conjecture 6.8 was established in [Z]. By Theorem 6.11, this also proves Conjecture 6.8. Thus, we obtain the second proof of Conjecture 4.4 for $w = w_0$ (the first one is given in the proof in Theorem 4.6). In the next section, we will prove Conjecture 4.4 for simple reflections.

6.3. Polynomiality for simple reflections. Let now $w = s_i, i \in I$, a simple reflection from the Weyl group.

Theorem 6.12. *Conjecture 6.9 holds for simple reflections $s_i \in W, i \in I$. Namely, for any $j \in I$, every eigenvalue of $X_{s_i(\omega_j)}^N(z)$ on a simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module is a polynomial in z .*

Applying Theorem 6.11, we obtain the following corollary.

Theorem 6.13. *Conjecture 4.4 holds for simple reflections $s_i \in W, i \in I$.*

To prove Theorem 6.12, we need some results from [He1, Sections 3.3, 3.4]. Let $k \in I$ and M a k -dominant monomial (that is all powers of the variables $Y_{k,a}$ in M are non-negative). An element $L_k(M)$ is defined in [He1] in the following way. Let $M^{(k)}$ be the expression

obtained from M by keeping only the factors $Y_{k,a}$. Then it corresponds to a simple module L over $U_{qk}(\widehat{sl}_2)$. Its q -character belongs to $M^{(k)} \cdot \mathbb{Z}[(Y_{k,bq_k^{-1}}Y_{k,bq_k})^{-1}]_{b \in \mathbb{C}^\times}$. By definition,

$$L_k(M) \in M \cdot \mathbb{Z}[(Y_{k,bq_k^{-1}}Y_{k,bq_k})^{-1}]_{b \in \mathbb{C}^\times}$$

is obtained from the q -character of L by replacing $M^{(k)}$ by M and each factor $(Y_{k,bq_k^{-1}}Y_{k,bq_k})^{-1}$ by $A_{k,b}^{-1}$.

Now let m be a dominant monomial. Then it follows from [He1, Proposition 3.1] that we have a unique decomposition

$$(6.39) \quad \chi_q(L(m)) = \sum_{M \text{ } k\text{-dominant}} \lambda_k(M) L_k(M)$$

with $\lambda_k(M) \geq 0$.

We are ready to prove Theorem 6.12.

Proof. For $j \neq i$, we have $X_{s_i(\omega_j)}^N = X_j^N(z)$ and the result is proved in [FH1] (see the last paragraph of Section 6.2). It remains to consider the case $j = i$. Then $T_{s_i(\omega_i)}(z)$ is given by explicit formulas in Example 5.7. Moreover, formula (5.27) implies that the desired polynomiality property is multiplicative with respect to taking tensor products of finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -modules. Hence it suffices to prove the polynomiality of the eigenvalues of $T_{s_i(\omega_i)}(z)$ on the fundamental representations $L(Y_{l,1})$, $l \in I$.

First assume that $l \neq i$. Then we have $T_{s_i}(Y_{l,1}) = Y_{l,1}$. Recall that a monomial m in $\chi_q(L(Y_{l,1}))$ is of the form

$$(6.40) \quad m = Y_{l,1} \prod_{k \in I, a \in \mathbb{C}^\times} A_{k,a}^{-v_{k,a}},$$

with the $v_{k,a} \geq 0$. By Proposition 5.8, in order to prove our statement about the eigenvalues, it suffices to show that the powers with which the variables $A_{k,a}^{s_i}$ appear in the factorization of the monomial $m(Y_{l,1}^{s_i})^{-1}$ are negative. The formulas in the proof of Lemma 4.1 give a relation between the $A_{k,a}$ and the $A_{k,a}^{s_i}$. In particular, if $k \neq i$, then $A_{k,a}^{s_i}$ appears once in the factorization of $A_{k,a}$. This implies that the $A_{k,a}^{s_i}$ with $k \neq i$ appear in the factorization of the monomial $m(Y_{l,1}^{s_i})^{-1}$ in negative powers. Hence, it remains to consider the case $k = i$ and to prove that the powers are negative in this case as well. The explicit formulas given in the proof of Lemma 4.1 show that we need to prove the following inequality:

$$(6.41) \quad v_{i,a} \leq \sum_{k \in I, C_{i,k} = -1} v_{k,aq_i^{-1}} + \sum_{k \in I, C_{i,k} = -2} (v_{k,aq^{-2}} + v_{k,a}) + \sum_{k \in I, C_{i,k} = -3} (v_{k,aq^{-3}} + v_{k,aq^{-1}} + v_{k,aq}).$$

Let us prove this inequality by induction on the length of the weight of $mY_{l,1}^{-1}$. Formula (6.41) is clear if m is the highest monomial, $m = Y_{l,1}$. Otherwise, it follows from the results of [FM] that there exists $p \in I$ such that m is not p -dominant. Consider the decomposition (6.39):

$$\chi_q(L(Y_{l,1})) = \sum_{M_p \text{ } p\text{-dominant}} \lambda_{M_p} L_p(M_p).$$

Then there is a p -dominant monomial M , such that m is a monomial $L_p(M)$ which occurs in this decomposition. By the induction hypothesis, we have the desired inequality for M .

If $p \neq i$, the inequality for M implies directly the inequality for m .

If $p = i$, we use explicit formulas for the q -characters of simple modules over $U_q(\widehat{sl}_2)$. These formulas imply that

$$v_{i,a}(mM^{-1}) \leq u_{i,aq_i^{-1}}$$

where $u_{i,aq_i^{-1}}$ is the multiplicity of $Y_{i,aq_i^{-1}}$ in M and $v_{i,a}$ is defined in formula (6.40). If we define $V_{r,b}$ now by the formula

$$M = Y_{l,1} \prod_{r,b} A_{r,b}^{-V_{r,b}},$$

then we have the following inequalities:

$$(6.42) \quad \begin{aligned} u_{i,aq_i^{-1}} &\leq -V_{i,a} - V_{i,aq_i^{-2}} + \sum_{r \in I, C_{i,k} = -1} V_{r,aq_i^{-1}} \\ &+ \sum_{r \in I, C_{i,r} = -2} (V_{r,aq^{-2}} + V_{r,a}) + \sum_{r \in I, C_{i,r} = -3} (V_{r,aq^{-3}} + V_{r,aq^{-1}} + V_{r,aq}). \end{aligned}$$

If $r \neq i$, then $V_{r,b} = v_{r,b}$. Since

$$v_{i,a} = V_{i,a} + v_{i,a}(mM^{-1}) \leq V_{i,a} + V_{i,aq_i^{-2}} + v_{i,a}(mM^{-1}),$$

we obtain the desired inequality for m . This completes the case $l \neq i$.

Finally, consider the case $l = i$. Then $T_{s_i}(Y_{i,1}) = Y_{i,1} A_{i,aq_i}^{-1}$. For any monomial m in $\chi_q(L(Y_{i,1}))$, one of the following three conditions hold:

$$m = Y_{i,1};$$

$$m = T_{s_i}(Y_{i,1});$$

$$m = T_{s_i}(Y_{i,1}) \prod_{k \in I, r > r_i} A_{k,q^r}^{-v_{k,q^r}}, \text{ with } v_{k,q^r} \geq 0.$$

The result is clear for the first two monomials. For the third one, we use the same argument as above. Namely, we need to show that the powers with which the variables $A_{k,a}^{s_i}$ appear in the factorization of the monomial $m(Y_{i,1}^{s_i})^{-1}$ are negative. As above, this is clear if $k \neq i$. For $k = i$, we establish the inequality (6.41) by induction on the length of the weight of $m(T_{s_i}(Y_{i,1}))^{-1}$. We have already seen the result for $m = T_{s_i}(Y_{i,1})$. Otherwise, there is $p \in I$ such that m is not p -dominant and there is a p -dominant monomial M , such that m is a monomial $L_p(M)$ which occurs in this decomposition of $\chi_q(L(Y_{i,1}))$. As $m \neq Y_{i,1}, T_{s_i}(Y_{i,1})$, we have that $M \neq Y_{i,1}$ (as $L_i(Y_{i,1}) = Y_{i,1} + T_{s_i}(Y_{i,1})$ and $L_p(Y_{i,1}) = Y_{i,1}$ for $p \neq i$). By the induction hypothesis, we have the desired inequality for M .

If $p \neq i$, the inequality for M implies directly the inequality for m .

If $p = i$, we have seen $v_{i,a}(mM^{-1}) \leq u_{i,aq_i^{-1}}$ with the same notations as above. If we set $V_{r,b}$ as above, we have also the inequalities (6.42). If $r \neq i$, then $V_{r,b} = v_{r,b}$. Since

$$v_{i,a} = -\delta_{a,q_i} + V_{i,a} + v_{i,a}(mM^{-1}) \leq V_{i,a} + V_{i,aq_i^{-2}} + v_{i,a}(mM^{-1}),$$

we obtain the desired inequality for m . □

7. TRANSFER-MATRICES AND BAXTER POLYNOMIALITY

In this section we introduce the transfer-matrices and particular examples of the transfer-matrices, the so-called generalized Baxter Q -operators. We will show that the $\mathcal{O}^*$ versions of these operators are in fact multi-parameter deformations of the X -series $X_{w(\omega_i)}(z)$. We expect that polynomiality of the renormalized series $X_{w(\omega_i)}^N(z)$, which we conjectured in the previous section, extends to polynomiality of the (suitably renormalized) generalized Baxter Q -operators. This is Conjecture 7.19. So far, it has been proved only for $w = e$, in [FH1]. In the special case $\mathfrak{g} = \mathfrak{sl}_2$, we recover the results of Baxter on the polynomiality of the eigenvalues of the celebrated Baxter Q -operator. More precisely, Baxter introduced his Q -operator in the XYZ model [Ba], whereas we are considering the analogous operator in the XXZ model (which appears in a certain limit of the XYZ model) as well as its generalizations to the XXZ-type models associated to $U_q(\widehat{\mathfrak{g}})$. See the Introduction of [FH1] for more details.

7.1. Definition of transfer-matrices. The universal R -matrix $\mathcal{R}$ of $U_q(\widehat{\mathfrak{g}})$ belongs to the completed tensor product $U_q(\widehat{\mathfrak{g}}) \widehat{\otimes} U_q(\widehat{\mathfrak{g}})$ (it is completed with respect to the $\mathbb{Z}$ -grading of $U_q(\widehat{\mathfrak{g}})$, and with a certain completion of the tensor product of the Cartan subalgebra). This completed tensor product acts on the tensor products of modules we consider below. The universal R -matrix satisfies the Yang-Baxter equation.

In fact, it is known that $\mathcal{R}$ belongs to $U_q(\widehat{\mathfrak{b}}) \widehat{\otimes} U_q(\widehat{\mathfrak{b}}_-)$, where $U_q(\widehat{\mathfrak{b}}_-)$ is the subalgebra generated by f_i and $k_i^{\pm 1}$ with $0 \leq i \leq n$. Hence for any $U_q(\widehat{\mathfrak{b}})$ -module V , we define the L -operator associated to V as

$$L_V(z) := (\pi_{V(z)} \otimes \text{Id})(\mathcal{R}) \in \text{End}(V)[[z]] \widehat{\otimes} U_q(\widehat{\mathfrak{g}}),$$

where $\pi_V(z) : U_q(\widehat{\mathfrak{b}}) \rightarrow \text{End}(V)[[z]]$ is the action of $U_q(\widehat{\mathfrak{b}})$ on V twisted by τ_z .

Let V be a $U_q(\widehat{\mathfrak{b}})$ -module which is Cartan-diagonalizable with finite-dimensional weight spaces (for example, this property holds if V is in the category $\mathcal{O}$ or in the dual category $\mathcal{O}^*$). Assume in addition that the ordinary weights of V are in $P \subset P_{\mathbb{Q}}$. For $g \in U_q(\widehat{\mathfrak{b}})$ (or in $\text{End}(V)$), the twisted trace of g on V is defined as follows:

$$\text{Tr}_{V,u}(g) = \sum_{\lambda \in P} \text{Tr}_{V_{\lambda}}(\pi_V(g)) \left(\prod_{i \in I} u_i^{\lambda_i} \right) \in \mathbb{C}[[u_i^{\pm 1}]]_{i \in I},$$

where $u = (u_i)$, $i \in I$, where each u_i is a formal variable and $\lambda_i \in \mathbb{Z}$ is defined by $\lambda(i) = q_i^{\lambda_i}$.

Definition 7.1. *The twisted transfer-matrix associated to V is*

$$t_V(z, u) := (\text{Tr}_{V,u} \otimes \text{Id})(L_V(z)) \in U_q(\widehat{\mathfrak{b}}_-)[[z, u_i^{\pm 1}]]_{i \in I}.$$

For $i \in I$, introduce the variable

$$v_i := \prod_{j \in I} u_j^{C_{j,i}}$$

which corresponds to the i th simple root. Then we have

$$t_V(z, u) \in U_q(\widehat{\mathfrak{b}}_-)[[z, v_i^{-1}]]_{i \in I}[[u_j^{\pm 1}]]_{j \in I} \text{ if } V \text{ is in } \mathcal{O},$$

$$t_V(z, u) \in U_q(\widehat{\mathfrak{b}}_-)[[z, v_i]]_{i \in I}[u_j^{\pm 1}]_{j \in I} \text{ if } V \text{ is in } \mathcal{O}^*,$$

Hence products of the twisted transfer-matrices of modules from the same category ($\mathcal{O}$ or $\mathcal{O}^*$) are well-defined.

If V is a finite-dimensional module (i.e. an object of $\mathcal{C}$), $t_V(z, u)$ is a polynomial in the variables $u_i^{\pm 1}$, and therefore the variables u_i can be specialized to any non-zero complex values. For example, the standard specialization is

$$u_i = q_i^{2 \sum_{j \in I} (C^{-1})_{j,i}} = q^{2 \sum_{j \in I} (DC^{-1})_{i,j}}.$$

This means $v_i = q_i^2$ because DC^{-1} is symmetric (and $u_1 = q$ if $\mathfrak{g} = \mathfrak{sl}_2$).

It can be proved as in [FR, Lemma 2] that for V, V' in the category $\mathcal{O}$ (or $\mathcal{O}^*$) whose weights are in P , and for any extension W of V and V' , we have

$$(7.43) \quad t_W(z, u) = t_V(z, u) + t_{V'}(z, u) \text{ and } t_{V \otimes V'}(z, u) = t_V(z, u)t_{V'}(z, u).$$

Hence $t_V(z, u)$ depends only of the class of V in the corresponding Grothendieck ring.

Theorem 7.2. [FH1] *For V, V' in the category $\mathcal{O}$ (or $\mathcal{O}^*$) whose weights are in $\bar{P} \subset \mathfrak{t}^\times$, we have*

$$t_V(z, u)t_{V'}(w, u) = t_{V'}(w, u)t_V(z, u).$$

7.2. X -series as limits of the transfer-matrices. Recall the notation Q^+ from section 2.1. Let V be an object of $\mathcal{O}$ (resp. $\mathcal{O}^*$) whose ordinary weights are in Q^+ (resp. $-Q^+$). Let us write

$$t_V(z, u) = \sum_{m \geq 0} t_V[m](u)z^m.$$

Then the Fourier coefficients $t_V[m](u)$, where $m \geq 0$ and V runs over the classes of objects of the category $\mathcal{O}$ (resp. $\mathcal{O}^*$) satisfying the above condition, are mutually commuting elements of the algebra $U(\widehat{\mathfrak{b}}_-)[[v_i^{-1}]]_{i \in I}$ (resp. $U(\widehat{\mathfrak{b}}_-)[[v_i]]_{i \in I}$). Hence they generate a commutative subalgebra of this algebra. It follows that every element of this subalgebra, as well as the algebra itself, have a well-defined limit when $v_i^{-1} = 0$ ($i \in I$) (resp. $v_i = 0$ ($i \in I$)).

In particular, let $V = R_{i,a}^+$, which is an object of $\mathcal{O}^*$ satisfying the above condition. We are interested in the limit of $t_{R_{i,a}^+}(z, u)$ when $v_j = 0, \forall j \in I$, which we will express simply as $v = 0$. This limit was computed in [FH1, Theorem 5.5].

Theorem 7.3. *For each $i \in I$, the limit of $t_{R_{i,a}^+}(z, u)$ at $v = 0$ is equal to $X_i(za)$.*

Thus, we see the transfer-matrices $t_{R_{i,a}^+}(z, u)$ simplify dramatically in the limit $v = 0$: all of their Fourier coefficients end up in the (commutative) Cartan-Drinfeld subalgebra $U_q(\widehat{\mathfrak{h}}_-)$ of $U_q(\widehat{\mathfrak{b}}_-)$!

The results of [FH1] allow us to describe the eigenvalues of both $t_{R_{i,a}^+}(z, u)$ and $X_i(za)$ on any $U_q(\widehat{\mathfrak{g}})$ -module N which is a tensor product of simple finite-dimensional modules. It is instructive to compare these eigenvalues.

The module N has a highest weight vector x whose ℓ -weight corresponds to a monomial $m = \prod_{j \in I, b \in \mathbb{C}^\times} Y_{j,b}^{u_{j,b}(m)} \in \mathcal{Y}$ with the ordinary weight ω . Therefore x is an eigenvector of $X_i(z)$. The corresponding eigenvalue $f_{i,m}(z)$ was found in [FH1]; it is given by formula

(6.36). Moreover, by [FM, Theorem 4.1], all other ℓ -weights of N correspond to monomials of the form

$$(7.44) \quad M = mA_{i_1, a_1}^{-1} A_{i_2, a_2}^{-1} \cdots A_{i_N, a_N}^{-1}$$

where $i_1, \dots, i_N \in I$ and $a_1, \dots, a_N \in \mathbb{C}^\times$.

This implies the following description the eigenvalues of $X_i(z)$

Proposition 7.4. *The eigenvalue of $f_{i,m}(z)^{-1}X_i(z)$ on the ℓ -weight subspace of N corresponding to the ℓ -weight M is equal to*

$$(7.45) \quad \prod_{1 \leq k \leq N, i_k = i} (1 - za_k^{-1}).$$

Now let us discuss the eigenvalues of $t_{R_{i,a}^+}(z, u), i \in I, a \in \mathbb{C}^\times$, on N . It is easy to see that $t_{R_{i,a}^+}(z, u) = t_{R_{i,1}^+}(za, u)$, so it is enough to consider the eigenvalues of $t_{R_{i,1}^+}(z, u), i \in I$.

Let λ be a weight of N and $ht_i(\omega - \lambda)$ the multiplicity of α_i in $\omega - \lambda$.

Theorem 7.5. [FH1] *The operator*

$$f_{i,m}(z)^{-1}t_{R_{i,1}^+}(z, u) \in ((\text{End}(N_\lambda))[[v_j]]_{j \in I})[z]$$

is a polynomial in z of degree $ht_i(\omega - \lambda)$.

In the limit $v = 0$, the operator $f_{i,m}(z)^{-1}t_{R_{i,1}^+}(z, u)$ restricted to the subspace $N_\lambda \subset N$ becomes the operator $(f_{i,m}(z))^{-1}X_i(z)$ restricted to the subspace $N_\lambda \subset N$. According to Theorem 7.5, every eigenvalue of the former operator is a polynomial in z of degree $ht_i(\omega - \lambda)$. Moreover, it is known that its roots (generically) correspond to solutions of the corresponding Bethe Ansatz equations (see [FH1]). On the other hand, the eigenvalues of the latter operator are given by the polynomial (7.45) of the same degree $ht_i(\omega - \lambda)$. The roots of this polynomial can be found directly from the corresponding ℓ -weight (see formula (7.44)).

Thus, we obtain that in the limit $v \rightarrow 0$ each eigenvector of $f_{i,m}(z)^{-1}t_{R_{i,1}^+}(z, u)$ tends to an ℓ -weight vector, while the corresponding set of solutions of the Bethe Ansatz equations (describing the eigenvalue) tends to the numbers a_i^{-1} in formula (7.45) encoded by this ℓ -weight according to formula (7.44).

We are going to introduce a large family of modules from the category $\mathcal{O}^*$ labeled by $w \in W$ and $i \in I$ (with the modules corresponding to $w = e$ being $R_{i,a}^+, i \in I$) for which we will conjecture the same polynomiality property (Conjecture 7.19).

7.3. The ℓ -weights $\Psi_{w(\omega_i), a}$. The first step is to recall the definition of the ℓ -weights $\Psi_{w(\omega_i), a}, i \in I, w \in W$, from [FH4]. The transfer-matrices corresponding to the simple modules associated to these weights are the generalized Baxter Q -operators introduced in [FH4] (up to a normalization). In the following subsections we will recall the TQ -relations which we proved in [FH1] and their extended versions which we conjectured in [FH4]. We will also formulate a new dual version of these relations for the category $\mathcal{O}^*$ (see Section 7.5).

Recall the monomial $Y_{w(\omega_i),1}$ introduced in Section 3.1. For $i \in I$ and $a \in \mathbb{C}^\times$, we defined the following monomials in [FH4]:

$$(7.46) \quad W_{i,a} = \begin{cases} Y_{i,a} & \text{if } d_i = d, \\ Y_{i,aq^{-1}}Y_{i,aq} & \text{if } d_i = d-1, \\ Y_{i,aq^{-2}}Y_{i,a}Y_{i,aq^2} & \text{if } d_i = d-2. \end{cases}$$

As established in [FH4], if $C_{k,i} < -1$, then $Y_{w(\omega_i),1}$ is a Laurent polynomial in $W_{k,b}$.

Definition 7.6. [FH4] For $i \in I$, $a \in \mathbb{C}^\times$, and $w \in W$, define an ℓ -weight $\Psi_{w(\omega_i),a}$ by the following formulas.

The ℓ -weight $\Psi_{w(\omega_i),1}$ is defined from the factorization of $Y_{w(\omega_i),1}$ as a product of the variables $Y_{k,b}^{\pm 1}$ by replacing

$Y_{k,b}$ by $\Psi_{k,b^{-1}}$ if $d_i = d_k$,

$Y_{k,b}$ by $\Psi_{k,b^{-1}q}\Psi_{k,b^{-1}q^{-1}}$ if $C_{i,k} = -2$,

$Y_{k,b}$ by $\Psi_{k,b^{-1}q^2}\Psi_{k,b^{-1}}\Psi_{k,b^{-1}q^{-2}}$ if $C_{i,k} = -3$,

$W_{k,b}$ by $\Psi_{k,b^{-1}}$ if $C_{k,i} < -1$.

Finally, we define $\Psi_{w(\omega_i),a}$ from $\Psi_{w(\omega_i),1}$ by the change of variable $z \mapsto za$.

For example, we have

$$Y_{s_i(\omega_i),1} = Y_{i,a}A_{i,aq_i}^{-1} \text{ and } \Psi_{s_i(\omega_i),a} = \tilde{\Psi}_{i,aq_i^{-2}}.$$

Recall the operators T'_i on $\mathcal{Y}$ introduced in Section 3.2.

Proposition 7.7. [FH4] We have

$$\Psi_{w(\omega_i),a} = \sigma \circ T'_w \circ \sigma(\Psi_{i,a}), \quad w \in W,$$

where $T'_w = T'_{i_1}T'_{i_2}\dots T'_{i_k}$ for any reduced decomposition $w = s_{i_1}\dots s_{i_k}$ and σ is given by formula (3.17).

Recall that we have the bar involution $i \mapsto \bar{i}$ on I satisfying $\omega_{\bar{i}} = -w_0(\omega_i)$.

Proposition 7.8. The family $\{\Psi_{w(\omega_i),a}\}_{i \in I, w \in W, a \in \mathbb{C}^\times}$ is preserved by σ . More precisely, we have

$$(7.47) \quad \sigma(\Psi_{w(\omega_i),a}) = \Psi_{ww_0(\omega_{\bar{i}}),a^{-1}q^{r^\vee h^\vee}}.$$

Proof. Consider the set of all extremal monomials $Y_{w(\omega_i),a}$ occurring in the q -characters of all fundamental representations $L(Y_{i,a})$, $i \in I$, $a \in \mathbb{C}^\times$. According to the results of [He2], this set is preserved by the automorphism sending $Y_{i,a} \mapsto Y_{i,a^{-1}}^{-1}$, $i \in I$, $a \in \mathbb{C}^\times$ (note that this automorphism was denoted by σ in [He2], but this is not the automorphism σ that we use in the present paper for the $Y_{i,a}$ variables). This automorphism has the same effect on the variables $Y_{i,a}$ as our σ has on the variables $\Psi_{i,a}$. Since the substitutions in Definition 7.6 are compatible with σ , we obtain the result.

To prove formula (7.47), we use the result established in [He2] that the image of the set of all monomials occurring in $\chi_q(L(Y_{i,a}))$ under the above automorphism is the set of monomials occurring in $\chi_q(L(Y_{i,a^{-1}q^{-r^\vee h^\vee}}))$. Since $Y_{w(\omega_i),a}$ is the unique monomial of weight $w(\omega_i)$ in $\chi_q(L(Y_{i,a}))$ and $w(\omega_i) = -ww_0(\omega_{\bar{i}})$, this implies that under this automorphism,

we have $Y_{w(\omega_i),a} \mapsto Y_{ww_0(\omega_i),a^{-1}q^{-r^\vee h^\vee}}$. We then obtain formula (7.47) by using Definition 7.6. $\square$

Example 7.9. For $\mathfrak{g} = sl_2$, we have $\sigma(\Psi_{\omega_1,a}) = \sigma(\Psi_{1,a}) = \Psi_{1,a^{-1}}^{-1} = \Psi_{-\omega_1,a^{-1}q^2}$ and $\sigma(\Psi_{-\omega_1,a}) = \sigma(\Psi_{1,aq^{-2}}^{-1}) = \Psi_{1,a^{-1}q^2} = \Psi_{\omega_1,a^{-1}q^2}$.

Next, we define a ring automorphism Ω of $\mathcal{Y}'$ by the formula

$$(7.48) \quad \Omega(\Psi_{i,a}) := \Psi_{i,a^{-1}}, \quad i \in I, \quad a \in \mathbb{C}^\times; \quad \Omega([\omega]) = [\omega], \quad \omega \in P.$$

Since σ is the composition of Ω and taking the inverse, the next statement follows from Proposition 7.8.

Proposition 7.10. *The image of the family $\{\Psi_{w(\omega_i),a}\}_{i \in I, a \in \mathbb{C}^\times, w \in W}$ under Ω is the family $\{\Psi_{w(\omega_i),a}^{-1}\}_{i \in I, a \in \mathbb{C}^\times, w \in W}$. More precisely,*

$$(7.49) \quad \Omega(\Psi_{w(\omega_i),a}) = \Psi_{ww_0(\omega_i),a^{-1}q^{r^\vee h^\vee}}^{-1}.$$

Example 7.11. For $\mathfrak{g} = sl_2$, we have $\Omega(\Psi_{\omega_1,a}) = \Omega(\Psi_{1,a}) = \Psi_{1,a^{-1}} = \Psi_{-\omega_1,a^{-1}q^2}^{-1}$ and $\Omega(\Psi_{-\omega_1,a}) = \Omega(\Psi_{1,aq^{-2}}^{-1}) = \Psi_{1,a^{-1}q^2}^{-1} = \Psi_{\omega_1,a^{-1}q^2}^{-1}$.

7.4. Extended TQ -relations. The following is the *Extended TQ -relations Conjecture* (in $K_0(\mathcal{O})$) which we formulated in [FH4].

Conjecture 7.12. *Let $w \in W$ and let V be a finite-dimensional simple $U_q(\widehat{\mathfrak{g}})$ -module. Replace every variable $Y_{i,a}, i \in I$, appearing in the q -character $\chi_q(V)$ with*

$$Y_{i,a} \mapsto [w(\omega_i)] \frac{[L(\Psi_{w(\omega_i),aq_i^{-1}})]}{[L(\Psi_{w(\omega_i),aq_i})]}.$$

By equating the resulting expression with $[V]$ and clearing the denominators, we obtain an algebraic relation in $K_0(\mathcal{O})$.

The generalized Baxter Q -operators $Q_{w(\omega_i),a}, i \in I, w \in W$, were defined in [FH4] as the transfer-matrices corresponding to the simple modules $L(\Psi_{w(\omega_i),a})$ in the category $\mathcal{O}$, divided by their ordinary characters. Hence they (and their eigenvalues) should satisfy the same relations as the classes of these simple modules in $K_0(\mathcal{O})$ – these are the relations stated in the above conjecture. This explains the term “ TQ -relations” (see the Introduction of [FH1] for more details on the origins of these relations).

For $w = w_0$, we proved Conjecture 7.12 in [FH1]. In the case when w is a simple reflection, $w = s_i, i \in I$, we proved this conjecture in [FH2, FHR].

These results motivated us to define in [FH3] the Weyl group action on an extension of $\mathcal{Y}$, such that the subring of its invariants in $\mathcal{Y}$ is equal to the image of the q -character homomorphism χ_q (for more details, see [FH4, Section 3.5]).

7.5. Dual extended TQ -relations. Analogous relations can be written for the classes of simple modules in the category $\mathcal{O}^*$. We call them the *Dual Extended TQ -relations* (in $K_0(\mathcal{O}^*)$).

Conjecture 7.13. *Let $w \in W$ and let V be a finite-dimensional simple $U_q(\widehat{\mathfrak{g}})$ -module. Replace every variable $Y_{i,a}, i \in I$, appearing in the q -character $\chi_q(V)$ with*

$$Y_{i,a} \mapsto [-w(\omega_i)] \frac{[L'(\Psi_{w(\omega_i),aq_i^{-1}}^{-1})]}{[L'(\Psi_{w(\omega_i),aq_i}^{-1})]}.$$

By equating the resulting expression with $[\tau_{q^{-2h \vee r \vee}}(V^)]$ and clearing the denominators, we obtain an algebraic relation in $K_0(\mathcal{O}^*)$.*

Proposition 7.14. *Conjecture 7.13 is equivalent to Conjecture 7.12.*

Proof. Recall from Proposition 2.10 that $L'(\Psi_{w(\omega_i),a}^{-1})^* \simeq L(\Psi_{w(\omega_i),a})$ and from Remark 2.11 that $(\tau_{q^{-2h \vee r \vee}}(V^*))^* \simeq V$. We also have $[\lambda]^* \simeq [-\lambda]$. Therefore the relations of Conjecture 7.12 are obtained by dualizing the relations of Conjecture 7.13. $\square$

Example 7.15. *Let $\mathfrak{g} = \mathfrak{sl}_2$ and $V = L(Y_{1,bq^2})$ with $\chi_q(Y_{1,bq^2}) = Y_{1,bq^2} + Y_{1,bq^4}^{-1}$. Then $V^* = L(Y_{1,bq^4})$ and using Example 7.11, we can compute the relation.*

For $w = e$, the substitution is $Y_{1,a} \mapsto [-\omega_1][L'(\Psi_{1,aq^{-1}}^{-1})]/[L'(\Psi_{1,aq}^{-1})]$. Then

$$[L(Y_{1,b})][L'(\Psi_{1,bq^3}^{-1})] = [-\omega_1][L'(\Psi_{1,bq}^{-1})] + [\omega_1][L'(\Psi_{1,bq^5}^{-1})].$$

For $w = s_1$, the substitution is $Y_{1,a} \mapsto [\omega_1][L'(\Psi_{1,aq^{-3}})]/[L'(\Psi_{1,aq^{-1}})]$. Then

$$[L(Y_{1,b})][L'(\Psi_{1,bq}^{-1})] = [\omega_1][L'(\Psi_{1,bq^{-1}})] + [-\omega_1][L'(\Psi_{1,bq^3}^{-1})].$$

Thus, we recover the examples discussed in [FH1, Section 4.3].

In order to relate the Dual Extended TQ -relations of Conjecture 7.13 to the polynomiality of the generalized Baxter operators discussed in the next subsection, we need to rewrite these relations in terms of the modules of the form $L'(\Omega(\Psi_{ww_0(\omega_j),b}), j \in I, b \in \mathbb{C}^\times)$.

Proposition 7.16. *Conjecture 7.13 is equivalent to the following statement: Replace every variable $Y_{i,a}, i \in I$, appearing in $\chi_q(V)$ with*

$$Y_{i,a} \mapsto [-w(\omega_i)] \frac{[L'(\Omega(\Psi_{ww_0(\omega_i),a^{-1}q^{r \vee h \vee} q_i}))]}{[L'(\Omega(\Psi_{ww_0(\omega_i),a^{-1}q^{r \vee h \vee} q_i^{-1}))]}.$$

By equating the resulting expression with $[\tau_{q^{-2h \vee r \vee}}(V^)]$ and clearing the denominators, we obtain an algebraic relation in $K_0(\mathcal{O}^*)$.*

Proof. Applying formula (7.49), we obtain the statement of the proposition. $\square$

For $w = w_0$, we proved the relations of Conjecture 7.13 in [FH1, Theorem 4.8]. The corresponding family of modules is

$$\{L'(\Omega(\Psi_{\omega_i,a^{-1}}))\}_{i \in I, a \in \mathbb{C}^\times} = \{L'(\Psi_{\omega_j,b})\}_{j \in I, b \in \mathbb{C}^\times}$$

(see [FH4, Example 3.8]), which is the family $\{R_{j,b}^+\}_{j \in I, b \in \mathbb{C}^\times}$ we discussed above.

As explained in [FH1], the relations of Conjecture 7.13 for $w = w_0$ imply the same relations on the joint eigenvalues (or spectra) of the corresponding transfer-matrices. This enables us to express the spectra of the transfer-matrices $t_V(z, u)$, where V is a simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module (these are the commuting *Hamiltonians* of the quantum

integrable model of XXZ type associated to $U_q(\widehat{\mathfrak{g}})$, in terms of the spectra of the transfer-matrices corresponding to $R_{j,b}^+$, $j \in I, b \in \mathbb{C}^\times$. In [FH1], we proved that the spectra of these transfer-matrices on the tensor products of simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -modules are in fact *polynomials*, up to an overall factor. Therefore, we obtain a description of the spectra of the Hamiltonians of the quantum integrable model of XXZ type associated to $U_q(\widehat{\mathfrak{g}})$ in terms of these polynomials. As explained in [FH1], this description is closely related to the description in terms of the solutions of the corresponding Bethe Ansatz equations, but is in fact more direct. (In the case of $\mathfrak{g} = \mathfrak{sl}_2$, this goes back to the results of Baxter on the polynomiality of his Q -operator [Ba].)

The discussion of the preceding paragraph concerns the TQ -relations in the case $w = w_0$ and their implications for the spectra of the XXZ type model associated to $U_q(\widehat{\mathfrak{g}})$. And now we want to generalize this to the case of an arbitrary $w \in W$.

Namely, we can view the family of modules $\{L'(\Omega(\Psi_{ww_0(\omega_j),b}))\}_{w \in W, j \in I, b \in \mathbb{C}^\times}$ as an extension of the family $\{R_{j,b}^+\}_{j \in I, b \in \mathbb{C}^\times}$ (corresponding to $w = w_0$) to all $w \in W$. As in the case $w = w_0$, given any $w \in W$, the corresponding TQ -relations, in the form stated in Proposition 7.16, imply the same relations on the spectra of the corresponding transfer-matrices. This enables us to express the spectra of the transfer-matrices $t_V(z, u)$, where V is a simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module, in terms of the spectra of the transfer-matrices corresponding to the modules $L'(\Omega(\Psi_{ww_0(\omega_j),b}))$ with a fixed $w \in W$.

In the next subsection we will discuss the polynomiality of the spectra of the transfer-matrices of the modules $L'(\Omega(\Psi_{w(\omega_j),b}))$, $w \in W, j \in I$. On the one hand, this gives us an alternative description of the spectra of the Hamiltonians of the quantum integrable model of XXZ type associated to $U_q(\widehat{\mathfrak{g}})$ in terms of these polynomials, for each $w \in W$. On the other hand, we will show (Proposition 7.17) that in a certain limit, the transfer-matrix of the module $L'(\Omega(\Psi_{w(\omega_j),b}))$ becomes the X -series $X_{w(\omega_i)}(zb)$. Thus, polynomiality of the spectra of these transfer-matrices is closely related to the polynomiality of the spectra of these X -series, which is one of the main themes of the present paper.

7.6. Polynomiality of the generalized Baxter operators. Denote by $t_{w(\omega_i)}(z, u)$ the transfer-matrix of the simple module $L'(\Omega(\Psi_{w(\omega_i),1}))$ in the category $\mathcal{O}^*$. By Proposition 7.10, these are the $\mathcal{O}^*$ -versions of the generalized Baxter operators (which are the transfer-matrices corresponding to the simple modules $L(\Psi_{w(\omega_i),a})$ in the category $\mathcal{O}$, see Section 7.4). By construction, the ordinary weights of $L'(\Psi_{w(\omega_i),1})$ belong to $Q^+ \subset Q$. Therefore, according to the results of Section 7.2, $t_{w(\omega_i)}(z, u) \in U(\widehat{\mathfrak{b}}_-)[[z, v_i]]_{i \in I}$ and hence it has a well-defined limit at $v = 0$.

The following result generalizes Theorem 7.3.

Proposition 7.17. *The limit of the transfer-matrix $t_{w(\omega_i)}(z, u)$ at $v = 0$ is equal to the X -series $X_{w(\omega_i)}(z)$.*

Proof. We use the same argument as in the proof of Theorem 7.3 (which is [FH1, Theorem 5.5] proved in [FH1, Section 7.1]). Namely, the transfer-matrix $t_{w(\omega_i)}(z)$ is a formal Taylor power series in the variables v_j with coefficients in $U_q(\widehat{\mathfrak{g}})[[z]]$. At the limit when all v_j tend to 0, we obtain the constant term which is the trace on the lowest weight vector of $L'(\Omega(\Psi_{w(\omega_i),1}))$ (of weight 0).

For weight reasons, only the abelian factor $\mathcal{R}^0$ can contribute to this trace (see [FH1, Section 7.1]), and so the limit depends only on the eigenvalues of the Drinfeld-Cartan generators on the lowest weight vector of $L'(\Omega(\Psi_{w(\omega_i),1}))$. To compute the latter, we use the argument similar to the one in [FH1, Section 5.3]. Namely, comparing Proposition 5.11 and Definition 7.6, we find that the factorization of the lowest ℓ -weight $\Omega(\Psi_{w(\omega_i),1})$ in terms of the $\Psi_{j,b}$'s is the same as the factorization of $X_{w(\omega_i)}(z)$ in terms of the $X_{\omega_j}(zb)$'s. $\square$

Note that in general $L'(\Omega(\Psi_{w(\omega_i),1}))$ is *not* isomorphic to the tensor product of prefundamental representations

$$(7.50) \quad M = \bigotimes_{j \in I, b \in \mathbb{C}^\times} L'(\Psi_{j,b}^{m_{j,b}^i}),$$

where we write $\Omega(\Psi_{w(\omega_i),1}) = \prod_{j \in I, b \in \mathbb{C}^\times} \Psi_{j,b}^{m_{j,b}^i}$. Indeed, $L'(\Omega(\Psi_{w(\omega_i),1}))$ is a submodule of the tensor product M , but M is *not* necessarily simple. In this case, the transfer matrices $t_M(z, u)$ and $\prod_{j \in I, b \in \mathbb{C}^\times} t_{\omega_j, b}(z, u)^{m_{j,b}^i}$ are not equal to each other. However, in the limit when the v_j 's tend to 0, they become equal as the contributions of all additional simple constituents in the tensor product (7.50) drop out.

Therefore, Proposition 7.17 implies the following result.

Proposition 7.18. *The limit of the transfer-matrix $t_M(z, u)$ at $v = 0$ is equal to $X_{w(\omega_i)}(z)$.*

This motivates the following conjecture. Let $L(m)$ be a simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module.

Conjecture 7.19. *The operator $(f_{i,m}(z))^{-1} t_{w(\omega_i)}(z, u)$ acting on any simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module $L(m)$ is a polynomial in z .*

Theorem 7.20. [FH1] *Conjecture 7.19 holds if w is the identity element.*

Remark 7.21. *An analogue of this statement, for a simple module in the category $\mathcal{O}$ corresponding to the longest element of the Weyl group, was proved in [Z]. It also implies Conjecture 6.8 for the longest element of the Weyl group because the limit of the corresponding transfer-matrix is the X -series associated to this element.*

Remark 7.22. *Using the dual extended TQ-relations from Section 7.5, we now obtain a conjectural description, for each fixed $w \in W$, of the spectra of the transfer-matrices $t_V(z, u)$, where V are simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -modules in terms of polynomial eigenvalues of the renormalized generalized Baxter operators $(f_{i,m}(z))^{-1} t_{w(\omega_i)}(z, u)$. This generalizes the description of the spectra of the $t_V(z, u)$'s in terms of the polynomial eigenvalues of $(f_{i,m}(z))^{-1} t_{R_{i,1}^+}(z, u)$, $i \in I$, proved in [FH1], which corresponds to the case $w = w_0$ (we note that it had been conjectured in [FR]). Thus, we obtain many different descriptions, labeled by elements $w \in W$, of the spectra of the Hamiltonians (the transfer-matrices $t_V(z, u)$) of the corresponding quantum integrable model of XXZ type.*

The manuscript has no associated data.

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