

WEYL GROUP SYMMETRY OF q -CHARACTERS

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ABSTRACT. We define an action of the Weyl group W of a simple Lie algebra $\mathfrak{g}$ on a completion of the ring $\mathcal{Y}$, which is the codomain of the q -character homomorphism of the corresponding quantum affine algebra $U_q(\widehat{\mathfrak{g}})$. We prove that the subring of W -invariants of $\mathcal{Y}$ is precisely the ring of q -characters, which is isomorphic to the Grothendieck ring of the category of finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$. This resolves an old puzzle in the theory of q -characters. We also identify the screening operators, which were previously used to describe the ring of q -characters, as the subleading terms of simple reflections from W in a certain limit. Our results have already found applications to the study of the category $\mathcal{O}$ of representations of the Borel subalgebra of $U_q(\widehat{\mathfrak{g}})$ in [FH3] and to the categorification of cluster algebras in [GHL].

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1. INTRODUCTION

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of rank n and G the corresponding simply-connected Lie group. Let $\text{Rep } G$ be the Grothendieck ring of finite-dimensional representations of G (equivalently, its Lie algebra $\mathfrak{g}$). Denote by T the Cartan subgroup of G . Attaching to each finite-dimensional representation V of G its character, i.e. the function $\chi_V : T \rightarrow \mathbb{C}$ defined by $\chi_V(t) = \text{Tr}_V(t)$, $t \in T$, we obtain an injective homomorphism of commutative algebras

$$\chi : \text{Rep } G \rightarrow \mathbb{Z}[T] \simeq \mathbb{Z}[y_i^{\pm 1}]_{i \in I}.$$

Here I denotes the set $\{1, 2, \dots, n\}$ and y_i is the i th fundamental weight. Together, these fundamental weights generate the lattice of homomorphisms $T \rightarrow \mathbb{C}^\times$. Moreover, it is well-known that the image of χ is isomorphic to the subring of invariants of $\mathbb{Z}[y_i^{\pm 1}]_{i \in I}$ under the action of the *Weyl group* W of G . This group is generated by the simple reflections s_i acting on the fundamental weights by the formula

$$(1.1) \quad s_i \cdot y_j := y_j a_i^{-\delta_{ij}},$$

where a_i is the monomial corresponding to the i th simple root,

$$(1.2) \quad a_i := \prod_{j \in I} y_j^{C_{ji}},$$

$C = (C_{ij})$ is the Cartan matrix of $\mathfrak{g}$, and δ_{ij} is the Kronecker delta.

In [FR], N. Reshetikhin and one of the authors introduced the notion of q -characters. These are the analogues of characters for finite-dimensional representations of the level 0 quotient $\mathcal{U}_q(\widehat{\mathfrak{g}})$ of the quantum affine algebra associated to the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ which is the affinization of $\mathfrak{g}$. We will assume throughout this paper that $q \in \mathbb{C}^\times$ is not a root of unity. The role of $\mathbb{Z}[y_i^{\pm 1}]_{i \in I}$ is now played by the ring of Laurent polynomials

$$(1.3) \quad \mathcal{Y} := \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times}$$

and the role of χ is played by the injective q -character homomorphism

$$(1.4) \quad \chi_q : \text{Rep}(\mathcal{U}_q(\widehat{\mathfrak{g}})) \rightarrow \mathcal{Y},$$

where $\text{Rep}(\mathcal{U}_q(\widehat{\mathfrak{g}}))$ is the Grothendieck ring of the category of finite-dimensional (type 1) representations of $\mathcal{U}_q(\widehat{\mathfrak{g}})$ (see Section 5) and

$$\chi_q(V) := \sum_m \dim(V_m) \cdot m.$$

Here the sum is over the set of all monomials m in $Y_{i,a}^{\pm 1}$, which are in one-to-one correspondence with the set of ℓ -weights of $\mathcal{U}_q(\widehat{\mathfrak{g}})$, and V_m is the corresponding ℓ -weight subspace of V . We will call the image of χ_q the *ring of q -characters*. It follows from the injectivity of χ_q that it is isomorphic to $\text{Rep}(\mathcal{U}_q(\widehat{\mathfrak{g}}))$.

The theory of q -characters has proved to be very useful in representation theory of quantum affine algebras, and it also plays an important role in other areas, such quantum integrable models, quiver varieties, and cluster algebras.

It is natural to ask whether there is an action of the Weyl group W on $\mathcal{Y}$, so that the subring of W -invariants is equal to the ring of q -characters. In fact, natural q -analogues of the monomials a_i given by formula (1.2) are known [FR]; these are the monomials

$$(1.5) \quad A_{i,a} := Y_{i,aq_i^{-1}} Y_{i,aq_i} \left(\prod_{\{j \in I | C_{j,i} = -1\}} Y_{j,a} \prod_{\{j \in I | C_{j,i} = -2\}} Y_{j,aq^{-1}} Y_{j,aq} \prod_{\{j \in I | C_{j,i} = -3\}} Y_{j,aq^{-2}} Y_{j,a} Y_{j,aq^2} \right)^{-1},$$

where $q_i = q^{d_i}$ and the d_i 's are the relatively prime positive integers such that the matrix $\text{diag}[d_1, \dots, d_n] \cdot C$ is symmetric. Using this formula, V. Chari [C] defined q -analogues of the simple reflections s_i given by formula (1.1) as the automorphisms of $\mathcal{Y}$ acting by (up to replacing q by q^{-1})

$$(1.6) \quad T_i \cdot Y_{j,a} := Y_{j,a} A_{i,aq_i^{-1}}^{-\delta_{i,j}}.$$

(closely related operators were also defined in [BP] in the framework of deformed W-algebras). However, as was shown in [C], the automorphisms T_i generate the braid group associated to $\mathfrak{g}$, *not* the Weyl group (in fact, each T_i has infinite order, whereas the simple reflections $s_i \in W$ have order 2). Furthermore, it is easy to see that the subring of T_i -invariants of $\mathcal{Y}$ is equal to $\mathbb{Z}$ (i.e. consists of the constant elements).

On the other hand, in [FR] another collection of operators, $S_i, i \in I$, called the *screening operators*, was introduced. This was motivated by the close relation between the ring of q -characters and the deformed $\mathcal{W}$ -algebra associated to $\mathfrak{g}$. It was conjectured in [FR] and proved in [FM] that the ring of q -characters coincides with the subring of invariants of the screening operators, so in a sense, they may be viewed as a replacement for the action of the simple reflections $s_i, i \in I$, and hence of the Weyl group, on $\mathbb{Z}[y_i^{\pm 1}]_{i \in I}$.

However, unlike simple reflection s_i , which are automorphisms of $\mathbb{Z}[y_i^{\pm 1}]_{i \in I}$, each screening operator S_i is a *derivation* acting from $\mathcal{Y}$ to a free module over $\mathcal{Y}$, and this creates a puzzling disparity between the theories of characters and q -characters. Nonetheless, it was generally believed that this is the best one can do.

In the present paper, we resolve this puzzle and introduce a genuine action of the Weyl group W such that the ring of q -characters gets identified with the subring of W -invariants in $\mathcal{Y}$. Moreover, we show that the screening operator S_i naturally arises as a subleading term in a certain limit of a natural one-parameter deformation of (a component of) the automorphism $\Theta_i, i \in I$, corresponding to the action of the simple reflection $s_i \in W$. This explains why S_i is a derivation. In addition, we show that Chari's automorphisms T_i generating the braid group also appear as the leading terms of the Θ_i 's in a different sense.

An important aspect of our construction is that the automorphisms Θ_i involve infinite series. Hence they do not preserve $\mathcal{Y}$ but rather map elements of $\mathcal{Y}$ to a direct sum Π of its completions. Our first main result is that these automorphisms extend to well-defined automorphisms of the ring Π where they generate an *action of the Weyl group W* . Our second main result is that under this action, *the subring of W -invariants in $\mathcal{Y}$ is equal to $\text{Rep}(\mathcal{U}_q(\widehat{\mathfrak{g}}))$ (i.e. the ring of q -characters of $\mathcal{U}_q(\widehat{\mathfrak{g}})$).*

Explicitly, the automorphism Θ_i is given by the formulas $\Theta_i \cdot Y_{j,a}^{\pm 1} := Y_{j,a}^{\pm 1}$, if $j \neq i$, and

$$(1.7) \quad \Theta_i \cdot Y_{i,a} := Y_{i,a} A_{i,aq_i}^{-1} \frac{\Sigma_{i,aq_i}^{-3}}{\Sigma_{i,aq_i}^{-1}},$$

where $\Sigma_{i,a}$ is a unique solution of the q -difference equation

$$(1.8) \quad \Sigma_{i,a} = 1 + A_{i,a}^{-1} \Sigma_{i,aq_i}^{-2}$$

in the above completion Π . Note that in the limit $q \rightarrow 1$, the numerator and the denominator in formula (1.7) cancel out and (1.7) becomes (1.1).

The element $\Sigma_{i,a}$ can be expanded as a formal series in two ways. The first expansion is

$$(1.9) \quad \sum_{k \geq 0} \prod_{0 < j \leq k} A_{i,aq_i}^{-1-2j+2} = 1 + A_{i,a}^{-1} (1 + A_{i,aq_i}^{-1} (1 + \dots)),$$

where the $k = 0$ term in the summation is defined to be 1. The second expansion is

$$(1.10) \quad - \sum_{k > 0} \prod_{0 < j \leq k} A_{i,aq_i}^{2j} = -A_{i,aq_i}^2 (1 + A_{i,aq_i}^4 (1 + \dots)).$$

Considering simultaneously different expansions of solutions of q -difference equations such as (1.8) is a crucial ingredient of our construction of the Weyl group action (see Definition 2.7 and Section 3).

There is an analogous phenomenon in the case of the ring of characters of finite-dimensional representations of $\mathfrak{g}$. Namely, suppose we want to adjoin to this ring the characters of infinite-dimensional representations of $\mathfrak{g}$ from the category $\mathcal{O}$. Recall that the objects of

this category have finite-dimensional weight subspaces, whose weights belong to a subset of the form

$$(1.11) \quad \bigcup_{j=1,\dots,N} \{ \lambda_j - \sum_{i \in I} n_i \alpha_i \mid n_i \geq 0 \}.$$

Hence, the characters of representations from the category $\mathcal{O}$ belong to a completion of the ring $\mathbb{Z}[y_i^{\pm 1}]_{i \in I}$. But the problem is that the action of the Weyl group W on $\mathbb{Z}[y_i^{\pm 1}]_{i \in I}$ does not extend to this completion. However, we can rectify this situation by introducing a similar category $\mathcal{O}_w$ for each element $w \in W$. Its objects have non-zero weight subspaces only for the weights from a subset whose image under w is of the form (1.11). The characters of representations from the category $\mathcal{O}_w$ belong to another completion of $\mathbb{Z}[y_i^{\pm 1}]_{i \in I}$, and it is easy to see that the action of W on $\mathbb{Z}[y_i^{\pm 1}]_{i \in I}$ naturally extends to the direct sum π of these completions for all $w \in W$.

In the limit $q \rightarrow 1$ our direct sum Π of completions of $\mathcal{Y}$ becomes π , and the automorphisms Θ_i reduce to the simple reflections $s_i \in W$ acting on π . Moreover, just like π , our Π has an interpretation in terms of characters of representations from an analogue of the category $\mathcal{O}$ for the quantum affine algebra $\mathcal{U}_q(\widehat{\mathfrak{g}})$ (and its twists by $w \in W$).

More precisely, this is the category $\mathcal{O}$ of representations of the subalgebra $\mathcal{U}_q(\widehat{\mathfrak{b}}) \subset \mathcal{U}_q(\widehat{\mathfrak{g}})$, where $\widehat{\mathfrak{b}}$ is a Borel subalgebra of $\widehat{\mathfrak{g}}$, which was introduced by M. Jimbo and one of the authors in [HJ]. It contains all finite-dimensional representations of $\mathcal{U}_q(\widehat{\mathfrak{g}})$ as well as many infinite-dimensional representations whose ordinary weight subspaces are finite-dimensional and their weights belong to a subset of the form (1.11). The q -character homomorphism χ_q given by formula (1.4) can be extended from $\text{Rep}(\mathcal{U}_q(\widehat{\mathfrak{g}}))$ to the Grothendieck ring $K_0(\mathcal{O})$ of this category if we also enlarge its target, $\mathcal{Y}$.

Our definition of the completion Π , explicit formulas (1.7) for the automorphisms Θ_i , and the statement that these automorphisms generate the Weyl group W were motivated by our investigation of some novel relations in $K_0(\mathcal{O})$ (which can also be viewed as relations between q -characters of representations from the category $\mathcal{O}$). These are the extensions of the generalized Baxter TQ -relations and the QQ -system established in our earlier works [FH1] and [FH2], respectively. (We note that this extended QQ -system was introduced and studied in [MRV1, MRV2] in the context of affineopers. Its Yangian version for simply-laced $\mathfrak{g}$ was introduced in [MV1, MV2], and the corresponding extended version was studied in [ESV, EV].) Both the extended TQ -relations and the extended QQ -systems are labeled by elements of the Weyl group W . We have conjectured them (and proved in some cases) in our recent work [FH3], in which we have used the Weyl group action defined in the present paper as a guiding principle.

In addition, our Weyl action plays an important role in the definition given in [GHL] of a new cluster algebra structure on the Grothendieck rings of representations of the so-called shifted quantum affine algebras. This is another application of the results of the present paper.

After we obtained the results of the present paper, we learned of the paper by R. Inoue [In] in which operators similar to one of the components of our Θ_i were constructed. Inoue showed that in the case that q is a *root of unity*, they satisfy the relations of the Weyl group W and that the ring of q -characters is contained in the subring of W -invariants (see also [IY]). For generic q , in Remark 4.15 of [In], Inoue asked whether these operators could in some sense satisfy the relations of the Weyl group and whether there is any connection between them and the screening operators. In the present paper we give affirmative answers

to both questions, and we also prove that the subring of W -invariant elements of $\mathcal{Y}$ is exactly the ring of q -characters (we note that Π , the direct sum of completions of $\mathcal{Y}$ discussed above which is crucial in our definition of the Weyl group action, does not appear in [In]).

The paper is organized as follows. In Section 2 we introduce completions of the ring $\mathcal{Y}$ labeled by $w \in W$ which we will need in order to define the Weyl group action. In Section 3 we introduce automorphisms Θ_i on the direct sum Π of these completions. Our first main result, Theorem 3.4, is that these operators generate an action of the Weyl group W on Π . The proof consists of demonstrating the relations $\Theta_i^2 = \text{Id}$ (this is Proposition 3.5) and the braid group relations, which are proved in Section 4 by reducing the statement to the case of rank 2 simple Lie algebras (see Theorem 4.3). Our proof for rank 2 Lie algebras given in Section 4.6 relies on a combinatorial study of the TQ -relations for those Lie algebras (established in [FH1]) at the level of q -characters; note that we do not use any information about representation theory. In Section 4.5, we also give a more direct proof for simply-laced $\mathfrak{g}$ that does not use the TQ -relations. In Section 5 we prove our second main result, Theorem 5.1, that the subring of W -invariants in $\mathcal{Y} \subset \Pi$ is equal to the subring $\text{Rep}(\mathcal{U}_q(\widehat{\mathfrak{g}}))$ of q -characters of $\mathcal{U}_q(\widehat{\mathfrak{g}})$. In order to do that, we relate the operators Θ_i to the screening operators. In Section 6 we study the relations between our W -action and other known symmetries. In particular, we introduce a q -analogue of the ring of rational functions equipped with an action of W that naturally appears in representation theory of $\mathfrak{g}$. In Section 7, we present explicit formulas for some elements of Π that can be used to give an alternative, purely combinatorial, proof of the braid relations.

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2. COMPLETIONS

In this section we introduce various completions of ring $\mathcal{Y}$ introduced in equation (1.3) which we will need in order to define the Weyl group action.

2.1. Lie algebra and Weyl group. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank n . We will use the notation and conventions of [Kac]. We denote $C = (C_{i,j})_{i,j \in I}$ its Cartan matrix where $I = \{1, \dots, n\}$. Let $\{\alpha_i\}_{i \in I}$, $\{\alpha_i^\vee\}_{i \in I}$, $\{\omega_i\}_{i \in I}$, $\{\omega_i^\vee\}_{i \in I}$, and $\mathfrak{h}$ be the simple roots, the simple coroots, the fundamental weights, the fundamental coweights, and a fixed Cartan subalgebra of $\mathfrak{g}$, respectively. We set $Q = \oplus_{i \in I} \mathbb{Z}\alpha_i$, $Q^+ = \oplus_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$, $P = \oplus_{i \in I} \mathbb{Z}\omega_i$. We have the set of roots $\Delta \subset Q$, which is decomposed as $\Delta = \Delta_+ \sqcup \Delta_-$ where $\Delta_\pm = \pm(\Delta \cap Q^+)$. Let $D = \text{diag}(d_1, \dots, d_n)$ be the unique diagonal matrix such that $B = DC$ is symmetric and d_i 's are relatively prime positive integers. We have the partial ordering on P defined by $\omega \leq \omega'$ if and only if $\omega' - \omega \in Q^+$. Let W be the Weyl group of $\mathfrak{g}$, generated by simple reflections s_i ($i \in I$).

Throughout this paper, we fix a non-zero complex number q which is not a root of unity. We set $q_i = q^{d_i}$.

2.2. Laurent polynomials. Following [FR], consider the ring $\mathcal{Y}$ of Laurent polynomials in variables in $Y_{i,a}$, $i \in I, a \in \mathbb{C}^\times$, over $\mathbb{Z}$ (see formula (1.3)). Let $\mathcal{M}$ be the multiplicative

group of monomials in $\mathcal{Y}$. We have the group homomorphism $\omega : \mathcal{M} \rightarrow P$ which assigns to $m = \prod_{i \in I, a \in \mathbb{C}^\times} Y_{i,a}^{u_{i,a}(m)}$ in $\mathcal{M}$ its $\mathfrak{g}$ -weight

$$\omega(m) := \sum_{i \in I, a \in \mathbb{C}^\times} u_{i,a}(m) \omega_i \in P.$$

For example, for $i \in I, a \in \mathbb{C}^\times$, $\omega(Y_{i,a}) = \omega_i$. We define $A_{i,a} \in \mathcal{M}$ by formula (1.5). Its weight is $\omega(A_{i,a}) = \alpha_i$.

2.3. Completions by formal series.

Definition 2.1. For $w \in W$, we define the free $\mathbb{Z}$ -module $\tilde{\mathcal{Y}}^w$ consisting of the formal series

$$(2.12) \quad \sum_{m \in S} a_m \cdot m, \quad a_m \in \mathbb{Z},$$

where S is any subset of the set $\mathcal{M}$ of monomials in $\mathcal{Y}$ such that for any $m \in S$ we have

$$(2.13) \quad w \cdot \omega(m) \in \bigcup_{j=1, \dots, N} \{ \lambda_j - \sum_{i \in I} n_i \alpha_i \mid n_i \geq 0 \}$$

for some integral weights $\lambda_j, j = 1, \dots, N$, and for any $\omega \in P$, we have

$$(2.14) \quad |\{m \in S \mid \omega(m) = \omega \text{ and } a_m \neq 0\}| < \infty.$$

The following result is straightforward.

Lemma 2.2. $\tilde{\mathcal{Y}}^w$ is a complete topological ring with respect to the natural topology induced by the partial ordering on the sets of weights of the form (2.13), and it is a completion of the ring $\mathcal{Y}$ with respect to this topology.

Remark 2.3. For example, if $w = e$, then $\tilde{\mathcal{Y}}^e$ contains

$$\mathcal{Y} \otimes_{\mathbb{Z}[A_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times}} \mathbb{Z}((A_{i,a}^{-1}))_{i \in I, a \in \mathbb{C}^\times}.$$

□

Similarly, we define a completion $(\mathbb{Z}[y_i^{\pm 1}]_{i \in I})^w$ of $\mathbb{Z}[y_i^{\pm 1}]_{i \in I}$. Namely, for each $\alpha = \sum_{i \in I} c_i \alpha_i \in \Delta$ set

$$(2.15) \quad a_\alpha := \prod_{i \in I} a_i^{c_i}.$$

Then

$$(\mathbb{Z}[y_i^{\pm 1}]_{i \in I})^w := \mathbb{Z}[y_i^{\pm 1}]_{i \in I} \otimes_{\mathbb{Z}[a_{w^{-1}(\alpha_i)}^{\pm 1}]_{i \in I}} \mathbb{Z}((a_{w^{-1}(\alpha_i)}^{-1}))_{i \in I}.$$

Let us set the weight of a monomial $\prod_{i \in I} y_i^{u_i}$ to be $\sum_{i \in I} u_i \omega_i$. The assignment $\varpi_w(Y_{i,a}) = y_i$ extends to a ring homomorphism

$$(2.16) \quad \varpi_w : \tilde{\mathcal{Y}}^w \rightarrow (\mathbb{Z}[y_i^{\pm 1}]_{i \in I})^w.$$

2.4. Uniqueness of solutions of q -difference equations. The ring $\tilde{\mathcal{Y}}^w$ admits automorphisms τ_a , $a \in \mathbb{C}^\times$, defined by $\tau_a(Y_{i,b}) = Y_{i,ab}$. A family $(U(a))_{a \in \mathbb{C}^\times}$ of elements in $\tilde{\mathcal{Y}}^w$ is said to be $\mathbb{C}^\times$ -equivariant if

$$U(ba) = \tau_b(U(a)) \text{ for any } a, b \in \mathbb{C}^\times.$$

Definition 2.4. Let $\tilde{G}^w$ be the subgroup of the group $(\tilde{\mathcal{Y}}^w)^\times$ of invertible elements of $\tilde{\mathcal{Y}}^w$ consisting of elements of the form $A \cdot S$ where $A \in \mathcal{M}$ and S is of the form

$$S = \pm 1 + \sum_{m \in \mathcal{M}; w(\omega(m)) < 0} a_m m, \quad a_m \in \mathbb{Z}.$$

We have a group homomorphism $\tilde{G}^w \rightarrow P$ sending $A \cdot S$ to the weight $\omega(A)$ of A . It will be called the weight of $A \cdot S$.

Note that any $\mathbb{C}^\times$ -equivariant family contained in $\tilde{G}^w$ has a well-defined weight. The following can be viewed as a uniqueness result for solutions of q -difference equations in $\tilde{\mathcal{Y}}^w$.

Lemma 2.5. Suppose that $\chi \in \tilde{\mathcal{Y}}^w$ is such that $\chi = \Psi \tau_{q^{-r}}(\chi)$ for some $r \in \mathbb{Z}$ and $\Psi \in \tilde{G}^w$ with weight in Δ . Then $\chi = 0$.

Proof. We give the proof in the case of $\tilde{\mathcal{Y}}^e$; the case of $\tilde{\mathcal{Y}}^w$ for a general element w is similar. Suppose $\chi \neq 0$ and consider a monomial in χ of maximal weight ν . If the weight α of Ψ is in Δ_- , then ν does not appear among the weights of the monomials in the expansion of $\Psi \tau_{q^{-r}}(\chi)$, which is a contradiction. And if $\alpha \in \Delta_+$, then we apply the same argument to the equation $(\Psi)^{-1} \chi = \tau_{q^{-r}}(\chi)$. $\square$

2.5. The ring Π . Consider the ring

$$(2.17) \quad \pi := \bigoplus_{w \in W} (\mathbb{Z}[y_i^{\pm 1}]_{i \in I})^w.$$

and the diagonal embedding $\mathbb{Z}[y_i^{\pm 1}] \hookrightarrow \pi$. The action of the simple reflection $s_i, i \in I$, on $\mathbb{Z}[y_j^{\pm 1}]_{j \in I}$ given by formula (1.1), naturally extends to $s_i^w : (\mathbb{Z}[y_j^{\pm 1}]_{j \in I})^w \rightarrow (\mathbb{Z}[y_j^{\pm 1}]_{j \in I})^{ws_i}$.

Hence we obtain the following.

Lemma 2.6. The action of the Weyl group W on $\mathbb{Z}[y_i^{\pm 1}]_{i \in I}$ generated by the simple reflections $s_i, i \in I$, naturally extends to π defined in equation (2.17), with s_i acting as $(s_i^w)_{w \in W}$.

We will now define a q -analogue of π ; namely, the ring

$$\Pi := \bigoplus_{w \in W} \tilde{\mathcal{Y}}^w.$$

It is equipped with a diagonal embedding $\mathcal{Y} \hookrightarrow \Pi$ and automorphisms $\tau_a, a \in \mathbb{C}^\times$. For each $w \in W$, denote by E_w the projection $\Pi \rightarrow \tilde{\mathcal{Y}}^w$. We also denote by ϖ the homomorphism $\Pi \rightarrow \pi$ which restricts to ϖ_w given by formula (2.16) on each summand $\tilde{\mathcal{Y}}^w$ in Π .

Definition 2.7. For $i \in I$ and $a \in \mathbb{C}^\times$, let

$$(2.18) \quad \Sigma_{i,a}^+ := \sum_{k \geq 0} \prod_{0 < j \leq k} A_{i,aq_i}^{-1} = 1 + A_{i,a}^{-1} (1 + A_{i,aq_i}^{-1} (1 + \dots)),$$

where the $k = 0$ term in the summation is defined to be 1, and

$$(2.19) \quad \Sigma_{i,a}^- := - \sum_{k > 0} \prod_{0 < j \leq k} A_{i,aq_i^2} = -A_{i,aq_i^2} (1 + A_{i,aq_i^4} (1 + \dots)).$$

Given $w \in W$, we define an element $\Sigma_{i,a}^w$ of $\tilde{\mathcal{Y}}^w$ as follows: $\Sigma_{i,a}^w = \Sigma_{i,a}^+$ if $w(\alpha_i) \in \Delta_+$ and $\Sigma_{i,a}^w = \Sigma_{i,a}^-$ if $w(\alpha_i) \in \Delta_-$. Finally, set

$$(2.20) \quad \Sigma_{i,a} := (\Sigma_{i,a}^w)_{w \in W} \in \Pi.$$

The next lemma follows directly from the definition and Lemma 2.5.

Lemma 2.8. *For each $w \in W$, $\Sigma_{i,a}^w$ is the unique solution in $\tilde{\mathcal{Y}}^w$ of the q -difference equation*

$$(2.21) \quad \Sigma_{i,a}^w = 1 + A_{i,a}^{-1} \Sigma_{i,aq_i^{-2}}^w.$$

In addition, $\Sigma_{i,a}^w$ is invertible in $\tilde{\mathcal{Y}}^w$ and $\Sigma_{i,a}$ is invertible in Π .

3. THE WEYL GROUP ACTION

In this section we introduce automorphisms Θ_i generating an action of the Weyl group on Π and state our first main result, Theorem 3.4.

Recall from Section 2.3 that each $\tilde{\mathcal{Y}}^w$ is a complete topological ring, which is a completion of $\mathcal{Y}$. Denote by $\mathcal{Y}^w$ the image of $\mathcal{Y}$ in $\tilde{\mathcal{Y}}^w$. We will use the same notation for the corresponding image of $\mathcal{Y}$ in Π .

Definition 3.1. *For each $i \in I$ and $w \in W$, we define the ring homomorphism*

$$\Theta_i^w : \mathcal{Y}^w \rightarrow \tilde{\mathcal{Y}}^{ws_i}$$

by $\Theta_i^w(Y_{j,a}^{\pm 1}) := Y_{j,a}^{\pm 1}$ if $j \neq i$ and

$$(3.22) \quad \Theta_i^w(Y_{i,a}) := Y_{i,a} A_{i,aq_i^{-1}}^{-1} \frac{\Sigma_{i,aq_i^{-3}}^{ws_i}}{\Sigma_{i,aq_i^{-1}}^{ws_i}}.$$

Lemma 3.2. *The homomorphism $\Theta_i^w, i \in I$, extends uniquely to a continuous homomorphism $\tilde{\mathcal{Y}}^w \rightarrow \tilde{\mathcal{Y}}^{ws_i}$.*

Proof. We need to show that for any formal series $\chi \in \tilde{\mathcal{Y}}^w$, the series obtained by replacing each monomial m in χ by $\Theta_i^w(m)$ is well-defined in $\tilde{\mathcal{Y}}^{ws_i}$. We do this in two steps. First, we replace each variable $Y_{j,a}$ in each monomial m of χ with $T_i(Y_{j,a}) = Y_{j,a} A_{i,aq_i^{-1}}^{-\delta_{i,j}}$ (see formula (1.6)). As the result, each monomial m of weight ω becomes a monomial of weight $s_i(\omega)$, so according to Definition 2.1 we obtain a well-defined series in $\tilde{\mathcal{Y}}^{ws_i}$. Next, we multiply each $T_i(Y_{i,a})$ obtained this way by the additional factor $\frac{\Sigma_{i,aq_i^{-3}}^{ws_i}}{\Sigma_{i,aq_i^{-1}}^{ws_i}}$. By Definition 2.7, the result is a well-defined element of $\tilde{\mathcal{Y}}^{ws_i}$. $\square$

We will use the same notation Θ_i^w for the homomorphism $\tilde{\mathcal{Y}}^w \rightarrow \tilde{\mathcal{Y}}^{ws_i}$. Taking the direct sum over all $w \in W$, we obtain the following.

Lemma 3.3. *For each $i \in I$, the homomorphisms*

$$\Theta_i^w : \tilde{\mathcal{Y}}^w \rightarrow \tilde{\mathcal{Y}}^{ws_i}, \quad w \in W,$$

combine into a continuous homomorphism

$$\Theta_i : \Pi \rightarrow \Pi.$$

The following is the first main result of this paper. Its proof will be given in Proposition 3.5 and in Section 4.

Theorem 3.4. *The homomorphisms $\Theta_i, i \in I$, generate an action of the Weyl group W on Π .*

The first step is to show that $\Theta_i^2 = \text{Id}$ for all $i \in I$.

Proposition 3.5. *Each endomorphism $\Theta_i, i \in I$, of Π is an involution.*

Proof. We first prove that $\Theta_i^2 = \text{Id}$ on each $\mathcal{Y}^w \subset \Pi, w \in W$. We have $\Theta_i^w(Y_{j,a}) = Y_{j,a}$ for all $j \neq i$. The homomorphism property of Θ_i^w implies that we only need to show that viewing $Y_{i,a}$ as an element of $\mathcal{Y}^w$, we have

$$(3.23) \quad \Theta_i^{ws_i} \circ \Theta_i^w(Y_{i,a}) = Y_{i,a},$$

which, according to formula (3.22) is equivalent to

$$(3.24) \quad \Theta_i^{ws_i} \left(Y_{i,a} A_{i,aq_i}^{-1} \frac{\Sigma_{i,aq_i}^{ws_i}}{\Sigma_{i,aq_i}^{ws_i}} \right) = Y_{i,a}.$$

Formula (3.22) (with w replaced by ws_i) also implies that

$$(3.25) \quad \Theta_i^{ws_i}(A_{i,a}^{-1}) = A_{i,aq_i}^{-2} \frac{\Sigma_{i,a}^w}{\Sigma_{i,aq_i}^{w-4}}.$$

Using (2.21) for the element $ws_i \in W$, we obtain that $\Theta_i^{ws_i}(\Sigma_{i,a}^{ws_i}) \in \tilde{\mathcal{Y}}^w$ is a solution of the q -difference equation

$$(3.26) \quad \Theta_i^{ws_i}(\Sigma_{i,a}^{ws_i}) = 1 + A_{i,aq_i}^{-2} \frac{\Sigma_{i,a}^w}{\Sigma_{i,aq_i}^{w-4}} \Theta_i^{ws_i}(\Sigma_{i,aq_i}^{ws_i}).$$

Note that

$$A_{i,aq_i}^{-2} \frac{\Sigma_{i,a}^w}{\Sigma_{i,aq_i}^{w-4r}} \in \tilde{G}^w$$

and has weight $\alpha_i \in P$. Therefore, Lemma 2.5 implies that $\Theta_i^{ws_i}(\Sigma_{i,a}^{ws_i})$ is a unique solution of equation (3.26) in $\tilde{\mathcal{Y}}^w$.

On the other hand, using formula (2.21) now for the element $w \in W$, we find that $1 - \Sigma_{i,a}^w$ is also a solution of (3.26) in $\tilde{\mathcal{Y}}^w$. Indeed,

$$1 + A_{i,aq_i}^{-2} \frac{\Sigma_{i,a}^w}{\Sigma_{i,aq_i}^{w-4}} (1 - \Sigma_{i,aq_i}^w) = 1 - A_{i,aq_i}^{-2} \frac{\Sigma_{i,a}^w}{\Sigma_{i,aq_i}^{w-4}} A_{i,aq_i}^{-1} \Sigma_{i,aq_i}^w = 1 - \Sigma_{i,a}^w.$$

Applying Lemma 2.5, we derive that the two solutions must be equal, i.e.

$$(3.27) \quad \Theta_i^{ws_i}(\Sigma_{i,a}^{ws_i}) = 1 - \Sigma_{i,a}^w = -A_{i,a}^{-1} \Sigma_{i,aq_i}^w.$$

Substituting this formula, as well as (3.22) and (3.25), into the LHS of (3.24), we obtain that it, and hence the LHS of (3.23), is equal to

$$Y_{i,a} A_{i,aq_i}^{-1} \frac{\Sigma_{i,aq_i}^w}{\Sigma_{i,aq_i}^w} \cdot A_{i,aq_i}^{-3} \frac{\Sigma_{i,aq_i}^w}{\Sigma_{i,aq_i}^w} \cdot \frac{-A_{i,aq_i}^{-1} \Sigma_{i,aq_i}^w}{-A_{i,aq_i}^{-1} \Sigma_{i,aq_i}^w} = Y_{i,a}.$$

This proves that $\Theta_i^2 = \text{Id}$ on each $\mathcal{Y}^w \subset \Pi$, $w \in W$. The continuity of Θ_i (see Lemma 3.2) implies that the same is true on the entire Π . This completes the proof. $\square$

The braid group relations between the Θ_i 's will be proved in the next section. We close this section with the statement about the compatibility of the actions of W on the completions of the rings of characters and q -characters.

Lemma 3.6. *The projection $\varpi : \Pi \rightarrow \pi$ intertwines the actions of W on the two rings given by Theorem 3.4 and Lemma 2.6.*

Proof. It follows from the definitions that

$$(3.28) \quad \varpi_{ws_i} \circ \Theta_i^w = s_i \circ \varpi_w.$$

This implies the statement of the lemma. $\square$

4. BRAID GROUP RELATIONS

In this section we complete the proof Theorem 3.4. Since we have already proved in Proposition 3.5 that the endomorphisms $\Theta_i, i \in I$, are involutions of Π , it suffices to prove that they satisfy the braid group relations $(\mathcal{R}_{i,j})$ for $i \neq j$. Using the continuity of Θ_i as in the proof of Proposition 3.5, it suffices to show that the braid relations are satisfied on the elements of $\mathcal{Y}^w \subset \Pi$ for all $w \in W$. Since each Θ_i is a ring automorphism of Π , it is sufficient to show that the braid relations are satisfied on the generators $Y_{k,a}$ of $\mathcal{Y}^w$.

In the above proof of Proposition 3.5 we explicitly tracked the component equations in $\tilde{\mathcal{Y}}^w$ corresponding to different $w \in W$. But in this section, whenever possible, we combine these components into a single equation in Π . We can go back and forth between the two presentations because each braid relation $(\mathcal{R}_{i,j})$ has the form $\Theta_i \Theta_j \dots \Theta_i \Theta_j = \Theta_j \Theta_i \dots \Theta_j \Theta_i$ or $\Theta_i \Theta_j \dots \Theta_i = \Theta_j \Theta_i \dots \Theta_j$. These automorphisms of Π map $\tilde{\mathcal{Y}}^w$ to $\tilde{\mathcal{Y}}^{w'}$, where $w' = ws_i s_j \dots s_i s_j = ws_j s_i \dots s_j s_i$ or $w' = ws_i s_j \dots s_i = ws_j s_i \dots s_j$. Since the map $W \rightarrow W$ given by $w \mapsto w'$ is a bijection, we can always disentangle the components of a combined relation corresponding to different $w \in W$.

In our computations below, unless stated otherwise, every element we consider is a sum of its components in $\tilde{\mathcal{Y}}^w$ for all $w \in W$. For instance, if we write $Y_{i,a}$, we mean the element of Π which is equal to the sum of the elements $Y_{i,a}$ contained in $\mathcal{Y}^w \subset \tilde{\mathcal{Y}}^w$ for all $w \in W$, and if we write $\Sigma_{i,a}$, we mean the sum of the elements $\Sigma_{i,a}^w \in \tilde{\mathcal{Y}}^w$ for all $w \in W$ (as in formula (2.20)). Every equation we write should be viewed as an equation in Π , i.e. a collection of the component equations in $\tilde{\mathcal{Y}}^w$ for all $w \in W$.

With this understanding, formulas (3.22), (3.25) and (3.27) are written as follows:

$$\Theta_i(Y_{i,a}) = Y_{i,a} A_{i,aq_i^{-1}}^{-1} \frac{\Sigma_{i,aq_i^{-3}}}{\Sigma_{i,aq_i^{-1}}}, \quad \Theta_i(A_{i,a}^{-1}) = A_{i,aq_i^{-2}} \frac{\Sigma_{i,a}}{\Sigma_{i,aq_i^{-4}}},$$

$$\Theta_i(\Sigma_{i,a}) = 1 - \Sigma_{i,a} = -A_{i,a}^{-1} \Sigma_{i,aq_i^{-2}}.$$

4.1. Some invariant elements.

Definition 4.1. *For $i \in I$, $a \in \mathbb{C}^\times$ and $k \geq 0$, define the element of $\mathcal{Y}$,*

$$T_{i,a}^{(k)} := Y_{i,a} Y_{i,aq_i^{-2}} \dots Y_{i,aq_i^{2(1-k)}} (1 + A_{i,aq_i}^{-1} (1 + A_{i,aq_i^{-1}}^{-1} (1 + \dots + A_{i,aq_i^{5-2k}}^{-1} (1 + A_{i,aq_i^{3-2k}}^{-1})))$$

$$(4.29) \quad = Y_{i,a} \cdots Y_{i,aq_i^{2(1-k)}} \sum_{0 \leq \alpha \leq k} V_{i,aq_i}^{(\alpha)},$$

where

$$V_{i,a}^{(\alpha)} := (A_{i,a} A_{i,aq_i^{-2}} \cdots A_{i,aq_i^{2-2\alpha}})^{-1}, \quad \alpha > 0, \quad V_{i,a}^{(0)} := 1.$$

We also define

$$V_{i,a}^{(\alpha)} = \left(V_{i,aq_i^{-2\alpha}}^{(-\alpha)} \right)^{-1}, \quad \alpha < 0.$$

For any $\alpha \in \mathbb{Z}$ we have the relations

$$(4.30) \quad V_{i,a}^{(\alpha+1)} = V_{i,a}^{(\alpha)} A_{i,aq_i^{-2\alpha}}^{-1} = V_{i,aq_i^{-2\alpha}}^{(\alpha)} A_{i,a}^{-1}.$$

It follows from Definition 2.7 that

$$(4.31) \quad \Sigma_{i,a}^w = \begin{cases} \sum_{\alpha \geq 0} V_{i,a}^{(\alpha)} & \text{if } w(\alpha_i) \in \Delta_+, \\ -\sum_{\alpha < 0} V_{i,a}^{(\alpha)} & \text{if } w(\alpha_i) \in \Delta_-. \end{cases}$$

Proposition 4.2. *Each element $T_{i,a}^{(k)}$ given by (4.1) is fixed by Θ_i .*

Proof. For $k = 1$, the image of $Y_{i,a}(1 + A_{i,aq_i}^{-1})$ under Θ_i is

$$\frac{Y_{i,a} A_{i,aq_i^{-1}}^{-1} \Sigma_{i,aq_i^{-3}} + Y_{i,a} \Sigma_{i,aq_i}}{\Sigma_{i,aq_i^{-1}}} = \frac{Y_{i,a} (\Sigma_{i,aq_i^{-1}} - 1) + Y_{i,a} (1 + A_{i,aq_i}^{-1} \Sigma_{i,aq_i^{-1}})}{\Sigma_{i,aq_i^{-1}}} = Y_{i,a} (1 + A_{i,aq_i}^{-1}).$$

Then we have the recurrence relation for $k \geq 1$:

$$T_{i,aq_i^2}^{(k+1)} = T_{i,a}^{(k)} T_{i,aq_i^2}^{(1)} - T_{i,aq_i^{-2}}^{(k-1)} (Y_{i,a} Y_{i,aq_i^2} A_{i,aq_i}^{-1}).$$

The result follows since $Y_{i,a} Y_{i,aq_i^2} A_{i,aq_i}^{-1} \in \mathbb{Z}[Y_{j,b}^{\pm 1}]_{j \neq i, b \in \mathbb{C}^\times}$ is fixed by Θ_i . $\square$

4.2. Reduction to the rank 2 Lie algebras. As we explained at the beginning of Section 4, in order to prove that a braid relation $(\mathcal{R}_{i,j})$ (with $i \neq j$ in I) holds, it is sufficient to verify that the relation $(\mathcal{R}_{i,j})$ applied to $Y_{k,a}$ holds for every $k \in I$ and $a \in \mathbb{C}^\times$. Denote the latter by $(\mathcal{R}_{i,j}(Y_{k,a}))$.

If $k \notin \{i, j\}$, then $\Theta_i(Y_{k,a}) = \Theta_j(Y_{k,a}) = Y_{k,a}$, and so $(\mathcal{R}_{i,j}(Y_{k,a}))$ holds. Hence we only need to check that $(\mathcal{R}_{i,j}(Y_{i,a}))$ and $(\mathcal{R}_{i,j}(Y_{j,a}))$ hold. Moreover, it follows from the definition of Θ_i and Θ_j that the result of applying them to $Y_{i,a}$ and $Y_{j,a}$ can be expressed in terms of iterated solutions of q -difference equations involving only $A_{i,b}^{\pm 1}$ and $A_{j,b}^{\pm 1}$.

Hence, the action of any successive composition of the automorphisms Θ_i, Θ_j on $Y_{i,a}$ and $Y_{j,a}$ for the Lie algebra $\mathfrak{g}$ coincides with the corresponding action for the rank 2 Lie algebra $\mathfrak{g}_{i,j}$ (associated to the simple roots α_i and α_j) if replace the variables $Y_{i,a}, Y_{j,a}, A_{i,b}^{\pm 1}$, and $A_{j,b}^{\pm 1}$ of type $\mathfrak{g}$ with the corresponding variables of type $\mathfrak{g}_{i,j}$. Therefore, if the relation $(\mathcal{R}_{i,j})$ holds for $\mathfrak{g}_{i,j}$, then it also holds for $\mathfrak{g}$. We thus obtain the following result.

Theorem 4.3. *Theorem 3.4 follows from the braid relation for all rank 2 semisimple Lie algebras.*

It remains to prove the braid relations for types $A_1 \times A_1, A_2, B_2$, and G_2 . In Sections 4.3 and 4.5, we prove them combinatorially for types $A_1 \times A_1$ and A_2 , respectively. This implies Theorem 3.4 for all simply-laced $\mathfrak{g}$. Then in Section 4.6 we prove them uniformly for types A_2, B_2 , and G_2 using the q -character versions of the generalized TQ -relations for these Lie algebras established in [FH1].

4.3. **Type $A_1 \times A_1$.** Suppose that $C_{i,j} = 0$. Then $(\mathcal{R}_{i,j})$ is

$$(4.32) \quad \Theta_j \Theta_i = \Theta_i \Theta_j.$$

Since $\Theta_j(A_{i,a}) = A_{i,a}$, we obtain that $\Theta_j(\Sigma_{i,1})$ satisfies the same relation (1.8) as $\Sigma_{i,1}$, and hence $\Theta_j(\Sigma_{i,1}) = \Sigma_{i,1}$ by Lemma 2.5. Therefore

$$(\Theta_j \Theta_i)(Y_{i,1}) = \Theta_i(Y_{i,1}) = (\Theta_i \Theta_j)(Y_{i,1}),$$

so $(\mathcal{R}_{i,j}(Y_{i,a}))$ holds. Applying the automorphism of $A_1 \times A_1$ exchanging i and j , we obtain that $(\mathcal{R}_{i,j}(Y_{j,a}))$ also holds, and so we are done.

To handle the cases with $C_{i,j} < 0$, we need some preliminary results.

4.4. **Image of the $\Sigma_{j,a}$.** Suppose that $C_{i,j} < 0$. We have

$$\Theta_i(A_{j,a}) = A_{j,a} A_{ij,a} \frac{\Sigma_{i,aq_i}^{-2-C_{i,j}}}{\Sigma_{i,aq_i}^{-2+C_{i,j}}} \text{ where } A_{ij,a} = \begin{cases} A_{i,aq_i^{-1}} & \text{if } C_{i,j} = -1, \\ A_{i,aq_i^{-2}} A_{i,a} & \text{if } C_{i,j} = -2, \\ A_{i,aq_i} A_{i,aq_i^{-1}} A_{i,aq_i^{-3}} & \text{if } C_{i,j} = -3. \end{cases}$$

In particular,

$$\Theta_i(\Sigma_{j,a}) = 1 + A_{j,a}^{-1} A_{ij,a}^{-1} \Theta_i(\Sigma_{j,aq_j^{-2}}) \frac{\Sigma_{i,aq_i}^{-2+C_{i,j}}}{\Sigma_{i,aq_i}^{-2-C_{i,j}}}.$$

The following formulas will be useful:

$$\Theta_i(\Sigma_{j,a}) = \frac{\Sigma_{ij,a}}{\Sigma_{i,a}^{(j)}} \text{ where } \Sigma_{i,a}^{(j)} = \begin{cases} \Sigma_{i,aq_i}^{-2-C_{i,j}} & \text{if } C_{j,i} = -1, \\ \Sigma_{i,aq_i^{-2}} \Sigma_{i,aq_i^{-4}} & \text{if } C_{j,i} = -2, \\ \Sigma_{i,aq_i^{-3}} \Sigma_{i,aq_i^{-5}} \Sigma_{i,aq_i^{-7}} & \text{if } C_{j,i} = -3, \end{cases}$$

and $\Sigma_{ij,a}$ is the unique solution of the q -difference equation

$$(4.33) \quad \Sigma_{ij,a} = \Sigma_{i,a}^{(j)} + A_{j,a}^{-1} A_{ij,a}^{-1} \Sigma_{ij,aq_j^{-2}}.$$

4.5. **Simply-laced types.** In this subsection, we prove the braid relations for type A_2 . By the argument of Section 4.2, this implies Theorem 3.4 for all simply-laced $\mathfrak{g}$.

Suppose that $C_{i,j} = C_{j,i} = -1$ and $d_i = d_j = 1$. Then $(\mathcal{R}_{i,j})$ is

$$(4.34) \quad \Theta_i \Theta_j \Theta_i = \Theta_j \Theta_i \Theta_j.$$

Applying the automorphism of A_2 exchanging i and j to the relation $(\mathcal{R}_{i,j}(Y_{i,a}))$, we obtain $(\mathcal{R}_{i,j}(Y_{j,a}))$. Hence it is sufficient to prove $(\mathcal{R}_{i,j}(Y_{i,a}))$, which is the following

Proposition 4.4.

$$(\Theta_i \Theta_j \Theta_i)(Y_{i,a}) = (\Theta_j \Theta_i \Theta_j)(Y_{i,a}).$$

Proof. Since $\Theta_i(Y_{j,a}) = Y_{j,a}$, we need to prove that

$$(\Theta_i \Theta_j \Theta_i)(Y_{i,a}) = (\Theta_j \Theta_i)(Y_{i,a}).$$

Equation (4.33), with i and j exchanged, reads in this case:

$$(4.35) \quad \Sigma_{ji,a} = \Sigma_{j,aq^{-1}} + A_{i,a}^{-1} A_{j,aq^{-1}}^{-1} \Sigma_{ji,aq^{-2}}.$$

We will prove that

$$(4.36) \quad \Theta_i(\Sigma_{ji,a}) = \Sigma_{ji,a},$$

so that we have the following commutative diagram:

$$\begin{array}{ccccc}
Y_{i,a} & \xrightarrow{\Theta_i} & Y_{i,a} A_{i,aq^{-1}}^{-1} \frac{\Sigma_{i,aq^{-3}}}{\Sigma_{i,aq^{-1}}} & \xrightarrow{\Theta_j} & Y_{j,aq^{-3}}^{-1} \frac{\Sigma_{ji,aq^{-3}}}{\Sigma_{ji,aq^{-1}}} \\
\Theta_j \downarrow & & & & \downarrow \Theta_i \\
Y_{i,a} & \xrightarrow{\Theta_i} & Y_{i,a} A_{i,aq^{-1}}^{-1} \frac{\Sigma_{i,aq^{-3}}}{\Sigma_{i,aq^{-1}}} & \xrightarrow{\Theta_j} & Y_{j,aq^{-3}}^{-1} \frac{\Sigma_{ji,aq^{-3}}}{\Sigma_{ji,aq^{-1}}}
\end{array}$$

First, Lemma 2.5 implies that

$$(4.37) \quad \Sigma_{i,a} \Sigma_{j,aq} = \Sigma_{ij,aq} + A_{j,aq}^{-1} \Sigma_{ji,a}$$

because both sides of (4.37) satisfy the same q -difference equation (for $U(a)$):

$$U(a) = A_{j,aq}^{-1} A_{i,a}^{-1} U(aq^{-2}) + (\Sigma_{i,a} + \Sigma_{j,aq} - 1).$$

Now, since $\Theta_i(\Sigma_{j,a}) = \Sigma_{ij,a} \Sigma_{i,aq^{-1}}^{-1}$, applying Θ_i to (4.37), we obtain

$$-A_{i,a}^{-1} \Sigma_{i,aq^{-2}} \Sigma_{ij,aq} \Sigma_{i,a}^{-1} = -\Sigma_{j,aq} A_{i,a}^{-1} \Sigma_{i,aq^{-2}} + A_{j,aq}^{-1} A_{i,a}^{-1} \Theta_i(\Sigma_{ji,a}) \Sigma_{i,aq^{-2}} \Sigma_{i,a}^{-1}.$$

and so

$$\Theta_i(\Sigma_{ji,a}) = A_{j,aq}(-\Sigma_{ij,aq} + \Sigma_{j,aq} \Sigma_{i,a}) = \Sigma_{ji,a}.$$

□

4.6. General types. The proof in the previous section is based on the q -difference equations (4.33) and (4.35). We expect that there is a similar proof for other types as well, but so far we have not been able to find the relevant equations for types B_2 and G_2 .

Hence we present here a different but uniform proof for the rank 2 simple types. So, suppose that $\mathfrak{g}$ is of type A_2 , B_2 or G_2 . The proof is based on the generalized Baxter TQ -relations proved in [FH1]. To explain this, let us consider the extension

$$\Pi' := \mathcal{Y}' \otimes_{\mathcal{Y}} \Pi$$

of Π where

$$\mathcal{Y}' := \mathbb{Z}[\Psi_{k,a}^{\pm 1}, [\omega]]_{k \in I, a \in \mathbb{C}^\times, \omega \in P} \supset \mathcal{Y}$$

and we set

$$(4.38) \quad Y_{k,a} = [\omega_k] \Psi_{k,aq_k^{-1}} \Psi_{k,aq_k}^{-1}.$$

Thus, $\mathcal{Y}'$ is the ring of Laurent polynomials in the algebraically independent variables $\Psi_{k,a}$ over the ring $\mathbb{Z}[P]$.

The weight of a Laurent monomial $[\omega] \Psi_{k_1,a_1}^{p_1} \cdots \Psi_{k_N,a_N}^{p_N}$ in $\mathcal{Y}'$ is defined to be ω .

Like Π , the ring Π' has components $\tilde{\mathcal{Y}}'^w$, $w \in W$, and the corresponding projections.

Next, we extend the operators Θ_k on Π to the operators Θ'_k on Π' given by the following formulas: for $\omega \in P$ and $a \in \mathbb{C}^\times$

$$(4.39) \quad \Theta'_k([\omega]) := [s_k(\omega)], \quad \Theta'_k(\Psi_{m,a}) := \begin{cases} \Psi_{m,a} & \text{if } m \neq k, \\ \tilde{\Psi}_{k,aq_k^{-2} \Sigma_{k,aq_k}^{-2}} & \text{if } m = k. \end{cases}$$

where $\tilde{\Psi}_{k,aq_k^{-2}}$, introduced in [FH2], is given by the formula

$$(4.40) \quad \tilde{\Psi}_{k,aq_k^{-2}} := \Psi_{k,aq_k^{-2}}^{-1} \cdot \prod_{m \in I, C_{k,m} = -1} \Psi_{m,aq_k^{-1}} \cdot \prod_{m \in I, C_{k,m} = -2} \Psi_{m,aq^{-2}} \Psi_{m,a} \cdot \prod_{m \in I, C_{k,m} = -3} \Psi_{m,aq^{-3}} \Psi_{m,aq^{-1}} \Psi_{m,aq}.$$

These operators are well-defined and according to formula (4.38), they are compatible with the operators Θ_k .

We set $I = \{i, j\}$ with $d_i \geq d_j$. We are going to prove that the operators Θ'_i and Θ'_j satisfy the braid relations $\mathcal{R}'_{i,j,\ell} = \mathcal{R}'_{i,j,r}$, where

$$\begin{aligned} \mathcal{R}'_{i,j,\ell} &:= \Theta'_j \Theta'_i \Theta'_j & \text{and} & \quad \mathcal{R}'_{i,j,r} := \Theta'_i \Theta'_j \Theta'_i \text{ in type } A_2, \\ \mathcal{R}'_{i,j,\ell} &:= \Theta'_j \Theta'_i \Theta'_j \Theta'_i & \text{and} & \quad \mathcal{R}'_{i,j,r} := \Theta'_i \Theta'_j \Theta'_i \Theta'_j \text{ in type } B_2, \\ \mathcal{R}'_{i,j,\ell} &:= \Theta'_j \Theta'_i \Theta'_j \Theta'_i \Theta'_j \Theta'_i & \text{and} & \quad \mathcal{R}'_{i,j,r} := \Theta'_i \Theta'_j \Theta'_i \Theta'_j \Theta'_i \Theta'_j \text{ in type } G_2. \end{aligned}$$

This will imply the sought-after braid relations.

The above relation is clearly satisfied on the elements $[\omega]$, $\omega \in P$. Let us now check this relation on the elements $\Psi_{i,a}$ and $\Psi_{j,a}$. Explicitly, formula (4.39) specializes to

$$\begin{aligned} \Theta'_i(\Psi_{i,a}) &= \Psi_{i,aq^{-2d_i}}^{-1} \Psi_{j,aq^{-d_i}} \Sigma_{i,aq^{-2d_i}}, & \Theta'_i(\Psi_{j,a}) &= \Psi_{j,a}, & \Theta'_j(\Psi_{i,a}) &= \Psi_{i,a}, \\ \Theta'_j(\Psi_{j,a}) &= \Psi_{j,aq^{-2}}^{-1} \Sigma_{j,aq^{-2}} \times \begin{cases} \Psi_{i,aq^{-1}} & \text{in type } A_2, \\ \Psi_{i,aq^{-2}} \Psi_{i,a} & \text{in type } B_2, \\ \Psi_{i,aq^{-3}} \Psi_{i,aq^{-1}} \Psi_{i,aq} & \text{in type } G_2. \end{cases} \end{aligned}$$

Introduce the following notation for $a \in \mathbb{C}^\times$:

$$(4.41) \quad \phi_{i,a}^\ell := \mathcal{R}'_{i,j,\ell}(\Psi_{i,a}), \quad \phi_{j,a}^\ell := \mathcal{R}'_{i,j,\ell}(\Psi_{j,a}), \quad \phi_{i,a}^r := \mathcal{R}'_{i,j,r}(\Psi_{i,a}), \quad \phi_{j,a}^r := \mathcal{R}'_{i,j,r}(\Psi_{j,a}).$$

It remains to prove that

$$(4.42) \quad \phi_{i,a}^\ell = \phi_{i,a}^r \quad \text{and} \quad \phi_{j,a}^\ell = \phi_{j,a}^r.$$

We will consider the projections onto $\tilde{\mathcal{Y}}'^e$ (the proof for the other components $\tilde{\mathcal{Y}}'^e, w \in W$, is similar).

Consider the sets of words

$$\begin{aligned} W_i &:= \{(i), (ji), (ijj)\}, & W_j &:= \{(j), (ij), (jij)\} \text{ in type } A_2, \\ W_j &:= \{(j), (ij), (jij), (ijij)\}, & W_i &:= \{(i), (ji), (ijj), (jijj)\} \text{ in type } B_2, \\ W_j &:= \{(j), (ij), (jij), (ijij), (jijij), (ijijij)\}, \\ W_i &:= \{(i), (ji), (ijj), (jijj), (ijijij), (jijijij)\} \text{ in type } G_2. \end{aligned}$$

Then Θ'_w , written as the product of the Θ'_i and Θ'_j according to the sequence w , is well-defined for any such word w . Note that we have $\mathcal{R}'_{i,j,\ell} = \Theta'_{w_j}$ and $\mathcal{R}'_{i,j,r} = \Theta'_{w_i}$, where w_j (resp. w_i) is the longest word in W_j (resp. W_i).

We introduce intermediate elements $\Sigma_{w,a} \in \Pi$ for each $w \in W_i \cup W_j$ and $a \in \mathbb{C}^*$. For w of length 1 or 2, $\Sigma_{w,a}$ has been defined above.

In type A_2 , we define $\Sigma_{iji,a}$ and $\Sigma_{jij,a}$ by the formulas

$$\Theta_i(\Sigma_{iji,a}) = \Sigma_{iji,a}, \quad \Theta_j(\Sigma_{ij,a}) = \Sigma_{jij,a}.$$

In type B_2 , we define $\Sigma_{iji,a}$, $\Sigma_{jji,a}$, $\Sigma_{ijj,a}$, and $\Sigma_{ijji,a}$ by the formulas

$$\Theta_i(\Sigma_{ji,a}) = \frac{\Sigma_{iji,a}}{\Sigma_{i,aq^{-2}}}, \quad \Theta_j(\Sigma_{iji,a}) = \Sigma_{jji,a}, \quad \Theta_j(\Sigma_{ij,a}) = \frac{\Sigma_{jjj,a}}{\Sigma_{j,aq^{-4}}}, \quad \Theta_i(\Sigma_{ijj,a}) = \Sigma_{ijji,a}.$$

In type G_2 , we define $\Sigma_{iji,a}$, $\Sigma_{jji,a}$, $\Sigma_{ijji,a}$, and $\Sigma_{ijijj,a}$ by the formulas

$$\begin{aligned} \Theta_i(\Sigma_{ji,a}) &= \frac{\Sigma_{iji,a}}{\Sigma_{i,aq^{-4}}\Sigma_{i,aq^{-2}}} & \Theta_j(\Sigma_{ijj,a}) &= \frac{\Sigma_{jjj,a}}{\Sigma_{j,aq^{-3}}}, \\ \Theta_i(\Sigma_{ijji,a}) &= \frac{\Sigma_{ijijj,a}}{\Sigma_{i,aq^{-6}}}, & \Theta_j(\Sigma_{ijijj,a}) &= \Sigma_{ijijji,a}, \end{aligned}$$

and we define $\Sigma_{jjj,a}$, $\Sigma_{ijij,a}$, $\Sigma_{ijijj,a}$, and $\Sigma_{ijijij,a}$ by the formulas

$$\begin{aligned} \Theta_j(\Sigma_{ij,a}) &= \frac{\Sigma_{jjj,a}}{\Sigma_{j,aq^{-4}}\Sigma_{j,aq^{-6}}}, \quad \Theta_i(\Sigma_{ijj,a}) = \frac{\Sigma_{ijij,a}}{\Sigma_{i,aq^{-7}}\Sigma_{i,aq^{-9}}\Sigma_{i,aq^{-11}}}, \\ \Theta_j(\Sigma_{ijij,a}) &= \frac{\Sigma_{ijijj,a}}{\Sigma_{j,aq^{-10}}}, \quad \Theta_i(\Sigma_{ijijj,a}) = \Sigma_{ijijji,a}. \end{aligned}$$

It follows from the above definitions that for each $w \in W_i$ (resp. $w \in W_j$), $\Theta'_w(\Psi_{i,a})\Sigma_{w,aq^{-2d_i}}^{-1}$ (resp. $\Theta'_w(\Psi_{j,a})\Sigma_{w,aq^{-2}}^{-1}$) is a Laurent monomial in the $\Psi_{k,b}^{\pm 1}$, $k = i, j$.

In type A_2 , the family $\Sigma_{ji,a}$ satisfies the q^2 -difference equation

$$\Sigma_{ji,a} = \Sigma_{j,aq^{-1}} + A_{i,a}^{-1}A_{j,aq^{-1}}^{-1}\Sigma_{ji,aq^{-2}}.$$

The family $\Sigma_{ij,a}$ satisfies the q^2 -difference equation

$$\Sigma_{ij,a} = \Sigma_{i,aq^{-1}} + A_{j,a}^{-1}A_{i,aq^{-1}}^{-1}\Sigma_{ij,aq^{-2}}.$$

Therefore, we have the following q^2 -difference equations

$$\begin{aligned} \Sigma_{iji,a}\Sigma_{i,aq^{-2}} &= \Sigma_{ij,aq^{-1}} + A_{j,aq^{-1}}^{-1}\Sigma_{i,a}\Sigma_{iji,aq^{-2}}, \\ \Sigma_{ijj,a}\Sigma_{j,aq^{-2}} &= \Sigma_{ji,aq^{-1}} + A_{i,aq^{-1}}^{-1}\Sigma_{j,a}\Sigma_{ijj,aq^{-2}}. \end{aligned}$$

In type B_2 , the family $\Sigma_{ji,a}$ satisfies the q^4 -difference equation

$$(4.43) \quad \Sigma_{ji,a} = \Sigma_{j,a} + A_{i,a}^{-1}A_{j,aq^{-2}}^{-1}A_{j,a}^{-1}\Sigma_{ji,aq^{-4}}.$$

The family $\Sigma_{ij,a}$ satisfies the q^2 -difference equation

$$(4.44) \quad \Sigma_{ij,a} = \Sigma_{i,aq^{-2}}\Sigma_{i,aq^{-4}} + A_{j,a}^{-1}A_{i,aq^{-2}}^{-1}\Sigma_{ij,aq^{-2}}.$$

Therefore, we have the following q^4 -difference equations

$$\begin{aligned} (4.45) \quad \Sigma_{iji,a}\Sigma_{i,aq^{-4}} &= \Sigma_{ij,a} + A_{i,aq^{-2}}^{-1}A_{j,aq^{-2}}^{-1}A_{j,a}^{-1}\Sigma_{i,a}\Sigma_{iji,aq^{-4}}, \\ \Sigma_{ijj,a}\Sigma_{j,aq^{-4}} &= \Sigma_{ji,a} + A_{i,aq^{-2}}^{-1}\Sigma_{ji,a}\Sigma_{ijj,aq^{-4}}, \end{aligned}$$

and the following q^2 -difference equations

$$\begin{aligned} (4.46) \quad \Sigma_{ijj,a}\Sigma_{j,aq^{-2}} &= \Sigma_{ji,aq^{-2}}\Sigma_{ji,aq^{-4}} + A_{i,aq^{-2}}^{-1}A_{j,aq^{-4}}^{-1}\Sigma_{j,a}\Sigma_{ijj,aq^{-2}}, \\ \Sigma_{ijij,a}\Sigma_{ij,aq^{-2}} &= \Sigma_{ijji,aq^{-2}}\Sigma_{ijji,aq^{-4}} + A_{j,aq^{-4}}^{-1}\Sigma_{ij,a}\Sigma_{ijij,aq^{-2}}. \end{aligned}$$

In type G_2 , the family $\Sigma_{ji,a}$ satisfies the q^6 -difference equation

$$(4.47) \quad \Sigma_{ji,a} = \Sigma_{j,aq} + A_{i,a}^{-1}A_{j,aq}^{-1}A_{j,aq^{-1}}^{-1}A_{j,aq^{-3}}^{-1}\Sigma_{ji,aq^{-6}},$$

The family $\Sigma_{ij,a}$ satisfies the q^2 -difference equation

$$(4.48) \quad \Sigma_{ij,a} = \Sigma_{i,aq^{-3}} \Sigma_{i,aq^{-5}} \Sigma_{i,aq^{-7}} + A_{j,a}^{-1} A_{i,aq^{-3}}^{-1} \Sigma_{ij,aq^{-2}},$$

Therefore, we have the following q^6 -difference equations

$$(4.49) \quad \Sigma_{iji,a} \Sigma_{i,aq^{-6}} = \Sigma_{ij,aq} + A_{i,aq^{-4}}^{-1} A_{i,aq^{-2}}^{-1} A_{j,aq^{-3}}^{-1} A_{j,aq^{-1}}^{-1} A_{j,aq}^{-1} \Sigma_{iji,aq^{-6}} \Sigma_{i,a},$$

$$(4.50) \quad \Sigma_{jiji,a} \Sigma_{j,aq^{-6}} = \Sigma_{jij,aq} + A_{i,aq^{-4}}^{-1} A_{i,aq^{-2}}^{-1} A_{j,aq^{-7}}^{-1} A_{j,aq^{-5}}^{-1} A_{j,aq^{-3}}^{-1} \Sigma_{jiji,aq^{-6}} \Sigma_{ji,a},$$

$$(4.51) \quad \Sigma_{ijiji,a} \Sigma_{ij,aq^{-6}} = \Sigma_{ijij,aq} + A_{i,aq^{-6}}^{-1} A_{j,aq^{-7}}^{-1} A_{j,aq^{-5}}^{-1} A_{j,aq^{-3}}^{-1} \Sigma_{ijiji,aq^{-6}} \Sigma_{iji,a},$$

$$(4.52) \quad \Sigma_{jijiji,a} \Sigma_{jiji,aq^{-6}} = \Sigma_{jijij,aq} + A_{i,aq^{-6}}^{-1} \Sigma_{jijiji,aq^{-6}} \Sigma_{jiji,a},$$

and the following q^2 -difference equations

$$(4.53) \quad \Sigma_{jij,a} \Sigma_{j,aq^{-2}} = \Sigma_{jij,aq^{-3}} \Sigma_{jij,aq^{-5}} \Sigma_{jij,aq^{-7}} + A_{j,aq^{-4}}^{-1} A_{i,aq^{-3}}^{-1} A_{j,aq^{-6}}^{-1} \Sigma_{jij,aq^{-2}} \Sigma_{j,a},$$

$$(4.54) \quad \Sigma_{ijij,a} \Sigma_{ij,aq^{-2}} = \Sigma_{ijij,aq^{-3}} \Sigma_{ijij,aq^{-5}} \Sigma_{ijij,aq^{-7}} + A_{j,aq^{-4}}^{-1} A_{i,aq^{-7}}^{-1} A_{j,aq^{-6}}^{-1} \Sigma_{ijij,aq^{-2}} \Sigma_{ij,a},$$

$$(4.55) \quad \Sigma_{jijij,a} \Sigma_{jij,aq^{-2}} = \Sigma_{jijij,aq^{-3}} \Sigma_{jijij,aq^{-5}} \Sigma_{jijij,aq^{-7}} + A_{j,aq^{-10}}^{-1} A_{i,aq^{-7}}^{-1} \Sigma_{jijij,aq^{-2}} \Sigma_{jij,a},$$

$$(4.56) \quad \Sigma_{ijijij,a} \Sigma_{ijij,aq^{-2}} = \Sigma_{ijijij,aq^{-3}} \Sigma_{ijijij,aq^{-5}} \Sigma_{ijijij,aq^{-7}} + A_{j,aq^{-10}}^{-1} \Sigma_{ijijij,aq^{-2}} \Sigma_{ijij,a}.$$

In all of these equations, the first factor in the second term on the right-hand side is a monomial in $A_{k,c}^{-1}$ whose weight is a negative root. This implies, by induction on the length of w , that

$$\Sigma_{w,a} \in \mathbb{Z}[[A_{k,c}^{-1}]]_{k \in I, c \in \mathbb{C}^\times}^{(1)}$$

where $\mathbb{Z}[[A_{k,c}^{-1}]]_{k \in I, c \in \mathbb{C}^\times}^{(1)}$ consists of all elements of $\mathbb{Z}[[A_{k,c}^{-1}]]_{k \in I, c \in \mathbb{C}^\times}$ whose highest weight term is equal to 1. Recalling formulas (4.41), we obtain the following result.

Lemma 4.5. *We have, for $k = i, j$,*

$$\phi_{k,a}^\ell \in \phi_{k,a,0}^\ell \cdot \mathbb{Z}[[A_{k,c}^{-1}]]_{k \in I, c \in \mathbb{C}^\times}^{(1)}, \quad \phi_{k,a}^r \in \phi_{k,a,0}^r \cdot \mathbb{Z}[[A_{k,c}^{-1}]]_{k \in I, c \in \mathbb{C}^\times}^{(1)},$$

where $\phi_{k,a,0}^\ell$ and $\phi_{k,a,0}^r$ are monomials in $\Psi_{i,b}^{\pm 1}$ and $\Psi_{j,b}^{\pm 1}$.

We will prove that the highest weight terms $\phi_{k,a,0}^\ell$ and $\phi_{k,a,0}^r$ are equal to each other for $k = i, j$. We will then use this result to prove that $\phi_{k,a}^\ell = \phi_{k,a}^r$ for $k = i, j$.

Consider the following elements of $\mathcal{Y} \subset \Pi$: in type A_2

$$T_{i,a} = Y_{i,a} + Y_{i,aq^2}^{-1} Y_{j,aq} + Y_{j,aq^3}^{-1}, \quad T_{j,a} = Y_{j,a} + Y_{j,aq^2}^{-1} Y_{i,aq} + Y_{i,aq^3}^{-1},$$

in type B_2

$$T_{i,a} = Y_{i,a} + Y_{i,aq^4}^{-1} Y_{j,aq} Y_{j,aq^3} + Y_{j,aq} Y_{j,aq^5}^{-1} + Y_{j,aq^3}^{-1} Y_{j,aq^5}^{-1} Y_{i,aq^2} + Y_{i,aq^6}^{-1},$$

$$T_{j,a} = Y_{j,a} + Y_{j,aq^2}^{-1} Y_{i,aq} + Y_{i,aq^5}^{-1} Y_{j,aq^4} + Y_{j,aq^6}^{-1},$$

and in type G_2

$$(4.57) \quad T_{i,a} = Y_{i,a} + Y_{i,aq^6}^{-1} Y_{j,aq} Y_{j,aq^3} Y_{j,aq^5} + Y_{j,aq} Y_{j,aq^3} Y_{j,aq^7}^{-1} + Y_{i,aq^4} Y_{j,aq} Y_{j,aq^5}^{-1} Y_{j,aq^7}^{-1}$$

$$+ Y_{i,aq^2} Y_{i,aq^4} Y_{j,aq^3}^{-1} Y_{j,aq^5}^{-1} Y_{j,aq^7}^{-1} + Y_{i,aq^{10}}^{-1} Y_{j,aq} Y_{j,aq^9} + Y_{i,aq^2} Y_{i,aq^{10}}^{-1} Y_{j,aq^3}^{-1} Y_{j,aq^9}$$

$$+ Y_{i,aq^8}^{-1} Y_{i,aq^4} + Y_{i,aq^8}^{-1} Y_{i,aq^{10}}^{-1} Y_{j,aq^5} Y_{j,aq^7} Y_{j,aq^9} + Y_{i,aq^8}^{-1} Y_{j,aq^5} Y_{j,aq^7} Y_{j,aq^{11}}^{-1}$$

$$\begin{aligned}
& + Y_{j,aq^5} Y_{j,aq^9}^{-1} Y_{j,aq^{11}}^{-1} + Y_{i,aq^6} Y_{j,aq^7}^{-1} Y_{j,aq^9}^{-1} Y_{j,aq^{11}}^{-1} + Y_{i,aq^{12}}^{-1}, \\
(4.58) \quad T_{j,a} &= Y_{j,a} + Y_{j,aq^2}^{-1} Y_{i,aq} + Y_{i,aq^7}^{-1} Y_{j,aq^4} Y_{j,aq^6} + Y_{j,aq^4} Y_{j,aq^8}^{-1} \\
& + Y_{j,aq^6}^{-1} Y_{j,aq^8}^{-1} Y_{i,aq^5} + Y_{i,aq^{11}}^{-1} Y_{j,aq^{10}} + Y_{j,aq^{12}}^{-1}.
\end{aligned}$$

Lemma 4.6. *The elements $T_{i,a}$ and $T_{j,a}$ are invariant under the action of Θ_i and Θ_j .*

Proof. The elements $T_{i,a}$ and $T_{j,a}$ are the q -characters of the i th and j th fundamental representations of $U_q(\widehat{\mathfrak{g}})$ (recall that we denote the two nodes of the Dynkin diagram of $\mathfrak{g}$ by i and j). This follows from the algorithm of [FM] (see [H, Section 8.4]). We prove their invariance under Θ_i and Θ_j following the argument of Theorem 5.1 below. Namely, by construction (see [FM, Corollary 5.7]), each of them can be written as a polynomial in

$$Y_{i,a}(1 + A_{i,aq_i}^{-1}) \quad \text{and} \quad Y_{j,a}^{\pm 1}$$

for various $a \in \mathbb{C}^\times$. We have $\Theta_i(Y_{j,a}^{\pm 1}) = Y_{j,a}^{\pm 1}$ by definition, and since $Y_{i,a}(1 + A_{i,aq_i}^{-1}) = T_{i,a}^{(1)}$ (see formula (4.1)), Proposition 4.2 implies that

$$\Theta_i(Y_{i,a}(1 + A_{i,aq_i}^{-1})) = Y_{i,a}(1 + A_{i,aq_i}^{-1}).$$

The proof of invariance under Θ_j is similar, and this completes the proof. $\square$

Next, recall the generalized Baxter TQ -relations proved in [FH1]. These relations appear in the Grothendieck ring of the category $\mathcal{O}$ introduced in [HJ], which is an extension of the category of finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$. Our proof below is based on a combinatorial study of the images of the TQ -relations under the q -character homomorphism; we will not use any information about representations from the category $\mathcal{O}$. They are as follows:

$$T_{k,a} = S_k([\omega_i], [\omega_j], \Psi_{i,b}, \Psi_{j,b})_{b \in \mathbb{C}^\times}, \quad k = i, j,$$

where the right hand side is obtained by substituting the right hand side of formula (4.38) for $Y_{i,a}$ and $Y_{j,a}$ in the defining formulas for $T_{i,a}$ and $T_{j,a}$.

Let us apply $\mathcal{R}'_{i,j,\ell}$ and $\mathcal{R}'_{i,j,r}$ to these equations. By Lemma 4.6, $T_{i,a}$ and $T_{j,a}$ are invariant under $\mathcal{R}'_{i,j,\ell}$ and $\mathcal{R}'_{i,j,r}$, and we have $\mathcal{R}'_{i,j,\ell}([\omega_k]) = \mathcal{R}'_{i,j,r}([\omega_k]) = [-\omega_{\bar{k}}]$ for $k = i, j$, where $\bar{k} = k$ in types B_2, G_2 and $\bar{k} \neq k$ in type A_2 . Hence we obtain the following system of two equations in Π on $\phi_{i,b}^{\ell/r}$ and $\phi_{j,b}^{\ell/r}$:

$$(4.59) \quad S_i([\omega_i], [\omega_j], \Psi_{i,b}, \Psi_{j,b})_{b \in \mathbb{C}^\times} = S_i([-\omega_{\bar{i}}], [-\omega_{\bar{j}}], \phi_{i,b}^{\ell/r}, \phi_{j,b}^{\ell/r})_{b \in \mathbb{C}^\times},$$

$$(4.60) \quad S_j([\omega_i], [\omega_j], \Psi_{i,b}, \Psi_{j,b})_{b \in \mathbb{C}^\times} = S_j([-\omega_{\bar{i}}], [-\omega_{\bar{j}}], \phi_{i,b}^{\ell/r}, \phi_{j,b}^{\ell/r})_{b \in \mathbb{C}^\times},$$

where notation ℓ/r means that the formula is valid if we put ℓ everywhere or r everywhere. We are going to derive from this system the desired equalities (4.42).

Recall that we are considering the projections onto $\mathcal{Y}^e$. Let $D = 3$ in type A_2 (resp $D = 6$ in type B_2 and $D = 12$ in type G_2).

It is clear that only the highest weight terms of

$$\mathcal{R}'_{i,j,\ell/r}(Y_{i,aq^D}^{-1}) = [\omega_i] \phi_{i,aq^{D+d_i}}^{\ell/r} (\phi_{i,aq^{D-d_i}}^{\ell/r})^{-1}$$

contribute to the highest weight term on the right hand side of (4.59). But the highest weight term on the left hand side of (4.59) is $Y_{i,a} = [\omega_i] \Psi_{i,aq^{-d_i}} \Psi_{i,aq^{d_i}}^{-1}$. Therefore, the highest weight terms of $\phi_{i,a}^{\ell}$ and $\phi_{i,a}^r$ are both equal to $\Psi_{i,aq^{-D}}^{-1}$.

Using equation (4.60) in the same way, we find that the highest weight terms of $\phi_{j,a}^\ell$ and $\phi_{j,a}^r$ are both equal to $\Psi_{j,aq^{-D}}^{-1}$.

Combining this with Lemma 4.5, we obtain the following result.

Lemma 4.7. *We have*

$$(4.61) \quad \phi_{i,b}^{\ell/r} \in \Psi_{i,bq^{-D}}^{-1} \cdot \mathbb{Z}[[A_{k,c}^{-1}]]_{k \in I, c \in \mathbb{C}^\times}^{(1)} \quad \phi_{j,b}^{\ell/r} \in \Psi_{j,bq^{-D}}^{-1} \cdot \mathbb{Z}[[A_{k,c}^{-1}]]_{k \in I, c \in \mathbb{C}^\times}^{(1)}.$$

We can now prove the equalities (4.42).

Lemma 4.8. *There is a unique solution $(\phi_{i,b}^{\ell/r}, \phi_{j,b}^{\ell/r})$ of the system (4.59), (4.60) in $\tilde{\mathcal{Y}}^e$ of the form (4.61).*

Proof. We have already shown the existence of such solutions in Lemma 4.7. Now we prove uniqueness. To simplify our notation, in this proof we will write $\phi_{i,b}$ (resp. $\phi_{j,b}$) for $\phi_{i,b}^{\ell/r}$ (resp. $\phi_{j,b}^{\ell/r}$).

By induction on descending weights, we can show that all lower weight terms in $\phi_{i,a}$ and $\phi_{j,a}$ are uniquely determined by equations (4.59) and (4.60). Indeed, let us expand

$$\phi_{i,b} = \sum_{m \geq 0} \phi_{i,b,m} \quad \phi_{j,b} = \sum_{m \geq 0} \phi_{j,b,m},$$

where $\phi_{i,b,0} = \Psi_{i,bq^{-D}}^{-1}$, $\phi_{j,b,0} = \Psi_{j,bq^{-D}}^{-1}$, and for $m > 0$, each expression $\phi_{i,b,m}$ (resp. $\phi_{j,b,m}$) is a finite linear combination of terms equal to the product of $\Psi_{i,bq^{-D}}^{-1}$ (resp. $\Psi_{j,bq^{-D}}^{-1}$) and a monomial of the form $A_{k_1,c_1}^{-1} \cdots A_{k_m,c_m}^{-1}$; note that the weight of such a term is equal to the sum of the corresponding m negative simple roots: $-\sum_{a=1}^m \alpha_{k_a}$. We will prove uniqueness of $\phi_{i,b,m}$, $\phi_{j,b,m}$ by induction on $m \geq 0$.

The terms $\phi_{i,b,0}$ and $\phi_{j,b,0}$ corresponding to $m = 0$ are fixed by formula (4.61) (which is our assumption in the present lemma); namely, they are equal to $\Psi_{i,bq^{-D}}^{-1}$ and $\Psi_{j,bq^{-D}}^{-1}$, respectively. Suppose now that we have found the terms $\phi_{i,b,m'}$ and $\phi_{j,b,m'}$ with $m' < m$. Consider the part of equation (4.59) comprising all terms whose weights are of the form $\omega_i - \gamma$, where γ is a sum of m simple roots. It is clear that only $\phi_{i,b,m'}$, $\phi_{j,b,m'}$ with $m' \leq m$ can contribute to this part of equation (4.59).

By computing the weights of these terms, we find that the contribution of $\phi_{i,b,m}$, $\phi_{j,b,m}$ to this part of equation (4.59) comes exclusively from $\mathcal{R}'_{i,j,\ell/r}(Y_{i,aq^D}^{-1}) = [\omega_i] \phi_{i,aq^{D+d_i}} \phi_{i,aq^{D-d_i}}^{-1}$, and hence this contribution is equal to $[\omega_i](\phi_{i,aq^{D+d_i,m}} - \phi_{i,aq^{D-d_i,m}})$.

Therefore, we obtain from equation (4.59) that $[\omega_i](\phi_{i,aq^{D+d_i,m}} - \phi_{i,aq^{D-d_i,m}})$ can be expressed in terms of $\phi_{i,b,m'}$, $\phi_{j,b,m'}$ with $m' < m$. This implies that $\phi_{i,b,m}$ is uniquely determined by $\phi_{i,b,m'}$, $\phi_{j,b,m'}$ with $m' < m$. Indeed, if there were two solutions $\phi_{i,b,m}^{(1)}$ and $\phi_{i,b,m}^{(2)}$, then their difference $\tilde{\phi}_{i,b,m}$ would satisfy the corresponding homogeneous equation $[\omega_i](\tilde{\phi}_{i,aq^{D+d_i,m}} - \tilde{\phi}_{i,aq^{D-d_i,m}}) = 0$. This equation implies that $\tilde{\phi}_{i,b,m}$ does not depend on b and hence is in $\mathbb{Z}[[\omega]]_{\omega \in P}$. But according to formula (4.61), $\tilde{\phi}_{i,b,m}$ is a linear combination of monomials of the form $\Psi_{i,bq^{-D}}^{-1} A_{k_1,c_1}^{-1} \cdots A_{k_m,c_m}^{-1}$, $m > 0$. Therefore, we obtain that $\tilde{\phi}_{i,b,m} = 0$, and so $\phi_{i,b,m}$ is uniquely determined by $\phi_{i,b,m'}$, $\phi_{j,b,m'}$ with $m' < m$.

Applying the same argument to the corresponding part of equation (4.60), we find that $\phi_{j,b,m}$ is also uniquely determined by $\phi_{i,b,m'}, \phi_{j,b,m'}$ with $m' < m$. This completes the inductive step and hence the proof. $\square$

This lemma implies the equalities (4.42). This completes the proof of the braid relation.

5. SUBRING OF W -INVARIANTS, q -CHARACTERS, AND SCREENING OPERATORS

In this section we prove that the subring of the invariants in $\mathcal{Y}$ of the action of the Weyl group on Π coincides with the ring of q -characters inside $\mathcal{Y}$, which is isomorphic to the Grothendieck ring of the category of finite-dimensional representations of $\mathcal{U}_q(\widehat{\mathfrak{g}})$. For that, we relate the operators Θ_i to the screening operators constructed in [FR] and use the results of [FM] that the subring of invariants of the screening operators in $\mathcal{Y}$ coincides with the ring of q -characters.

5.1. Quantum affine algebras and their finite-dimensional representations. In this section we collect some definitions and results on quantum affine algebras and their representations. We refer the reader to [CP] for a canonical introduction, and to [CH, L] for more recent surveys on this topic.

Let $\widehat{\mathfrak{g}}$ be the Kac–Moody Lie algebra of untwisted affine type associated to $\mathfrak{g}$. The quantum affine algebra $\mathcal{U}_q(\widehat{\mathfrak{g}})$ is a Hopf algebra over $\mathbb{C}$ defined in terms of the Drinfeld–Jimbo generators $e_i, f_i, k_i^{\pm 1}$ ($0 \leq i \leq n$); see, e.g., [FH1, Section 2.1]. We are interested in its level 0 quotient, which we denote by $\mathcal{U}_q(\widehat{\mathfrak{g}})$. It has a presentation [Dr, Be, Da] in terms of the Drinfeld generators

$$x_{i,r}^{\pm} \ (i \in I, r \in \mathbb{Z}), \quad \phi_{i,\pm m}^{\pm} \ (i \in I, m \geq 0), \quad k_i^{\pm 1} \ (i \in I).$$

We will use the generating series ($i \in I$):

$$\phi_i^{\pm}(z) = \sum_{m \geq 0} \phi_{i,\pm m}^{\pm} z^{\pm m} = k_i^{\pm 1} \exp \left(\pm (q_i - q_i^{-1}) \sum_{m > 0} h_{i,\pm m} z^{\pm m} \right).$$

The algebra $\mathcal{U}_q(\widehat{\mathfrak{g}})$ has a $\mathbb{Z}$ -grading defined by $\deg(x_{i,m}^{\pm}) = \deg(\phi_{i,m}^{\pm}) = m$ for $i \in I, m \in \mathbb{Z}$, and $\deg(k_i^{\pm 1}) = 0$ for $i \in I$. For $a \in \mathbb{C}^{\times}$, there is a corresponding automorphism τ_a of $\mathcal{U}_q(\widehat{\mathfrak{g}})$ so that for an element g of degree $m \in \mathbb{Z}$ satisfies $\tau_a(g) = a^m g$.

For $i \in I$, the action of k_i on any object of the category $\mathcal{F}$ of finite-dimensional representations of $\mathcal{U}_q(\widehat{\mathfrak{g}})$ is diagonalizable with eigenvalues in $\pm q^{d_i \mathbb{Z}}$. Without loss of generality, we can assume that $\mathcal{F}$ is the category of *type 1* finite-dimensional representations (see [CP]), i.e. we assume that for any object of $\mathcal{F}$, the eigenvalues of k_i are in $q^{d_i \mathbb{Z}}$ for $i \in I$. The simple objects of $\mathcal{F}$ have been classified by Chari–Pressley (see [CP]). The simple objects are parametrized by n -tuples of polynomials $(P_i(u))_{i \in I}$ satisfying $P_i(0) = 1$ (they are called *Drinfeld polynomials*).

For $\omega \in P$, the *weight space* V_{ω} of an object V in $\mathcal{F}$ is the set of *weight vectors* of weight ω , i.e. of vectors $v \in V$ satisfying $k_i v = q^{(d_i \omega(\alpha_i^{\vee}))} v$ for any $i \in I$. Thus, we have the weight space decomposition

$$V = \bigoplus_{\omega \in P} V_{\omega}.$$

Let us define the ℓ -weight decomposition which is its refinement.

Since we consider representations of the level-zero quotient $\mathcal{U}_q(\widehat{\mathfrak{g}})$, the Drinfeld–Cartan elements $h_{i,r}$, $i \in I, r \in \mathbb{Z} \setminus \{0\}$, commute on any object V of $\mathcal{F}$. Since the elements $h_{i,r}$

also commute with the elements k_i , $i \in I$, every object in $\mathcal{F}$ can be decomposed as a direct sum of generalized eigenspaces of the $h_{i,r}$ and k_i .

More precisely, it follows from the Frenkel–Reshetikhin theory of q -characters [FR] that the eigenvalues of the $h_{i,r}$ and k_i can be *encoded* by *monomials* $m \in \mathcal{M}$. Given $m \in \mathcal{M}$ and an object V in $\mathcal{F}$, let V_m be the corresponding generalized eigenspace of V (also called ℓ -weight spaces); thus,

$$V = \bigoplus_{m \in \mathcal{M}} V_m.$$

If $v \in V_m$, then v is a weight vector of weight $\omega(m) \in P$.

The q -character homomorphism is an injective ring homomorphism

$$\chi_q : \text{Rep}(\mathcal{U}_q(\mathfrak{g})) \rightarrow \mathcal{Y}, \quad \chi_q(V) = \sum_{m \in \mathcal{M}} \dim(V_m) m.$$

If $V_m \neq \{0\}$ we say that m is an ℓ -weight of V .

A monomial $m \in \mathcal{M}$ is said to be *dominant* if $u_{i,a}(m) \geq 0$ for any $i \in I, a \in \mathbb{C}^\times$. Given a simple object V in $\mathcal{F}$, let $M(V)$ be the *highest weight monomial* of $\chi_q(V)$, i.e. such that $\omega(M(V))$ is maximal for the partial ordering on P . It is known that $M(V)$ is dominant and characterizes the isomorphism class of V (it is, in fact, equivalent to the data of the Drinfeld polynomials). Conversely, to a dominant monomial M is associated a unique (up to an isomorphism) simple object $L(M)$ in $\mathcal{F}$. For $i \in I$ and $a \in \mathbb{C}^\times$, we set $V_i(a) := L(Y_{i,a})$; this is the i th fundamental representation of $\mathcal{U}_q(\widehat{\mathfrak{g}})$.

For example, the q -character of the fundamental representation $V_{1,a} = L(Y_{1,a})$ of $\mathcal{U}_q(\widehat{sl}_2)$ is

$$\chi_q(L(Y_{1,a})) = Y_{1,a} + Y_{1,aq^2}^{-1}.$$

It was proved in [FR, FM] that for a simple module $L(m)$ we have

$$\chi_q(L(m)) \in m\mathbb{Z}[A_{i,a}^{-1}]_{i \in I, a \in \mathbb{C}^\times}.$$

5.2. Screening operators and the ring of W -invariants. In this subsection we link the operators Θ_i generating the action of the Weyl group to the screening operators S_i introduced in [FR, Section 7]. Recall that in Section 3 we denoted by $\mathcal{Y}^w$ the image of $\mathcal{Y}$ in the completion $\widetilde{\mathcal{Y}}^w$ and in the direct sum Π of these completions for all $w \in W$. Then in Definition 3.1 we define the ring homomorphisms $\Theta_i^w : \mathcal{Y}^w \rightarrow \widetilde{\mathcal{Y}}^{ws_i}$ using formula (3.22). Now we define the ring homomorphisms

$$\Theta_i : \mathcal{Y} \rightarrow \widetilde{\mathcal{Y}}^e, \quad i \in I,$$

by $\Theta_i(Y_{j,a}^{\pm 1}) = Y_{j,a}^{\pm 1}$ if $j \neq i$ and

$$(5.62) \quad \Theta_i(Y_{i,a}) = Y_{i,a} A_{i,aq_i^{-1}}^{-1} \frac{\sum_{i,aq_i^{-3}}^e}{\sum_{i,aq_i^{-1}}^e}.$$

The connection between Θ_i and Θ_i is as follows: Θ_i is the composition of the isomorphism $\mathcal{Y} \simeq \mathcal{Y}^{s_i}$ and the homomorphism $\Theta_i^{s_i} : \mathcal{Y}^{s_i} \rightarrow \widetilde{\mathcal{Y}}^e$ from Definition 3.1.

Now let h be a formal variable and set

$$\widetilde{\Sigma}_{i,a} := h \sum_{i,a}^e \text{ for } i \in I \text{ and } a \in \mathbb{C}^\times.$$

We then have

$$\begin{aligned}\Theta_i(Y_{i,a}) &= Y_{i,a} A_{i,aq_i}^{-1} \frac{\Sigma_{i,aq_i}^e}{\Sigma_{i,aq_i}^e} = Y_{i,a} - h \frac{Y_{i,a}}{\widetilde{\Sigma}_{i,aq_i}^{-1}}, \\ \Theta_i(Y_{i,a}^{-1}) &= Y_{i,a}^{-1} A_{i,aq_i}^{-1} \frac{\Sigma_{i,aq_i}^e}{\Sigma_{i,aq_i}^e} = Y_{i,a}^{-1} + h \frac{Y_{i,a}^{-1} A_{i,aq_i}^{-1}}{\widetilde{\Sigma}_{i,aq_i}^{-3}}.\end{aligned}$$

Let us also set

$$\mathcal{Y}_{i,h} := \mathcal{Y}[\widetilde{\Sigma}_{i,a}^{-1}, h]_{a \in \mathbb{C}^\times}.$$

Then the above formulas give rise to a ring homomorphism

$$\Theta_{i,h} : \mathcal{Y} \rightarrow \mathcal{Y}_{i,h}.$$

Consider the limit $h \rightarrow 0$. Note that the relation $\widetilde{\Sigma}_{i,a} = h + A_{i,a}^{-1} \widetilde{\Sigma}_{i,aq_i}^{-2}$ then becomes $\widetilde{\Sigma}_{i,a} = A_{i,a}^{-1} \widetilde{\Sigma}_{i,aq_i}^{-2}$. Therefore,

$$(5.63) \quad \Theta_{i,h} = \text{Id} + h S_i + h^2(\dots),$$

where $S_i : \mathcal{Y} \rightarrow \mathcal{Y}_i$ is a derivation, and $\mathcal{Y}_i$ is the $\mathcal{Y}$ -module generated by the $\widetilde{\Sigma}_{i,a}^{-1}$ with the relation $\widetilde{\Sigma}_{i,a}^{-1} = A_{i,a} \widetilde{\Sigma}_{i,aq_i}^{-1}$ and

$$S_i(Y_{j,a}) = -\delta_{i,j} Y_{i,a} \widetilde{\Sigma}_{i,aq_i}^{-1}, \quad S_i(Y_{j,a}^{-1}) = \delta_{i,j} Y_{i,a}^{-1} \widetilde{\Sigma}_{i,aq_i}^{-1}.$$

By denoting $S_{i,a} = -\widetilde{\Sigma}_{i,aq_i}^{-1}$, we obtain the following relation in the limit $h \rightarrow 0$:

$$S_{i,aq_i^2} = A_{i,aq_i} S_{i,a},$$

and so S_i gets identified with the screening operator defined in [FR]. Thus, the screening operator S_i appears in the limit of a one-parameter deformation of the operator Θ_i defined using formula (5.62).

We will now use this to prove that the ring of q -characters (equivalently, the Grothendieck ring of the category of finite-dimensional representations of $\mathcal{U}_q(\widehat{\mathfrak{g}})$) is equal to the subring of W -invariants in $\mathcal{Y}$ embedded diagonally into Π .

Theorem 5.1. *The image of the q -character homomorphism χ_q in $\mathcal{Y}$ is equal to the subring of invariants of the diagonal subspace $\mathcal{Y} \subset \Pi$ under the action of $\Theta_i, i \in I$, i.e.*

$$\text{Im}(\chi_q) = \bigcap_{i \in I} \mathcal{Y}^{\Theta_i}.$$

Equivalently, the Grothendieck ring of the category of finite-dimensional representations of $\mathcal{U}_q(\widehat{\mathfrak{g}})$ is isomorphic to the subring of invariants in $\mathcal{Y}$ of the action of the Weyl group on Π .

Proof. First, we prove that the elements in $\text{Im}(\chi_q)$ are invariant under Θ_i for each $i \in I$. Indeed, let $i \in I$ and $L \in \text{Im}(\chi_q)$. According to [FM, Corollary 5.7], L is a polynomial in

$$Y_{i,a}(1 + A_{i,aq_i}^{-1}) \quad \text{and} \quad Y_{j,a}^{\pm 1}, j \neq i$$

for various $a \in \mathbb{C}^\times$. We have $\Theta_i(Y_{j,a}^{\pm 1}) = Y_{j,a}^{\pm 1}$ by definition, and since $Y_{i,a}(1 + A_{i,aq_i}^{-1}) = T_{i,a}^{(1)}$ (see formula (4.1)), Proposition 4.2 implies that

$$\Theta_i(Y_{i,a}(1 + A_{i,aq_i}^{-1})) = Y_{i,a}(1 + A_{i,aq_i}^{-1}).$$

Thus, we find that $\text{Im}(\chi_q)$ is contained in $\bigcap_{i \in I} \mathcal{Y}^{\Theta_i}$.

Conversely, let L be an element in the intersection $\bigcap_{i \in I} \mathcal{Y}^{\Theta_i}$. Then $\Theta_i(L) = L$, and therefore $\Theta_{i,h}(L) = L$, for all $i \in I$. But then formula (5.63) implies that $S_i(L) = 0$ for all $i \in I$. Hence $L \in \bigcap_{i \in I} \text{Ker}_{\mathcal{Y}} S_i$. By [FM, Theorem 5.1],

$$\text{Im}(\chi_q) = \bigcap_{i \in I} \text{Ker}_{\mathcal{Y}} S_i.$$

Hence $L \in \text{Im}(\chi_q)$. This completes the proof. $\square$

Remark 5.2. The subring of W -invariants in Π is larger than the ring of q -characters. This is already so for $\mathfrak{g} = \mathfrak{sl}_2$. Indeed, according to formula (3.27) the element $\Sigma_{1,a}(1 - \Sigma_{1,a})$ of Π is invariant under Θ_1 . But it is not in $\mathcal{Y}$. $\square$

6. RELATION WITH OTHER SYMMETRIES

In this section we study the relation of the action of the Weyl group W defined above with other known symmetries. In particular, we introduce a q -analogue of a natural ring of rational functions carrying an action of W .

6.1. Ring of rational fractions. Recall that in Section 2.3 we defined completions

$$(\mathbb{Z}[y_j^{\pm 1}]_{j \in I})^w, \quad w \in W$$

of $\mathbb{Z}[y_j^{\pm 1}]_{j \in I}$. In Lemma 2.6 we extended the natural action of the Weyl group W on $\mathbb{Z}[y_i^{\pm 1}]_{i \in I}$ (generated by the simple reflections $s_i, i \in I$, given by formula (1.1)) to the direct sum π of these completions defined in equation (2.17) (into which $\mathbb{Z}[y_i^{\pm 1}]_{i \in I}$ is embedded diagonally).

Now we introduce a ring $\mathcal{R}$ which lies in-between $\mathbb{Z}[y_i^{\pm 1}]_{i \in I}$ and π , and is preserved by the action of W . Namely, we set

$$(6.64) \quad \mathcal{R} := \mathbb{Z}[y_i^{\pm 1}, (1 - a_{\alpha}^{-1})^{-1}]_{i \in I, \alpha \in \Delta_+}.$$

We have a natural embedding

$$(6.65) \quad \mathcal{R} \hookrightarrow (\mathbb{Z}[y_i^{\pm 1}]_{i \in I})^w, \quad w \in W$$

obtained by replacing $(1 - a_{\alpha}^{-1})^{-1}$ with

$$\sum_{n \geq 0} a_{\alpha}^{-n} \quad \text{if } \alpha \in w^{-1}(\Delta_+)$$

and with

$$-\sum_{n > 0} a_{\alpha}^n \quad \text{if } \alpha \in w^{-1}(\Delta_-).$$

Thus, we have natural embeddings

$$(6.66) \quad \mathbb{Z}[y_i^{\pm 1}]_{i \in I} \subset \mathcal{R} \subset \pi,$$

where the last embedding is diagonal.

Example 6.1. For $\mathfrak{g}$ of type A_2 with $I = \{i, j\}$, we have $\frac{1}{(1-a_{\alpha_j}^{-1})(1-a_{\alpha_i+\alpha_j}^{-1})}$ in $\mathcal{R}$. This is $\varpi(\Sigma_{ji,a})$. Its expansion in $(\mathbb{Z}[y_i^{\pm 1}]_{i \in I})^w$ is

$$(6.67) \quad \begin{cases} \sum_{0 \leq \beta \leq \alpha} a_j^{-\alpha} a_i^{-\beta} & \text{if } w(\alpha_j) \in \Delta_+ \text{ and } w(\alpha_i + \alpha_j) \in \Delta_+, \\ -\sum_{0 \leq \beta, \alpha < \beta} a_j^{-\alpha} a_i^{-\beta} & \text{if } w(\alpha_j) \in \Delta_- \text{ and } w(\alpha_i + \alpha_j) \in \Delta_+, \\ -\sum_{\beta < 0, \beta \leq \alpha} a_j^{-\alpha} a_i^{-\beta} & \text{if } w(\alpha_j) \in \Delta_+ \text{ and } w(\alpha_i + \alpha_j) \in \Delta_-, \\ \sum_{\alpha < \beta < 0} a_j^{-\alpha} a_i^{-\beta} & \text{if } w(\alpha_j) \in \Delta_- \text{ and } w(\alpha_i + \alpha_j) \in \Delta_-. \end{cases}$$

These correspond to the different cases in (7.75).

Let us extend the action of simple reflections $s_i, i \in I$, on $\mathbb{Z}[y_i^{\pm 1}]_{i \in I}$ given by formula (1.1), to $\mathcal{R}$ as follows. For each $\alpha \in \Delta_+$, the element $1 - a_\alpha$ is invertible in $\mathcal{R}$ with the inverse

$$(1 - a_\alpha)^{-1} = -a_\alpha^{-1}(1 - a_\alpha^{-1})^{-1}.$$

Since Δ is stable under W , the action of W on $\mathbb{Z}[y_i^{\pm 1}]_{i \in I}$ extends to $\mathcal{R}$. Moreover, we obtain that the embeddings (6.66) commute with the action of W .

The ring $\mathcal{R}$ is meaningful from the point of view of representation theory of the Lie algebra $\mathfrak{g}$ because the characters of $\mathfrak{g}$ -modules from the category $\mathcal{O}$ and its twists $\mathcal{O}_w, w \in W$, discussed in the introduction, may be viewed as elements of $\mathcal{R}$ expanded in a particular completion $(\mathbb{Z}[y_i^{\pm 1}]_{i \in I})^w$ of $\mathbb{Z}[y_i^{\pm 1}]_{i \in I}$ (that's because the character of every module from the category $\mathcal{O}$ is known to be a linear combination of the characters of Verma modules). Finite-dimensional $\mathfrak{g}$ -modules correspond to elements of the ring $\mathbb{Z}[y_i^{\pm 1}]_{i \in I}$ itself.

We will now define a q -analogue $\bar{\mathcal{Y}} \subset \Pi$ of $\mathcal{R} \subset \pi$ and a q -analogue of the embeddings (6.65). It will then follow from Lemma 3.6 that the homomorphism $\bar{\mathcal{Y}} \rightarrow \mathcal{R}$ obtained by restriction of the homomorphism $\varpi : \Pi \rightarrow \pi$ intertwines the actions of W on $\bar{\mathcal{Y}}$ and $\mathcal{R}$.

6.2. Solutions of q -difference equations. Recall the group $\tilde{G}^w$ from Definition 2.4.

Lemma 6.2. Fix an integer $r \in \mathbb{Z}$ and let $G(a), H(a), F(a)$ be families of $\mathbb{C}^\times$ -equivariant elements in $\tilde{G}^w$ of respective weights $\alpha \in \Delta, 0$ and 0 . Then there is a unique $\mathbb{C}^\times$ -equivariant family $U(a)$ of elements in $\tilde{\mathcal{Y}}^w$ such that

$$(6.68) \quad F(a)U(a) = H(a) + G(a)U(aq^{-r}) \text{ for any } a \in \mathbb{C}^\times.$$

Moreover, we have $U(a) \in \tilde{G}^w$ for any $a \in \mathbb{C}^\times$.

Proof. We give the proof for $\tilde{\mathcal{Y}}^e$; the proof for an arbitrary $\tilde{\mathcal{Y}}^w$ is analogous. Since $H(a)/F(a)$ has weight 0 and $G(a)/F(a)$ has weight α , dividing by $F(a)$, we can assume without loss of generality that $F(a) = 1$. Next, we can assume without loss of generality that $H(a) = 1$. Indeed, the above equation with $F(a) = 1$ is equivalent to the equation

$$U'(a) = 1 + G'(a)U'(aq^{-r})$$

where

$$U'(a) = U(a)/H(a), \quad G'(a) = G(a)H(aq^{-r})/H(a),$$

and $G'(a) \in \tilde{G}^e$ has the same weight as $G(a)$. So let us set $F(a) = H(a) = 1$.

First, let us prove the existence and uniqueness of the solution $U(a) \in \tilde{G}$ of the equation

$$(6.69) \quad U(a) = 1 + G(a)U(aq^{-r})$$

provided that $G(a)$ is a $\mathbb{C}^\times$ -equivariant family of weight $\alpha \in \Delta$.

Let us factorize

$$U(a) = u(a)\tilde{U}(a) \text{ and } G(a) = g(a)\tilde{G}(a)$$

where $u(a), g(a)$ are the highest weight monomials of $U(a), G(a)$, respectively. Let $\gamma = \omega(u(a))$ be the weight of $u(a)$ and let us look at the terms of highest weight in (6.69). It is clear from (6.69) that the highest weight could only be 0, γ or $\gamma + \alpha$.

If $\alpha \in \Delta_+$, then the highest weight is $\gamma + \alpha > \gamma$. Then we must have $\gamma + \alpha = 0$ and $0 = 1 + g(a)u(aq^{-r})$, so we find that $\gamma = -\alpha$ and $u(a) = -g(aq^r)^{-1}$. Equation (6.69) can therefore be written as

$$\tilde{G}(a)^{-1}(g(aq^r)^{-1}\tilde{U}(a) + 1 - \tilde{G}(a)) = \tilde{U}(aq^{-r}) - 1.$$

Denote by $(\tilde{U}(a))_\lambda$ the term of weight λ in the expansion of $\tilde{U}(a)$. The above equation gives rise to a system of recurrence relations for $\lambda < 0$:

$$(\tilde{U}(aq^{-r}))_\lambda = \sum_{\mu \leq 0} (\tilde{G}(a)^{-1})_\mu \left(g(aq^r)^{-1} (\tilde{U}(a))_{\lambda-\mu+\alpha} + (1 - \tilde{G}(a))_{\lambda-\mu} \right).$$

All the terms $(\tilde{U}(a))_\nu$ appearing on the right hand side have weights $\nu = \lambda - \mu + \alpha > \lambda$. Hence these equations determine uniquely all the terms $(\tilde{U}(a))_\lambda, \lambda < 0$, of $\tilde{U}(a)$.

If $\alpha \in \Delta_-$, then the highest weight must be $\gamma = 0$ and we must have $u(a) = 1$. Equation (6.69) becomes

$$\tilde{U}(a) - 1 = g(a)\tilde{G}(a)\tilde{U}(aq^{-r}).$$

Hence for $\lambda < 0$, we have recurrence relations

$$(\tilde{U}(a))_\lambda = \sum_{\mu \leq 0} g(a)(\tilde{G}(a))_\mu (\tilde{U}(aq^{-r}))_{\lambda-\mu-\alpha}.$$

This again determines uniquely all the terms $(\tilde{U}(a))_\lambda, \lambda < 0$, of $\tilde{U}(a)$.

It remains to prove that any solution of (6.69) is necessarily of this form. Suppose that there are two solutions $U(a)$ and $V(a)$ in $\tilde{\mathcal{Y}}^w$. Then the difference $D(a) = U(a) - V(a)$ satisfies $D(a) = G(a)D(aq^{-r})$. By Lemma 2.5, we have $D(a) = 0$. This completes the proof. $\square$

6.3. The algebra $\bar{\mathcal{Y}}$. We define by induction a sequence of $\mathcal{Y}$ -subalgebras of Π :

$$\mathcal{Y}^0 \hookrightarrow \mathcal{Y}^1 \hookrightarrow \mathcal{Y}^2 \hookrightarrow \dots$$

together with subgroups of invertible elements $G^m \subset (\mathcal{Y}^m)^\times$ such that

- (1) the algebra $\mathcal{Y}^m$ is invariant under the automorphisms τ_a ($a \in \mathbb{C}^\times$),
- (2) for any $w \in W$, we have $E_w(G^m) \subset \tilde{G}^w$.

We will also define

$$(6.70) \quad (G^m)^{adm} := \{g \in G^m \mid \forall w \in W, \varpi_w(E_w(g)) = a_\alpha \text{ for some } \alpha \in \Delta\},$$

where a_α is given by formula (2.15).

First, we define $\mathcal{Y}^0$ (resp. G^0) as the image of $\mathcal{Y}$ (resp. $\mathcal{M}$) in Π under the diagonal embedding. Suppose now that $G^m \subset \mathcal{Y}^m$ have been defined. Let $g \in (G^m)^{adm}$, $a \in \mathbb{C}^\times$, $r \in \mathbb{Z} \setminus \{0\}$, and $w \in W$. By Lemma 6.2, there is a unique invertible $f_{g,r}^w(a) \in \tilde{G}^w$ such that

$$(6.71) \quad f_{g,r}^w(a) = 1 + E_w(\tau_a(g))f_{g,r}^w(aq^{-r}).$$

Hence $f_{g,r}(a) = (f_{g,r}^w(a))_{w \in W}$ is the unique invertible solution in Π of

$$(6.72) \quad f_{g,r}(a) = 1 + \tau_a(g)f_{g,r}(aq^{-r}).$$

We define $\mathcal{Y}^{m+1}$ as the $\mathcal{Y}^m$ -subalgebra of Π generated by the elements $(f_{g,r}(a))^{\pm 1}$ obtained in this way, and G^{m+1} as the subgroup of invertible elements in $\mathcal{Y}^{m+1}$ generated by these $f_{g,r}(a)$ and by G^m .

Example 6.3. *The q -difference equation (1.8) is of the form (6.72). Hence there is an element of G^1 representing its unique solution; namely, $\Sigma_{i,a}$ introduced in formula (2.20). Note that $\varpi_w(\Sigma_{i,a}^w)$ is the expansion of $(1 - a_{\alpha_i}^{-1})^{-1}$ in negative powers of a_{α_i} if $w(\alpha_i) \in \Delta_+$, and $\varpi_w(\Sigma_{i,a}^w)$ is the expansion of the same rational function $(1 - a_{\alpha_i}^{-1})^{-1}$ in positive powers of a_{α_i} if $w(\alpha_i) \in \Delta_-$.*

Definition 6.4. *The algebra $\bar{\mathcal{Y}} \subset \Pi$ (resp. the group $G \subset (\Pi)^\times$) is the union of the algebras $\mathcal{Y}^m$ (resp. of the groups G^m), $m \geq 0$.*

It follows from the definition that $\bar{\mathcal{Y}}$ contains the diagonal subalgebra $\mathcal{Y} \subset \Pi$. Hence $\bar{\mathcal{Y}}$ is a completion of $\mathcal{Y}$.

Thus, we have a sequence of embeddings

$$(6.73) \quad \mathcal{Y} \subset \bar{\mathcal{Y}} \subset \Pi.$$

Recall the homomorphism $\varpi : \Pi \rightarrow \pi$ introduced in Section 2.5 and the ring $\mathcal{R}$ defined by formula (6.64) which we have realized as a subring of π , see equation (6.66). The following result shows that the sequence (6.73) is mapped by ϖ to the sequence (6.66).

Lemma 6.5. *The restriction of the homomorphism $\varpi : \Pi \rightarrow \pi$ to $\bar{\mathcal{Y}}$ yields a homomorphism*

$$(6.74) \quad \bar{\mathcal{Y}} \rightarrow \mathcal{R}.$$

Proof. We need to show that for any $g \in \bar{\mathcal{Y}}$, there is an element of $\mathcal{R}$ such that for any $w \in W$, $\varpi_w(E_w(g))$ is the image of this element in $(\mathbb{Z}[y_i^{\pm 1}]_{i \in I})^w$ under the embedding (6.65).

We prove this statement for $g \in \mathcal{Y}^m$ by induction on $m \geq 0$. This is clear on $\mathcal{Y}^0 = \mathcal{Y}$. Suppose it is true on $\mathcal{Y}^m$. Then it suffices to prove this statement for the $f_{g,r}(a)$, where $g \in (G^m)^{adm}$, $r \in \mathbb{Z} \setminus \{0\}$, $a \in \mathbb{C}^\times$. But for each $w \in W$, we have

$$\varpi_w(E_w(f_{g,r}(a))) = (1 - \varpi_w(E_w(g)))^{-1},$$

which is in $\mathcal{R}$ according to condition (6.70). The statement of the lemma follows. $\square$

Next, we show that the action of W on Π preserves $\bar{\mathcal{Y}}$ and the homomorphism (6.74) commutes with the action of W .

Proposition 6.6. *The operators Θ_i preserve the subalgebra $\bar{\mathcal{Y}} \subset \Pi$.*

Proof. It suffices to prove by induction on $m \geq 0$ that $\Theta_i(G^m) \subset G^{m+1}$. By definition, this is true for $m = 0$. Suppose that $\Theta_i(G^m) \subset G^{m+1}$. Since $s_i(\Delta) = \Delta$, we have $\Theta_i((G^m)^{adm}) \subset (G^{m+1})^{adm}$.

Let $g \in (G^m)^{adm}$, $r \neq 0$ and $a \in \mathbb{C}^\times$. Then $\Theta_i(g) \in (G^{m+1})^{adm}$ and there is a unique $f_{\Theta_i(g),r}(a) \in \mathcal{Y}^{m+2}$ such that

$$f_{\Theta_i(g),r}(a) = 1 + \tau_a(\Theta_i(g))f_{\Theta_i(g),r}(aq^{-r}).$$

The uniqueness implies that $\Theta_i(f_{g,r}(a)) = f_{\Theta_i(g),r}(a)$ and moreover $f_{\Theta_i(g),r}(a) \in G^{m+2}$. This concludes the proof. $\square$

Proposition 6.7. *The homomorphism $\bar{\mathcal{Y}} \rightarrow \mathcal{R}$ given by (6.74) commutes with the action of W .*

Proof. We have shown in Lemma 6.5 that the homomorphism $\varpi : \Pi \rightarrow \pi$ restricts to a homomorphism $\bar{\mathcal{Y}} \rightarrow \mathcal{R}$. The former commutes with the action of W by Lemma 3.6. Proposition 6.6 shows that W preserves $\bar{\mathcal{Y}} \subset \Pi$. Hence the result. $\square$

This is consistent with the fact [FR] that ϖ sends the subring of q -characters in $\mathcal{Y} \subset \bar{\mathcal{Y}}$ to the subring of characters of finite-dimensional representations of $\mathfrak{g}$ in $\mathbb{Z}[y_i^{\pm 1}]_{i \in I} \subset \mathcal{R}$ and gives a precise sense in which the W -action on $\bar{\mathcal{Y}}$ generated by the Θ_i 's is a “ q -deformation” of the W -action on $\mathcal{R}$.

Remark 6.8. Consider, for example, the homomorphism $\varpi : \bar{\mathcal{Y}} \rightarrow \mathcal{R}$ in the case of sl_2 . Then we have $\varpi(\Sigma_{1,a}) = (1 - y^{-2})^{-1}$ and

$$\varpi(\Theta_1(\Sigma_{1,a})) = \varpi(1 - \Sigma_{1,a}) = 1 - \frac{1}{1 - y^{-2}} = \frac{1}{1 - y^2} = s_1 \left(\frac{1}{1 - y^{-2}} \right) = s_1(\varpi(\Sigma_{1,a})).$$

$\square$

The action of the Weyl group on $\mathbb{Z}[y_i^{\pm 1}]_{i \in I}$ is faithful. Hence we obtain the following.

Corollary 6.9. *The W -action on Π is faithful.*

6.4. Braid group action. Let $\bar{\mathcal{M}}$ be the subgroup of invertible elements of the ring $\bar{\mathcal{Y}}$ generated by the multiplicative group $\mathcal{M}$ of monomials in $\mathcal{Y}$ (see Section 2.2) and by $G^m, m \geq 0$ (see Section 6.3). By construction, Θ_i defines an automorphism of the group $\bar{\mathcal{M}}$.

Introduce the truncation homomorphism $\Lambda : \bar{\mathcal{M}} \rightarrow \mathcal{M}$ which assigns to $P \in \bar{\mathcal{M}}$ the unique element $\Lambda(P) \in \mathcal{M}$ such that $E_e(P) = \Lambda(P) \cdot \bar{P}$, where $\bar{P} \in \mathbb{Z}[[A_{j,b}^{-1}]]_{j \in I, b \in \mathbb{C}^\times}$ has constant term ± 1 . Thus, one can think of $\Lambda(P)$ as the leading monomial of $E_e(P)$. The following result is obtained by a straightforward calculation.

Proposition 6.10. *The restriction of $\Lambda \circ \Theta_i$ to $\mathcal{M}$ is the Chari operator T_i from [C] given by formula (1.6).*

Remark 6.11. (i) More precisely, we obtain Chari's operators if we replace q with q^{-1} .

(ii) It was shown in [C] (see also [BP]) that the operators T_i generate an action of the braid group associated to $\mathfrak{g}$, but not the Weyl group. Indeed, T_i has infinite order. From the point of view of the above proposition, the reason is that Λ and Θ_i do not commute.

(iii) In the sl_2 case, we have proved that $\chi_q(L(Y_{1,a}))$ is invariant under Θ_1 . But it is clearly not invariant under T_1 (moreover, $T_1(\chi_q(L))$ is not even in the image of χ_q). In fact, it is easy to see that the subring of invariants of T_i in $\mathcal{Y}$ is trivial, i.e. is equal to $\mathbb{Z}$ (consists of the constant elements of $\mathcal{Y}$).

(iv) It is proved in [C] that if $w = s_{i_1} \cdots s_{i_N}$ is a reduced expression for $w \in W$ and m' the lowest weight monomial of a simple module $L(m)$, then $T_{i_1} \cdots T_{i_N}(m')$ is a monomial occurring in $\chi_q(L(m))$ with multiplicity 1 (this is stated in [C] with m instead of m' as q^{-1} is used instead of q in the definition of T_i). In fact, this exhausts all monomials in $\chi_q(L(m))$ of extremal weights (but in general there are many other monomials in $\chi_q(L(m))$).

(v) In a different setting, a possible relation between a Weyl group action and the Chari braid group action is discussed in [In, Remark 4.16] via tropicalization. This could be related to the restriction of $\Lambda \circ \Theta_i$ in our Proposition.

(vi) We want to mention that in [KKOP] an action of the braid group is constructed on a quantized Grothendieck ring of a certain category of finite-dimensional representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$. However, as far as we can see, it is unrelated to the actions we consider in this paper.

7. EXPANSIONS

In this section we present explicit formulas for some elements of Π related to the braid relations for Lie algebras of rank 2. These formulas are not used in this paper, but they can be used to give a purely combinatorial proof of the braid relations, and we have also used them in [FH3].

7.1. Type A_2 . Here are explicit formulas for the w -components $E_w(\Sigma_{ji,a})$, $w \in W$, of the solution of equation (4.35) (which we have used in the alternative proof of the braid relations in the simply-laced case given in Section 4.5) in terms of the monomials $V_{k,a}^{(\alpha)}$ introduced in Definition 4.1:

$$(7.75) \quad E_w(\Sigma_{ji,a}) = \begin{cases} \sum_{0 \leq \beta \leq \alpha} V_{j,aq^{-1}}^{(\alpha)} V_{i,a}^{(\beta)} & \text{if } w(\alpha_j) \in \Delta_+ \text{ and } w(\alpha_i + \alpha_j) \in \Delta_+, \\ -\sum_{0 \leq \beta, \alpha < \beta} V_{j,aq^{-1}}^{(\alpha)} V_{i,a}^{(\beta)} & \text{if } w(\alpha_j) \in \Delta_- \text{ and } w(\alpha_i + \alpha_j) \in \Delta_+, \\ -\sum_{\beta < 0, \beta \leq \alpha} V_{j,aq^{-1}}^{(\alpha)} V_{i,a}^{(\beta)} & \text{if } w(\alpha_j) \in \Delta_+ \text{ and } w(\alpha_i + \alpha_j) \in \Delta_-, \\ \sum_{\alpha < \beta < 0} V_{j,aq^{-1}}^{(\alpha)} V_{i,a}^{(\beta)} & \text{if } w(\alpha_j) \in \Delta_- \text{ and } w(\alpha_i + \alpha_j) \in \Delta_-. \end{cases}$$

To prove the first formula, write $E_w(\Sigma_{ji,a})$ as a sum of terms of increasing degrees in $A_{i,aq^{-2m}}^{-1}$ and compute the corresponding coefficients by induction using the q -difference equation (4.35). One checks the other formulas in a similar way.

We can deduce formula (4.36) directly from these expansions and hence obtain an alternative proof of the braid relations in the simply-laced case.

7.2. Type B_2 . Let us suppose that $C_{i,j} = -1$, $C_{j,i} = -2$, $d_i = 2$ and $d_j = 1$. As in Section 7.1, one finds, using the relevant q -differences equations, the following explicit formulas:

$$(7.76) \quad E_e(\Sigma_{ij,a}) = \sum_{0 \leq \beta \leq \text{Min}(2\alpha, 2\alpha'+1)} V_{i,aq^{-2}}^{(\alpha)} V_{i,aq^{-4}}^{(\alpha')} V_{j,a}^{(\beta)},$$

$$(7.77) \quad E_e(\Sigma_{iji,a}) = \sum_{0 \leq \gamma \leq \beta/2 \leq \alpha} V_{i,aq^{-2}}^{(\alpha)} V_{j,a}^{(\beta)} V_{i,a}^{(\gamma)},$$

$$(7.78) \quad E_e(\Sigma_{ji,a}) = \sum_{0 \leq 2\beta \leq \alpha} V_{j,a}^{(\alpha)} V_{i,a}^{(\beta)},$$

$$(7.79) \quad E_e(\Sigma_{jij,a}) = \sum_{0 \leq \beta \leq \alpha/2, 0 \leq \beta' \leq (\alpha+1)/2, 0 \leq \gamma \leq \text{Min}(1+2\beta, 2\beta')} V_{j,aq^{-4}}^{(\alpha)} V_{i,aq^{-4}}^{(\beta)} V_{i,aq^{-2}}^{(\beta')} V_{j,a}^{(\gamma)}.$$

7.3. Type G_2 . Let us suppose that $C_{i,j} = -1$, $C_{j,i} = -3$, $d_i = 3$ and $d_j = 1$. We have

$$(7.80) \quad E_e(\Sigma_{ji,a}) = \sum_{0 \leq 3\epsilon \leq \delta} V_{j,aq}^{(\delta)} V_{i,a}^{(\epsilon)},$$

$$(7.81) \quad E_e(\Sigma_{ij,a}) = \sum_{0 \leq \epsilon \leq \text{Min}(3\delta_1, 3\delta_2+1, 3\delta_3+2)} V_{i,aq^{-3}}^{(\delta_1)} V_{i,aq^{-5}}^{(\delta_2)} V_{i,aq^{-7}}^{(\delta_3)} V_{j,a}^{(\epsilon)}.$$

By a similar argument, we obtain the following expansions (with appropriate conditions on the domains of summation):

$$E_e(\Sigma_{iji,a}) = \sum V_{i,aq^{-2}}^{(\gamma)} V_{i,aq^{-4}}^{(\gamma')} V_{j,aq}^{(\delta)} V_{i,a}^{(\epsilon)},$$

$$\begin{aligned}
E_e(\Sigma_{jijj,a}) &= \sum V_{j,aq^{-3}}^{(\beta)} V_{i,aq^{-2}}^{(\gamma)} V_{i,aq^{-4}}^{(\gamma')} V_{j,aq}^{(\delta)} V_{i,a}^{(\epsilon)}, \\
E_e(\Sigma_{ijjji,a}) &= \sum V_{i,aq^{-6}}^{(\alpha)} V_{j,aq^{-3}}^{(\beta)} V_{i,aq^{-2}}^{(\gamma)} V_{i,aq^{-4}}^{(\gamma')} V_{j,aq}^{(\delta)} V_{i,a}^{(\epsilon)}, \\
E_e(\Sigma_{jijj,a}) &= \sum V_{j,aq^{-4}}^{(\gamma_1)} V_{j,aq^{-6}}^{(\gamma_2)} V_{i,aq^{-3}}^{(\delta_1)} V_{i,aq^{-5}}^{(\delta_2)} V_{i,aq^{-7}}^{(\delta_3)} V_{j,a}^{(\epsilon)}, \\
E_e(\Sigma_{ijjj,a}) &= \sum V_{i,aq^{-7}}^{(\beta_1)} V_{i,aq^{-9}}^{(\beta_2)} V_{i,aq^{-11}}^{(\beta_3)} V_{j,aq^{-4}}^{(\gamma_1)} V_{j,aq^{-6}}^{(\gamma_2)} V_{i,aq^{-3}}^{(\delta_1)} V_{i,aq^{-5}}^{(\delta_2)} V_{i,aq^{-7}}^{(\delta_3)} V_{j,a}^{(\epsilon)}, \\
E_e(\Sigma_{jjij,a}) &= \sum V_{j,aq^{-10}}^{(\alpha)} V_{i,aq^{-7}}^{(\beta_1)} V_{i,aq^{-9}}^{(\beta_2)} V_{i,aq^{-11}}^{(\beta_3)} V_{j,aq^{-4}}^{(\gamma_1)} V_{j,aq^{-6}}^{(\gamma_2)} V_{i,aq^{-3}}^{(\delta_1)} V_{i,aq^{-5}}^{(\delta_2)} V_{i,aq^{-7}}^{(\delta_3)} V_{j,a}^{(\epsilon)}.
\end{aligned}$$

Using the above expansions, one can give a purely combinatorial proof of the braid relations for B_2 and G_2 . However, since we have already given a uniform proof of these relations in Section 4.6, we will omit the details.

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