On the Brunn-Minkovski inequality in sub-Riemannian geometry

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Riemannian geometry and Generalized Functions
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Joint work with

This is based on joint works with

- Luca Rizzi (Institut Fourier, Univ. Grenoble-Alpes)

→ Main references:

**BR-17** DB, L. Rizzi,
*Sub-Riemannian interpolation inequalities*,
→ Preprint Arxiv, 2017

**BR-18** DB, L. Rizzi,
*Sub-Riemannian Bakry-Emery curvature: comparison and model spaces*,
→ Soon on Arxiv!
Outline

1. Introduction
2. The sub-Riemannian case
3. Few ideas from the proof
4. What are model spaces?

Davide Barilari (IMJ-PRG, Paris Diderot)
SR Brunn-Minkovski inequality
October 4-5, 2018
Euclidean Brunn-Minkowski

\( A, B \subset \mathbb{R}^n \) non-empty measurable bounded sets

**Minkowski sum:** \( A + B = \{ z \mid z = a + b, \ a \in A, \ b \in B \} \)

**Brunn-Minkowski Inequality:**

\[
\text{vol}(A + B)^{1/n} \geq \text{vol}(A)^{1/n} + \text{vol}(B)^{1/n}
\]

Here \( \text{vol} \) is the Lebesgue measure in \( \mathbb{R}^n \)
Euclidean Brunn-Minkowski

$A, B \subset \mathbb{R}^n$ non-empty measurable bounded sets

**Minkowski interpolation:** $(1 - t)A + tB = \{z \mid z = (1 - t)a + tb, \ a \in A, \ b \in B\}$

![Diagram showing Minkowski interpolation](image)

**Brunn-Minkowski Inequality:**

$$\text{vol}((1 - t)A + tB)^{1/n} \geq (1 - t)\text{vol}(A)^{1/n} + t\text{vol}(B)^{1/n} \quad \forall t \in [0, 1]$$

Here $\text{vol}$ is the Lebesgue measure in $\mathbb{R}^n$
Geometric inequalities have often a functional counterpart

**Theorem** (*+$\infty$*-mean Borell-Brascamp-Lieb inequality)

*Fix* $t \in [0, 1]$. Let $f, g, h : \mathbb{R}^n \to \mathbb{R}$ *be non-negative and integrable.* Assume that for every $x, y \in \mathbb{R}^n$

$$h((1 - t)x + ty) \geq \max \{f(x), g(y)\}. \quad (1)$$

Then,

$$\|h\|_{L^1}^{1/n} \geq (1 - t)\|f\|_{L^1}^{1/n} + t\|g\|_{L^1}^{1/n}, \quad (2)$$

- one could restrict to $(x, y) \in A \times B$
- $A, B \subset \mathbb{R}^n$ Borel subsets such that $\int_A f \, dm = \|f\|_{L^1}$ and $\int_B g \, dm = \|g\|_{L^1}$.

→ generalized to other $p$-mean inequalities
(from Prékopa-Leindler to Borell-Brascamp-Lieb)
Generalization to Riemannian: a necessary condition

Denote \( Z_t(A, B) := (1 - t)A + tB \) the \( t \)-interpolating set

\textbf{Brunn-Minkowski Inequality:}

\[
\text{vol}(Z_t(A, B))^{1/n} \geq (1 - t)\text{vol}(A)^{1/n} + t\text{vol}(B)^{1/n} \quad \forall t \in [0, 1]
\]

- notice for \( A = \{x\} \) and \( B = B_r(y) \) a ball.

\[
\text{vol}(Z_t(x, B_r(y))) \geq t^n \text{vol}(B_r(y)) \quad \forall t \in [0, 1]
\]

- in general this implies a control on the ratio

\[
\frac{\text{vol}(Z_t(x, B_r(y)))}{\text{vol}(B_r(y))} \geq t^n
\]

\( \rightarrow \) measure contraction along geodesics, curvature
Distortion coefficient

(M, g) Riemannian manifold, vol Riemannian volume measure

\[ \beta_t(x, y) := \limsup_{r \to 0} \frac{\text{vol}(Z_t(x, B_r(y)))}{\text{vol}(B_r(y))}, \quad \forall x, y \in M, \; t \in [0, 1] \]

- \( \beta_1(x, y) = 1 \) and \( \beta_0(x, y) = 0 \). Important: \( \beta_t(x, y) \sim t^n \) for \( t \to 0 \).
- \( \beta_t(x, y) \) depends on the geodesics joining \( x \) with \( y \)
- Computable in terms of Jacobi fields.
Riemannian Brunn-Minkowski

\((M, g)\) complete Riem. manifold, \(A, B\) non-empty Borel sets

\[
Z_t(A, B) := \{\gamma(t) \mid \gamma: [0, 1] \to M \text{ geodesic s.t. } \gamma(0) \in A, \gamma(1) \in B\}
\]

Theorem (Cordero-Erausquin, McCann, Schmuckenschläger - 2001)

Assume \((M, g)\) complete Riem. manifold with \(\text{Ric} \geq 0\). Then

\[
\text{vol}(Z_t(A, B))^{1/n} \geq (1 - t)\text{vol}(A)^{1/n} + t\text{vol}(B)^{1/n}
\]

- If \(\text{Ric} \geq K\) the inequality holds with modified coefficients
- It can be used to define Ricci bounds for m.m.s. (Sturm, Lott-Villani, ...)
A limiting procedure: the Heisenberg group

Define on $\mathbb{R}^3$

\[ X_1 = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z}, \quad X_3^\varepsilon = \varepsilon \frac{\partial}{\partial z} \]

- $(\mathbb{R}^3, g^\varepsilon)$ Riemannian structure for $\varepsilon > 0$ with $\{X_1, X_2, X_3^\varepsilon\}$ o.n. frame.

The sequence of curvatures is unbounded from below:

- $D^\varepsilon = \text{span}\{X_1, X_2, X_3^\varepsilon\} \to D = \text{span}\{X_1, X_2\}$
- $\text{Sec}^\varepsilon(v_1, v_2) \to -\infty$ for all $v_1, v_2 \in D$
- $\text{Ric}^\varepsilon(v) \to -\infty$ for all $v \in D$

As metric spaces $(\mathbb{R}^3, d^\varepsilon) \to (\mathbb{R}^3, d_{SR})$ (in the Gromov-Hausdorff sense)

- Cannot prove directly BM by taking limits of Ricci bounded structures
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Sub-Riemannian geometry

Sub-Riemannian structure

- $M$ smooth, connected manifold
- $D \subseteq TM$ distribution of constant rank $k \leq n$
  - Hörmander condition: $\text{Lie}(D)|_x = T_x M$ for all $x \in M$
- $g$ smooth scalar product on $D$

Admissible curve: $\gamma : [0, 1] \to M$ such that $\dot{\gamma}(t) \in D_{\gamma(t)}$

$$\ell(\gamma) = \int_0^1 \|\dot{\gamma}(t)\| dt$$

Sub-Riemannian distance: (or Carnot-Carathédory)

$$d_{SR}(x, y) = \inf \{ \ell(\gamma) \mid \gamma \text{ admissible joining } x \text{ with } y \}$$

Chow-Rashevskii: $d_{SR} < +\infty$ and $(M, d_{SR})$ has the same topology of $M$
Brunn-Minkowski on the Heisenberg group

The standard Brunn-Minkowski inequality $BM(0, N)$:

$$\text{vol}(Z_t(A, B))^{\frac{1}{N}} \geq (1 - t)\text{vol}(A)^{\frac{1}{N}} + t\text{vol}(B)^{\frac{1}{N}},$$

Theorem (Juillet - 2009)

The Heisenberg group $\mathbb{H}_3$, equipped with Lebesgue measure:
- satisfy the MCP$(0, N)$ for $N \geq 5$
- does not satisfy any $BM(0, N)$

$\Rightarrow$ Geodesic dimension (Agrachev, DB, Rizzi - 2013)

Theorem (Balogh, Kristály, Sipos - 2016)

The Heisenberg group $\mathbb{H}_3$, equipped with Lebesgue measure, satisfy

$$\text{vol}(Z_t(A, B))^{\frac{1}{3}} \geq (1 - t)^{\frac{5}{3}}\text{vol}(A)^{\frac{1}{3}} + t^{\frac{5}{3}}\text{vol}(B)^{\frac{1}{3}}, \quad \forall t \in [0, 1]$$
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$$\text{vol}(Z_t(A, B)) \frac{1}{N} \geq (1 - t)\text{vol}(A) \frac{1}{N} + t\text{vol}(B) \frac{1}{N},$$

**Theorem (Juillet - 2009)**

*The Heisenberg group $\mathbb{H}_3$, equipped with Lebesgue measure:*

- *satisfy the MCP(0, N) for N ≥ 5: roughly $\text{vol}(Z_t(X, B)) \geq t^5 \text{vol}(B)$*
- *does not satisfy any BM(0, N)*

$\Rightarrow$ **Geodesic dimension** (Agrachev, DB, Rizzi - 2013)

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*The Heisenberg group $\mathbb{H}_3$, equipped with Lebesgue measure, satisfy*

$$\text{vol}(Z_t(A, B)) \frac{1}{3} \geq (1 - t)^\frac{5}{3} \text{vol}(A) \frac{1}{3} + t^\frac{5}{3} \text{vol}(B) \frac{1}{3}, \quad \forall t \in [0, 1]$$
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The standard Brunn-Minkowski inequality $BM(0, N)$:

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The Heisenberg group $\mathbb{H}_3$, equipped with Lebesgue measure:

- **satisfy the** MCP$(0, N)$ **for** $N \geq 5$: roughly $\text{vol}(Z_t(x, B)) \geq t^5\text{vol}(B)$
- **does not satisfy any** $BM(0, N)$

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Towards SR interpolation inequalities

For the Heisenberg group (and higher dimensional versions):
- Juillet $\Rightarrow$ standard BM is not the right one
- Balogh-Kristály-Sipos $\Rightarrow$ some modified BM holds

Do general sub-Riemannian structures support interpolation inequalities? (with weights that may depend on geometry)

Main results
- interpolation inequalities for ideal sub-Riemannian structures
- new examples of sharp BM (Grushin plane, some Carnot groups)
- regularity results for the sub-Riemannian distance
Main assumption: ideal structures

Definition (Ideal structure)

A sub-Riemannian structure is ideal if \((M, d_{SR})\) is complete and it admits **no singular minimizing geodesics**

- **singular minimizer**: cf talk by Ludovic Rifford
- True for the generic sub-Riemannian structure with \(\text{rank } D \geq 3\)
  - \([\text{Chitour, Jean, Trélat - 2006}]\)
- True for all contact structures

In this case, geodesics are described by a Hamiltonian flow on \(T^*M\)

- \(H\) is quadratic on fibers but **degenerate**

\[
H(p, x) = \frac{1}{2} \sum_{i,j=1}^{n} g^{ij}(x)p_ip_j, \quad g^{ij}(x) \text{ is degenerate}
\]

- not immediate replace Levi-Civita connection / tensor curvature (in general)
The Heisenberg sphere

Even without singular minimizers things are not trivial

Sub-Riemannian spheres are not smooth, even for small radii
Asymptotics of distortion coefficients

Sub-Riemannian distortion coefficient

\[ \beta_t(x, y) := \limsup_{r \to 0} \frac{m(Z_t(x, B_r(y)))}{m(B_r(y))}, \quad \forall x, y \in M, \ t \in [0, 1] \]

- **Riemannian case:** \( \beta_t(x, y) \sim t^n \)

Theorem (Agrachev, DB, Rizzi - 2013)

Fix \( x \in M \). Then for a.e. \( y \in M \) one has

\[ \beta_t(x, y) \sim t^{\mathcal{N}(x)}, \]

for some \( \mathcal{N}(x) \in \mathbb{N} \) such that \( \mathcal{N}(x) > n \).

- \( \mathcal{N}(x) \) is the **geodesic dimension at** \( x \)
  - definable in terms of directional Lie brackets
  - it is bigger also than the Hausorff dimension
Sub-Riemannian BM with weights

Given $A, B \subset M$ Borel and $t \in [0, 1]$

$$\beta_t(A, B) := \inf \{ \beta_t(x, y) \mid (x, y) \in A \times B \}$$

**Theorem (Barilari, R. - 2017)**

Let $(M, D, g)$ be an ideal $n$-dim sub-Riemannian manifold, $m$ smooth measure. For all $A, B \subset M$ Borel and $t \in [0, 1]$

$$m(Z_t(A, B))^{1/n} \geq \beta_{1-t}(B, A)^{1/n} m(A)^{1/n} + \beta_t(A, B)^{1/n} m(B)^{1/n}$$

- Particular case of more general sub-Riemannian interpolation inequalities
- Functional inequalities à la Borell-Brascamp-Liebb
- $\beta_t(x, y)$ explicitly computable in terms of Hamiltonian flow
Sub-Riemannian BM with weights

Given $A, B \subset M$ Borel and $t \in [0, 1]$

$$\beta_t(A, B) := \inf \{ \beta_t(x, y) \mid (x, y) \in A \times B \}$$

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- Particular case of more general sub-Riemannian interpolation inequalities
- Difficulties: absence of standard Jacobi fields, degenerate Hamiltonian
- $\beta_t(x, y)$ **explicitly computable** in terms of Hamiltonian flow
  - Notice that IF $\beta_t(x, y) \geq t^n$ then linear weights in $t$, but ...
Equivalence of inequalities

\[ m(Z_t(A, B))^{1/n} \geq \beta_{1-t}(B, A)^{1/n} m(A)^{1/n} + \beta_t(A, B)^{1/n} m(B)^{1/n} \]

- Interesting case: \( \beta_t(x, y) \geq t^N \) for some \( N \) (\( \rightarrow \) hence \( N \geq N(x) \))

Corollary

Let \((M, D, g)\) be an ideal \( n \)-dim sub-Riemannian manifold, \( m \) smooth measure. Let \( N > 0 \). The following are equivalent:

(i) bound on the distortion coefficient:
\[ \beta_t(x, y) \geq t^N \]

(ii) the modified Brunn-Minkowski inequality:
\[ m(Z_t(A, B))^{1/n} \geq (1 - t)^{N/n} m(A)^{1/n} + t^{N/n} m(B)^{1/n} \]

(iii) the measure contraction property \( \text{MCP}(0, N) \):
\[ m(Z_t(x, B)) \geq t^N m(B) \]
Application to some Carnot groups

Theorem (Rifford - 2014, Rifford, Badreddine - 2018)

There exists $N > 0$ such that

$$\beta_t(x, y) \geq t^N \quad \forall t \in [0, 1]$$

(a) for every compact 2-step sub-Riemannian manifold ($\rightarrow D + [D, D] = TM$)

(b) a class of 3-step Carnot group ($\rightarrow D + [D, D] + [X, [D, D]] = TM$)

Conjecture: for Carnot groups best exponent = geodesic dimension?

Theorem (Barilari, R. - 2017)

For any generalized H-type Carnot group of dimension $n$ and rank $k$, equipped with the Lebesgue measure, for all Borel subsets $A, B$ we have the sharp inequality

$$\frac{1}{n} \text{vol}(Z_t(A, B)) \geq (1 - t) \frac{k+3(n-k)}{n} \text{vol}(A) + t \frac{k+3(n-k)}{n} \text{vol}(B),$$

$\rightarrow$ not necessarily ideal (tensorization: Ritoré-Nicolàs - 2017)
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For any generalized H-type Carnot group of dimension $n$ and rank $k$, equipped with the Lebesgue measure, for all Borel subsets $A, B$ we have the sharp inequality

$$\text{vol}(Z_t(A, B))^{\frac{1}{n}} \geq (1 - t)^{\frac{k+3(n-k)}{n}} \text{vol}(A)^{\frac{1}{n}} + t^{\frac{k+3(n-k)}{n}} \text{vol}(B)^{\frac{1}{n}}$$

\hfill \rightarrow \text{not necessarily ideal (tensorization: Ritoré-Nicolàs - 2017)}
Application to some Carnot groups

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**Theorem (Rifford - 2014, Rifford, Badreddine - 2018)**

There exists $N > 0$ such that

$$\beta_t(x, y) \geq t^N \quad \forall t \in [0, 1]$$

a) for every compact 2-step sub-Riemannian manifold $D$ such that $\mathcal{D} + [D, D] = TM$

b) a class of 3-step Carnot group $\mathcal{D}$ such that $\mathcal{D} + [D, D] + [X, [D, D]] = TM$

**Conjecture:** for Carnot groups best exponent $= \text{geodesic dimension}$?

**Theorem (Barilari, R. - 2017)**

For any generalized H-type Carnot group of dimension $n$ and rank $k$, equipped with the Lebesgue measure, for all Borel subsets $A, B$ we have the **sharp** inequality

$$\text{vol}(Z_t(A, B)) \frac{1}{n} \geq (1 - t) \frac{k+3(n-k)}{n} \text{vol}(A) \frac{1}{n} + t \frac{k+3(n-k)}{n} \text{vol}(B) \frac{1}{n},$$

→ not necessarily ideal (tensorization: Ritoré-Nicolàs - 2017)
Application to the Grushin plane

Rank-varying structure on $M = \mathbb{R}^2$, equipped with Lebesgue measure

$$X_1 = \partial_x, \quad X_2 = x\partial_y$$

Well defined geodesic m.m.s. (almost Riemannian, with $\text{Curv} \to -\infty$).

**Theorem (Barilari, R. - 2017)**

The distortion coefficient of Grushin satisfies the following sharp inequality

$$\beta_t(x, y) \geq t^5, \quad \forall t \in [0, 1]$$

which is equivalent to the Brunn-Minkowski inequality:

$$\text{vol}(Z_t(A, B))^{1/2} \geq (1 - t)^{5/2}\text{vol}(A)^{1/2} + t^{5/2}\text{vol}(B)^{1/2}$$

- **Gap** between the geodesic dimension and the best $N$

$$N(x) = \begin{cases} 2 & \text{in the Riemannian region} \\ 4 & \text{otherwise} \end{cases}$$
Regularity of distance

$(M, D, g)$ complete (sub-)Riemannian structure. Fix $x \in M$.

Theorem (Agrachev - 2009, Rifford-Trélat - 2006)

The set of points where $d^2_{SR}(x, \cdot)$ is smooth is open and dense in $M$.

The cut locus $\text{cut}(x)$ is the complement of the set of smooth points.
Regularity of distance

The proof of the main inequality implies the following characterization

**Theorem (Barilari, R. - 2017)**

Let \((M, D, g)\) be an ideal sub-Riemannian manifold. Let \(y \neq x\). Then \(y \in \text{cut}(x)\) if and only if \(f = d_{SR}^2(x, \cdot)\) fails to be semiconvex at \(y\):

\[
\inf_{0 < |v| < 1} \frac{f(y + v) + f(y - v) - 2f(y)}{|v|^2} = -\infty
\]

→ “one cannot put a parabola below the graph of the distance”

- Extends an analogue result in the Riemannian case [CEMS, 2001]
- Differentiability of transport map [FR, 2008]
- Sharp → there are non-ideal structures where \(d_{SR}^2(x, \cdot)\) is locally semiconvex at the cut locus (it fails to be semiconcave)
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**Theorem (Barilari, R. - 2017)**

Let $(M, D, g)$ be an ideal sub-Riemannian manifold. Let $y \neq x$. Then $y \in \text{cut}(x)$ if and only if $f = d^2_{SR}(x, \cdot)$ fails to be semiconvex at $y$:

$$\inf_{0 < |v| < 1} \frac{f(y + v) + f(y - v) - 2f(y)}{|v|^2} = -\infty$$

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- Extends an analogue result in the Riemannian case [CEMS,2001]
- Differentiability of transport map [FR,2008]
- Sharp → there are non-ideal structures where $d^2_{SR}(x, \cdot)$ is locally semiconvex at the cut locus (it fails to be semiconcave) → role of abnormal minimizers
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**Idea of the proof**

**Step 0.** Optimal transport problem: \( \mu_0, \mu_1 \in \mathcal{P}_c(M) \)

\[
\inf_{T \# \mu_0 = \mu_1} \frac{1}{2} \int_M d_{SR}^2(x, T(x)) dm(x)
\]

- Maps \( T : M \to M \) that realizes the inf are **optimal transport maps**
- Points are transported along geodesics \( x \mapsto T(x) \)
- To prove BM \( \Rightarrow \) choose \( (\mu_0, \mu_1) = (\chi_A, \chi_B) \). The interpolating measure \( \mu_t \) gives a lower bound for the measure of \( Z_t(A, B) \)
Idea of the proof


**Theorem (Figalli, Rifford - 2010)**

Let $\mu_0 \in \mathcal{P}_c^{ac}(M)$, $\mu_1 \in \mathcal{P}_c(M)$. Assume $\text{supp}(\mu_0) \cap \text{supp}(\mu_1) = \emptyset$.

- There exists a unique optimal transport map $T : M \rightarrow M$ such that $T_\# \mu_0 = \mu_1$, given by
  
  $$T(x) = \exp_x(d_x \psi),$$

  where $\psi : M \rightarrow \mathbb{R}$ is locally semiconvex.

- For $\mu_0$-a.e. $x \in M$ there exists a unique geodesic joining $x$ with $T(x)$:
  
  $$T_t(x) = \exp_x(td_x \psi), \quad \forall t \in [0, 1].$$

- **Ideal** $\Rightarrow$ semiconvexity of $\psi$
Idea of the proof

**Step 2.** Geodesics interpolation between $\mu_0$ and $\mu_1$ at time $t \in [0, 1]$:

$$\mu_t := (T_t)_#\mu_0,$$

with

$$T_t(x) = \exp_x(td_x\psi)$$

$\psi$ Alexandrov second differentiability theorem $\Rightarrow T_t(x)$ is $m$-a.e. differentiable

if $|\det(d_xT_t)| > 0$ m-a.e.

$$\mu_t = \rho_t m, \quad \rho_t(T_t(x)) = \frac{\rho_0(x)}{|\det(d_xT_t)|}$$

**Step 3.** The differential $d_xT_t : T_xM \to T_{T_t(x)}M$

$$d_xT_t = \pi_* \circ e^{t\vec{H}} \circ d_x^2\psi$$

- use the natural symplectic structure on $T^*M$ and Darboux moving frames
- avoid the classical machinery of connection and parallel transport

**Step 4.** Jacobian inequality: i.e., interpolation inequality for $\det d_xT_t$
Idea of the proof

Step 2. Geodesics interpolation between $\mu_0$ and $\mu_1$ at time $t \in [0, 1]$:  

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\mu_t := (T_t)_\# \mu_0, \quad \text{with} \quad T_t(x) = \exp_x (td_x \psi)
$$

$\psi$ Alexandrov second differentiability theorem $\Rightarrow T_t(x)$ is m-a.e. differentiable

if $|\det(d_x T_t)| > 0 \text{ m-a.e.}$  

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\mu_t = \rho_t m, \quad \rho_t(T_t(x)) = \frac{\rho_0(x)}{|\det(d_x T_t)|}
$$

Step 3. The differential $d_x T_t : T_x M \to T_{T_t(x)} M$

$$
d_x T_t = \pi_* \circ e_{T_t}^* \circ d_x \psi
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- use the natural symplectic structure on $T^* M$ and Darboux moving frames
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- avoid the classical machinery of connection and parallel transport

**Step 4.** Jacobian inequality: i.e., interpolation inequality for $\det d_xT_t$
Concentration inequality

The proof implies:

- \( T(x) \notin \text{cut}(x) \) for \( \mu_0 \)-a.e. \( x \in M \)
- \( \det(d_x T_t) > 0 \) for all \( t \in [0, 1) \) and \( \mu_t = \rho_t m \)
- The Jacobian inequality holds on the whole \( [0, 1] \)

Theorem (Barilari, R. - 2017)

Let \((D, g)\) be an ideal sub-Riemannian structure on \( M \), and \( \mu_0, \mu_1 \in \mathcal{P}_c^{ac}(M) \). Let \( \rho_t = d\mu_t / dm \). For all \( t \in [0, 1] \), it holds

\[
\frac{1}{\rho_t(T_t(x)))^{1/n}} \geq \frac{\beta_{1-t}(T(x), x)^{1/n}}{\rho_0(x)^{1/n}} + \frac{\beta_t(x, T(x))^{1/n}}{\rho_1(T(x))^{1/n}}, \quad \mu_0 - \text{a.e. } x \in M.
\]

If \( \mu_1 \) is not absolutely continuous, an analogous result holds, provided that \( t \in [0, 1) \), and that the second term on the right hand side is omitted.

- Borell-Brascamp-Lieb, \( p \)-mean inequality, Brunn-Minkowski follow
Outline

1. Introduction
2. The sub-Riemannian case
3. Few ideas from the proof
4. What are model spaces?
Comparison: the Riemannian case

$$m(Z_t(A, B))^{1/n} \geq \beta_{1-t}(B, A)^{1/n} m(A)^{1/n} + \beta_t(A, B)^{1/n} m(B)^{1/n}$$

- Distortion coefficients are in general difficult to compute,
- a bound on the geometry gives a bound in terms of model spaces.

Theorem

Let \((M, g)\) be a \(n\)-dimensional Riemannian, with \(m = \text{vol}_g\) Riemannian volume. Assume that \(\text{Ric}_g \geq K\). Then for \((x, y) \notin \text{cut}(M)\) we have

$$\beta_t(x, y) \geq \beta_{t(K,n)}(x, y),$$

- \(\beta_{t(K,n)}(x, y)\) distortion coefficient of model: constant sectional curvature \(K\) and dimension \(n\).
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- \(\beta_t^{(K,n)}(x, y)\) distortion coefficient of model: constant sectional curvature \(K\) and dimension \(n\).
Assume the Riemannian manifold \((M, g)\) endowed with arbitrary smooth measure \(m = e^{-V} \text{vol}_g\)

- Bakry-Emery Ricci tensor

\[
\text{Ric}^m_N := \text{Ric}_g + \nabla^2 V - \frac{\nabla V \otimes \nabla V}{N - n},
\]

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- This inequality is weaker than the one is possible to obtain (\(\rightarrow\) the one defining \(\text{CD}(K, N)\) spaces)
- i.e., the latter cannot be obtained plugging this inequality into the main one.
- this can be generalized to sub-Riemannian (could not expect \(\text{CD}(K, N)\))
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Explicit formula for the coefficient appearing in the right-hand side of (3)

$$\beta_t^{(K,n)}(x, y) = \begin{cases} 
  t \left( \frac{\sin(t\alpha)}{\sin(\alpha)} \right)^{n-1} & \text{if } K > 0, \\
  t^n & \text{if } K = 0, \\
  t \left( \frac{\sinh(t\alpha)}{\sinh(\alpha)} \right)^{n-1} & \text{if } K < 0,
\end{cases} \quad \alpha = \sqrt{\frac{|K|}{n-1}} d(x, y). \quad (6)$$

- only depends on $d(x, y)$
- the $(n - 1) \to$ no curvature in direction of the geodesic
- Jacobi equation in parallel transported frame

$$\ddot{J}_i + R_{ij}(t)J_j = 0$$

where $R_{ij}(t) = R(X_i, \dot{\gamma}, \dot{\gamma}, X_j)$.

- constant curvature $R = \text{diag}(K, K, \ldots, K, 0)$
The problem of models

When do we have $\beta_t(x, y) \geq \beta_t^{\text{model}}$?

- In the above example models are given by Riemannian space forms
- No reason to be good models also for the sub-Riemannian case
- do not depend only on $d(x, y)$ but on the whole trajectory

Problems in the SR case: What are models? What is curvature?

We propose an approach from the viewpoint of control theory:
- Curvature: invariant extracted from derivatives of the sub-Riemannian distance
- Models: simple optimal control problems
Linear Quadratic problems as models

Variational problems in $\mathbb{R}^n$

\[ \dot{x} = Ax + Bu \]

with minimization of a quadratic cost

\[ \frac{1}{2} \int_0^1 (u^*u - x^*Qx) dt \rightarrow \min \]

Bracket generating: $\exists m \geq 0$ such that $\text{rank}(B, AB, \ldots, A^m B) = n$

Optimal trajectories solve a Hamiltonian system:

\[ H(p, x) = \frac{1}{2} (p^*BB^*p + 2p^*Ax + x^*Qx) \]

For all LQ problems that we use, minimizers exist and are unique.
LQ distortion

Definition (LQ distortion coefficients)

\[ \beta_{t}^{A,B,Q}(x,y) := \limsup_{r \to 0} \frac{|Z_{t}(x, B_{r}(y))|}{|B_{r}(y)|}, \quad x, y \in \mathbb{R}^{n} \]

- It does not depend on \( x, y \) (the Hamiltonian flow is linear)
- Very simple to compute

Example: Harmonic oscillator

No drift (\( A = 0 \)), no constraint on velocity (\( B = 1 \)), isotropic potential (\( Q = \kappa \mathbb{1} \)):

\[ H(p, x) = \frac{1}{2}(|p|^{2} + \kappa |x|^{2}) \]

\[ \Rightarrow \beta_{t}^{A,B,Q} = \text{Riemannian distortion coefficients!} \]
What is curvature? (sketchy)

Fact/definition:

To any SR geodesic $\gamma$ (+ technical assumptions) we associate

- two constant matrices $A$ and $B \to$ structure of Lie derivatives along geodesics
- a curvature operator, quadratic form $\mathcal{R}_\gamma(t) : T_{\gamma(t)}M \to \mathbb{R}$ for $t \in [0, T]$.

In the Riemannian case: $A = 0$, $B = I$, $\mathcal{R}_\gamma(t)(X) = R^\nabla(\dot{\gamma}_t, X, X, \dot{\gamma}_t)$

Given the operator $\mathcal{R}_\gamma(t)$ and a smooth measure $m$ one can define a Bakry-Emery sub-Riemannian tensor

$$\mathcal{R}^m,N_{\gamma(t)} = \mathcal{R}_\gamma(t) - \frac{\dot{\rho}(t)}{k} \Pi_{\gamma(t)} - \frac{n}{N-n} \frac{\rho^2(t)}{k^2} \Pi_{\gamma(t)}.$$  \hspace{1cm} (7)

Recall that $n = \dim M$ and $k = \dim D$.

- in Riemannian $\rho(t) = -g(\nabla V, \dot{\gamma}(t))$ for $m = e^{-V} \text{vol}_g$. 

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Final comparison

In terms of the Bakry-Emery SR curvature $\mathcal{R}_{\gamma(t)}^{m,N}$ we have the following comparison

**Theorem**

*Let $(x, y) \notin \text{cut}(M)$ and assume that the unique length-minimizer joining $x$ and $y$ is associated with matrices $A, B$.  

(a) If there exists $N > n$ and $Q$ such that $\frac{1}{N} \mathcal{R}_{\gamma(t)}^{m,N} \geq \frac{1}{n} Q$ for every $t \in [0, T]$, then

$$\beta_t(x, y)^{\frac{1}{N}} \geq (\beta_t^{A,B,Q})^{\frac{1}{n}}$$

(8)

Assume now that $\rho = 0$.  

(b) If there exists $Q$ such that $\mathcal{R}_{\gamma(t)} \geq Q$ for every $t \in [0, T]$, then

$$\beta_t(x, y) \geq \beta_t^{A,B,Q}$$

(9)*
THANKS FOR YOUR ATTENTION!