Comparison theorems for conjugate points in sub-Riemannian geometry

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→ Main Reference:


→ Other references:


Outline

1. Introduction and motivation
2. Geodesic growth vector and LQ models
3. Jacobi fields revisited and Directional curvature
4. Main results and few examples
5. Applications to 3D unimodular Lie groups
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What do we mean by comparison theorem?

Let $M$ be a Riemannian manifold:

Comparison between a property on $M$ w.r.t. some model space:
- local property = sectional curvature, Ricci curvature
- model spaces = space forms ($\mathbb{R}^n$, $S^n$, $H^n$)

Many examples of these results:
- Bonnet-Myers theorem $\rightarrow$ diameter
- Bishop-Gromov inequality $\rightarrow$ volumes
- Spectral Gap inequality $\rightarrow$ first eigenvalue of Laplacian
- and also many geometric inequalities (Poincaré, Li-Yau, Sobolev, etc.)

In this talk we will focus on comparison on conjugate points.
Examples of comparison theorems

- $M$ a Riemannian manifold.
- $\text{Sec}(v, w) =$ sectional curvature of the plane $v \wedge w = R(v, w, v, w)$.
- $\text{Ric}(v) =$ trace $\text{Sec}(v, \cdot)$.

**Theorem (Riemannian comparison for conjugate points)**

Let $\gamma$ be a unit speed geodesic:

**(L)** If for all $t$ and unit $v \perp \dot{\gamma}(t)$

$$\text{Sec}(\dot{\gamma}(t), v) \geq \kappa > 0$$

then $\gamma(t)$ has a conjugate point at time $t_c(\gamma) \leq \pi / \sqrt{\kappa}$.

**(U)** If for all $t$ and unit $v \perp \dot{\gamma}(t)$

$$\text{Sec}(\dot{\gamma}(t), v) \leq 0$$

then $\gamma(t)$ has no conjugate points, i.e. $t_c(\gamma) = +\infty$. 
Examples of comparison theorems

- $M$ a Riemannian manifold.
- $\text{Sec}(v, w) =$ sectional curvature of the plane $v \wedge w = R(v, w, v, w)$.
- $\text{Ric}(v) =$ trace $\text{Sec}(v, \cdot)$.

**Theorem (Riemannian comparison for conjugate points)**

*Let $\gamma$ be a unit speed geodesic:*

**AL** If for all $t$

$$\text{Ric}(\dot{\gamma}(t)) \geq \kappa > 0$$

then $\gamma(t)$ has a finite first conjugate time $t_c(\gamma) \leq \pi / \sqrt{\kappa}$.

**U** If for all $t$ and unit $v \perp \dot{\gamma}(t)$

$$\text{Sec}(\dot{\gamma}(t), v) \leq 0$$

then $\gamma(t)$ has no conjugate points, i.e. $t_c(\gamma) = +\infty$.

→ Proof: uses theory of Jacobi fields.
Some ideas

The first conjugate time $t_c(\gamma)$ is the infimum of $T > 0$ such that there exists a Jacobi field

$$J(t) = \left. \frac{\partial}{\partial s} \gamma_s(t) \right|_{s=0}$$

such that $J(0) = J(T) = 0$.

- Jacobi equation for Jacobi fields

$$\ddot{J}_i(t) + R_{ik}(t)J_k(t) = 0$$

where $J_1(t), \ldots, J_n(t)$ are $n$ independent Jacobi fields along the geodesics and

$$R_{ij}(t) = \text{Riem}(\dot{\gamma}(t), f_i(t), \dot{\gamma}(t), f_j(t))$$

where $f_1(t), \ldots, f_n(t)$ is parallelly transported frame along $\gamma$.

- When $M$ has constant curvature $R(t) = \kappa \mathbb{I}$ and one gets the solutions $x(t)$ of the equation

$$\ddot{x} + \kappa x = 0$$
Some ideas

The first conjugate time $t_c(\gamma)$ is the infimum of $T > 0$ such that there exists a Jacobi field

$$J(t) = \left. \frac{\partial}{\partial s} \right|_{s=0} \gamma_s(t)$$

such that $J(0) = J(T) = 0$.

- **Jacobi equation for Jacobi fields**

  $$\ddot{J}(t) + R(t)J(t) = 0$$

  where $J(t) = (J_1(t), \ldots, J_n(t))$ are $n$ independent Jacobi fields along the geodesics and

  $$R(t) = \text{Riem}(\dot{\gamma}(t), \cdot, \dot{\gamma}(t), \cdot)$$

  is the **directional curvature** written in a parallely transported frame.

- **When $M$ has constant curvature $R(t) = \kappa \mathbb{I}$** and one gets the solutions $x(t)$ of the equation

  $$\ddot{x} + \kappa x = 0$$
Figure: Conjugate points: where we lose local optimality
Motivation

We want to expand these ideas to sub-Riemannian geometry.

→ **Difficulties**
  - No canonical connection and/or parallel transport
  - Definition of sub-Riemannian curvature (sectional, Ricci)
  - What are model spaces?

→ **Main ideas:**
  - Sub-Riemannian problem is an affine optimal control problem
  - Models: Linear- Quadratic problem with potential
  - Potential plays the role of the curvature
  - Write the analogue of Jacobi equation
  - Try to simplify them as much as possible → curvature
Why LQ optimal control problems?

Optimal control problem in $M = \mathbb{R}^n$ with $k$ controls:

$$\dot{x} = Ax + Bu, \quad \leftarrow \text{Kalman condition}$$

$$J_T(x_u(\cdot)) = \frac{1}{2} \int_0^T (|u|^2 - x^* Q x) \, dt \rightarrow \min$$

The Hamiltonian function $H : T^* \mathbb{R}^n \rightarrow \mathbb{R}$ is

$$H(p, x) = \frac{1}{2} p^* BB^* p + p^* Ax + \frac{1}{2} x^* Q x$$

**Hamilton equations**

$$\begin{cases}
\dot{p} = -A^* p - Q x \\
\dot{x} = BB^* p + Ax
\end{cases} \quad \text{(\*)}$$

The **conjugate time** $t_c$ is the smallest $T > 0$ such that $\exists$ solution of (\*) such that $x(0) = x(T) = 0$. 
Why LQ optimal control problems?

**Facts**

- \( t_c \) depends only on \( A, B, Q \).
- For \( t < t_c \) there exists a **unique optimal** solution joining \( x_0 \) and \( x_1 \) in time \( t \).
- For \( t > t_c \) there are **no optimal** solution joining \( x_0 \) and \( x_1 \) in time \( t \).

**Example.** Consider the case of a free particle in \( \mathbb{R}^n \) with potential

\[
\dot{x} = u, \quad J_T(x_u(\cdot)) = \frac{1}{2} \int_0^T |u|^2 - x^* Q x \, dt.
\]

In this case the Hamilton equations are equivalent to \((A = 0 \text{ and } B = I)\)

\[
\begin{cases}
    \dot{p} = -Qx \\
    \dot{x} = p
\end{cases} \quad \Leftrightarrow \quad \ddot{x} + Qx = 0
\]

- These are precisely the equation of a Riemannian Jacobi field.
- If \( Q = \kappa I \) we get the conjugate time \( t_c = \pi / \sqrt{\kappa} \).

→ The potential \( Q \) represents the **directional curvature**.
What to do: main ideas

Consider a SR geodesic $\gamma(t)$ (+ some assumptions on the geodesic)

We associate with it

- A “directional curvature” $\mathcal{R}_\gamma(t) : T\gamma(t)M \times T\gamma(t)M \to \mathbb{R}$
- suitable adaptation of the Jacobi fields/equations
- a LQ control problem with $k = \text{dim } \mathcal{D}$ control.
- Related to the linearization of the control system along the geodesic
- A quadratic cost with potential $Q$ that represents the bound for $\mathcal{R}_\gamma(t)$.

Such that in the Riemannian case:

- $\mathcal{R}_\gamma(t)(v) = \text{Sec}(v, \dot{\gamma}(t))$
- $\dot{x} = u$ and $J_T = \frac{1}{2} \int u^2 - x^*Qx \, dt$
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Affine optimal control problems

(Dynamic) Let us consider a smooth affine control system on a manifold $M$

$$\dot{x} = f(x, u) = X_0(x) + \sum_{i=1}^{k} u_i X_i(x), \quad x \in M, u \in \mathbb{R}^k.$$ 

- we call $D_x = \text{span}_x\{X_1, \ldots, X_k\}$ the distribution.
- we assume $\text{Lie}_x\{(\text{ad}^j X_0) X_i, i = 1, \ldots, k, j \in \mathbb{N}\} = T_x M$ for all $x \in M$.

(Cost) Given a Tonelli Lagrangian $L : M \times \mathbb{R}^k \to \mathbb{R}$ we define the cost at time $T$ as the functional

$$J_T(u) := \int_0^T L(\gamma_u(t), u(t)) dt,$$

For two given points $x_0, x_1 \in M$ and $T > 0$, we define the value function

$$S_T(x_0, x_1) = \inf\{J_T(u) \mid u \text{ admissible, } \gamma_u(0) = x_0, \gamma_u(T) = x_1\},$$
Sub-Riemannian geometry

The (sub-)Riemannian case corresponds to the case when

- the system is driftless \((X_0 = 0)\)
- \(k < n\) (\(k = n\) corresponds to Riemannian)
- the cost is quadratic
- Hörmander condition: \(\text{Lie}_x\{X_1, \ldots, X_k\} = T_x M\) for all \(x \in M\)

\[
\dot{x} = \sum_{i=1}^{k} u_i X_i(x), \quad x \in M, u \in \mathbb{R}^k.
\]

\[
J_T(u) := \frac{1}{2} \int_0^T \|\dot{\gamma}(t)\|^2 dt, \quad S_T(x_0, x_1) = \frac{1}{2T} d^2(x_0, x_1)
\]

→ The cost is induced by a scalar product such that \(X_1, \ldots, X_k\) are orthonormal.
→ \(d(\cdot, \cdot)\) Carnot-Caratheodory distance, \(d\) is finite and continuous.
→ maximized Hamiltonian

\[
H(p, x) = \frac{1}{2} \sum_{i=1}^{k} \langle p, X_i(x) \rangle^2
\]
Exponential map

Two kind of extremals

- Abnormals: critical point of the end point map.
- Normals: projection of the flow of $\vec{H}$.

Theorem (PMP)

Let $M$ be a SR manifold and let $\gamma: [0, T] \rightarrow M$ be a normal minimizer. There exists a Lipschitz curve $\lambda: [0, T] \rightarrow T^*M$, with $\lambda(t) \in T^*_\gamma(t)M$, such that

$$\dot{\lambda}(t) = \vec{H}(\lambda(t)).$$

- $\lambda(t) = e^{t\vec{H}}(\lambda_0)$ → parametrized by initial covectors $\lambda_0 \in T^*_{x_0}M$
- $\gamma(t) = \pi(\lambda(t))$

The exponential map starting from $x_0$ as

$$\text{Exp}_{x_0} : \mathbb{R}^+ \times T^*_{x_0}M \rightarrow M, \quad \text{Exp}_{x_0}(t, \lambda_0) = \pi(e^{t\vec{H}}(\lambda_0)) = \gamma(t).$$
Geodesic growth vector

Let $\gamma$ be a normal geodesic. Let $T \in X_0 + D$ an admissible extension of $\dot{\gamma}$.

Geodesic flag

$$F^i_\gamma(t) = \text{span}\{[T, \ldots, [T, X]]|_\gamma(t) | \forall X \in \Gamma(D), \quad j = 0, \ldots, i - 1\}$$

For all $t$ this defines a flag

$$F^1_\gamma(t) \subset F^2_\gamma(t) \subset \ldots \subset T_{x_0}M$$

- Does not depend on the choice of $T$
- $F^1_\gamma(t) = D_{\gamma(t)}$.

Geodesic growth vector

$$G_\gamma(t) = \{k_1(t), k_2(t), \ldots\}, \quad k_i(t) = \text{dim } F^i_\gamma(t)$$

$\rightarrow$ For an LQ problem $k_i = \text{rank}\{B, AB, \ldots, A^{i-1}B\}$. 
A normal geodesic is

- **equiregular** if \( \dim \mathcal{F}_\gamma^i(t) \) does not depend on \( t \)
- **ample** if \( \exists m > 0 \) s.t. \( \mathcal{F}_\gamma^m(t) = T_{x_0} M \)

- “Microlocal Hörmander condition”. \( \mathcal{G}_\gamma = \{k_1, \ldots, k_m\} \)

→ Related with controllability of the linearised system around \( \gamma \)
- Ample \( \Rightarrow \) \( \gamma \) is not abnormal (even \( \gamma|_{[0,t]} \) for all \( t \)).
- the linearized system along \( \gamma \) is controllable for all \( T > 0 \).

Let \( \mathcal{G}_\gamma = \{k_1, k_2, \ldots, k_m\} \)

**Lemma**

*For an equiregular ample geodesic the sequence \( \{k_i - k_{i-1}\}_i \) is decreasing.*
Young diagram of the geodesic

Let $\gamma$ be an ample, equiregular geodesic, with $G_{\gamma} = \{k_1, k_2, \ldots, k_m\}$

- $k_1 = \text{dim } D_{\gamma(t)}$
- $k_i - k_{i-1}$: new “directions” obtained with Lie derivative in direction of $\dot{\gamma}$
- Ample geodesics: $\#$ boxes $= \text{dim } M$ ($\rightarrow$ generic condition)
- Length of the rows $\{n_1, \ldots, n_k\}$
Young diagram of the geodesic

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Young diagram of the geodesic

Let $\gamma$ be an ample, equiregular geodesic, with $G_\gamma = \{k_1, k_2, \ldots, k_m\}$

$$\begin{array}{c c c c}
 n_1 & & \cdots & \\
 n_2 & & \cdots & \\
 \vdots & \vdots & \vdots & \\
 n_{k-1} & & & \\
 n_k & & & \\
\end{array}$$

- $k_1 = \dim D_\gamma(t)$
- $k_i - k_{i-1}$: new “directions” obtained with Lie derivative in direction of $\dot{\gamma}$
- ample geodesics: $\# \text{ boxes} = \dim M \ (\rightarrow \text{generic condition})$
- Length of the rows $\{n_1, \ldots, n_k\}$

For LQ problems: $\{n_1, \ldots, n_k\} = \text{Kronecker/controllability indices}$. 
LQ models

Given an ample and equiregular geodesic with indices $n_1, \ldots, n_k$

$LQ(n_1, \ldots, n_k; Q)$ is an LQ optimal control problem in $\mathbb{R}^n$ with

- $k$ controls
- $A, B$ corresponds to the Brunovsky normal form having indices $n_1, \ldots, n_k$
  $\rightarrow$ coupling of $k$ scalar equations $y^{(n_i)} = u_i$ for $i = i, \ldots, k$.
- constant potential $Q$

We denote by $t_c(n_1, \ldots, n_k; Q)$ its conjugate time

- a priori $t_c(n_1, \ldots, n_k; Q)$ may be $+\infty$
  $\rightarrow$ this always happens, for instance, when $Q = 0$. 
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Jacobi fields revisited

- $\gamma(t) = \pi(\lambda(t)) = \pi \circ e^{t\vec{H}}(\lambda_0)$, where $\lambda_0 \in T^*M$ initial covector of $\gamma$
- $\vec{H} \in \text{Vec}(T^*M)$ Hamiltonian vector field

For any variation $\lambda_s \in T^*_{x_0}M$ of $\lambda_0$ we define the vector field along $\lambda(t)$:

$$X(t) := \frac{d}{ds} \bigg|_{s=0} e^{t\vec{H}}(\lambda_s) \in T_{\lambda(t)}(T^*M)$$

$J(t) = \pi_*X(t)$ is a Jacobi field along the geodesic $\gamma(t) = \pi \circ \lambda(t)$

$$J(t) := \frac{d}{ds} \bigg|_{s=0} \gamma_s(t) = \frac{d}{ds} \bigg|_{s=0} \pi(e^{t\vec{H}}(\lambda_s)) \in T_{\gamma(t)}(M)$$

The first conjugate time $t_c(\gamma)$ is the smallest $T > 0$ such that there exists a Jacobi field along $\gamma$ such that $J(0) = J(T) = 0$.

$\rightarrow$ If $\gamma$ not abnormal, then $\gamma$ loses local optimality at time $t_c(\gamma)$

$\rightarrow$ No connection needed.
Figure: from “A.Agrachev, Y.Sachkov, Control Theory from the geometric viewpoint.”
Moving frame along the extremal

Aim: recover Jacobi equation, and generalize it to the sub-Riemannian setting

- $\sigma$ is the symplectic form on $T^*M$

A frame along the extremal $\lambda(t)$:

$$E^i_{\lambda(t)}, F^j_{\lambda(t)} \in T_{\lambda(t)}(T^*M), \quad i, j = 1, \ldots, n$$

With the following properties:

- $\text{ver}_{\lambda(t)} = \ker \pi_*|_{\lambda(t)} = \text{span}\{E^i_{\lambda(t)}, i = 1, \ldots, n\}$
- It is a Darboux frame:

$$\sigma(E^i, E^j) = 0, \quad \sigma(F^i, F^j) = 0, \quad \sigma(E^i, F^j) = \delta_{ij}$$

$\rightarrow$ The projections $\pi_* F^i_{\lambda(t)}$ define a set of $n$ vector fields along $\gamma(t) = \pi(\lambda(t))$. 
Hamilton equations for the Jacobi fields

Jacobi field written in the moving frame along the extremal

\[ X(t) = \sum_{i=1}^{n} p_i(t) E^i_{\lambda(t)} + x_i(t) F^i_{\lambda(t)} \]

The field \( X(t) \) is associated with a curve \( t \mapsto (p(t), x(t)) \in \mathbb{R}^{2n} \) such that

\[
\begin{align*}
\dot{p} &= -A_t^* p - Q_t x \\
\dot{x} &= B_t B_t^* p + A_t x
\end{align*}
\]

for some matrices \( A_t, B_t, Q_t \) such that rank \( B_t = k \) and \( Q_t = Q_t^* \)

These are Hamilton equations in \( \mathbb{R}^{2n} \) for the time-dependent Hamiltonian

\[
H(p, x) = \frac{1}{2} p^* B_t B_t^* p + p^* A_t x + \frac{1}{2} x^* Q_t x
\]

→ The correspondence depends on the choice of the Darboux moving frame
Canonical frame

In the sub-Riemannian case, there exists a preferred choice:

- “Jacobi equation” = Hamilton equation for a LQ problem

**Theorem (Agrachev-Zelenko 2002, Zelenko-Li 2009)**

For any ample, equiregular geodesic $\gamma(t)$ with indices $n_1, \ldots, n_k$ there exists a canonical moving frame along $\lambda(t)$ such that

- $A_t, B_t$ are constant, with $A, B$ in Brunovski normal form
- $Q_t$ has particular algebraic symmetries (equations as simple as possible)

- This “replaces” the parallel transport along $\gamma$
- In the Riemannian case this procedure gives the equations

$$\begin{cases}
\dot{p} = -Q_t x \\
\dot{x} = p
\end{cases} \quad \Leftrightarrow \quad \ddot{x} + Q_t x = 0$$
Directional curvature

Denote $f_i(t) := \pi_* F^i_{\lambda(t)} \in T_{\gamma(t)}M$ the vector fields on $\gamma$.

$$T_{\gamma(t)}M = \text{span}\{f_1(t), \ldots, f_n(t)\}.$$  

Sub-Riemannian directional curvature

The formula

$$\mathcal{R}_{\gamma(t)}(f_i, f_j) := [Q_t]_{ij}$$

defines a well posed quadratic form

$$\mathcal{R}_{\gamma(t)} : T_{\gamma(t)}M \times T_{\gamma(t)}M \to \mathbb{R}.$$  

- In the Riemannian case

$$\mathcal{R}_{\gamma(t)}(\nu) = \text{Sec}(\nu, \dot{\gamma}(t))$$  

- $\mathcal{R}_{\gamma(t)}$ can be nicely expressed for contact manifold.
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Microlocal comparison theorem

Theorem (DB, L. Rizzi, ’14)

Let $\gamma$ be an ample, equiregular geodesic, with indices $n_1, \ldots, n_k$. Then

(L) if $R_\gamma(t) \geq Q_+$ for all $t$, then $t_c(\gamma) \leq t_c(n_1, \ldots, n_k; Q_+)$,

(U) if $R_\gamma(t) \leq Q_-$ for all $t$, then $t_c(n_1, \ldots, n_k; Q_-) \leq t_c(\gamma)$.

- The first conjugate time of a LQ problem gives an estimate for the first conjugate time along the geodesic.
- The LQ problem with Brunovsky normal form and constant potential is a model (i.e. we have equality).
- In SR case there are no example where the curvature $R_\gamma(t)$ is equal for all geodesics ($\rightarrow$ model spaces out of SR).
- We can “take out the direction of motion” (dimensional reduction).
Microlocal comparison theorem

Theorem (DB, L.Rizzi, '14)

Let \( \gamma \) be an ample, equiregular geodesic, with indices \( n_1, \ldots, n_k \). Then

- (L) if \( R_\gamma(t) \geq Q_+ \) for all \( t \), then \( t_c(\gamma) \leq t_c(n_1, \ldots, n_k; Q_+) \),
- (U) if \( R_\gamma(t) \leq Q_- \) for all \( t \), then \( t_c(n_1, \ldots, n_k; Q_-) \leq t_c(\gamma) \).

Corollary (Constant curvature along \( \gamma \))

Assume that \( R_\gamma(t) = Q \) for all \( t \), then \( t_c(\gamma) = t_c(n_1, \ldots, n_k; Q) \).

Corollary (Negative curvature)

Assume that \( R_\gamma(t) \leq 0 \) for all \( t \), then \( t_c(\gamma) = +\infty \).

- These are matrix inequalities.
- Can be reduced to scalar with the “averaging” procedure. (\( \rightarrow \) if I have time)
Question: when does $t(n_1, \ldots, n_k; Q) < +\infty$?

Hamiltonian vector field of the LQ problem: $\tilde{H}(p, x) = \begin{pmatrix} -A^* & -Q \\ BB^* & A \end{pmatrix} \begin{pmatrix} p \\ x \end{pmatrix}$

Theorem (Agrachev - Rizzi - Silveira, 2014)

The following are equivalent

- LQ optimal control problem has finite conjugate time
- $\tilde{H}$ has at least one Jordan block of odd size with purely imaginary eigenvalue.

- computation of $t_c(n_1, \ldots, n_k, Q)$ reduces to an algebraic question
- there is no (evident) explicit formula for arbitrary $Q$ and $n \gg 1$.
- could be simplified with the “averaging” procedure. (→ if I have time)
Example: Riemannian case

- For all $\gamma$ we have $G_\gamma = \{\dim M\}$ $\implies$ Indices: $\{1, 1, \ldots, 1\}$
- Moreover $\mathcal{R}_{\gamma(t)}(v) = \text{Sec}(\dot{\gamma}(t), v)$

Assume that $\mathcal{R}_{\gamma(t)} = \text{Sec}(\dot{\gamma}(t), v) \geq \kappa > 0$ for all unit $v \in T_{\gamma(t)}M$. Then

$$t_c(\gamma) \leq t_c(1, \ldots, 1; \kappa \mathbb{I}) = \pi/\sqrt{\kappa}$$

Indeed $LQ(1, \ldots, 1; \kappa \mathbb{I})$ is the $n$-dimensional harmonic oscillator

$$H(p, x) = \frac{1}{2}(|p|^2 + \kappa|x|^2), \quad t_c(1, \ldots, 1; \kappa) = \begin{cases} +\infty & \kappa \leq 0 \\ \frac{\pi}{\sqrt{\kappa}} & \kappa > 0 \end{cases}$$

Assume that $\mathcal{R}_{\gamma(t)} = \text{Sec}(\dot{\gamma}(t), v) \leq 0$ for all unit $v \in T_{\gamma(t)}M$. Then

$$t_c(\gamma) \geq t_c(1, \ldots, 1; 0) = +\infty.$$
Model example: Heisenberg group

- For all $\gamma$ we have $G_\gamma = \{2, 3\} \implies$ Kronecker indices: $\{2, 1\}$
- Geodesic $\gamma$ with initial covector $\lambda = (h_0, h_1, h_2)$.
  → Recall that $h_0 := \langle \lambda, Z \rangle$ is constant.

$$
\gamma(t) = \begin{pmatrix}
  h_0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
  Q
\end{pmatrix}
\text{constant along the extremal!}
$$

$LQ(2, 1; Q)$ is a LQ problem in $\mathbb{R}^3$, with Hamiltonian

$$
H(p, x) = \frac{1}{2} p_1^2 + p_2 x_1 + \frac{1}{2} h_0^2 x_1^2
$$

$$
t_c(2, 1; Q) = \begin{cases}
  +\infty & h_0 = 0 \\
  \frac{2\pi}{|h_0|} & h_0 \neq 0
\end{cases}
$$

Let $\gamma$ be a geodesic with initial covector $\lambda$, then $t_c(\gamma) = \begin{cases}
  +\infty & h_0 = 0 \\
  \frac{2\pi}{|h_0|} & h_0 \neq 0
\end{cases}$
Model Example: SU(2) and SL(2)

- For all $\gamma$ we have $\mathcal{G}_{\gamma} = \{2, 3\} \implies$ Kronecker indices: $\{2, 1\}$
- Geodesic $\gamma$ with initial covector $\lambda = (h_0, h_1, h_2)$.
- Recall that $h_0 := \langle \lambda, Z \rangle$ is constant.

$$
\mathcal{R}^{SU(2)}_{\gamma(t)} = \begin{pmatrix}
    h_0^2 + 1 & 0 & 0 \\
    0 & 0 & 0 \\
    0 & 0 & 0
\end{pmatrix}, \quad \mathcal{R}^{SL(2)}_{\gamma(t)} = \begin{pmatrix}
    h_0^2 - 1 & 0 & 0 \\
    0 & 0 & 0 \\
    0 & 0 & 0
\end{pmatrix},
$$

→ We recover [Boscain, Rossi - 2008]:

**SU(2)** Every geodesic has conjugate time $t_c(\gamma) = \frac{2\pi}{\sqrt{h_0^2 + 1}}$.

**SL(2)** Let $\gamma$ be a geodesic with initial covector $\lambda$, then

$$
t_c(\gamma) = \begin{cases} 
+\infty & |h_0| \leq 1 \\
\frac{2\pi}{\sqrt{h_0^2 - 1}} & |h_0| > 1
\end{cases}
$$
Averaging - sub-Riemannian setting

- Collect all directions with the same controllability indices.

\[
\begin{array}{cccc}
\Gamma_1 & \Gamma_2 & \Gamma_3 & \ldots & \Gamma_\ell \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\end{array}
\]

\(r\) rows of length \(\ell\)

- Boxes, rows \(\implies\) generalized boxes, rows
- Average of \(\mathfrak{H}_\lambda(t)\) w.r.t. directions in a gen. box \(\implies\) Ricci of the gen. box
- Riemannian case: 1 gen. box \(\implies\) 1 Ricci
Averaging - sub-Riemannian setting (2)

For a gen. row $\Gamma = \{\Gamma_1, \ldots, \Gamma_\ell\}$, define the Ricci curvatures

$$\text{Ric}_{\gamma(t)}(\Gamma_j) := \sum_{i \in \Gamma_j} \mathcal{R}_{\gamma(t)}(f_i, f_i), \quad j = 1, \ldots, \ell$$

We have 1 comparison theorem for each gen. row

**Theorem (DB, L.Rizzi, ’14)**

*Let $\gamma(t)$ be an ample, equiregular geodesic. Assume that, for $\Gamma = \{\Gamma_1, \ldots, \Gamma_\ell\}$

$$\frac{1}{r} \text{Ric}_{\gamma(t)}(\Gamma_j) \geq \kappa_j, \quad \forall j = 1, \ldots, \ell$$

Then $t_c(\gamma) \leq t_c(\ell; Q)$, where $Q = \text{diag}\{\kappa_1, \ldots, \kappa_\ell\}$*
Sub-Riemannian Bonnet-Myers Theorem

- $M$ complete, connected sub-Riemannian manifold
- All the minimizing geodesics have the same growth vector

**Theorem (Sub-Riemannian Bonnet-Myers)**

Assume that there exists a gen. row $\Gamma = \{\Gamma_1, \ldots, \Gamma_\ell\}$ and constants $\kappa_1, \ldots, \kappa_\ell$ such that, for every geodesic,

$$\frac{1}{r} \text{Ric}_{\gamma(t)}(\Gamma_j) \geq \kappa_j, \quad j = 1, \ldots, \ell$$

Then, if the polynomial

$$P_{\kappa_1, \ldots, \kappa_\ell}(x) = x^{2\ell} + \sum_{j=0}^{\ell-1} \kappa_{\ell-j} x^{2j} (-1)^{\ell-j-1}$$

has at least one simple imaginary root, the manifold is compact, has finite diameter $\leq t(\ell; \kappa_1, \ldots, \kappa_\ell)$. Moreover its fundamental group is finite.
Contact structures on 3D unimodular Lie Groups

- $M$ is a unimodular, simply connected Lie group, $\dim M = 3$
- 1-form $\omega$ is the *contact form*. Distribution: $\Delta = \ker \omega$
- left-invariant sub-Riemannian structure $(\Delta, \langle \cdot | \cdot \rangle)$
- $X_1, X_2$ left-invariant orthonormal frame for $(\Delta, \langle \cdot | \cdot \rangle)$
- $X_0$ Reeb vector field: $X_0 \in \ker d\omega$, $\omega(X_0) = 1$
- Normalization $d\omega|_{\Delta}$ is the area element
- Structural constants: $[X_i, X_j] = \sum_{\ell=0}^{2} c_{ij}^{\ell}X_{\ell}$

**Theorem (Agrachev, Barilari - 2012)**

*The equivalence classes of isometric contact structures on 3D unimodular Lie groups are classified by two invariants: $\chi \geq 0, \kappa \in \mathbb{R}$.*

Up to rescaling $\chi^2 + \kappa^2 = 1$. 
The equivalence classes of isometric contact structures on 3D unimodular Lie groups are classified by two invariants $\chi, \kappa \in \mathbb{R}$.

Up to rescaling and reflections $\chi^2 + \kappa^2 = 1$ and $\chi \geq 0$. 
Some known results (case $\chi = 0$)

$h_0 := \langle \lambda, X_0 \rangle$ is always a constant along the extremal

**Theorem (Boscain, Rossi - 2008)**

*Let $\gamma$ be a geodesic on $\text{SL}(2)$, $\text{SU}(2)$:

- $\text{SL}(2)$ ($\kappa = -1$): $t_c(\gamma) = \begin{cases} +\infty & h_0^2 \leq 1 \\ \frac{2\pi}{\sqrt{h_0^2 - 1}} & h_0^2 > 1 \end{cases}$

- $\text{SU}(2)$ ($\kappa = 1$): $t_c(\gamma) = \frac{2\pi}{\sqrt{h_0^2 + 1}}$*
Some new results \((\chi > 0)\)

Let \(\chi > 0\). There exists a left-invariant orthonormal frame \(X_1, X_2\) such that

\[
\begin{align*}
[X_1, X_0] &= (\chi + \kappa)X_2, \\
[X_2, X_0] &= (\chi - \kappa)X_1, \\
[X_2, X_1] &= X_0
\end{align*}
\]

Moreover the function \(E : T^* M \to \mathbb{R}\) is a constant of the motion

\[
E = \frac{h_0^2}{2\chi} + h_2^2, \quad h_i(\lambda) := \langle \lambda, X_i \rangle
\]

Theorem (Barilari, Rizzi - 2014)

Let \(M\) be a 3D unimodular Lie group with a left-invariant sub-Riemannian structure, with \(\chi > 0\) and \(\kappa \in \mathbb{R}\). Then there exists \(\overline{E} = \overline{E}(\chi, \kappa)\) such that every length parametrised geodesic \(\gamma\) with \(E(\gamma) \geq \overline{E}\) has a finite conjugate time.
Final Comments

Other results obtained:
- Proof of a Ricci-type “average” comparison result
  \[ \rightarrow \] Reduction to (more than one) scalar inequalities.
- Bonnet-Myers result (diameter estimate with \( t_c \) of LQ models).
- New results about conjugate points for unimodular 3D Lie groups

Technical points in the proofs
- Conjugate points = blow up time of a Riccati equation
- Comparison of solution for Matrix Riccati equations
  \[ \rightarrow \] This is highly extendable to other comparison results
- Difficult technical point: how to “average”? 
  \[ \rightarrow \] Collect all directions with the same controllability indices.

Good and bad points
- The method is quite general (no restriction on the sub-Riemannian structure)
- It could be very complicated to compute (and bound) \( H_\gamma(t) \)
THANKS FOR YOUR ATTENTION!