Heat kernel asymptotics at the cut locus for Riemannian and sub-Riemannian manifolds

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Outline

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2 Sub-Riemannian geometry: regularity of $d^2$ and the heat equation

3 Main results

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Motivation

Sub-Riemannian geometry: regularity of $d^2$ and the heat equation

Main results

Some results for generic metrics

Introduction

\[ (Hypo)\text{-elliptic operators} \leftrightarrow (Sub)\text{-Riemannian metrics} \]

Main motivation:

- understand the interplay between
- the analysis of the diffusion processes on the manifold (heat equation)
- the geometry of these spaces (distance, geodesics, curvature)

Problem: relating

- analytic properties of the heat kernel $p_t(x, y)$ (small time asymptotics)
- geometry underlying (properties of distance and geodesics joining $x$ and $y$)

→ In particular: what happens for $p_t(x, y)$ when $y \in \text{Cut}(x)$?

→ What happens “generically”?
Heat equation on $\mathbb{R}^2$

- The classical heat equation on $\mathbb{R}^2$

$$\partial_t \psi(t, x) = (\partial^2_{x_1} + \partial^2_{x_2}) \psi(t, x)$$

- The fundamental solution, or *heat kernel*, of this equation

$$p_t(x, y) = \frac{1}{4\pi t} \exp \left( - \frac{|x - y|^2}{4t} \right)$$

$\rightarrow$ Every solution such that $\psi(0, x) = \phi(x)$ is of the form

$$\psi(t, x) = \int_{\mathbb{R}^2} p_t(x, y) \phi(y) dy$$

$\rightarrow$ $p_t(\cdot, y)$ corresponds to the solution with initial datum Dirac $\delta_y$. 
Heat equation on $\mathbb{S}^2$

- The heat equation on the sphere $\mathbb{S}^2$
  \[ \partial_t \psi(t, x) = \Delta \psi(t, x) \]

  where $\Delta$ is the Laplace Beltrami operator → elliptic operator.

- It is natural to expect that
  \[ p_t(x, y) \sim \frac{1}{4\pi t} \exp \left( -\frac{d(x, y)^2}{4t} \right) \]

- This is true everywhere but at the antipodal point \( \hat{x} \), where
  \[ p_t(x, \hat{x}) \sim \frac{1}{4\pi t^{3/2}} \exp \left( -\frac{d(x, y)^2}{4t} \right) \]

→ Here and in what follows

\[ f(t) \sim g(t) \iff f(t) = g(t)[C + o(1)], \quad C \neq 0 \]
Heat vs Cut locus

Naive idea:
the heat diffuses along geodesics
- only one optimal geodesic reaches $y$
- $\hat{x}$ is the point where all geodesics meet
- $\hat{x} = \text{Cut}(x) = \text{Conj}(x)$
- the function $x \mapsto d^2(x, \cdot)$
  is not smooth at $\hat{x}$

→ even in this simple example it is easy to see how the structure of the geodesics is related with the heat kernel asymptotics.
A complete proof on cut and conjugate locus has been proved only in 2004.
(even if first works about geodesics on ellipsoids dates back to Jacobi)

From Wikipedia:
By Cffk (Own work) [CC-BY-SA-3.0 (http://creativecommons.org/licenses/by-sa/3.0)]
Surfaces of revolution

For a metric on $S^2$ of the form $dr^2 + m^2(r)d\theta^2$ such that

- symmetric w.r.t. the equator
- non-singularity condition at the equator [i.e. $K'' \neq 0$]
- Typical example: ellipsoid of revolution
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**Theorem (D.B., J.Jendrej, ’13)**

*Fix $x \in M$ along the equator and let $y$ be a cut-conjugate point with respect to $x$. Then we have*

$$p_t(x, y) \sim \frac{1}{t^{5/4}} e^{-d^2(x,y)/4t}, \quad \text{for } t \to 0.$$
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**Theorem (D.B., Jendrej)**

Fix $x \in M$ along the equator and let $y$ be a cut-conjugate point with respect to $x$. Then we have

$$p_t(x, y) \sim \frac{1}{t^{1+1/4}} e^{-d^2(x,y)/4t}, \quad \text{for } t \to 0.$$ 

For the standard sphere $S^2$

$$p_t(x, y) \sim \frac{1}{t^{1+1/2}} e^{-d^2(x,y)/4t}$$

$x = \text{nord}, \ y = \text{sud}$
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2 Sub-Riemannian geometry: regularity of $d^2$ and the heat equation

3 Main results

4 Some results for generic metrics
Sub-Riemannian geometry

Definition

A sub-Riemannian manifold is a triple \((M, D, \langle \cdot, \cdot \rangle)\), where

(i) \(M\) manifold, \(C^\infty\), dimension \(n \geq 3\);

(ii) \(D\) vector distribution of rank \(k < n\), i.e. \(D_x \subset T_x M\) subspace \(k\)-dim. that is bracket generating: \(\text{Lie}_x D = T_x M\).

(iii) \(\langle \cdot, \cdot \rangle_x\) inner product on \(D_x\), smooth in \(x\).

A curve \(\gamma : [0, T] \rightarrow M\) is horizontal if \(\dot{\gamma}(t) \in \Delta_{\gamma(t)}\).

For a horizontal curve \(\gamma : [0, T] \rightarrow M\) its length is

\[
\ell(\gamma) = \int_0^T \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle} \, dt.
\]
We can define the sub-Riemannian distance as
\[ d(x, y) = \inf \{ \ell(\gamma) \mid \gamma(0) = x, \gamma(T) = y, \gamma \text{ horizontal} \}. \]

- The bracket generating condition implies
  1. \( d(x, y) < +\infty \) for all \( x, y \in M \).
  2. topology \((M, d) = \) manifold topology.

**Question:** Regularity of \( d^2 \)? Relation with minimizing admissible curves?

For a minimizing curve we can define
- **Conjugate locus:** where geodesics lose local optimality
- **Cut locus:** where geodesics lose global optimality (and \( d^2 \) is not smooth)
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Regularity of $d^2$

Consider geodesics starting from $x \in M$

- geodesics lose optimality arbitrarily close to $x$
- $f(\cdot) = \frac{1}{2} d^2(x, \cdot)$ is not smooth at $x$

- $f : M \to \mathbb{R}$ is $C^\infty$ on an open and dense set $\Sigma(x)$ [A.Agrachev, 2009]

$x \notin \Sigma(x)$ and $\text{Cut}(x) \subset M \setminus \Sigma(x)$

$\Sigma(x) = \{ y \in M \mid \exists! \text{ non-abnormal and non-conjugate minimizer from } x \text{ to } y \}$

$\to$ for simplicity: assume no minimizing abnormal extremals.
Regularity of $d^2$

Consider geodesics starting from $x \in M$

- geodesics lose optimality **arbitrarily close** to $x$
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Conjugate points and Exponential map

- Normal minimizer are projection of the flow of $\vec{H}$.

**Theorem (PMP)**

Let $M$ be a SR manifold and let $\gamma : [0, T] \rightarrow M$ be a minimizer. \exists Lipschitz curve $\lambda : [0, T] \rightarrow T^* M$, with $\lambda(t) \in T_{\gamma(t)}^* M$, such that $\dot{\lambda}(t) = \vec{H}(\lambda(t))$.

- $\lambda(t) = e^{t\vec{H}}(\lambda_0)$ \rightarrow parametrized by initial covectors $\lambda_0 \in T_{x_0}^* M$
- $\gamma(t) = \pi(\lambda(t))$

The **exponential map** starting from $x_0$ as

$$\text{Exp}_{x_0} : T_{x_0}^* M \rightarrow M, \quad \text{Exp}_{x_0}(\lambda_0) = \pi(e^{\vec{H}}(\lambda_0)).$$

- $\text{Exp}_{x_0}(t\lambda_0) = \gamma(t).$ \leftarrow by homogeneity of $H$

**Fact:**

- $\bar{t}$ first conjugate time along $\gamma \Rightarrow \bar{t}\lambda_0$ is a critical point of $\text{Exp}_{x_0}$. 
We introduce the SR Laplacian operator $\Delta$ to define

$$\partial_t \psi(t, x) = \Delta \psi(t, x)$$

→ If $X_1, \ldots, X_k$ is an orthonormal basis for $\mathcal{D}$ we set

$$\Delta \phi = \text{div}(\nabla \phi), \quad \nabla \phi = \sum_i X_i(\phi)X_i$$

$$\Delta = \sum_{i=1}^k X_i^2 + (\text{div } X_i)X_i$$

→ sum of squares + 1st order term that depends on the volume

We need to fix a volume $\mu$!
SR Laplacian

We introduce the SR Laplacian operator $\Delta$ to define

$$\partial_t \psi(t, x) = \Delta \psi(t, x)$$

→ If $X_1, \ldots, X_k$ is an orthonormal basis for $\mathcal{D}$ we set

$$\Delta_\mu \phi = \text{div}_\mu (\nabla \phi), \quad \nabla \phi = \sum_i X_i(\phi)X_i$$

$$\Delta = \sum_{i=1}^k X_i^2 + (\text{div}_\mu X_i)X_i$$

→ sum of squares + 1sr order term that depends on the volume

We need to fix a volume $\mu$!
The sub-Riemannian heat equation on a complete manifold $M$

\[
\begin{aligned}
\frac{\partial \psi}{\partial t}(t, x) &= \Delta \psi(t, x), \quad \text{in } (0, \infty) \times M, \\
\psi(0, x) &= \varphi(x), \quad x \in M, \quad \varphi \in C_0^\infty(M).
\end{aligned}
\]  

(*

**Theorem (Hörmander)**

*If* $\{X_1, \ldots, X_k\}$ *are bracket generating, then* $\Delta$ *is hypoelliptic.***

The problem (*) has unique solution for $\varphi \in C_0^\infty(M)$

\[
\psi(t, x) := e^{t\Delta} \varphi(x) = \int_M p_t(x, y) \varphi(y) d\mu(y), \quad \varphi \in C_0^\infty(M),
\]

where $p_t(x, y) \in C^\infty$ is the heat kernel associated with $\Delta$. 
Results on the asymptotic of $p_t(x, y)$

Fix $x, y \in M$, $\dim M = n$:

**Theorem (Main term, Leandre, '87)**

\[
\lim_{t \to 0} 4t \log p_t(x, y) = -d^2(x, y)
\]

**Theorem (Smooth points, Ben Arous, '88)**

Assume $y \in \Sigma(x)$, then

\[
p_t(x, y) \sim \frac{1}{t^{n/2}} \exp \left( -\frac{d^2(x, y)}{4t} \right)
\]

**Facts**

1. In Riemannian geometry $x \in \Sigma(x)$, in sub-Riemannian it is not true!
2. The on-the-diagonal expansion indeed is different.
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**Theorem (On the diagonal, Ben Arous, ’89)**

*We have the expansion*

\[
p_t(x, x) \sim \frac{1}{t^{Q/2}}
\]

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Assume $y \in \Sigma(x)$, then

$$p_t(x, y) \sim \frac{1}{t^{n/2}} \exp \left( -\frac{d^2(x, y)}{4t} \right)$$ (1)

**Questions**

1. What happens in (1) if $y \in \text{Cut}(x)$?
2. Can we relate the expansion of $p_t(x, y)$ with the properties of the geodesics joining $x$ to $y$?
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\begin{itemize}
  \item \textbf{Cut/Conjugacy vs Asymptotics}
\end{itemize}

\textbf{Theorem (D.B., Boscain, Neel,’12)}

Let $M$ be an $n$-dimensional complete SR manifold, $\mu$ smooth volume. Let $x \neq y$ and assume that every optimal geodesic joining $x$ to $y$ is strongly normal.

- If $x$ and $y$ are not conjugate

  $$p_t(x, y) = \frac{C}{t^{n/2}} e^{-d^2(x,y)/4t} (1 + O(t)),$$

- If $x$ and $y$ are conjugate along at least one minimal geodesic

  $$\frac{C}{t^{(n/2)+(1/4)}} e^{-d^2(x,y)/4t} \leq p_t(x, y) \leq \frac{C'}{t^{n-(1/2)}} e^{-d^2(x,y)/4t},$$

\begin{itemize}
  \item we can detect only points that are cut and conjugate.
  \item If we are cut but not conjugate the constant $C$ changes.
\end{itemize}
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→ we can detect only points that are cut and conjugate.

→ If we are cut but not conjugate the constant $C$ changes.
Case of a 2-dim surface

The theorem in the case of a 2-dim Riemannian surface says that

- If \( x \) and \( y \) are not conjugate
  \[
  p_t(x, y) = \frac{C}{t} e^{-d^2(x, y)/4t} (1 + O(t)),
  \]

- If \( x \) and \( y \) are conjugate along at least one minimal geodesic
  \[
  \frac{C}{t^{5/4}} e^{-d^2(x, y)/4t} \leq p_t(x, y) \leq \frac{C'}{t^{3/2}} e^{-d^2(x, y)/4t},
  \]

→ all cases are between the ellipsoid and the sphere.

→ they correspond to the “minimal” and “maximal” degeneration for a conjugate point on a surface.
Refinement

If $\gamma(t) = \text{Exp}_{x}(t\lambda)$ joins $x$ and $y$ we say that
- $\gamma$ is conjugate of order $r$ if $\text{rank}(D\lambda \text{Exp}_{x}) = n - r$

**Theorem (D.B., Boscain, Charlot, Neel,'13)**

Let $M$ be an $n$-dimensional complete SR manifold, $\mu$ smooth volume. Let $x \neq y$ and assume that the only optimal geodesic joining $x$ to $y$ is conjugate of order $r$.

- Then there exist positive constants, such that for small $t$

$$\frac{C}{t^{\frac{n}{2} + \frac{r}{4}}} e^{-d^2(x,y)/4t} \leq p_t(x, y) \leq \frac{C'}{t^{\frac{n}{2} + \frac{r}{2}}} e^{-d^2(x,y)/4t},$$

→ This result can give estimates on the order of conjugacy of a point in the cut locus once you know the heat kernel (roughly, how much it is symmetric)
Example: Heisenberg

In the Heisenberg group the Heat kernel is explicit (here \( q = (x, y, z) \))

\[
p_t(0, q) = \frac{1}{(4\pi t)^2} \int_{-\infty}^{\infty} \frac{\tau}{\sinh \tau} \exp \left(-\frac{x^2 + y^2}{4t} - \frac{\tau}{\tanh \tau} \right) \cos \left(\frac{Z\tau}{t} \right) d\tau.
\]

and gives the asymptotics for cut-conjugate points \( \zeta = (0, 0, z) \)

\[
p_t(0, \zeta) \sim \frac{1}{t^2} \exp \left(-\frac{\pi z}{t} \right) = \frac{1}{t^2} \exp \left(-\frac{d^2(0, \zeta)}{4t} \right)
\]

Remark: The fact that \( \frac{4}{2} > \frac{3}{2} \) confirm the fact that the points \( \zeta = (0, 0, z) \) are not smooth points. What is the meaning?
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and gives the asymptotics for cut-conjugate points $\zeta = (0, 0, z)$

$$p_t(0, \zeta) \sim \frac{1}{t^{\frac{3}{2}}} \exp \left( -\frac{\pi z}{t} \right) = \frac{1}{t^{\frac{3}{2}}} \exp \left( -\frac{d^2(0, \zeta)}{4t} \right).$$

**Remark:** The fact that $\frac{4}{2} > \frac{3}{2}$ confirm the fact that the points $\zeta = (0, 0, z)$ are not smooth points. What is the meaning?
Example: Heisenberg

In the Heisenberg group we had the asymptotics for cut-conjugate points $\zeta = (0, 0, z)$

$$p_t(0, \zeta) \sim \frac{1}{t^{\frac{4}{2}}} \exp \left( -\frac{\pi z}{t} \right) = \frac{1}{t^{\frac{4}{2}}} \exp \left( -\frac{d^2(0, \zeta)}{4t} \right)$$

Remark: This a consequence of the fact that there exists a one parametric family of optimal trajectories (varying the angle), hence the hinged energy function is actually a function of two variables, being constant on the midpoints.
Idea of the proof: What happens at non good point?

Let \( x, y \in M \) with \( y \in \text{Cut}(x) \) and write

\[
p_t(x, y) = \int_M p_{t/2}(x, z)p_{t/2}(z, y) d\mu(z)
\]

Idea: \( z \in \Sigma(x) \cap \Sigma(y) \) and apply Ben-Arous expansion

\[
p_{t/2}(x, z)p_{t/2}(z, y) \sim \frac{1}{t^n} \exp \left( -\frac{d^2(x, z) + d^2(z, y)}{4t} \right)
\]

This led to the study of an integral of the kind

\[
p_t(x, y) = \frac{1}{t^n} \int_M c_{x, y}(z) \exp \left( -\frac{h_{x, y}(z)}{2t} \right) d\mu(z)
\]

where \( h_{x, y} \) is the hinged energy function

\[
h_{x, y}(z) = \frac{1}{2} \left( d^2(x, z) + d^2(z, y) \right).
\]

→ the asymptotic is given by the behavior of \( h_{x, y} \) near its minimum. 
(Laplace method)
Properties of $h_{x,y}$ hinged energy function

**Lemma**

Let $\Gamma$ be the set of midpoints of the minimal geodesics joining $x$ to $y$. Then $\min h_{x,y} = h_{x,y}(\Gamma) = d^2(x, y)/4$.

- A minimizer is called strongly normal if any piece of it is not abnormal.

**Theorem (D.B., Boscain, Neel,'12)**

Let $\gamma$ be a strongly normal minimizer joining $x$ and $y$. Let $z_0$ be its midpoint. Then

(i) $y$ is conjugate to $x$ along $\gamma$ $\iff$ $\text{Hess}_{z_0} h_{x,y}$ is degenerate.

(ii) The dimension of the space of perturbations for which $\gamma$ is conjugate
      is equal to $\dim(\ker \text{Hess}_{z_0} h_{x,y})$.

**Remark:** $\text{Hess} h_{x,y}$ is never degenerate along the direction of the geodesic!
Hinged vs Asymptotics

- To have the precise asymptotic one need that the expansion of $h_{x,y}$ is diagonal in some coordinates.

**Theorem (D.B., Boscain, Neel, ’12)**

Assume that, in a neighborhood of the midpoints of the strongly normal geodesic joining $x$ to $y$ there exists coordinates such that

$$h_{x,y}(z) = \frac{1}{4} d^2(x, y) + z_1^{2m_1} + \ldots + z_n^{2m_n} + o(|z_1|^{2m_1} + \ldots + |z_n|^{2m_n})$$

Then for some constant $C > 0$

$$p_t(x, y) = \frac{1}{t^{n-\sum_i \frac{1}{2m_i}}} \exp \left( - \frac{d^2(x, y)}{4t} \right) (C + o(1)).$$

Note: $h_{x,y}$ non degenerate ($m_i = 2$) $\rightarrow$ the exponent is $n/2$
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Then for some constant $C > 0$

$$p_t(x, y) = \frac{1}{t^{n-\sum_i \frac{1}{2m_i}}} \exp \left( -\frac{d^2(x, y)}{4t} \right) (C + o(1)).$$

**Note:** $h_{x,y}$ non degenerate ($m_i = 2$) $\rightarrow$ the exponent is $n/2$
Remarks

Nevertheless there are at least two cases that simplifies the analysis:

1. If we have symmetry, then a one-parametric family of optimal trajectories then $h_{x,y}$ is constant along the trajectory of midpoints.

2. If there is only one degenerate direction then $h_{x,y}$ is always diagonalizable.

**Lemma (Splitting Lemma - Gromoll, Meyer, ’69)**

Let $h : \mathbb{R}^n \to \mathbb{R}$ smooth such that $h(0) = dh(0) = 0$ and that $\dim \ker d^2 h(0) = 1$. Then there exists coordinates such that

$$h(z) = z_1^2 + \ldots + z_{n-1}^2 + \psi(z_n), \quad \text{where} \quad \psi(z_n) = O(z_n^4).$$
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Exponential map as a Lagrangian map

- A fibration $\pi : E \to N$ is **Lagrangian** if $E$ is a symplectic manifold and each fiber is Lagrangian.

- A **Lagrangian map** is a smooth map $f : M \to N$ between manifolds of the same dimension obtained by composition of a Lagrangian immersion $i : M \to E$ and a projection $\pi : E \to N$.

$$f : M \xrightarrow{i} E \xrightarrow{\pi} N.$$  

The exponential map $\text{Exp}_{x_0}$ is a **Lagrangian map**

$$\text{Exp}_{x_0} : T^* M \to M, \quad \text{Exp}_{x_0} = \pi \circ e^\vec{H} \big|_{T^*_{x_0} M}$$

It is the composition of

- Lagrangian immersion $e^\vec{H} : T^*_{x_0} M \to T^* M$
- a projection $\pi : T^* M \to M$
Normal form of generic singularities of Lagrangian maps

Theorem (Arnold’s school)

Let \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a generic Lagrangian singularity at \( x_0 \). Then there exist changes of coordinates around \( x_0 \) and \( f(x_0) \) such that in the new coordinates \( x_0 = f(x_0) = 0 \) and:

- if \( n = 1 \), \( f \) is the map \( x \mapsto x^2 \) \( (A_2) \)
- if \( n = 2 \) then \( f \) is the map \((x, y) \mapsto (x^3 + xy, y)\) \( (A_3) \)
  or a suspension of the previous one;
- if \( n = 3 \) then \( f \) is the map
  \[
  (x, y, z) \mapsto (x^4 + xy^2 + xz, y, z) \\
  (x, y, z) \mapsto (x^2 + y^2 + xz, xy, z) \\
  (x, y, z) \mapsto (x^2 - y^2 + xz, xy, z)
  \]
  or a suspension of the previous ones; \( (A_4) \) \( (D_4^+) \) \( (D_4^-) \)
Normal form of generic singularities of Lagrangian maps

Theorem (Arnold’s school)

- *if* $n = 4$ *then* $f$ *is the map*
  
  $$(x, y, z, t) \mapsto (x^5 + xy^3 + xz^2 + xt, y, z, t) \quad (A_5)$$
  
  $$(x, y, z, t) \mapsto (x^3 + y^2 + x^2z + xt, xy, z, t) \quad (D_5^+)$$
  
  $$(x, y, z, t) \mapsto (-x^3 + y^2 + x^2z + xt, xy, z, t) \quad (D_5^-)$$
  
  *or a suspension of the previous ones;*

- *if* $n = 5$ *then* $f$ *is the map*

  $$(x, y, z, t, u) \mapsto (x^6 + xy^4 + xz^3 + xt^2 + xu, y, z, t, u) \quad (A_6)$$
  
  $$(x, y, z, t, u) \mapsto (x^4 + y^2 + x^3z + xt^2 + xu, xy, z, t, u) \quad (D_6^+)$$
  
  $$(x, y, z, t, u) \mapsto (-x^4 + y^2 + x^3z + xt^2 + xu, xy, z, t, u) \quad (D_6^-)$$
  
  $$(x, y, z, t, u) \mapsto (x^2 + xyz + ty + ux, y^3 + x^2z, z, t, u) \quad (E_6^+)$$
  
  $$(x, y, z, t, u) \mapsto (x^2 + xyz + ty + ux, -y^3 + x^2z, z, t, u) \quad (E_6^-)$$
  
  *or a suspension of the previous ones.*

**Question**: which ones can appear as optimal singularities?

(i.e. as normal forms of Riemannian exponential maps at a cut-conjugate point?)
A3 singularity vs Exponential map

Let us consider the A3 singularity

\[ \Phi : (x, y) \mapsto (x^3 + xy, y) \]

The set of critical points is

\[ C = \{ \det D\Phi = 0 \} \iff \{ 3x - y^2 = 0 \} \iff \{(t, 3t^2), t \in \mathbb{R}\} \]

The image of this set

\[ \Phi(C) = \{(-2t^3, 3t^2)\} = \{y^3 = (27/4)x^2\} \]

It corresponds to the cut-conjugate point on the ellipsoid!
Lagrangian generic vs Riemannian generic

Let $M$ be a smooth manifold and $\mathcal{G}$ be the set of all complete Riemannian metrics endowed with the $C^\infty$ Whitney topology.

- We say that for a generic Riemannian metric on $M$ the property (P) holds if the property (P) is satisfied on an open and dense subset of the set $\mathcal{G}$.

$\rightarrow$ Singularities of generic Riemannian exponential maps are generic Lagrangian singularities.

- Weinstein ('68), Wall ('76) and Janesko-Mostowski ('95).

**Theorem**

Let $M$ be a smooth manifold with $\dim M \leq 5$, and fix $x \in M$. For a generic Riemannian metric on $M$, the singularities of the exponential map $\text{Exp}_x$ are those listed in the previous Theorem.
Elimination of singularities

→ One can eliminate all the singularities but three of them if one restricts to optimal ones (i.e. along minimizing geodesics)

Theorem (DB, U. Boscain, G. Charlot, R. Neel)

Let $M$ be a smooth manifold, $\dim M \leq 5$, and $x \in M$. For a generic Riemannian metric on $M$ and any minimizing geodesic $\gamma$ from $x$ to $y$ we have that $\gamma$ is

- either non-conjugate,
- $A_3$-conjugate,
- or $A_5$-conjugate.

Notice that

- $A_3$ appears only for $\dim M \geq 2$
- $A_5$ can only appear for $\dim M \geq 4$.
→ in dimension 2 and 3 there is only “one kind” of generic cut-conjugate point.
Consequences

Corollary

Let $M$ be a smooth manifold, $\dim M = n \leq 5$, and $x \in M$. For a generic Riemannian metric on $M$ the only possible heat kernel asymptotics are:

(i) No minimal geodesic from $x$ to $y$ is conjugate

$$p_t(x, y) = \frac{C + O(t)}{t^{n/2}} \exp \left(-\frac{d^2(x, y)}{4t}\right),$$

(ii) At least one min. geod. is $A_3$-conjugate but none is $A_5$-conjugate

$$p_t(x, y) = \frac{C + O(t^{1/2})}{t^{n/2+1/4}} \exp \left(-\frac{d^2(x, y)}{4t}\right),$$

(iii) At least one min. geod. is $A_5$-conjugate

$$p_t(x, y) = \frac{C + O(t^{1/3})}{t^{n/2+1/6}} \exp \left(-\frac{d^2(x, y)}{4t}\right).$$

→ consistent with the results obtained on surfaces of revolution.
What is possible for non generic surfaces?

Theorem (D.B., Boscain, Charlot, Neel, ’13)

For any integer $r \geq 3$, any positive real $\alpha$, and any real $\beta$, there exists a smooth metric on $S^2$ and $x \neq y$ such that

$$p_t(x, y) = \frac{1}{t^{\frac{3}{2} - \frac{1}{2r}}} e^{-d^2(x,y)/4t} (\alpha + t^{1/r} \beta + o(t^{1/r})).$$

- the existence of such expansions is not so surprising.
- the “big-O” term is computed and cannot in general be improved.
- we do see expansions in fractional powers of $t$ (and not integer)
Idea of the proof

Let $\gamma(t) = \text{Exp}_x(t\lambda_0)$ join $x$ and $y$ and conjugate

Singularity of $\text{Exp}_x$ at $\lambda_0 \iff$ Singularity of $h_{x,y}$ at midpoint $z_0$

Use two crucial facts:

- If $\gamma$ is minimizing there exists a variation $\lambda(s)$ such that $y(s) = \text{Exp}_x(\lambda(s))$ satisfies $y(s) - y = O(s^3)$ in a coordinate system.
- Assume $\text{rank}(D\lambda\text{Exp}_x) = n - 1$. Then

$$h_{x,y}(z) = \frac{d^2(x, y)}{4} + z_1^2 + \cdots + z_{n-1}^2 + z_n^m$$

where $m = \max\{k \in \mathbb{N} \mid y(s) - y = s^k v + o(t^k), v \neq 0\}$ for all variations $y(s) = \text{Exp}_x(\lambda(s))$. 
3D contact case

For the generic 3D contact case [Agrachev, Gauthier et al.,'96]

- close to the diagonal only singularities of type $A_3$ appear, accumulating to the initial point.
- The local structure of the conjugate locus is
  - either a suspension of a four-cusp astroid (at generic points)
  - or a suspension of a “six-cusp astroid” (along some special curves).
- for the four-cusp case, two of the cusps are reached by cut-conjugate geodesics,
- in the six-cusp case this happens for three of them.

→ Notice that the conjugate locus at a generic point looks like a suspension of the first conjugate locus that one gets on a Riemannian ellipsoid
Motivation

Sub-Riemannian geometry: regularity of $d^2$ and the heat equation

Main results

Some results for generic metrics

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Sub-Riemannian geometry

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**Theorem**

Let $M$ be a smooth manifold of dimension $3$. Then for a generic 3D contact sub-Riemannian metric on $M$, every $x$, and every $y$ (close enough to $x$) we have

(i) If no minimal geodesic from $x$ to $y$ is conjugate then

$$p_t(x, y) = \frac{C + O(t)}{t^{3/2}} \exp \left( - \frac{d^2(x, y)}{4t} \right),$$

(ii) If at least one minimal geodesic from $x$ to $y$ is conjugate then

$$p_t(x, y) = \frac{C + O(t^{1/2})}{t^{7/4}} \exp \left( - \frac{d^2(x, y)}{4t} \right),$$

Moreover, there are points $y$ arbitrarily close to $x$ such that case (ii) occurs.

- exponents of the form $N/4$, for integer $N$, were unexpected in the 90s literature for points close enough.
Motivation
Sub-Riemannian geometry: regularity of $d^2$ and the heat equation
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