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PERIODIC BILLIARD TRAJECTORIES IN RIGHT TRIANGLES AND RIGHT-ANGLED TETRAHEDRA

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This paper is dedicated to periodic billiard trajectories in right triangles and right-angled tetrahedra. We construct a specific type of periodic trajectories and show that the trajectories of this type fill the right triangle entirely. Then we establish the instability of all known types of periodic trajectories in right triangles. Finally, some of these results are generalized to the n-dimensional case and are given a mechanical interpretation.

1. Introduction

Polygonal billiards are a rather difficult object of study despite the simplicity of their description. The question *whether there exists at least one periodic trajectory in every triangle* (*more generally, in every polygon*) is still open. The main difficulty consists in the fact that for any n there exist some triangles where every periodic trajectory has more than n links (for a proof see [3] p. 12 or [4] p. 259). (In contrast, the *Birkhoff theorem* asserts that in a convex region with a smooth boundary there exists a periodic trajectory with n links for any $n \geqslant 2$; see [3], [4], [6]).

Below we list some known facts about periodic trajectories in triangles. We will need two definitions.

Definition. A trajectory in a triangle is called *perpendicular* if it meets a side of the triangle being perpendicular to it.

Definition. A periodic trajectory in a triangle is *stable* if for any small perturbation of the triangle, the triangle obtained contains a periodic trajectory close to the initial one. (We say that two trajectories are close if they have the same number of reflection points and the corresponding reflection points lie on the same edge of the triangle close to each other.)

Figure 1 sums up the information known about periodic trajectories in triangles. It is explained in the two following items.

1. In every *acute* triangle there exists a periodic trajectory with three links (called the *Fagnano trajectory*) and a bundle of 6-link trajectories parallel to it. These trajectories are stable. Moreover, the set of the acute triangles can be divided into infinitely many regions to which odd numbers are assigned, as in the right-hand upper part of Figure 1. The triangles that correspond to regions marked by k contain periodic trajectories with any odd number of links from 3 to k

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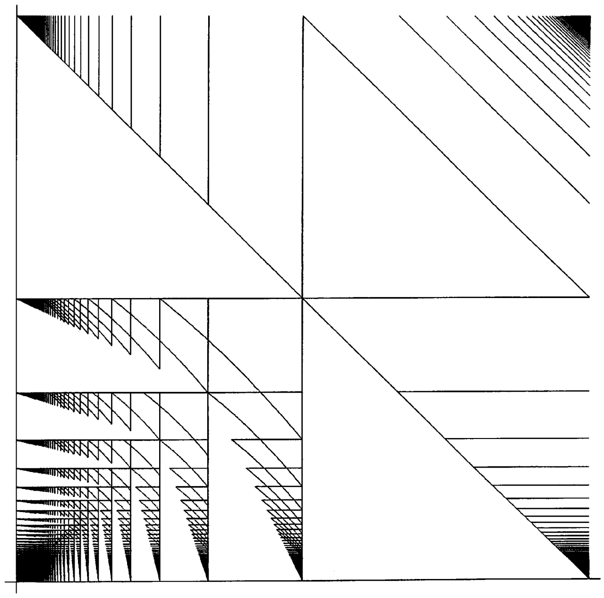

Fig. 1

and the corresponding bundles of parallel trajectories with the doubled number of links. These trajectories are also stable. The number k tends to infinity as the smallest angle of the triangle tends to 0.

It is not known whether any acute triangle contains *infinitely many* different types of periodic trajectories.

2. Galperin (see [3]) found a massive (i.e., with a strictly positive measure) open subset of the set of *obtuse* triangles such that every triangle of the subset contains a periodic trajectory. These trajectories are stable. The subset is a union of "triangles" with one curvilinear side (in [3] they are called "hang-gliders"). In the left-hand lower part of Fig. 1 we put an odd number k in every hang-glider. In a hang-glider marked by k, every point corresponds to a triangle containing a periodic trajectory with k links surrounded by a bundle of periodic trajectories with $2k$ links. If a point belongs to several hang-gliders, the triangle it represents contains several types of periodic trajectories corresponding to all these hang-gliders.

3. The diagonal of the square $\pi/2 \times \pi/2$ in Figure 1 corresponds to the right triangles. Periodic trajectories in generic (*irrational*) right triangles were first discovered in [3]. These are trajectories of a specific kind, the so-called "mirror" trajectories (see Section 3). The authors of the paper [3] assert without proof that these trajectories are *unstable*.

4. A separate class of triangles is that of *rational* triangles, i.e., such that all their angles are rational multiples of π. (Everything we say in this item is also true of rational polygons.) Although all rational triangles contain periodic trajectories, we did not show them in Figure 1: the corresponding points simply fill all the square densely. Billiard trajectories in rational triangles have the following properties.

   (a) Any *perpendicular* trajectory is *periodic* except if it hits a vertex of the polygon (in which case it is said to be singular) – see [3]. It is shown in [1] that perpendicular trajectories fill all the triangle *except perhaps a finite number of points.*

(b) The famous Masur theorem [5] asserts that *the set of directions of periodic orbits is dense in the circle S^1 of velocity directions.* However, Masur's result gives no indication about the location of periodic orbits.

(c) Recently the following strengthening of Masur's theorem was obtained (see [1]). Consider a point in the phase space of the triangle and an arbitrarily small neighborhood of this point. For an arbitrarily big integer N, this neighborhood contains a phase point corresponding to a periodic trajectory whose period is greater than N. In particular, it follows from this result that the set of "periodic points" in the phase space of a rational triangle is everywhere dense in it.

Nothing is known about the stability of periodic trajectories in rational triangles except for some specific cases (see for example Items 1 and 2).

5. In the recent paper [2] it is proved that *almost all perpendicular trajectories in a right triangle are periodic*. For rational right triangles this coincides with the result we cited in Item 4a. The paper says nothing about stability of these trajectories. The authors prove that the footpoints of *nonperiodic* perpendicular trajectories form a set of measure 0 on the sides of the triangle. It is unknown whether this set is finite or not (the authors' conjecture is that there is *at most one* such trajectory for each of the three directions).

The structure of this article is as follows.

In *Section 2* we prove that through every point of a right triangle passes a periodic trajectory. This is the main result of the article. To prove it, we construct a special type of periodic trajectories (we call them *S-trajectories*) and show that they cover the triangle entirely.

In *Section 3* we describe mirror periodic trajectories first introduced in [3]. Call α the smaller acute angle of the triangle. We show that for $\alpha \leqslant \pi/6$ there are infinitely many bundles of mirror trajectories while for $\alpha > \pi/6$ their number is given by the formula

$$k = \lfloor -\frac{2\sin^2\alpha}{4\cos^2\alpha - 3} + \frac{1}{2} \rfloor,$$

where $\lfloor x \rfloor$ denotes the greatest integer strictly less than x.

In *Section 4* we prove the instability of all known types of periodic trajectories, namely, of S-trajectories, mirror trajectories, and perpendicular trajectories.

In *Section 5* we consider billiards in right-angled tetrahedra of any dimension. An n-dimensional right-angled tetrahedron is the region that an oblique hyperplane cuts out from an n-dimensional octant. We prove that every right-angled tetrahedron contains an n-dimensional bundle of periodic trajectories. To do it, we first find a way of constructing periodic trajectories in the tetrahedron starting from periodic trajectories in rhombuses, then show that the trajectories obtained belong to n-dimensional bundles of periodic trajectories.

Finally, in *Section 6* the results obtained are applied to the mechanical system "*two elastic masses on a segment*", whose configuration space is a right triangle. Some results (either known earlier or proved in this article) are reformulated in terms of mechanics.

2. Periodic trajectories cover the triangle

In this section we prove the following theorem.

Theorem 1. *Through every point of a right traingle passes a periodic trajectory.*

We will construct two bundles of periodic trajectories and show that they cover the triangle entirely. (Actually, our construction yields more than two bundles of periodic trajectories, but only two are used in the proof; see Remark 2 at the end of the section.)

Remark 1. It follows from the results of the article [2] (described in Item 5 of the introduction) that the set of points not covered by periodic trajectories has the zero measure. We prove that this set is actually empty.

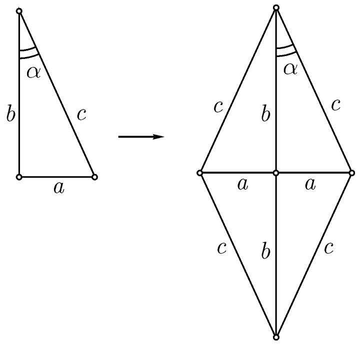

Fig. 2

All the constructions of periodic trajectories in right triangles were so far obtained in two steps. First, reflecting the triangle in its legs, we obtain a rhombus whose each diagonal is composed of two legs of the triangle (see Fig. 2). (For shortness, the vertices of the acute angles of the rhombus will be called its *extremities*.) Then we reflect the rhombus in its sides either following a straight line assumed to be an unfolding of a periodic trajectory (for a definition of an unfolding see for example [3]), or constructing a corridor of rhombuses in such a way that the last rhombus is parallel to the initial one. Then we can join two identical points in these rhombuses by a segment which, if entirely contained in the corridor, is the unfolding of a periodic trajectory.

Usually the trajectories that get into the vertex of an angle are considered singular and are said to end at this vertex. But the right angle, having the form π/n, is a removable singularity (see [4]): if two close trajectories hit different sides of the angle their unfoldings in the rhombus remain close and therefore the trajectories themselves remain close. It is possible to make two close trajectories remain close after leaving the angle even if one of them gets into the vertex. To achieve this, we assume that a particle getting into the vertex is merely reflected back. In other words, the unfolding of a trajectory that gets into the vertex of the right angle continues straight forward in the rhombus.

Denote the smaller acute angle of the triangle by α. Then the acute angle of the rhombus is equal to 2α. Since the rhombus is symmetric, its reflection in a side is equivalent to a rotation around its extremity by an angle $+2\alpha$ or -2α, depending on the direction of rotation.

We will prove Theorem 1 by the following construction shown in Figure 3. First reflect the initial rhombus in its sides n times so as to make it turn *counterclockwise* by $2\alpha \cdot n$ around the vertex B; then n more times so that it turns *clockwise*, by $-2\alpha \cdot n$, around the vertex C. The number n can change from 1 to $\lceil \pi/2\alpha \rceil$, where $\lceil x \rceil$ means the smallest integer greater than or equal to x. Later we will denote the maximal value of n by N, i.e., we put $N = \lceil \pi/2\alpha \rceil$. After such reflections the first and the last rhombuses will be parallel to each other (vertical in Fig. 3). Note that the corridor of rhombuses shown in Figure 3 has a center of symmetry. It coincides with the center O_n of the nth rhombus (that is, of the rhombus on which we change the direction of rotation).

Let us trace the maximally wide bundle of segments parallel to the line $O_0O_nO_{2n}$ and entirely contained in the rhombuses constructed. We will call this bundle a *strip*. All the segments of the strip (except its borders) are unfoldings of periodic trajectories in $\triangle BO_0A_1$ (Fig. 3). Later we call these trajectories *S-trajectories* (because the corridor resembles the letter S). Every n yields its own strip.

Obvious symmetry arguments show that for every $k \geqslant 1$, O_0O_{2k-1} is perpendicular to BA_k while O_0O_{2k} is perpendicular to BO_k. This allows us to prove two propositions.

Proposition 1. *S-trajectories are perpendicular.*

Proof. The trajectories of the nth strip are parallel to O_0O_n. Therefore they meet either the longer leg BA_k or the hypotenuse BO_k of the triangle being perpendicular to it.

Proposition 2. *The nth strip is inclined at an anlge $n\alpha$ with respect to the line O_0A_1.*

Proof. For an even n, $n = 2k$, the sides of the angle $\angle A_1O_0O_n$ are perpendicular to the sides of the angle $\angle O_0BO_k$. Therefore $\angle A_1O_0O_n = \angle O_0BO_k = n\alpha$. Similarly, for an odd n, $n = 2k-1$, the sides of the angle $\angle A_1O_0O_n$ are perpendicular to the sides of the angle $\angle O_0BA_k$ and we have again $\angle A_1O_0O_n = \angle O_0BA_k = n\alpha$.

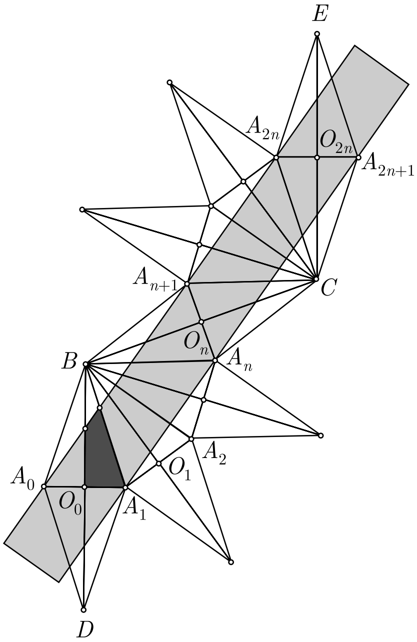

Fig. 3

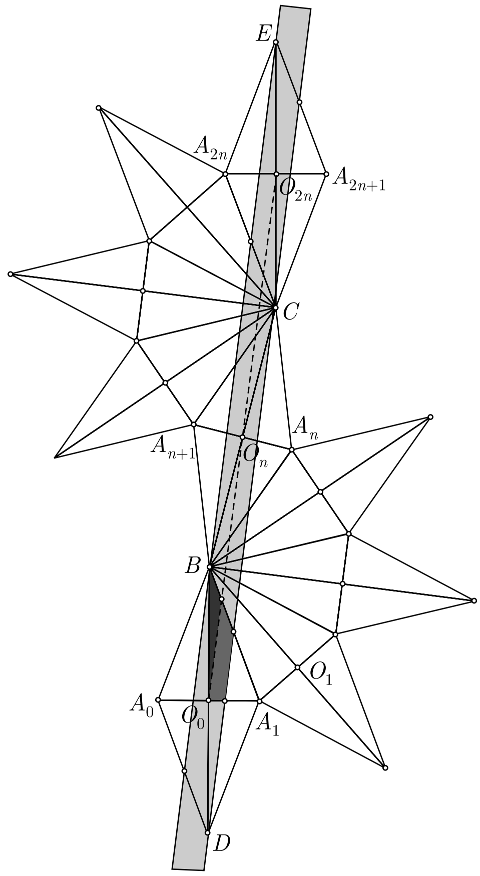

Fig. 4

Let us show that the strips corresponding to the two maximal values of n ($n = N - 1$ and $n = N$) cover the triangle BO_0A_1 entirely.

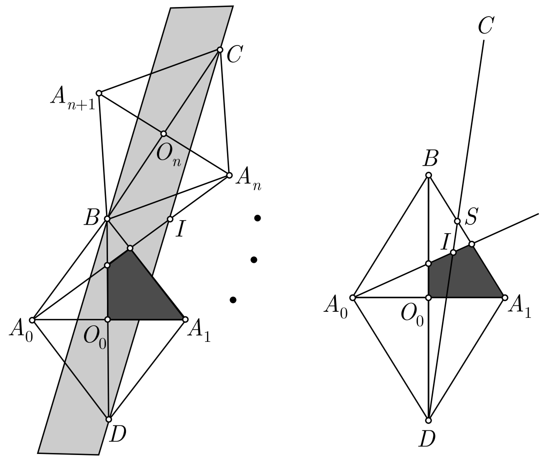

Fig. 5

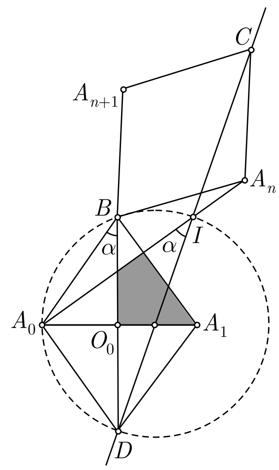

Fig. 6

Figure 3 represents the first strip (with $n = N - 1$). It is bounded by the lines $A_0A_{n+1}A_{2n}$ and $A_1A_nA_{2n+1}$. Indeed, it cannot by wider (otherwise its boundary would have passed either further to the left than A_0 or further to the right than A_{2n+1}). And it is entirely contained in the rhombuses because the polygons $BA_0A_1 \ldots A_nA_{n+1}$ and $CA_nA_{n+1} \ldots A_{2n}A_{2n+1}$ are *convex*.

The part of the triangle BO_0A_1 covered by this strip is shaded black in Fig. 3.

Figure 4 shows the second strip (corresponding to $n = N$). By similar arguments one obtains that it is bounded by the lines BE and DC. The part that it covers is shaded black in Fig. 4.

In Fig. 3, $n = N - 1$, while in Fig. 4, $n = N$. Therefore the lines A_0A_{n+1}, O_0O_n, and A_1A_n in Figure 3 coincide with the lines A_0A_n, O_0O_{n-1}, and A_1A_{n-1} respectively in Figure 4.

We now have to show that the two black parts cover the triangle entirely. Look at Figure 5. (In this figure $n = N$ as in Figure 4.) We see that the triangle BO_0A_1 is entirely covered iff the lines CD and A_0A_n intersect outside it, as in Fig. 5a, and *not* inside it, as in Fig. 5b.

Now, the lines A_0A_n and DC are inclined at angles $(N-1)\alpha$ and $N\alpha$ with respect to the line O_0A_1. Therefore the angle between them equals α, i.e., $\angle A_0ID = \alpha$. Since $\angle A_0BD = \alpha$ as well, it follows that the points D, A_0, B, and I lie on the same circle (see Fig. 6). The angle $\angle DA_0B$ is the obtuse angle of the rhombus. Therefore the circle circumscribed around $\triangle DA_0B$ lies entirely outside the rhombus. The point I is on this circle, therefore it lies outside the rhombus and in particular outside $\triangle BO_0A_1$.

Thus the triangle is entirely covered by the two strips considered.

■

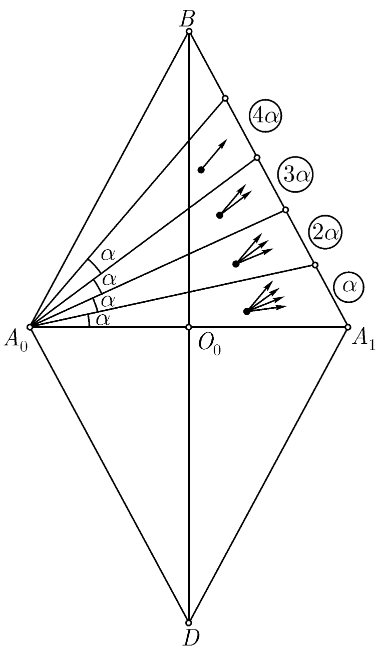

Fig. 7

Remark 2. We have proved that two strips of S-trajectories corresponding to the two maximal values of n cover the triangle BO_0A_1. The other strips are unnnecessary because, as one can easily see, for $n \leqslant N-1$ the part of $\triangle BO_0A_1$ covered by the $(n-1)$st strip is entirely contained in the part covered by the nth strip. The upper boundaries of the strips with $n \leqslant N-1$ go from A_0 at angles α, 2α, 3α, $\ldots$, $(N-1)\alpha$ (see Fig. 7). They divide the triangle into N parts. For $n \leqslant N-1$ and any k satisfying $n \leqslant k \leqslant N-1$, a trajectory launched from a point of the nth part at an angle $k\alpha$ is periodic.

The Nth part, that is, the uppermost one, is covered by trajectories of the last strip (the one with slope $N\alpha$).

3. “Mirror” periodic trajectories

To start with, let us describe the construction of mirror trajectories. In Fig. 8 the vertical rhombus A rotates first by 2α around the upper extremity, then by -2α around the lower one, thus giving a new vertical rhombus denoted by B. Now rotate the rhombus B by -2α about the lower extremity, then by 2α around the upper one. The vertical rhombus C thus obtained is the image of the initial rhombus A under a horizontal translation. The same two operations are further applied alternately to the rhombuses C, D, E, etc. Thus we obtain a corridor of rhombuses. The shorter diagonals of the *vertical* rhombuses are thought of as mirrors. (This explains the name “mirror periodic trajectories”.) A beam launched from a point of the very first mirror to the identical point of an upper mirror will move periodically in the corridor between the mirrors. If entirely contained in the union of rhombuses, it will form an unfolding of a periodic trajectory.

Let us number the upper rhombuses by numbers 1, 2, 3, etc. (Fig. 8). Denote the center of the very first rhombus by O and the center of the nth upper rhombus by O_n for every n. The beginning of every beam is parallel to the line OO_n for some n. The number n uniquely determines the type of the trajectory.

Later on we will speak of trajectories of *type number* n meaning that their beginning is parallel to OO_n.

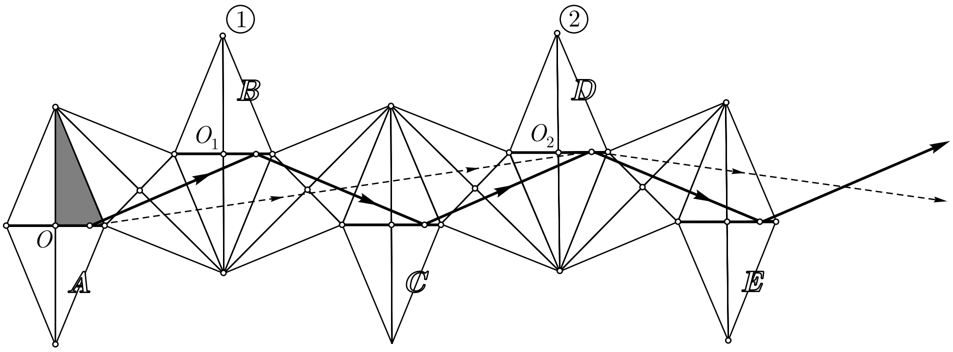

Fig. 8

We are now in position to formulate the theorem this section is dedicated to.

Theorem 2. *If $\alpha \leqslant \pi/6$ there exist mirror trajectories of type n for all n. If $\alpha > \pi/6$ there exist mirror trajectories of type n only for*

$$n < -\frac{2\sin^2\alpha}{4\cos^2\alpha - 3} + \frac{1}{2}\,.$$

Proof. The first part of the theorem deals with the case $\alpha \leqslant \pi/6$. Consider the strip bounded by two horizontal lines, the first containing the upper mirrors, the second the lower ones (see Fig. 9). The beams for all n are contained in this strip. It is obvious that the point Q lies below the strip. K also lies below the strip because $\angle ONP + \angle PNK \geqslant \pi/3 + 2\pi/3 = \pi$. Similarly, all the other lower extremeties of rhombuses lie below the strip and their upper extremities, above the strip. Thus the strip and therefore the beams for all n are entirely contained in the corridor of rhombuses. This means proved that for $\alpha \leqslant \pi/6$, there exist mirror trajectories of all types.

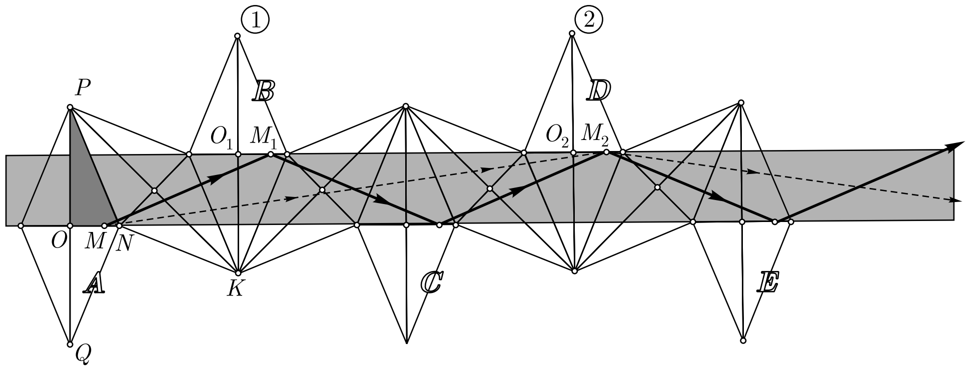

Fig. 9

To prove the second part of the theorem, we will find for every n the angle φ_n at which the trajectories of the nth type are inclined with respect to the line ON (Fig. 10). This angle is equal to the angle between the horizontal line ON and the segment OO_n. It is clear that $\angle O_1ON = \alpha$ because the sides of the angle O_1ON are perpendicular to those of the angle $\angle OPN$. Thus, $\varphi_1 = \alpha$. The projection of the segment OO_n on the vertical axis is equal to that of OO_1, whereas its projection on the horizontal axis is $2n - 1$ times as big as that of OO_1. Thus $\tan\varphi_n = (\tan\varphi_1)/(2n - 1)$, and

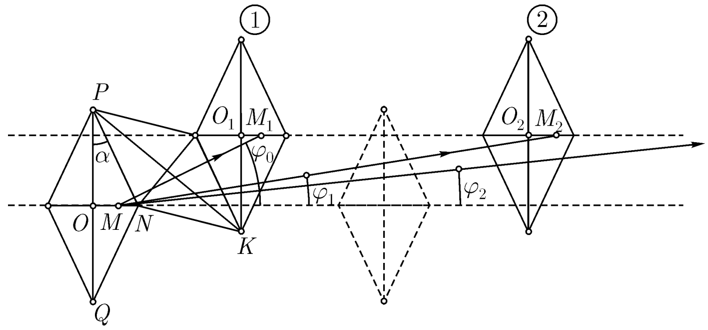

Fig. 10

therefore

$$\varphi_n = \arctan\left(\frac{1}{2n-1}\tan\alpha\right). \tag{3.1}$$

Although we use this formula only in the second part of the proof, it holds for $\alpha \leqslant \pi/6$ as well as for $\alpha > \pi/6$.

Suppose that the smaller acute angle of the triangle is equal to $\pi/6 + \varepsilon$, $0 < \varepsilon \leqslant \pi/12$ (see Fig. 11). Then the vertex K lies above the line ON; the angles $\angle RNK$ and $\angle NRK$ equal 3ε (because $\angle ONP + \angle PNK = \pi - 3\varepsilon$).

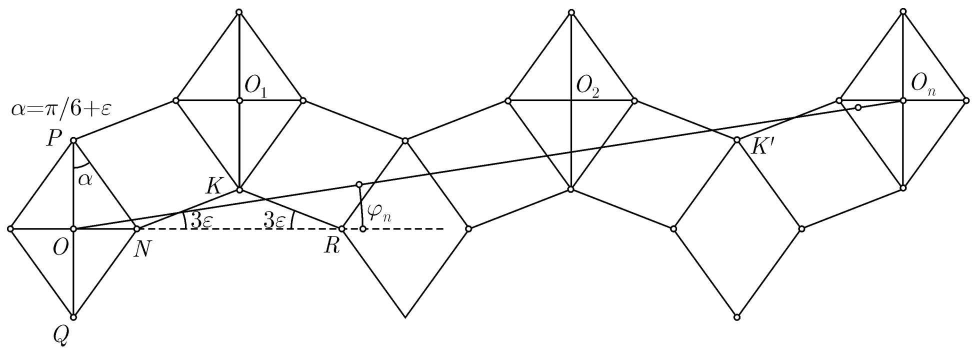

Fig. 11

Mirror trajectories of type n exist if and only if the line OO_n passes above K. Indeed, if it passes above K, it follows from symmetry arguments that it passes below K' (see Fig. 11). Therefore it passes below all the upper extremities of rhombuses and above all the lower extremities. Thus it is contained in the corridor of rhombuses and is surrounded by a bundle of trajectory unfoldings. Inversely, if the line OO_n passes through or below K (and therefore through or above K'), any line parallel to it will pass either below K or above K' and will not be an unfolding of a trajectory.

Thus mirror trajectories of type n exist if and only if $\varphi_n > \angle KON$, where φ_n is determined by formula 3.1. Denoting the shorter leg of the triangle by a, the longer one by b, and the hypotenuse by c we get:

$$\angle KON = \arctan\left(\frac{c\sin(3\varepsilon)}{a + c\cos(3\varepsilon)}\right) = \arctan\left(\frac{\sin(3(\alpha-\pi/6))}{a/c + \cos(3(\alpha-\pi/6))}\right) =$$

$$\arctan\left(\frac{\sin(3\alpha-\pi/2)}{\sin\alpha + \cos(3\alpha-\pi/2)}\right) = -\arctan\left(\frac{\cos(3\alpha)}{\sin\alpha + \sin(3\alpha)}\right), \quad \text{therefore}$$

$$\varphi_n > \angle KON \Longleftrightarrow$$

$$\arctan\left(\frac{1}{2n-1}\tan\alpha\right) > -\arctan\left(\frac{\cos(3\alpha)}{\sin\alpha+\sin(3\alpha)}\right) \Longleftrightarrow$$

$$\frac{1}{2n-1}\tan\alpha > -\frac{\cos(3\alpha)}{\sin\alpha+\sin(3\alpha)} \Longleftrightarrow$$

$$2n-1 < -\tan\alpha\frac{\sin\alpha+\sin(3\alpha)}{\cos(3\alpha)} \Longleftrightarrow$$

$$n < -\frac{1}{2}\left(\tan\alpha\frac{\sin\alpha+\sin(3\alpha)}{\cos(3\alpha)}-1\right) = -\frac{2\sin^2\alpha}{4\cos^2\alpha-3}+\frac{1}{2}\,.$$

The total number of trajectories is

$$k = \lfloor -\frac{2\sin^2\alpha}{4\cos^2\alpha-3}+\frac{1}{2}\rfloor, \tag{3.2}$$

where $\lfloor x\rfloor$ denotes the greatest integer strictly less than x.

The theorem is proved. ■

Remark 3. If α is close to $\pi/4$ and the triangle is almost isosceles the series contains only one type of mirror trajectories. One can deduce from formula (3.2) that this situation occurs when

$$\arctan(\sqrt{3/5}) \leqslant \alpha \leqslant \pi/4 \quad .$$

In the next section we will need the following proposition concerning mirror trajectories.

Proposition 3. *The period of a mirror trajectory of the nth type consists of $10n-4$ links.*

Proof. Denote by M the point from which the beam is launched, and by M_n (far all n) the corresponding point of the nth upper mirror (see Fig.10). In the unfolding, a period of a trajectory of nth type is represented by the segment MM_n. Therefore we have to prove that the sides and diagonals of the rhombuses divide this segment into $10n-4$ parts. It suffices to note that the segment MM_1 is indeed divided into $10\cdot 1-6=4$ parts and that we add 10 new parts by changing MM_n to MM_{n+1}. ■

4. Instability of periodic trajectories

In this section we prove the following theorem.

Theorem 3. A. *Mirror periodic trajectories are unstable.*
B. *Perpendicular periodic trajectories are unstable.*
C. *S-trajectories are unstable.*

(Recall that S-trajectories are the ones constructed in the proof of Theorem 1.)

We consider the three assertions separately.

A. Instability of the mirror trajectories. We will use the following criterion of instability introduced in [3]. Denote by a_1, a_2, a_3, etc the sides that the particle meets. Consider the *alternating sum* of these sides, i.e., the expression $a_1 - a_2 + a_3 - \ldots \pm a_N$. Here the sum is taken over one period if the length of the period is even, and over two periods if this length is odd. The criterion asserts that *the trajectory is stable if and only if the alternating sum vanishes:*

$$a_1 - a_2 + a_3 - \ldots \pm a_N = 0,$$

i.e., if each letter occurs with a $+$ as many times as with a $-$. The sequence $a_1 a_2 \ldots a_N$ is called the *code word* of the trajectory.

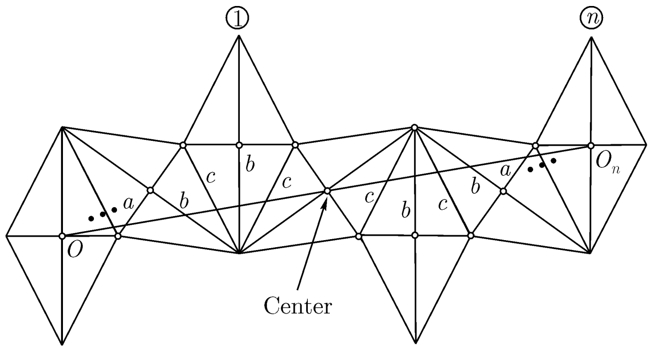

Fig. 12

Denote the smaller leg of the triangle by a, the longer one by b, and the hypotenuse by c. Thus the sides of rhombuses are denoted by c and the halves of diagonals by a and b. Let us find the alternating sum of letters for mirror trajectories of the nth type. The length of the period is even (see Proposition 3), so the sum is taken over one period. The unfolding of the trajectory is obtained from the line OO_n by a translation (see Fig. 12). First we find the code word for the line OO_n (without taking into account the points where it passes through centers of rhombuses). Since the line is always symmetric with respect to the center of one of the rhombuses, the code word has the form $X\overline{X}$, where X is a word and $\overline{X}$ is X read backwards. Now let us move the line OO_n slightly to the left so that it becomes a real unfolding of a trajectory.

If n is even ($n = 2k$), the middle rhombus is inclined to the left. Therefore the code word will become $W = abXba\overline{X}$. The length of this word is $10n + 6$ (see Proposition 3). Therefore the length of X equals $(10n + 6 - 4)/2 = 10k + 1$, which is odd. Thus the alternating sum of letters of W is equal to

$$a - b + X - b + a - \overline{X} = 2a - 2b,$$

where X and $\overline{X}$ denote the alternating sum of the letters composing them.

For an odd n, $n = 2k + 1$, the middle rhombus is inclined to the right, so after the shift of the line OO_n, the word becomes $W = abXab\overline{X}$, where the length of X (equal to $(10n + 6 - 4)/2 = 10k + 6$) is even. The alternating sum equals

$$a - b + X + a - b + \overline{X} = 2a - 2b.$$

(Note that if the length of X is even, we have $X - \overline{X} = 0$, while if the length is odd, we have $X + \overline{X} = 0$.)

We see that the alternating sum is always equal to $2a - 2b$.

Similarly one can prove that if the unfolding is obtained from the line OO_n by a shift *to the right*, the ulternating sum equals $2b - 2a$. The sum is different from 0, therefore mirror trajectories are unstable. ■

B. Instability of the perpendicular trajectories. We will use the following theorem from [3]: *if a perpendicular periodic trajectory in a polygon is stable then the number of its links is a multiple of 4.* We will show that the number of links in a perpendicular trajectory in a right triangle is never divisible by 4, which will imply its instability. It suffices to consider the case where the angle α is incommensurable with π, because if there existed a stable periodic trajectory for an α commensurable with π, we would be able, by a small tranformation of the triangle, to obtain such a trajectory for an incommensurable α as well.

According to [2] every perpendicular trajectory has the following form. Consider the unfolding of the trajectory such that the trajectory is perpendicular to a side or diagonal of the first rhombus. Let us assign a number to each of its rhombuses: the initial rhombus has the number 0; a rhombus turned by an angle $n \cdot 2\alpha$ with respect to the initial one, has the number n. (The same number can be assigned to several rhombuses.) Note that since α is incommensurable with π, the position of every rhombus determines its number unambiguously, and that the numbers of two consecutive rhombuses differ by 1. The particle launched from the rhombus number 0 goes through rhombuses with positive numbers until it meats another rhombus with number 0. Instead of continuing the unfolding in the same direction, we now imagine that the particle is reflected by the segment that is perpendicular to the trajectory. After that, it returns to the initial rhombus following its own track.

We will consider the first half of the track, before the reflection. We say that the particle goes *forward* or *backward* depending on whether the numbers of rhombuses it goes through increase or decrease. (These notions are also borrowed from [2].) Since the particle starts its motion going forward (from rhombus 0 to rhombus 1) and finishes it going backward (from rhombus 1 to rhombus 0), the total number of places where it changes direction is *odd.* Now note that if a particle goes through a rhombus without changing direction, then this rhombus contains *two* links of the trajectory (Fig. 13a). In contrast, when the particle changes direction inside a rhombus, the rhombus contains *three* of its links (Fig. 13b).

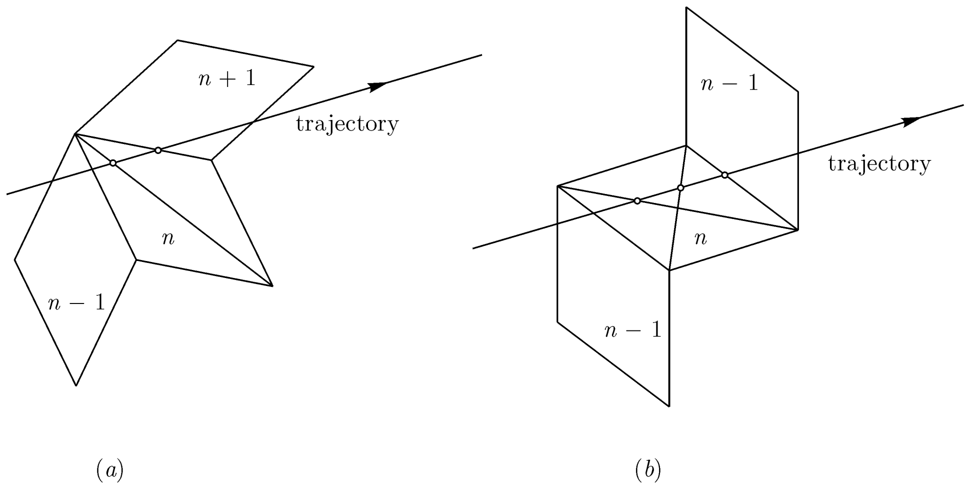

Fig. 13

Since the number of rhombuses where it changes direction is odd, the total number of links in the first half of the track is also odd. The whole period contains twice as many sides as its first half, so their total number cannot be divisible by 4. The instability is proved. ■

C. Instability of the S-trajectories. Since the S-trajectories are perpendicular (see Proposition 1), it follows that they are unstable. Using the same method as in Part A one can show that the alternating sum for S-trajectories is equal either to $2a - 2b$ or to $2b - 2a$. ■

5. Periodic trajectories in right-angled tetrahedra

A right-angled tetrahedron in $\mathbf{R}^n$ is a pyramid that an oblique hyperplane of codimension 1 cuts out from a hyperoctant (Fig. 14a). In this section we construct some periodic trajectories in right-angled tetrahedra. We prove the following theorem.

Theorem 4. *In every n-dimensional right-angled tetraherdron there exists at least one n-dimensional bundle of periodic trajectories.*

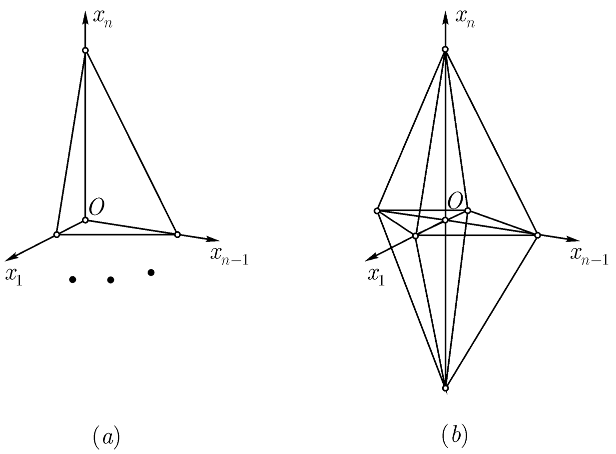

Fig. 14

Proof. As in the planar case, the first step of the construction is to reflect the tetrahedron in its $n-1$ right-angled hyperfaces in order to get a symmetrical figure in $\mathbf{R}^n$: a bipyramid, which we will call a *rhomboid* (Fig. 14b). We denote by O its center of symmetry. Similarly to the case of right triangles, we do not consider singular the trajectories that get into the right-angled vertex of the tetrahedron or in its right-angled edge of any dimension. Instead, we assume that the trajectory continues in such a way that its unfolding is a straight line in the rhomboid.

The next part of the proof is dedicated to constructing a periodic trajectory in the rhomboid.

Let us take two opposite vertices A and B of the rhomboid. (It is convenient to imagine the line AB as vertical.) Every 2-dimensional plane P passing through A and B intersects the rhomboid in a rhombus that we will call R.

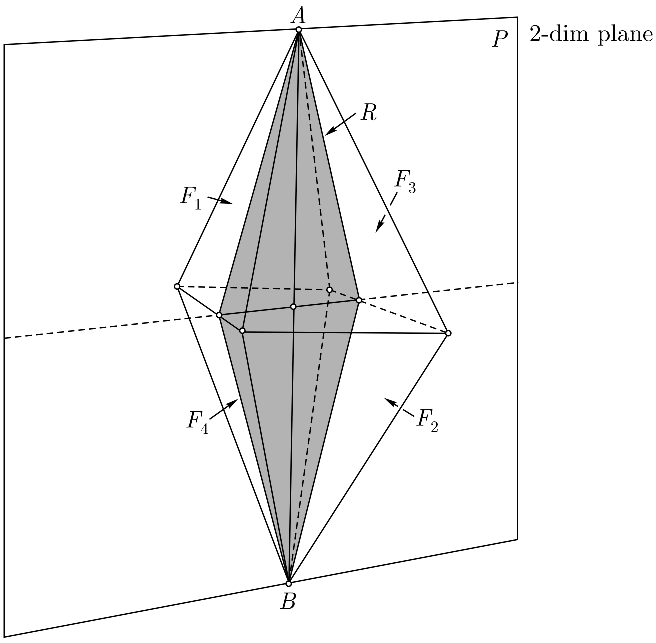

Fig. 15

Consider a particular case (shown in Fig. 15) when the plane P is perpendicular to one of the hyperfaces of the rhomboid. Let us denote this hyperface by F_1. Then P is at the same time perpendicular to three other hyperfaces, namely to the hyperface F_2 obtained from F_1 by the central symmetry of center O, and to the two hyperfaces F_3 and F_4 obtained from F_1 and F_2 by the symmetry with respect to the line AB. Now consider a billiard particle moving inside the rhombus R. Suppose it hits a hyperface F_i ($i = 1, \ldots, 4$) of the romboid. Since the plane P containing the rhombus is perpendicular to this hyperface, the particle will stay in the rhombus after the collision. Therefore the trajectory of the particle is entirely contained in the rhombus. In other words, any billiard trajectory in the rhombus is at the same time a billiard trajectory in the rhomboid. We know that there is at least one periodic trajectory in any rhombus (for example a mirror trajectory). Therefore we can claim that there is at least one periodic trajectory in the rhomboid.

Now we will prove that every periodic trajectory T in the rhombus R is surrounded by an n-dimensional bundle of periodic trajectories.

To begin with, every periodic trajectory in the rhombus is surrounded by a 2-*dimensional* bundle of periodic trajectories lying in the same rhombus.

Now let us take a small segment l contained in the trajectory T and shift it in any direction in such a way that it remains parallel to itself. Denote the segment obtained by l'. We must prove that the billiard trajectory containing l' is periodic, provided the shift is small enough. It is convenient to present the shift of l as a composition of two shifts: $s_{\|}$ and $s_{\perp}$. The first one is parallel to the plane P and takes l to a segment $\tilde{l}$ contained in the rhombus R. If $s_{\|}$ is small enough, the trajectory $\widetilde{T}$ containing the segment $\tilde{l}$ is periodic because it belongs to the 2-dimensional bundle surrounding T. The second shift, $s_{\perp}$, is perpendicular to P and takes $\tilde{l}$ to l'.

Consider the plane P' containing l' and parallel to P. It is perpendicular to the hyperfaces F_i ($i = 1, \ldots, 4$). Therefore if, moving in the plane P' inside the rhomboid, a particle hits one of these hyperfaces, it will stay in P' after the collision. The lines in which P' intersects the hyperplanes containing the hyperfaces F_1, F_2, F_3, and F_4, form a rhombus R' that is the image of R under the shift $s_{\perp}$ (see Fig. 16). The rhombus R' is not entirely contained in the rhomboid: small neighborhoods of its upper and lower vertices lie outside it. These neighborhoods can be made as small as we wish by taking a sufficiently small $s_{\perp}$. Applying the shift $s_{\perp}$ to the trajectory $\widetilde{T}$ in R, we get a periodic trajectory in R', which we will denote by T'. Note that T' contains the segment l'. Since T' is periodic, it does not come infinitely close to the upper or the lower vertices of the rhombus. Therefore by taking a sufficiently small $s_{\perp}$, the parts of R' that lie outside the rhomboid can be made not to intersect with T'. This means that T' is entirely contained in the rhomboid. Therefore the billiard particle in the rhomboid launched along l' will follow the trajectory T' hitting only the hyperfaces F_1, F_2, F_3, and F_4. Thus T', originally constructed as a billiard trajectory in R', proves to be at the same time a billiard trajectory in the rhomboid. (In contrast, if T' were not entirley contained in the rhomboid, it could not be a billiard trajectory in it. At some point, the particle would hit a hyperface of the rhomboid passing inside the rhombus R'. It would then be reflected in an unpredictable way and leave the trajectory T'.)

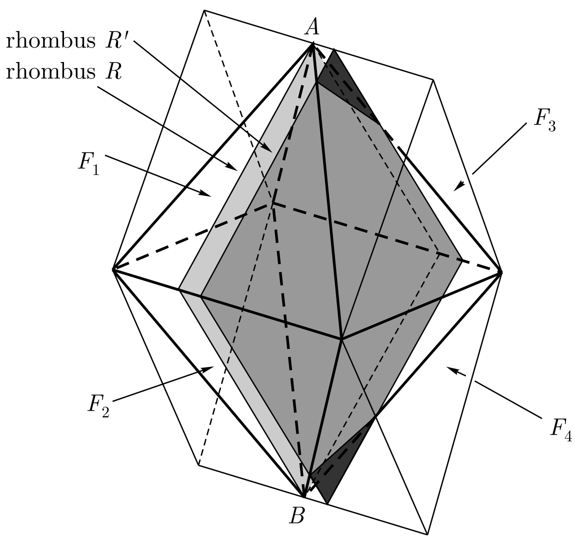

Fig. 16

We have proved that the segment l' obtained from l by a sufficiently small shift in any direction, is contained in a periodic trajectory. This means that every trajectory T is surrounded by an n-dimensional bundle of periodic trajectories.

Thus any right-angled tetrahedron contains at least one n-dimensional bundle of periodic trajectories. ■

Remark 4. A right-angled tetrahedron can contain much more than one bundle of periodic trajectories. For example, if the acute angle of the rhombus R is less than $\pi/3$ (and thus the smaller acute angle of the corresponding right triangle is less than $\pi/6$), the rhombus contains infinitely many distinct combinatorial types of periodic trajectories (mirror trajectories from Section 3). All of them are surrounded by n-dimensional bundles of periodic trajectories.

Remark 5. The rhombus R contains the line AB formed by 1-dimensional edges of the tetrahedra. Therefore for every trajectory in this rhombus, the corresponding trajectory in the tetrahedron passes many times through an edge. Although we agreed not to consider such trajectories singular, this would have been a rather unpleasant fact if we had not proved the existence of n-dimensional bundles (in a bundle, the majority of the trajectories do not pass through any singularities).

6. The mechanical interpretation

In this section we reformulate in terms of mecanics some results (either known earlier or proved in this article) concerning billiard trajectories in right triangles.

Consider two elastic masses moving on the segment $[0, 1]$ and colliding with each other and with the ends of the segment. We denote the masses by m_1 and m_2, their initial positions by x_1 and x_2, and their initial velocities by v_1 and v_2. We suppose that the second particle is situated further to the right than the first one, that is $(0 \leqslant x_1 \leqslant x_2 \leqslant 1)$.

Theorem 5. *Let* $\arctan\left(\sqrt{m_1/m_2}\right)$ *be commensurable with* π*. Then the periodic orbits in such a system are everywhere dense in the phase space. Morever, for given* x_1, x_2, v_1, *and* v_2, *an arbitrarily small* ε, *and an arbitrarily great integer* N, *it is possible to change* x_1, x_2, v_1, *and* v_2 *by less than* ε *so that the motion of the points be periodic and the period be greater than* N *(that is, the number of bounces in the period is greater than* N*).*

Theorem 6. *For any* m_1, m_2, x_1, x_2, *and* $v_1 \neq 0$ *there exists a* v_2 *such that the motion is periodic.*

Theorem 7. *Suppose* $x_1 = 0$, $m_1 \geqslant 3m_2$*. Then for any* x_2 *and* $v_1 \neq 0$ *there exists a* v_2 *such that the motion is periodic. Moreover, the period can be made as big as we wish.*

Theorem 8. *Suppose* $x_1 = 0$, $v_2 = 0$, $v_1 \neq 0$*. (The masses* m_1 *and* m_2 *are arbitrary.) Then for almost all* x_2 *the motion of the system is periodic.*

Proofs. Theorems 5, 6, 7, and 8 are proved by considering the configuration space of the system, i.e., the right triangle with acute angle $\alpha = \arctan\left(\sqrt{m_1/m_2}\right)$ (see [6]).

Before proving the theorems, we will explain why the movement of the masses is equivalent to the movement of a particle in such a triangle.

Put $y_1 = \sqrt{m_1}\, x_1$, $y_2 = \sqrt{m_2}\, x_2$. We will represent the position of the masses by the point with coordinates (y_1, y_2) in the plane. This point will play the role of a billiard particle. Its velocity vector $\vec{u}$ equals $\vec{u} = (u_1, u_2) = (\sqrt{m_1}\, v_1, \sqrt{m_2}\, v_2)$.

The configuration space of our system is determined by the conditions

$$0 \leqslant x_1 \leqslant 1 \Longleftrightarrow 0 \leqslant y_1 \leqslant \sqrt{m_1},$$

$$0 \leqslant x_2 \leqslant 1 \Longleftrightarrow 0 \leqslant y_2 \leqslant \sqrt{m_2},$$

$$x_1 \leqslant x_2 \Longleftrightarrow \sqrt{m_1}\, y_1 \leqslant \sqrt{m_2}\, y_2.$$

In other words, the configuration space is the right triangle with legs equal to $\sqrt{m_1}$ and $\sqrt{m_2}$ (see Fig. 17). A collision of a mass with an extremity of the segment corresponds to a collision of the billiard particle with a leg of the triangle, while a collision between the masses corresponds to a collision of the particle with the hypotenuse.

What we have to show is that the collisions of the particle obey the reflection law. This is almost obvious for a collision with a leg. Indeed, suppose that, say, the first mass collides with the left extremity of the segment. Then v_2 (and therefore u_2) do not change, while v_1 (and therefore u_1) change the sign. This means that the particle is reflected in the left leg of the triangle accordingly to the reflection law. The case of the other leg is similar. Now consider a collision between the masses. We denote by v_1', v_2', u_1', u_2', and $\vec{u}'$ the values of v_1, v_2, u_1, u_2, and $\vec{u}$ after the collision. Besides, we denote by $\vec{h}$ the vector represented by the hypotenuse of the triangle, $\vec{h} = (\sqrt{m_1}, \sqrt{m_2})$. The conservation of energy gives

$$m_1v_1^2 + m_2v_2^2 = m_1{v'}_1^2 + m_2{v'}_2^2 \iff$$
$$(\sqrt{m_1}\,v_1)^2 + (\sqrt{m_2}\,v_2)^2 = (\sqrt{m_1}\,v_1')^2 + (\sqrt{m_2}\,v_2')^2 \iff$$
$$\|\vec{u}\| = \|\vec{u}'\|.$$

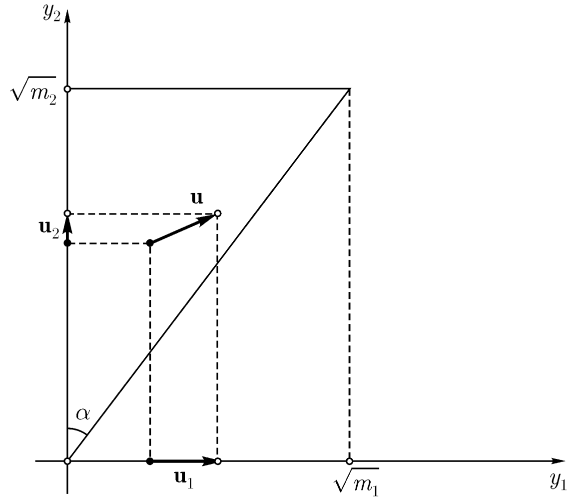

Fig. 17

The conservation of momentum gives

$$m_1v_1 + m_2v_2 = m_1{v'}_1 + m_2{v'}_2 \iff$$
$$\sqrt{m_1}\cdot(\sqrt{m_1}\,v_1) + \sqrt{m_2}\cdot(\sqrt{m_2}\,v_2) = \sqrt{m_1}\cdot(\sqrt{m_1}\,v_1') + \sqrt{m_2}\cdot(\sqrt{m_2}\,v_2') \iff$$
$$\vec{h}\cdot\vec{u} = \vec{h}\cdot\vec{u}'.$$

Thus after the collision, the velocity vector $\vec{u}$ conserves both its norm and the angle that it makes with the hypotenuse of the triangle. This means that the collision obeys the reflection law.

We have established the equivalence between the mecanical system of two masses on a segment and the billiards in the right triangle with acute angles equal to $\arctan\left(\sqrt{m_1/m_2}\right)$ and $\arctan\left(\sqrt{m_2/m_1}\right)$. The proofs of Theorems 5, 6, 7, and 8 follow from this equivalence.

Proof of Theorem 5. It is shown in [1] for a billiard particle in a rational triangle, that for given ε and N and for any initial position and velocity of the particle, the position and the velocity can be changed by less than ε in such a way that the trajectory of the particle becomes periodic and the period is greater than N (see Item 4(c) of the introduction). Theorem 5 follows from this result.

Proof of Theorem 6. Theorem 6 follows from Theorem 1. Theorem 1 asserts that the billiard particle can be launched from every point of a right triangle so that the trajectory be periodic. Moreover, it follows from the construction that the particle can be launched in a direction not parallel to the legs of the triangle. Now it suffices to take v_2 such that the slope of the trajectory at our point is equal to the slope of $\vec{u}$ in Fig. 17, i.e.,

$$\frac{u_2}{u_1} = \frac{\sqrt{m_2}\,v_2}{\sqrt{m_1}\,v_1} = \text{slope of the trajectory}.$$

Since $v_1 \neq 0$ such a v_2 always exists. ■

Proof of Theorem 7. To prove this theorem it suffices to consider the mirror trajectories of Section 3 launched from the smaller leg. Since $m_1 \geqslant 3m_2$, the smaller acute angle α of the triangle satisfies $\alpha = \arctan\sqrt{m_2/m_1} \leqslant \arctan(1/\sqrt{3}) = \pi/6$. Therefore there are infinitely many types of mirror periodic trajectories and the period can be made as big as we wish. (An easy computation shows that if we want the period to be equal to $10N-4$, we must take $v_2 = -(2N-1)\sqrt{m_2/m_1}\cdot v_1$). ■

Proof of Theorem 8. The motion of the system corresponds to perpendicular trajectories. According to [2] almost all such trajectories and therefore motions of the system are periodic. ■

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