Enumeration of almost polynomial rational functions with given critical values

Dmitri Panov*, Dimitri Zvonkine†

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Abstract

1 Rational functions and minimal factorizations of permutations

Let \( f : \mathbb{C} \to \mathbb{C} \) be a rational function of degree \( n \) in one complex variable. A critical point of \( f \) is a point \( z \in \mathbb{C} \) such that \( f'(z) = 0 \). Its degree is the number \( a \geq 2 \) such that \( f \) looks like \( f(z) = z^a \) in the neighborhood of the critical point. A critical value of \( f \) is its value at a critical point. (Note that we do not count poles as critical points.)

Definition 1.1 A rational function \( f \) is called simple if every critical value of \( f \) has exactly one critical preimage. It is called almost polynomial if the sum of orders of its poles is smaller than the degree of each critical point.

*IPDE, IHES, Le Bois-Marie, 35, route de Chartres F-91440, Bures-sur-Yvette, France. E-mail: panov@ihes.fr.

†Institut Mathématique de Jussieu, Université Paris VI, 175, rue du Chevaleret, 75013 Paris, France. E-mail: zvonkine@math.jussieu.fr. The second author was partially supported by EAGER - European Algebraic Geometry Research Training Network, contract No. HPRN-CT-2000-00099 (BBW) and by the Russian Foundation of Basic Research grant 02-01-22004.
Thus an almost polynomial rational function has a numerator of a much bigger degree than the denominator.

Our goal is to find the number of simple almost polynomial rational functions with fixed orders of poles and fixed critical values of fixed degrees, a problem first considered by V. Arnold [1].

The general problem of enumerating ramified coverings of the sphere with fixed ramification types can, in some sense, be solved using the representation theory of the symmetric group; however the answer one obtains is a rather complicated sum over the irreducible representations. In particular, there is still no simple criterion allowing one to determine whether the number of coverings is equal to 0 or not.

For (not necessarily simple) polynomials the problem can be reduced to a combinatorial problem solved by I. P. Goulden, D. M. Jackson in [3]. Later, when the relation to polynomials was discovered, their formula was reproved in [6], [7], and [5] (appendix by D. Zagier) by three different methods. In particular, there is still no simple criterion allowing one to determine whether the number of coverings is equal to 0 or not.

A. Goupil and G. Schaeffer [4] generalized Goulden and Jackson’s result on polynomials to meromorphic functions with a unique pole on Riemann surfaces of any genus, but their answer is not as explicit. For other results on the enumeration of ramified coverings and their relation to the intersection theory on moduli spaces see [2], [8] and the references therein.

All rational functions we consider in the sequel are simple and almost polynomial.

A rational function $f$ can be viewed as a ramified covering of the Riemann sphere by the Riemann sphere. Going around a critical value in the image we obtain a permutation of the sheets in the preimage (the monodromy of the covering). It follows from Riemann’s existence theorem that the problem of counting rational functions can be reformulated in terms of permutations (for details see [5], chapter 2).

Let $\sigma \in S_n$ be a permutation of $n = \deg f$ elements.

**Definition 1.2** A product $\sigma_k \ldots \sigma_1 = \sigma$ is called a minimal factorization of $\sigma$ if (i) the group generated by $\sigma_1, \ldots, \sigma_k$ acts transitively on the set $\{1, \ldots, n\}$ and (ii) the total number of cycles in the permutations $\sigma_1, \ldots, \sigma_k, \sigma$ equals $kn - n + 2$. 
Here $\sigma$ corresponds to the monodromy of $f$ at $\infty$, while the $\sigma_j$’s are the monodromies at the critical values. The conditions guarantee that the ramified covering determined by the permutations $\sigma_1, \ldots, \sigma_k$, $\sigma$ is (i) connected and (ii) of genus 0 (by the Riemann-Hurwitz formula). Rather than counting the cycles of the permutations, it is more natural to consider their defects: a defect being $n$—(the number of cycles). If the Euler characteristic of the covering surface equals $\chi$, the sum of defects of the corresponding monodromies equals $2n - \chi$. Hence the total defect $2n - 2$ of a minimal factorization is the smallest possible for a transitive factorization, which explains the word “minimal”.

For shortness, we will say that a rational function $f$ is of type $(a_1, \ldots, a_k)$, $(p_1, \ldots, p_c)$ if it has $c$ poles of orders $p_1, \ldots, p_c$ and $k$ critical points of degrees $a_1, \ldots, a_k$. (The sum $p = p_1 + \ldots + p_c$ satisfies $p < a_j$ for all $j$.) Similarly, we say that a minimal factorization is of type $(a_1, \ldots, a_k)$, $(p_1, \ldots, p_c)$ if $\sigma$ has cycles of lengths $p_1, \ldots, p_c, n - p$, while each $\sigma_j$ is an $a_j$-cycle, i.e., has exactly one cycle of length $a_j$ the other points being fixed.

To a minimal factorization of type $(a_1, \ldots, a_k)$, $(p_1, \ldots, p_c)$ we can assign a colored graph with oriented edges called a constellation. It is obtained in the following way. Take $n$ numbered vertices. For each $j$, $1 \leq j \leq k$ form an oriented polygon using the vertices from the cycle of $\sigma_j$. The edges of the polygon are colored in “color” $j$. Now forget the numbers of the vertices.

It is clear that the constellation allows one to reconstitute the permutations $\sigma_1, \ldots, \sigma_k$, $\sigma$ up to a common conjugation.

**Definition 1.3** A constellation is a connected graph whose edges are oriented and colored in colors from 1 to $k$, obtained by gluing together $k$ oriented polygons with colors 1, $\ldots, k$ at some of their vertices. A vertex of a polygon cannot be glued to another vertex of the same polygon.

To sum up, we now have three equivalent problems: given $(a_1, \ldots, a_k)$, $(p_1, \ldots, p_c)$, count the number of rational functions, or of minimal factorizations, or of constellations of this type. The three numbers differ by simple combinatorial factors.

We will need some more remarks on constellations.

A constellation coming from a minimal factorization has a natural embedding into a sphere (see [5]). It is given by the following conditions: (i) The orientations of the edges determine the counterclockwise orientation on each
polygon; (ii) If we choose any vertex and enumerate the colors of the polygons surrounding it in the counterclockwise order starting from the smallest color we obtain an increasing sequence of colors.

Cutting the sphere along the edges of the embedded constellation we obtain \( k + c + 1 \) pieces homeomorphic to open discs. Among them, \( k \) correspond to the polygons and the \( c + 1 \) others to the cycles of \( \sigma \). The piece corresponding to the long cycle (of length \( n - p \)) will be called the exterior face; the other \( c \) pieces will be called interior faces or just faces. Note that not all vertices of a face are contained in the corresponding cycle of \( \sigma \). Those which are will be called the essential vertices, see Figure 1.

![Figure 1: The color of each polygon is marked inside it. The essential vertices of the central face are shown in black; these vertices form a cycle of \( \sigma_5 \ldots \sigma_1 \).](image)

From now on we will always assume that the constellations are embedded into a sphere.

Note that the somewhat bizarre condition \( p < a_j \) has a simple interpretation for constellations: it means that every polygon must have at least one edge on the exterior face. However the algebraic meaning of this condition remains mysterious.

## 2 The main theorem

Let \( p_1, \ldots, p_c \) be \( c \geq 0 \) positive integers (orders of poles), \( \sum p_i = p \). Fix an integer \( n > p \) (degree of the almost polynomial). Let \( a_1, \ldots, a_k \) be \( k \geq 2 \) more positive integers (multiplicities of the critical points) satisfying \( a_j > p \) for all \( j \) and \( \sum a_j = n + k + c - 1 \) (the Riemann-Hurwitz formula). Denote by \( |\text{Aut}\{p_1, \ldots, p_c\}| \) the number of permutations \( s \) of \( c \) elements such that \( p_i = p_{a(i)} \) for all \( i \). For instance, \( |\text{Aut}\{4, 4, 3, 2, 2, 2, 2, 1, 1\}| = 2! \cdot 1! \cdot 5! \cdot 2! \).

Consider a permutation \( \sigma \in S_n \) with cycle type \( (p_1, \ldots, p_c, n - p) \).
Theorem 1 The number of minimal factorizations of \( \sigma \) into \( a_j \)-cycles equals

\[
\frac{(k + c - 2)!}{(k - 2)!} \prod_{c} p^2 (n - p)^{k-1}.
\]

The number of constellations as well as the number of rational functions with fixed critical values of type \((a_1, \ldots, a_k), (p_1, \ldots, p_c)\) equals

\[
\frac{1}{|\text{Aut}\{p_1, \ldots, p_c\}|} \frac{(k + c - 2)!}{(k - 2)!} \prod_{c} p (n - p)^{k-2}.
\]

The three assertions of the theorem are equivalent. Indeed, the number of constellations and the number of rational functions of a given type coincide, which follows from Riemann's existence theorem (see [5] for more details). On the other hand, to obtain a minimal factorization of \( \sigma \) from a given constellation, we must number the vertices of the constellation in such a way that the product \( \sigma_k \cdots \sigma_1 \) equals \( \sigma \). It is easy to see that there are

\[
|\text{Aut}\{p_1, \ldots, p_c\}| \cdot (n - p) \cdot \prod_{c} p (n - p)^{k-2}.
\]

ways to do that. (This number is also the number of permutations that commute with \( \sigma \).)

In the next two sections we prove the theorem for constellations.

3 Assembling a constellation

We are going to prove the assertion of the theorem on the number of constellations. We start by labeling the faces of the constellations so as to make them distinguishable, which kills the \(|\text{Aut}|\) factor. We must show that the number of such constellations with labeled faces equals

\[
\frac{(k + c - 2)!}{(k - 2)!} \prod_{c} p (n - p)^{k-2}.
\]

We proceed by induction on the number \( c \) of faces in the constellations.

For \( c = 0 \), the constellations have no (interior) faces, i.e., they are "trees" glued of a given set of polygons. Such constellations (also called "cacti") were enumerated in [3], [6], and [7]. The answer one obtains is \( n^{k-2} \).

Suppose the formula is established for constellations with \( \leq c \) faces and let us add one more face. A polygon of the constellation is a neighbor of the
(c + 1)st face if it has at least one edge in common with this face. A face of the constellation is a neighbor of the (c+1)st face if its bounding polygons are all neighbors of this face. The (c + 1)st face can have any number $2 \leq m \leq k$ of neighboring polygons and any number $0 \leq d \leq c$ of neighboring faces as in Figure 2. (Note that a polygon that has only one vertex in common with the face is not considered a neighbor.) The condition $p < a_j$ implies that each neighboring face is bounded by exactly two polygons.

Figure 2: The (c + 1)st face in this figure has 4 neighboring polygons and 5 neighboring faces.

To construct a constellation with $c + 1$ faces we must make the following choices.

1. Choose the numbers $m$ and $d$.
2. Choose $m$ polygons among $k$ and $d$ faces among $c$ to be the neighbors of the (c + 1)st face. This gives

$$\binom{k}{m} \binom{c}{d}$$

choices. Denote by $D \subset \{1, \ldots, c\}$ the set of the $d$ neighboring faces.

3. Form the (c + 1)st face using the $m$ chosen polygons. This is done in the following way. Denote the essential vertices of the face by $V_1, \ldots, V_{p.c+1}$. We must describe the colors and the order of the edges that will form the intervals between $V_i$ and $V_{i+1}$ for each $i$. We claim that such a disposition of edges is uniquely determined once we have (arbitrarily) assigned to each of the $m$ polygons the interval $V_iV_{i+1}$ where its first edge will appear (as
we go around the face in the clockwise direction). Indeed, the disposition of the edges can be obtained as follows. (a) For each polygon assigned to the interval $V_i V_{i+1}$ take one edge of its color. (b) Order these edges in the increasing order of colors. (c) In the case if either the biggest color used in $V_{i-1} V_i$ is smaller than the smallest color in the list for $V_i V_{i+1}$ or if there are no polygons assigned to $V_i V_{i+1}$, add, at the beginning of the list of colors for $V_i V_{i+1}$, the last color used in $V_{i-1} V_i$. An example of this algorithm is shown in Figure 3.

![Figure 3](image)

Figure 3: How to obtain a face from a list of colors assigned to each interval $V_i V_{i+1}$.

It is easy to see that the vertices $V_i$ are indeed the essential vertices and that this is the unique way to achieve this.

Thus the number of ways to form the $(c+1)$st face using $m$ given polygons is $p_{c+1}^m$.

4. Now we must choose the positions of the $d$ faces that were chosen to be the neighbors of the $(c+1)$st face. There are $m$ “clefts” between the polygons where these faces can be placed. Several faces can appear in the same cleft. In this case they must be ordered (starting from the face closest to the $(c+1)$st one). Thus there are

$$\frac{(d+m-1)!}{(m-1)!}$$

ways to choose the positions of the faces.

5. Each of the neighboring faces has exactly two bounding polygons. We now choose how many edges of the face will belong to the polygon, say, on
its left. For the $i$th face there are $p_i$ choices. Thus we obtain a factor

$$\prod_{i \in D} p_i,$$

This product, of course, depends on the particular choice of the $d$ neighboring faces. However we will soon see that the remaining part of the formula contains the complementary product

$$\prod_{i \notin D} p_i,$$

and thus it is not necessary to take a sum over all possible choices.

6. We have assembled together all the polygons and faces that are neighbors of the $(c + 1)$st face. They form a subconstellation $K$ of the total constellation. Now we are going to consider $K$ as a unique polygon. Indeed, we are going to show that the remaining polygons are attached to $K$ in the same way as they would be attached to a unique polygon. In particular, the number of ways to attach them is the same. This will allow us to precede by induction.

Denote by $r = p_{c+1} + \sum_{i \in D} p_i$ the sum of lengths of the faces of $K$ and by $v$ the total number of vertices of $K$. These $v$ vertices are acted upon by $m$ cyclic permutation corresponding to the polygons. The product of these $m$ permutations splits the vertices into $d + 2$ cycles of lengths $p_i, i \in D, p_{c+1},$ and $v - r$. The vertices of the last cycle will be called the essential exterior vertices of $K$.

Now consider a constellation $R$ formed by the $k - m$ polygons not used in $K$ and, in addition, one $(v - r)$-gon of color 0. Suppose that the faces of $R$ have lengths $p_i, i \notin D$. We are going to replace the $(v - r)$-gon in $R$ by the constellation $K$.

Let $p' = p_1 + \ldots + p_c$ and $p = p_1 + \ldots + p_{c+1}$.

7. First we establish a one-to-one correspondence between the vertices of the $(v - r)$-gon and the essential exterior vertices of $K$ preserving their cyclic order. There are $v - r$ ways of doing that, which will account for a factor $v - r$ in the final formula.

In principle, the number $v - r$ is different for different choices of the $m$ polygons and the $d$ faces. However we will soon see that this number appears in the total sum only as a linear factor. Therefore it will be a posteriori justified to replace it by its average over the possible choices of $m$ polygons and $d$ faces.
The average value $\langle v - r \rangle$ of $v - r$ equals

$$\frac{m}{k}(n + k + c) - m - d - p_{c+1} - \frac{d}{c}p' = \frac{mn + mc - kp_{c+1}}{k} - d \frac{p' + c}{c}.$$ 

Indeed, the total number of edges in the $k$ polygons equals $n + k + c$ by the Euler formula. Choosing randomly $m$ of the $k$ polygons we obtain an average of $\frac{m}{k}(n + k + c)$ edges in the subconstellation $K$. Since $K$ has $d + 1$ faces and $m$ polygons, its average number of vertices is $\langle v \rangle = \frac{m}{k}(n + k + c) - m - d$, again by the Euler formula. Now, $K$ has $d + 1$ faces, one of which is always of length $p_{c+1}$ and the other $d$ are chosen randomly from $c$ possibilities. Thus $\langle r \rangle = p_{c+1} + \frac{d}{c}p'$, whence we obtain $\langle v - r \rangle$.

8. Let us go around the exterior face of the constellation $K$ in the counterclockwise direction. The colors of the edges we meet will be increasing in each interval between two consecutive essential exterior vertices. Then, as we pass an essential vertex, the number of the color jumps down.

Suppose we are given a new polygon of some color $j$ that does not appear in the constellation $K$. We want to attach this polygon to the exterior of $K$ in such a way that the cyclic order of colors around each vertex remains increasing (see the remarks after Definition 1.3). It is easy to see that in each interval between two consecutive essential vertices $V_i$ (included) and $V_{i+1}$ (excluded) there is a unique vertex to which the polygon can be attached.

Now, the essential exterior vertices of $K$ are in a one-to-one correspondence with the vertices of a $(v - r)$-gon in the new constellation $R$. We want to replace the $(v - r)$-gon in $R$ by the subconstellation $K$. To do that, for each polygon $j$ attached to some vertex of the $(v - r)$-gon, we take the corresponding essential exterior vertex $V_i$ of $K$ and attach our polygon to the unique possible vertex between $V_i$ and $V_{i+1}$.

It is easy to see that the faces of $R$, even those that have been modified by our operation, will still have the same number of essential vertices as before. Thus there is a unique way to substitute the $(v - r)$-gon in $R$ by the subconstellation $K$. The result is the constellation we were trying to assemble.

9. It remains to choose the constellation $R$. By the induction assumption, there are

$$\frac{(k - m - 1 + c - d)!}{(k - m - 1)!} (n - p)^{k - m - 1} \prod_{i \in D} p_i$$

choices. Indeed, the constellation $R$ has $k - m$ polygons, $c - d$ (interior) faces
of lengths $p_i$, $i \notin D$, and the length of its exterior face is $n - p$ (the same as in the constellation that we are assembling).

4 Computing the sum

The result of our investigation is that the number of constellations (with labeled faces) is given by the following sum:

$$S = p_1 \cdots p_c \sum_{m=2}^{k} \binom{k}{m} (n-p)^{k-m-1} P_{c+1}^{m-1} \times$$

$$\times \sum_{d=0}^{c} \binom{c}{d} \frac{(d + m - 1)! (k - m - 1 + c - d)!}{(m-1)! (k - m - 1)!} \left( \frac{mn + mc - kp_{c+1}}{k} - d \frac{p' + c}{c} \right).$$

The rest of the proof is a sequence of elementary but rather cumbersome computations.

We first compute the subsum over $d$ using the elementary relations

$$\sum_{d=0}^{c} \binom{a + d}{d} \binom{b - d}{c - d} = \binom{a + b + 1}{c},$$

$$\sum_{d=0}^{c} d \binom{a + d}{d} \binom{b - d}{c - d} = (a+1) \binom{a + b + 1}{c - 1}$$

with $a = m - 1$, $b = k + c - m - 1$. We find that the subsum is equal to

$$\frac{(k + c - 1)!}{(k-1)!} \left( \frac{mn + mc - kp_{c+1}}{k} \right) - m \frac{(k + c - 1)!}{k!} (p' + c)$$

$$= \frac{(k + c - 1)!}{k!} [m(n-p') - kp_{c+1}].$$

Substituting this into the initial sum we obtain

$$S = p_1 \cdots p_c \frac{(k + c - 1)!}{k!} \sum_{m=2}^{k} \binom{k}{m} (n-p)^{k-m-1} P_{c+1}^{m-1} [m(n-p') - kp_{c+1}].$$
This sum can now be evaluated using two more elementary identities

\[
\sum_{m=2}^{k} m \binom{k}{m} x^{k-m-1} y^{m-1} = k \left[ \frac{(x + y)^{k-1}}{x} - x^{k-2} \right]
\]

\[
\sum_{m=2}^{k} \binom{k}{m} x^{k-m-1} y^{m} = \frac{(x + y)^{k}}{x} - x^{k-1} - k x^{k-2} y,
\]

with \( x = n - p \), \( y = p_{c+1} \). We obtain

\[
S = p_1 \ldots p_c \frac{(k + c - 1)!}{k!} \left\{ k \left[ \frac{(n - p')^{k-1}}{n - p} - (n - p)^{k-2} \right] (n - p') - \\
- k \left[ \frac{(n - p')^{k}}{n - p} - (n - p)^{k-1} - kp_{c+1}(n - p)^{k-2} \right] \right\}
\]

\[
= p_1 \ldots p_c \frac{(k + c - 1)!}{(k - 1)!} (n - p)^{k-2} \left\{ -(n - p') + (n - p) + kp_{c+1} \right\}
\]

\[
= p_1 \ldots p_{c+1} \frac{(k + c - 1)!}{(k - 2)!} (n - p)^{k-2}.
\]

This is precisely the formula of Theorem 1 with \( c \) replaced by \( c + 1 \). The theorem is proved.

References


