# Descriptive set theory
# Chapter 1-General topology

Descriptive set theory is the study of the definable subsets of the Polish topological spaces. In this theory, sets are classified in hierarchies, according to their topological complexity. This is a way of differentiating the objects of a big collection. Another motivation is as follows. In mathematics, an important concern is to prove existence results. Saying that there is an object satisfying a property is equivalent to say that the set of the objects satisfying this property is nonempty. And one way to prove that a set is nonempty is to prove that it is complex. Another concern is to look for simple characterizations of some properties. Proving that a property is complex is one way to rule out the possibility to characterize it simply. Descriptive set theory has been one of the main areas of research in set theory for more than one century now. Moreover, its concepts ans results are being used in diverse fields of mathematics, such as mathematical logic, combinatorics, topology, real and harmonic analysis, functional analysis, measure and probability theory, potential theory, ergodic theory, operator algebras, and topological groups and representations.

## 1 Topological spaces

Topology is a part of mathematics giving a precise meaning to the intuitive notions of "being close", "being the limit". The goal of this chapter is to recall the notions and results from topology that will be used later, not to give a course on topology.

**Definition 1.1** *Let $X$ be a set. A* **topology** *on $X$ is a set $\tau$ of subsets of $X$, the* **open sets***, such that*

*(a) the empty set and the whole set $X$ are open,*

*(b) the intersection of finitely many open sets is open,*

*(c) the union of any family of open sets is open.*

*If $\tau$ is a topology on $X$, then $(X, \tau)$ is called a* **topological space***. If there is no ambiguity, we will also say that $X$ is a topological space.*

**Notation.** If $X$ is a set, then $2^X$ is the set of all subsets of $X$. It is also the **discrete** topology on $X$.

We now see how to construct an open set, starting with an arbitray subset of the ambient space.

**Definition 1.2** *Let $X$ be a topological space, $x \in X$, and $S$ be a subset of $X$.*

*(a) A* **neighborhood** *of $x$ is a subset of $X$ containing an open set containing $x$.*

*(b) $S$ is* **closed** *if the complement of $S$ is open. The* **closure** *of $S$ is the intersection $\overline{S}$ of the closed sets containing $S$. The set $S$ is* **dense** *in $X$ if $\overline{S} = X$.*

*(c) The* **interior** *of $S$ is the union $Int(S)$ of the open sets contained in $S$.*

Note that $S$ is open (resp., closed) if and only if $S$ is equal to $Int(S)$ (resp., $\overline{S}$). The following notion is necessary to understand an important result in descriptive set theory, the perfect set theorem.

**Definition 1.3** *Let $X$ be a topological space, and $x \in X$.*

*(a) The point $x$ is* **isolated** *if $\{x\}$ is an open set.*

*(b) The space $X$ is* **perfect** *if it has no isolated point.*

**Definition 1.4** *Let $(X, \tau), (Y, \sigma)$ be topological spaces, and $f : X \to Y$ be a function.*

*(a) We say that $f$ is* **continuous** *if, for each $O$ in $\sigma$, $f^{-1}(O)$ is in $\tau$, i.e., if the pre-image of any open set is open.*

*(b) We say that $f$ is* **open** *if, for each $O$ in $\tau$, $f[O]$ is in $\sigma$, i.e., if the image of any open set is open.*

*(c) We say that $f$ is a* **homeomorphism** *if $f$ is a bijection from $X$ onto $Y$ and $f, f^{-1}$ are continuous.*

The continuous functions are the ones preserving the topology.

# 2 Operations

The purpose of this section is to see how to build new topological spaces out of old ones.

## 2.1 Induced topology

Given a topological space $X$ and a subset $S$ of $X$, there is a natural way to equip $S$ with a topology.

**Definition 2.1** *Let $(X, \tau)$ be a topological space, and $S \subseteq X$. The* **induced topology** *$\tau_S$ on $S$ is defined by $\tau_S := \{O \cap S \mid O \in \tau\}$.*

One can check that this defines indeed a topology.

**Exercise.** Let $X, Y$ be topological spaces, and $f : X \to Y$ be a function. Prove that $f$ is continuous if and only if its co-restriction $\overline{f} : X \to f[X]$ is continuous, $f[X]$ being equipped with the topology induced by that of $Y$.

## 2.2 Sum topology

**Notation.** Let $X, Y$ be sets. The **sum** of $X$ and $Y$ is the set $X \oplus Y := (\{0\} \times X) \cup (\{1\} \times Y)$.

**Definition 2.2** *Let $(X, \tau), (Y, \sigma)$ be topological spaces. The* **sum topology** *on $X \oplus Y$ is*

$$\{(\{0\} \times O') \cup (\{1\} \times O'') \mid O' \in \tau \wedge O'' \in \sigma\}.$$

One can check that this defines indeed a topology. Note that the definition of the topological sum of two spaces can be extended to an arbitrary sums.

**Exercise.** Prove that if $X, Y, Z$ are topological space and $f_X : X \to Z$, $f_Y : Y \to Z$ are continuous, then the function $f : X \oplus Y \to Z$ defined by $f(0, x) := f_X(x)$ and $f(1, y) := f_Y(y)$ is continuous for the sum topology, and that conversely every continuous map $f : X \oplus Y \to Z$ can be decomposed likewise.

### 2.3 Product topology

The following notion will be used to construct two very important spaces in descriptive set theory, the Cantor space and the Baire space.

**Definition 2.3** *Let $\big((X_i, \tau_i)\big)_{i\in I}$ be a family of topological spaces. The* **product topology** *on $\Pi_{i\in I}\ X_i$ is the topology whose open sets are the unions of the products of the form $\Pi_{i\in I}\ O_i$, where $O_i \in \tau_i$ for each $i \in I$, and $O_i = X_i$ except for finitely many $i$'s in $I$.*

**Exercise.** Let $\big((X_i, \tau_i)\big)_{i\in I}$ be a family of topological spaces, $i \in I$, and $\pi_i : \Pi_{i\in I}\ X_i \to X_i$ defined by $\pi_i\big((x_i)_{i\in I}\big) := x_i$ be the $i$'th projection. Prove that $\pi_i$ is continuous for the product topology.

In fact, the product topology is the smallest topology making all the projections continuous.

## 3 Generating topologies

**Exercise.** Let $X$ be a set, and $(\tau_i)_{i\in I}$ be a non-empty family of topologies on $X$. Then $\bigcap_{i\in I}\ \tau_i$ is a topology on $X$. It is the biggest topology contained in each $\tau_i$, for the inclusion.

**Definition 3.1** *Let $X$ be a set, $\mathcal{F} \subseteq 2^X$, and $\mathcal{T}_{\mathcal{F}}$ be the set of all topologies on $X$ containing $\mathcal{F}$. The topology* **generated** *by $\mathcal{F}$ is $\bigcap_{\tau\in\mathcal{T}_{\mathcal{F}}}\ \tau$. It is denoted by $\tau(\mathcal{F})$.*

Note that $\mathcal{T}_{\mathcal{F}}$ is not empty since it contains $2^X$, and $\tau(\mathcal{F})$ is the smallest topology containing $\mathcal{F}$.

**Definition 3.2** *Let $(X, \tau)$ be a topological space.*

*(a) A* **basis for the topology** *of $X$ is a collection $\mathcal{B} \subseteq \tau$ with the property that every open set is the union of elements of $\mathcal{B}$ (by convention, the empty union gives the empty set).*

*(b) A* **subbasis for the topology** *of $X$ is a collection $\mathcal{S} \subseteq \tau$ with the property that the set of finite intersections of sets in $\mathcal{S}$ is a basis for the topology of $X$.*

**Exercise.** Let $X$ be a set, and $\mathcal{B} \subseteq 2^X$. Then $\mathcal{B}$ is a basis for a topology on $X$ if and only if the intersection of two members of $\mathcal{B}$ can be written as a union of members of $\mathcal{B}$ and $X = \bigcup\{B \mid B \in \mathcal{B}\}$.

**Proposition 3.3** *Let $X$ be a set, and $\mathcal{F} \subseteq 2^X$. Then $\tau(\mathcal{F})$ is the set of unions of finite intersections of elements of $\mathcal{F}$ (by convention, the empty intersection gives $X$).*

We now relativize the notion of a basis to the points of the ambient space.

**Definition 3.4** *Let $X$ be a topological space, and $x \in X$.*

*(a) A family $\mathcal{U}$ of neighborhoods of $x$ is a* **neighborhood basis for** *$x$ if for every neighborhood $N$ of $x$, there is $U \in \mathcal{U}$ such that $U \subseteq N$.*

*(b) $X$ is* **first countable** *if all its elements have a countable neighborhood basis.*

**Exercise.** Any countable product of first countable spaces is first countable.

**Notation.** We denote by $\omega$ the set $\mathbb{N}$ of natural numbers, sometimes viewed as an ordinal or a cardinal. A sequence $(x_n)_{n\in\omega}$ of points of a set $X$ will sometimes be denoted by $(x_n)$, when there is no ambiguity.

We now define the basic notion of a limit of a sequence of points of a topological space.

**Definition 3.5** *Let $X$ be a topological space. A sequence $(x_n)$ of points of $X$ is* **convergent** *if there is $x\in X$ such that, for every open neighborhood $O$ of $x$ there is $N\in\omega$ such that, for all $n\geq N$, $x_n\in O$. In this case, we say that $(x_n)$* **converges to** *$x$ and that $x$ is the* **limit** *of $(x_n)$.*

**Exercise.** Let $X$ be a first countable space, and $S\subseteq X$. Then $\overline{S}$ is the set of limits of sequences of elements of $S$. In particular, $S$ is closed if and only if every convergent sequence of elements of $S$ onverges to an element of $S$.

**Proposition 3.6** *Let $X,Y$ be topological spaces, $X$ being first countable, and $f:X\rightarrow Y$ be a function. Then $f$ is continuous if and only if, for every $x\in X$ and every sequence $(x_n)$ converging to $x$, $\big(f(x_n)\big)$ converges to $f(x)$.*

**Proof.** Assume that $f$ is continuous and $O$ is an open neighborhood of $f(x)$. Then $f^{-1}(O)$ is an open neighborhood of $x$, which gives $N\in\omega$ such that, for all $n\geq N$, $x_n\in f^{-1}(O)$. Then $f(x_n)\in O$ if $n\geq N$, showing that $\big(f(x_n)\big)$ converges to $f(x)$. Conversely, assume that $C$ is a closed subset of $Y$. It is enough to prove that $S:=f^{-1}(C)$ is closed. By the previous exercise, it is enough to check that every convergent sequence $(x_n)$ of elements of $S$ converges to an element of $S$. Let $x$ be the limit of $(x_n)$. Note that $f(x_n)\in C$ for each $n$. By assumption, $\big(f(x_n)\big)$ converges to $f(x)$. Thus $f(x)$ is in $C$ which is closed, and we are done. $\square$

In the Cantor space and the Baire space, there will be a combinatorially very nice basis for the topology, having the following property.

**Definition 3.7** *Let $X$ be a topological space, and $S\subseteq X$. We say that*

*(a) $S$ is* **clopen** *if $S$ is closed and open,*

*(b) $X$ is* **zero-dimensional** *if there is a basis for the topology of $X$ made of clopen sets.*

# 4 Notions of separation

The notion of separation of sets is crucial in descriptive set theory.

**Definition 4.1** *Let $X$ be a topological space. We say that $X$ is*

*(a) $T_1$ if every singleton of $X$ is closed,*

*(b)* **Hausdorff** *if every two distinct points of $X$ have disjoint open neighborhoods,*

*(c)* **regular** *if, for any $x$ in $X$ and any open neighborhood $N$ of $x$, there is an open neighborhood $O$ of $x$ such that $\overline{O}\subseteq N$,*

*(c)* **normal** *if, for any disjoint closed subsets $C,F$ of $X$, there are disjoint open subsets $O,U$ of $X$ such that $C\subseteq O$ and $F\subseteq U$.*

Note that every Hausdorff space is $T_1$, and every zero-dimensional space is regular.

**Lemma 4.2** *(Urysohn) Let $X$ be a normal space, and $C, F$ be disjoint closed subsets of $X$. Then there is a continuous function $f: X \to [0,1]$ such that $f(x)=0$ if $x \in C$, and $f(x)=1$ if $x \in F$.*

**Proof.** We first construct a family $(O_q)_{q \in \mathbb{Q} \cap [0,1]}$ of open subsets of $X$ containing $C$ and contained in $X \setminus F$ such that $\overline{O_q} \subseteq O_r$ if $q < r$. Let $(q_n)$ be an enumeration of $\mathbb{Q} \cap [0,1]$ with $q_0 := 0$ and $q_1 := 1$. We proceed by induction on $n$. We first set $O_1 := X \setminus F$ and choose $O_0$ such that $C \subseteq O_0 \subseteq \overline{O_0} \subseteq O_1$. We now want to construct $O_{q_{n+1}}$, with $n \geq 1$. Take $k, l \leq n$ such that $q_k \leq q_{n+1}$ is maximal, and $q_l \geq q_{n+1}$ is minimal. By normality, we get $O_{q_{n+1}}$ with $O_{q_k} \subseteq O_{q_{n+1}} \subseteq \overline{O_{q_{n+1}}} \subseteq O_{q_l}$.

We now set $f(x) := \inf(\{1\} \cup \{q \in \mathbb{Q} \cap [0,1] \mid x \in O_q\})$. Note that $f(x) \leq q$ if $x \in O_q$, and $f(x) \geq q$ if $x \notin O_q$. Thus $f(x)=0$ if $x \in C$, and $f(x)=1$ if $x \in F$. If $b \in (0,1]$, then $f^{-1}\big([0,b)\big) = \bigcup_{q<b} O_q$. If now $a \in [0,1)$, then $f^{-1}\big((a,1]\big) = \bigcup_{q>a} X \setminus \overline{O_q}$. Thus $f$ is continuous. □

# 5 Metrizability, the Baire theorem

An important method to construct a topology is to use a metric.

**Definition 5.1** *Let $X$ be a set, and $d: X^2 \to [0,\infty)$ be a function. We say that $d$ is a* **metric** *on $X$ if, for $x, y, z \in X$,*

*(a) $d(x,y)=0$ if and only if $x=y$,*

*(b) $d(x,y)=d(y,x)$,*

*(c) $d(x,y) \leq d(x,z)+d(z,y)$.*

*We say that $(X,d)$ is a* **metric space** *if $X$ is a set and $d$ is a metric on $X$.*

**Example.** $X=\mathbb{R}$, equipped with $d$ defined by $d(x,y) := |x-y|$, is a metric space.

We can associate a topology to a metric. In order to do this, we need the following notation.

**Notation.** Let $(X,d)$ be a metric space, $x \in X$, and $r > 0$. The **open ball** $B_d(x,r)$ of center $x$ and radius $r$ consists in all the points which are $r$-close to $x$: $B_d(x,r) := \{y \in X \mid d(x,y) < r\}$. When the metric $d$ is clear from the context, we will skip the index and simply write $B(x,r)$ for the open ball of center $x$ and radius $r$. We also define the **closed ball** of center $x$ and radius $r$ by $B_d(x,r] := \{y \in X \mid d(x,y) \leq r\}$.

**Definition 5.2** *Let $(X,\tau)$ be a topological space. We say that $X$ is* **metrizable** *if there is a metric $d$ on $X$ such that the open balls form a basis for $\tau$.*

**Remarks.** (a) If $X$ is a metrizable space with witness $d$, then $X$ is first countable since, for any $x$ in $X$, $\big(B(x, \frac{1}{n+1})\big)_{n \in \omega}$ is a countable neighborhood basis for $x$.

(b) If the topology of $X$ is defined by a metric $d$, then $(x_n)$ converges to $x$ if and only if for every $\eta > 0$ there is $N \in \omega$ such that, for all $n \geq N$, $d(x, x_n) < \eta$.

**Exercise.** Let $(X,d)$ be a metric space, and $d'(x,y) = \min\big(1, d(x,y)\big)$. Then $d'$ is a metric which also defines the topology induced by $d$.

**Exercise.** Let $X$ be a metrizable space, with witness $d$. Then any closed ball $B_d(x,r]$ is closed.

**Proposition 5.3** *Every metrizable space is Hausdorff and regular, and thus* $T_1$.

**Proof.** Let $X$ be a metrizable space, with witness $d$. If $x \neq y$ are in $X$, then the open balls $B(x, \frac{d(x,y)}{2})$, $B(y, \frac{d(x,y)}{2})$ are disjoint open neighborhoods of $x, y$ respectively. Thus $X$ is Hausdorff. If now $N$ is an open neighborhood of $x$, then there is $r > 0$ such that $B(x, r) \subseteq N$. We set $O := B(x, \frac{r}{2})$, so that $O$ is an open neighborhood of $x$. If $y \in \overline{O}$, then $d(x, y) \leq \frac{r}{2} < r$ since $B_d(x, \frac{r}{2}]$ is closed, so that $y \in N$.□

**Exercise.** Prove that every metrizable space is normal.

**Proposition 5.4** *Let* $(X, d)$ *be a metric space, and* $S$ *be a nonempty subset of* $X$*. Then the map* $x \mapsto d(x, S) := inf_{y \in S}\ d(x, y)$ *is continuous.*

**Proof.** This comes from the fact that, for all $x, y \in X$, $d(x, y) \geq |d(x, S) - d(y, S)|$. □

**Theorem 5.5** *Any countable product of metrizable spaces is metrizable.*

**Proof.** Let $(X_n)$ be a sequence of metrizable spaces, and $(d_n)$ be a sequence of metrics such that $d_n$ is a witness for the metrizability of $X_n$, for each $n$. We set

$$d\big((x_n), (y_n)\big) := \Sigma_{n \in \omega}\ 2^{-n} \text{min}\big(1, d_n(x_n, y_n)\big).$$

We saw that $(x_n, y_n) \mapsto \text{min}\big(1, d_n(x_n, y_n)\big)$ is a metric on $X_n$ defining the topology induced by $d_n$, for each $n$. This implies that $d$ is a metric on $\Pi_{n \in \omega}\ X_n$. In order to see that $d$ defines the product topology, we need to show that the identity map on $\Pi_{n \in \omega}\ X_n$ is a homeomorphism when on one side we put the product topology and on the other side the topology induced by $d$. Since $\Pi_{n \in \omega}\ X_n$ is first countable for the product topology, by Proposition 3.6, we only need to check that for a sequence $\big((x_n^m)_{n \in \omega}\big)_{m \in \omega}$ of elements of $\Pi_{n \in \omega}\ X_n$ and $(x'_n) \in \Pi_{n \in \omega}\ X_n$, $\big((x_n^m)_{n \in \omega}\big)_{m \in \omega}$ converges to $(x'_n)$ in the product topology if and only if $d\big((x_n^m), (x'_n)\big)$ tends to 0. By the dominated convergence theorem for series, $d\big((x_n^m), (x'_n)\big)$ tends to 0 if and only if $d_n(x_n^m, x'_n)$ tends to 0 for all $n$, which holds if and only if $\big((x_n^m)_{n \in \omega}\big)_{m \in \omega}$ tends to $(x'_n)$ in the product topology as desired. □

**Definition 5.6** *Let* $(X, d)$ *be a metric space, and* $S$ *be a subset of* $X$*. The* **diameter** *of* $S$ *is*

$$diam_d(S) := sup_{x, y \in S}\ d(x, y).$$

*A sequence* $(S_n)$ *of subsets of* $X$ *has* **vanishing diameters** *if* $\big(diam_d(S_n)\big)$ *converges to* $0$.

The following two notions will be crucial to define the notion of a Polish space.

**Definition 5.7** *Let* $(X, d)$ *be a metric space. A sequence* $(x_n)$ *of points of* $X$ *is* **Cauchy** *if for every* $\eta > 0$ *there is* $N \in \omega$ *such that, for all* $m, n \geq N$, $d(x_m, x_n) < \eta$.

One can check that if $(X, d)$ be a metric space and $(x_n)$ is convergent for the topology defined by $d$, then $(x_n)$ is Cauchy. The converse does not hold in general, for instance with $X = (0, 1)$ and $d$ defined by $d(x, y) := |x - y|$, where the sequence $(\frac{1}{n+1})$ is Cauchy but not convergent. This leads to the following definition.

**Definition 5.8** *A metric space* $(X, d)$ *is* **complete** *if every Cauchy sequence is convergent.*

**Exercise.** Let $X$ be a set, and $l^\infty(X, \mathbb{R})$ be the set of bounded functions from $X$ into $\mathbb{R}$. We set $d(f, g) := \sup_{x\in X} |f(x) - g(x)|$. Prove that $d$ is a metric on $l^\infty(X, \mathbb{R})$ and $\big(l^\infty(X, \mathbb{R}), d\big)$ is complete.

**Theorem 5.9** *Let* $(X, d)$ *be a metric space. Then* $(X, d)$ *is complete if and only if every decreasing sequence of nonempty closed sets with vanishing diameters has nonempty intersection, in fact a singleton.*

**Proof.** Assume first that $(X, d)$ is complete. Fix a sequence $(C_n)$ as in the statement. Consider a sequence $(x_n)$ such that, for all $n \in \omega$, $x_n \in C_n$. Let us show that $(x_n)$ is Cauchy. Given $\eta > 0$, find $N \in \omega$ such that $\text{diam}(C_N) < \eta$. Then, for all $m, n \geq N$, $\{x_m, x_n\} \subseteq C_N$, so that $d(x_m, x_n) < \eta$ by definition of the diameter. We conclude that $(x_n)$ is Cauchy. By completeness we can consider the limit $x \in X$ of the sequence $(x_n)$. Since each $C_n$ is closed and contains all the $x_m$'s for $m \geq n$, $x$ belongs to every $C_n$.

Conversely, let $(x_n)$ be a Cauchy sequence. We consider, for every $n$, the nonempty closed set $C_n := \overline{\{x_m \mid m \geq n\}}$. As $(x_n)$ is Cauchy, the sequence of sets $(C_n)$ has vanishing diameters. Define $x \in X$ by $\{x\} := \bigcap_{n\in\omega} C_n$. Given $\eta > 0$, find $N$ such that $\text{diam}(C_N) < \eta$. Then $d(x, x_n) < \eta$ for all $n \geq N$ because $x_n \in C_n$ and $x \in C_n$, so $(x_n)$ converges to $x$ as desired. □

The following crucial result, known as Baire's theorem, gives a necessary condition for being complete.

**Theorem 5.10** *(Baire) Let* $(X, d)$ *be a complete metric space, and* $(O_n)$ *be a sequence of dense open subsets of* $X$*. Then* $\bigcap_{n\in\omega} O_n$ *is dense in* $X$*.*

**Proof.** We need to show that every nonempty open set $O$ meets $\bigcap_{n\in\omega} O_n$. As $O_0$ is dense, $O_0 \cap O$ is a nonempty open set. This gives an open ball $B_0 \subseteq O_0 \cap O$. By shrinking the radius of $B_0$ if necessary, we may assume that $\overline{B_0} \subseteq O_0 \cap O$ and $\text{diam}_d(B_0) \leq 1$. Now $B_0 \cap O_1$ is a nonmepty open set, which gives an open ball $B_1 \subseteq B_0 \cap O_1$. Again by shrinking its radius we may assume $\overline{B_1} \subseteq B_0 \cap O_1$ and $\text{diam}_d(B_1) \leq \frac{1}{2}$. We continue this construction by induction: assuming that for $n \geq 1$ we have build an open ball $B_n$, then the set $B_n \cap O_{n+1}$ is a nonempty open set, and we find an open ball $B_{n+1} \subseteq B_n \cap O_{n+1}$. By shrinking its radius we may assume that $B_{n+1} \subseteq B_n \cap O_{n+1}$ and $\text{diam}_d(B_{n+1}) \leq \frac{1}{2^{n+1}}$. Now observe that $(B_n)$ is a decreasing sequence of closed sets with vanishing diameters. Since $(X, d)$ is complete, there is $x \in X$ such that $\bigcap_{n\in\omega} B_n = \{x\}$ by Theorem 5.9. Since $B_0 \subseteq O$, $x \in O$, and since $B_n \subseteq O_n$ for every $n$, $x \in \bigcap_{n\in\omega} O_n$. Thus $O \cap \bigcap_{n\in\omega} O_n \neq \emptyset$ as desired. □

# 6 Countability

**Definition 6.1** *A topological space is*

*(a)* **second countable** *if there is a countable basis for its topology,*

*(b)* **separable** *if it has a countable dense subset.*

Note that a second countable space is separable.

**Exercise.** Let $X$ be an infinite set. We equip $l^\infty(X,\mathbb{R})$ with the metric defined above, which defines a topology. Prove that this space is not separable.

**Exercise.** (Lindelöf's lemma) Let $X$ be a second countable topological space. Then every open cover of $X$ contains a countable subcover: if $(O_i)_{i\in I}$ is an open cover of $X$, then there exists $J\subseteq I$ countable such that $X\subseteq\bigcup_{i\in J} O_i$.

**Exercise.** Any countable product of separable topological spaces is separable for the product topology. Any countable sum of separable topological spaces is separable for the sum topology.

**Lemma 6.2** *Every second countable regular space is normal.*

**Proof.** Let $C,F$ be disjoint closed subsets of a second countable regular space $X$. If $x\in C$, then by regularity we can find an open neighborhood $O_x$ of $x$ such that $\overline{O_x}\cap F=\emptyset$. As $X$ is second countable, by Lindelöf's lemma, we get $J\subseteq C$ countable such that $C\subseteq\bigcup_{x\in J} O_x$. We can enumerate our open sets with $\omega$, which gives a sequence $(O_i)$ of open subsets of $X$ with $C\subseteq\bigcup_{i\in\omega} O_i$ and $\overline{O_i}\cap F=\emptyset$. Similarly, we get a sequence $(U_i)$ of open subsets of $X$ with $F\subseteq\bigcup_{i\in\omega} U_i$ and $\overline{U_i}\cap C=\emptyset$. We then set $O'_n:=O_n\setminus(\bigcup_{i\leq n} \overline{U_i})$ and $U'_n:=U_n\setminus(\bigcup_{i\leq n} \overline{O_i})$. These sets are open, $(O'_n)$ covers $C$, and $(U'_n)$ covers $F$. Now $O':=\bigcup_{n\in\omega} O'_n$ is an open set containing $C$, $U':=\bigcup_{n\in\omega} U'_n$ is an open set containing $F$, and $O',U'$ are disjoint. □

We are now ready to prove the important Urysohn Metrization Theorem.

**Theorem 6.3** *(Urysohn) Let $X$ be a topological space. Then the following are equivalent:*

*(a) $X$ is metrizable and separable,*

*(b) $X$ is $T_1$, regular and second countable.*

**Proof.** (a) $\Rightarrow$ (b) We saw that $X$ is Hausdorff and regular, and thus $T_1$. Let $(x_n)$ be a dense sequence in $X$, and $d$ be a metric defining the topology of $X$. Then $\big(B(x_n,\frac{1}{p+1})\big)_{n,p\in\omega}$ is a countable basis for the topology of $X$.

(b) $\Rightarrow$ (a) We already noticed that $X$ is separable. By Lemma 6.2, $X$ is normal. Let $(O_n)$ be a basis for its topology. If $x\in O_n$, then we can find $m$ such that $x\in O_m\subseteq\overline{O_m}\subseteq O_n$, by regularity of $X$. In other words, $\{\overline{O_m}\mid\overline{O_m}\subseteq O_n\}$ is a cover of $O_n$. If $\overline{O_m}\subseteq O_n$, then Lemma 4.2 provides a continuous function $f_{m,n}:X\to[0,1]$ such that $f_{m,n}(x)=0$ if $x\in\overline{O_m}$, and $f_{m,n}(x)=1$ if $x\notin O_n$. Note that $x\in O_n$ if and only if there is $m$ such that $f_{m,n}(x)<1$. Let

$$I:=\{(m,n)\in\omega^2\mid\overline{O_m}\subseteq O_n\}.$$

We can define $\Phi:X\to[0,1]^I$ by $\Phi(x)(m,n):=f_{m,n}(x)$. Note that $\Phi$ is continuous. If $x\neq y\in X$, then we can find $n$ such that $x\in O_n$ and $y\notin O_n$, and $m$ such that $(m,n)\in I$ and $x\in O_m$. Thus $f_{m,n}(x)=0$ and $f_{m,n}(y)=1$. This shows that $\Phi$ is one-to-one. Now note that

$$\Phi[O_n]=\Phi[X]\cap\bigcup_{m\in\omega,(m,n)\in I}\pi^{-1}_{m,n}\big([0,1)\big),$$

so that $\Phi$ is open onto its range, and thus a homeomorphism onto its range. Thus $X$ is metrizable. □

**Lemma 6.4** *Let $(Y,d)$ be a separable metric space, and $\mathcal{U}$ be a family of nonempty open subsets of $Y$. Then $\mathcal{U}$ has a* **point-finite refinement** *$\mathcal{V}$, i.e., $\mathcal{V}$ is a family of nonempty open sets with $\bigcup\mathcal{V}=\bigcup\mathcal{U}$, $\forall V\in\mathcal{V}\ \exists U\in\mathcal{U}\ V\subseteq U$, and $\forall y\in Y\ \{V\in\mathcal{V}\mid y\in V\}$ is finite. Moreover, given $\eta>0$, we can also assume that diam$(V)<\eta$ if $V\in\mathcal{V}$.*

**Proof.** As $Y$ is second countable, let $(O_n)$ be a sequence of open sets such that $\bigcup_{n\in\omega}O_n=\bigcup\mathcal{U}$ and $\forall n\in\omega\ \exists U\in\mathcal{U}\ O_n\subseteq U$. Furthermore, given $\eta>0$, we can always assume that diam$(O_n)<\eta$. Next write $O_n=\bigcup_{p\in\omega}O^n_p$ with $O^n_p$ open, $O^n_p\subseteq O^n_{p+1}$, and $\overline{O^n_p}\subseteq O_n$. Put $V_m:=O_m\setminus(\bigcup_{n<m}\overline{O^n_m})$. First we claim that $\bigcup_{n\in\omega}V_n=\bigcup_{n\in\omega}O_n$. Indeed, if $x\in\bigcup_{n\in\omega}O_n$ and $m$ is least with $x\in O_m$, then $x\in V_m$. Clearly, $V_m\subseteq O_m$. Finally, if $x\in O_n$, then $x\in O^n_p$ for some $p$, so $x\notin V_m$ if $m>p,n$. Let $\mathcal{V}:=\{V_n\mid V_n\neq\emptyset\}$. □

# 7 Compactness

The notion of compactness is a crucial smallness property.

**Definition 7.1** *A topological space is* **compact** *if every open cover of $X$ contains a finite subcover.*

**Example.** Every finite topological space is compact.

**Exercise.** Every closed subspace of a compact topological space is compact.

**Exercise.** Let $X$ be a compact space, $Y$ be a topological space, and $f:X\to Y$ be a continuous function. Then $f[X]$ is compact. In particular, if $Y=\mathbb{R}$, equipped with the usual topology, then $f$ is **bounded**, i.e., $f[X]$ has finite diameter. Conclude that $\mathbb{R}$ is not compact.

**Theorem 7.2** *Let $X$ be a Hausdorff compact space. Then $X$ is regular.*

**Proof.** Let $x\in X$ and $N$ be an open neighborhood of $x$. If $y\notin N$, then there are disjoint open sets $O_y$ and $U_y$ with $x\in O_y$ and $y\in U_y$ since $X$ is Hausdorff. Note that $(U_y)_{y\in X\setminus N}$ is an open cover of $X\setminus N$. As $X\setminus N$ is closed in the compact space $X$, $X\setminus N$ is compact. This gives a finite subset $F$ of $X\setminus N$ such that $(U_y)_{y\in F}$ is an open cover of $X\setminus N$. It remains to set $O:=\bigcap_{y\in F}O_y$ since $x\in O$ and $\overline{O}\subseteq N$. □

**Remark.** This argument shows that if $X$ is a Hausdorff space, $K$ is a compact subspace of $X$, and $x\in X\setminus K$, then we can find disjoint open subsets $O,U$ of $X$ such that $x\in O$ and $K\subseteq U$. This shows that any compact subset of $X$ is a closed subset of $X$. This fact implies the following result.

**Exercise.** Prove that every Hausdorff compact space is normal.

**Exercise.** (a) Let $X$ be a compact space, an $\mathcal{F}$ be a family of open subsets of $X$. Suppose that $\mathcal{F}$ **separates points**, meaning that for every $x\neq y$ there are disjoint $O,U\in\mathcal{F}$ such that $x\in O$ and $y\in U$. Show that $\mathcal{F}$ generates the topology of $X$.

(b) Deduce that every countable compact Hausdorff space is metrizable.

**Theorem 7.3** *Let $X$ be a compact Hausdorff space, $Y$ be a Hausdorff space and $f : X \to Y$ be a one-to-one and continuous function. Then $f$ is a homeomorphism onto its range.*

We now study the compactness in metric spaces.

**Theorem 7.4** *(Heine) Let $(X, d_X)$ be a metric compact space, $(Y, d_Y)$ be a metric space, and $f : X \to Y$ be a continuous function. Then $f$ is* **uniformly continuous***, which means that for each $\eta > 0$, we can find $\delta > 0$ such that $d_X(x, y) < \delta$ implies $d_Y\big(f(x), f(y)\big) < \eta$.*

**Proof.** We set, for each $x \in X$, $\beta_{x,\eta} := \frac{1}{2}\delta_{x,\frac{\eta}{2}}$, where $\delta_{x,\frac{\eta}{2}}$ is given by the continuity of $f$ at $x$. Note that $\big(B(x, \beta_{x,\eta})\big)_{x\in X}$ is an open cover of $X$. As $X$ is compact, we get a finite subcover $\big(B(z, \beta_{z,\eta})\big)_{z\in F}$. We set $\delta := \min_{x\in F}\ \beta_{x,\eta}$. If $d_X(x, y) < \delta$, then we choose $z \in F$ such that $d_X(x, z) < \beta_{z,\eta}$. Then $d_X(y, z) < \delta_{z,\frac{\eta}{2}}$, thus $d_Y\big(f(x), f(y)\big) < \eta$. □

**Definition 7.5** *A metric space is* **precompact** *if, for each $\eta > 0$, we can cover $X$ with finitely many open balls of radius $\eta$.*

**Proposition 7.6** *Let $(X, d)$ be a precompact space. Then $X$ is separable and second countable.*

**Proof.** Note that $X$ is separable, and thus second countable by Theorem 6.3. □

**Theorem 7.7** *(Bolzano-Weierstrass) Let $(X, d)$ be a metric space. The following are equivalent:*

*(a) $X$ is compact,*

*(b) every sequence has a converging subsequence.*

**Proof.** (a) ⇒ (b) Let $(x_n)$ be a sequence of points of $X$, and $C_n := \overline{\{x_m \mid m \geq n\}}$. Then $(C_n)$ is a decreasing sequence of nonempty closed subsets of $X$. Thus it has a nonempty intersection, by compactness. Any point in the intersection is the limit of a subsequence of $(x_n)$.

(b) ⇒ (a) We first prove that $(X, d)$ is precompact. We argue by contradiction, which gives $\eta > 0$. Pick $x_0 \in X$. Then we can find, for each natural number $n$, $x_{n+1} \in X \setminus \bigcup_{i\leq n}\ B(x_i, \eta)$. This gives a sequence $(x_n)$ such that $d(x_m, x_n) \geq \eta$ if $m \neq n$. Such a sequence has no converging subsequence, which is absurd.

By Proposition 7.6, $X$ is second countable. Let $(O_i)_{i\in I}$ be an open cover of $X$. The Lindelöf lemma gives $J \subseteq I$ countable such that $X \subseteq \bigcup_{i\in J}\ O_i$, in other words we may assume that $J = \omega$. This gives a finite subcover, since otherwise we can construct a sequence $(y_j)_{j\in\omega}$ such that $y_j$ is in $X \setminus (\bigcup_{i<j}\ O_i)$. Such a sequence has no converging subsequence, which is absurd. □

**Theorem 7.8** *Let $(X, d)$ be a metric space. The following are equivalent:*

*(a) $X$ is compact,*

*(b) $(X, d)$ is complete and precompact.*

**Proof.** (a) ⇒ (b) By Theorem 5.9, in order to see the completeness, it is enough to prove that every decreasing sequence $(C_n)$ of nonempty closed sets with vanishing diameters has nonempty intersection. By compactness, it is enough to prove that if $F \subseteq \omega$ is finite, then $\bigcap_{n\in F}\ C_n$ is nonempty. Let $M := \max_F$. It remains to note that $C_M \subseteq \bigcap_{n\in F}\ C_n$.

Assume now that $\eta > 0$. Note that $\big(B(x, \eta)\big)_{x\in X}$ is an open cover. The compacness gives a finite subcover. Thus $X$ is precompact.

(b) ⇒ (a) Let $(x_n)$ be a sequence of elements of $X$. As $X$ is precompact, we can cover it with finitely many open balls of radius $1$. One of them, say $B_0$, contains infinitely many elements of $(x_n)$. This gives $\varphi_0 : \omega \to \omega$ strictly increasing such that $x_{\varphi_0(n)} \in B_0$ for each $n$. As $X$ is precompact, we can cover it with finitely many open balls of radius $\frac{1}{2}$. One of them, say $B_1$, contains infinitely many elements of $(x_{\varphi_0(n)})$. This gives $\varphi_1 : \omega \to \omega$ strictly increasing such that $x_{\varphi_0 \circ \varphi_1(n)} \in B_1$ for each $n$. We iterate this process, which gives a sequence $(B_p)$ of open balls of radius $\frac{1}{2^p}$ and a sequence $(\varphi_p)$ of strictly increasing functions from $\omega$ into itself such that $x_{\varphi_0 \circ \cdots \circ \varphi_p(n)} \in B_p$ for each $n$. We set, for $n \in \omega$, $\varphi(n) := \varphi_0 \circ \cdots \circ \varphi_n(n)$. Note that $\varphi : \omega \to \omega$ strictly increasing. Let us show that $(x_{\varphi(n)})$ is Cauchy. Let $n \in \omega$, and $p, q \geq n$. Note that $x_{\varphi(p)}, x_{\varphi(q)} \in B_n$, so that $d(x_{\varphi(p)}, x_{\varphi(q)}) < \frac{1}{2^{n-1}}$. Thus $(x_n)$ converges since $X$ is complete. It remains to apply Theorem 7.7. □

**Exercise.** Let $(X, d)$ be a complete metric space. Prove that a subset $Y$ of $X$ is precompact if and only if $\overline{Y}$ is compact.

**Theorem 7.9** *Any countable product of metrizable compact spaces is metrizable compact.*

**Proof.** Let us do the infinite case. Let $(X_n)$ be a countable family of metrizable compact spaces. By Theorem 5.5, $\Pi_{n\in\omega}\, X_n$ is metrizable. By Theorem 7.7, it is enough to prove that any sequence of elements of $\Pi_{n\in\omega}\, X_n$ has a converging subsequence.

So let $(f_m)_{m\in\omega}$ be a sequence of elements of $\Pi_{n\in\omega}\, X_n$. We inductively build a sequence $(\varphi_n)$ of increasing functions from $\omega$ into itself and elements $x_n$ of $X_n$ such that, for each $n$, the sequence $\big(f_{\varphi_0 \cdots \varphi_n(m)}(n)\big)_{m\in\omega}$ converges to $x_n$. We first apply the compactness of $X_0$ to find $\varphi_0$ increasing and $x_0 \in X_0$ such that $\big(f_{\varphi_0(m)}(0)\big)$ converges to $x_0$. Then, $\varphi_0, \cdots, \varphi_n$ and $x_0, \cdots, x_n$ having been built, we apply the compactness of $X_{n+1}$ to the sequence $\big(f_{\varphi_0 \cdots \varphi_n(m)}(n+1)\big)_{m\in\omega}$ to find $\varphi_{n+1}$ increasing and $x_{n+1} \in X_{n+1}$ such that $\big(f_{\varphi_0 \cdots \varphi_{n+1}(m)}(n+1)\big)_{m\in\omega}$ converges to $x_{n+1}$.

Now consider the map $\varphi : \omega \to \omega$ given by $\varphi(m) := \varphi_0 \cdots \varphi_m(m)$. Let us prove that $(f_{\varphi(m)})_{m\in\omega}$ converges to $(x_n)$. We need to show that, for all $n \in \omega$, $\big(f_{\varphi(m)}(n)\big)_{m\in\omega}$ converges to $x_n$. So fix $n \in \omega$ and let $V$ be a neighborhood of $x_n$. By the definition of $\varphi_n$, $\big(f_{\varphi_0 \cdots \varphi_n(m)}(n)\big)_{m\in\omega}$ tends to $x_n$. So there is $M \in \omega$ such that, for all $m \geq M$, $f_{\varphi_0 \cdots \varphi_n(m)}(n) \in V$. Up to replacing $M$ by $\max(M, n+1)$, we may as well assume that $M > n$. Now, for all $m \geq M$, $\varphi_{n+1} \cdots \varphi_m(m) \geq m$, and hence $f_{\varphi_0 \cdots \varphi_n \varphi_{n+1} \cdots \varphi_m(m)}(n) \in V$, which by definition means $f_{\varphi(m)}(n) \in V$ as desired. □

**Remark.** Tychonov proved that any product of compact spaces is compact. The proof we gave here in the metrizable case avoids the use of the axiom of choice.