Chapter 2-Polish spaces

1 Definition and examples

As we saw in the introduction, descriptive set theory is the study of the definable subsets of the Polish topological spaces. We now define this notion.

Definition 1.1 *A topological space X is*

(a) **completely metrizable** *if there is a metric d defining the topology of X such that (X,d) is complete,*

(b) **Polish** *if it is separable and completely metrizable.*

Definition 1.2 *A subset S of a topological space X is G_δ if S is the intersection of a countable family of open subsets of X. We say that S is F_σ if the complement of S is G_δ.*

Proposition 1.3 *Let X be a metrizable space, and C be a closed subset of X. Then C is G_δ.*

Proof. Let d be a metric defining the topology of X. Note that

$$C=\bigcap_{n\in\omega}\ \{x\in X\mid d(x,C)<\frac{1}{n+1}\},$$

so we are done since $x\mapsto d(x,C)$ is continuous. □

Proposition 1.4 *The class of*

(a) completely metrizable spaces is closed under countable products and topological sums,

(b) Polish spaces is closed under countable products and countable topological sums.

Proof. Let $(X_i)_{i\in I}$ be a sequence of completely metrizable spaces.

(a) We saw that $\Pi_{i\in I}\ X_i$ is metrizable if I is countable. Assume that the X_i's are completely metrizable. The proof of this fact shows that if $\big((x_i^n)_{i\in I})\big)_{n\in\omega}$ is Cauchy, then $(x_i^n)_{n\in\omega}$ is Cauchy for each $i\in I$, so that $(x_i^n)_{n\in\omega}$ converges to $x_i\in X_i$. Now $\big((x_i^n)_{i\in I})\big)_{n\in\omega}$ converges to $(x_i)_{i\in I}$. Thus $\Pi_{i\in I}\ X_i$ is completely metrizable.

Let d_i be a metric on X_i defining its topology. We set $d\big((i,x),(j,y)\big):=d_i(x,y)$ if $i=j$, 1 otherwise. Then d is a metric on $\oplus_{i\in I}\ X_i$ defining its topology, and we can chek that it is complete.

(b) We apply (a) and the fact that the class of separable spaces is closed under countable products and countable topological sums. □

Examples. Prove that the following spaces are Polish:

- the space ω of natural numbers, equipped with the discrete topology.

- the space $\mathbb{R}$ of real numbers, equipped with the usual topology.

- the **Cantor space** $\mathcal{C} := \{0,1\}^\omega := 2^\omega$, equipped with the product topology of the discrete topologies.

- the **Baire space** $\mathcal{N} := \omega^\omega$, equipped with the product topology of the discrete topologies.

- the **Hilbert cube** $[0,1]^\omega$, equipped with the product topology of the usual topologies.

- the $\mathbb{R}^\omega$, equipped with the product topology of the usual topologies.

- any separable Banach space.

- Let X be a metrizable compact space, Y be a Polish space with witness d, and $\mathcal{C}(X,Y)$ be the space of continuous functions from X into Y. Note that the formula

$$d_u(f,g) := \sup_{x\in X} d\big(f(x),g(x)\big)$$

defines a metric on $\mathcal{C}(X,Y)$, the **uniform metric**.

Theorem 1.5 *Let X be a metrizable compact space, and Y be a Polish space. Then $\mathcal{C}(X,Y)$, equipped with the topology defined by the uniform metric, is Polish.*

Proof. If (f_n) is Cauchy, then $\big(f_n(x)\big)$ is Cauchy for each n, and thus converges to $f(x)\in Y$. Note that f is continuous and (f_n) converges to f. Thus $\mathcal{C}(X,Y)$ is completely metrizable. It remains to see that it is separable. Fix a metric d_X on X defining its topology, and set

$$C_{m,n} := \Big\{ f\in\mathcal{C}(X,Y) \mid \forall x,y\in X \ \Big(d_X(x,y) < \frac{1}{m+1} \Rightarrow d\big(f(x),f(y)\big) < \frac{1}{n+1} \Big) \Big\}.$$

Choose $X_m \subseteq X$ finite with $X \subseteq \bigcup_{x\in X_m} B(x, \frac{1}{m+1})$. Let $D_{m,n} \subseteq C_{m,n}$ countable such that, for every $f \in C_{m,n}$ and every $\eta > 0$, there is $g \in D_{m,n}$ with $d\big(f(y),g(y)\big) < \frac{\eta}{3}$ for each $y \in X_m$. Then $\bigcup_{m,n\in\omega} D_{m,n}$ is dense in $\mathcal{C}(X,Y)$. Indeed, if $f \in \mathcal{C}(X,Y)$ and $\eta > 0$, then let $n > \frac{3}{\eta}$ and m such that $f \in C_{m,n}$ (which is possible since f is uniformly continuous). Let $g \in D_{m,n}$ be such that $d\big(f(y),g(y)\big) < \frac{1}{n+1}$ for each $y \in X_m$. If $x \in X$, then let $y \in X_m$ with $d_X(x,y) < \frac{1}{m+1}$. Then $d\big(f(x),g(x)\big) < \eta$. Thus $d_u(f,g) \leq \eta$. □

Theorem 1.6 *Every separable metrizable space is homeomorphic to a subspace of the Hilbert cube.*

Proof. Let (X,d) be a separable metric space. We saw that we may assume that $d(x,y) \leq 1$ for any x,y. Let $\{x_n \mid n\in\omega\}$ be dense in X. We define $f : X \to [0,1]^\omega$ by $f(x)(n) := d(x,x_n)$. Note that f is one-to-one and continuous. It remains to show that $f^{-1} : f[X] \to X$ is continuous. Assume that $\big(f(x^m)\big)$ converges to $f(x)$, i.e., $\big(d(x^m,x_n)\big)_{m\in\omega}$ converges to $d(x,x_n)$ for all n. Fix $\eta > 0$ and n such that $d(x,x_n) < \eta$. Let $M \in \omega$ such that, for each $m \geq M$, $d(x^m,x_n) < \eta$. If $m \geq M$, then $d(x^m,x) < 2\eta$. Thus (x^m) converges to x. □

An important class of Polish spaces is that of the metrisable compact spaces.

Theorem 1.7 *Let X be a compact Haudorff topological space. Then the following are equivalent:*

(a) X is Polish,

(b) X is homeomorphic to a subspace of the Hilbert cube,

(c) X is metrizable,

(d) X is second countable.

Proof. (a) $\Rightarrow$ (d) We know that metrizable separable spaces are second-countable.

(d) $\Rightarrow$ (b) As X is Hausdorff, X is T_1. As X is Hausdorff compact, X is regular. As X is also second countable, X is metrizable separable. It remains to apply Theorem 1.6.

(b) $\Rightarrow$ (c) This follows from the metrizability of $[0,1]^\omega$ along with the fact that subspaces of metrizable spaces are metrizable.

(c) $\Rightarrow$ (a) Let d be a metric inducing the topology of X. We know that (X,d) must be complete, precompact and thus separable. $\square$

2 Polish subspaces of Polish spaces

Notation. Let X be a topological space, (Y,d) be a metric space, $S \subseteq X$, and $f: S \to Y$. Then

$$osc_f(x) := \inf\{\text{diam}(f[O \cap S]) \mid O \text{ open neighborhood of } x\}$$

is the **oscillation of f at x.**

Theorem 2.1 *(Kuratowski) Let X be a metrizable space, Y be a completely metrizable space, $S \subseteq X$ and $f: S \to Y$ be a continuous function. Then we can find a G_δ subset G of X with $S \subseteq G \subseteq \overline{S}$ and $g: G \to Y$ continuous extending f.*

Proof. We set $G := \overline{S} \cap \{x \in X \mid osc_f(x) = 0\}$. Note that

$$osc_f(x) = 0 \Leftrightarrow \forall n \in \omega \text{ there is an open neighborhood } O \text{ of } x \text{ with diam}(f[O \cap S]) < \frac{1}{n+1},$$

so that $\{x \in X \mid osc_f(x) = 0\}$ is G_δ, as well as the closed set $\overline{S}$, by Proposition 1.3. Thus G is G_δ and contained in $\overline{S}$. If $x \in S$, then $x \in \overline{S}$ and $osc_f(x) = 0$ since f is continuous.

Now let $x \in G$. As $x \in \overline{S}$, there is a sequence (x_n) of points of S converging to x. Then $\big(\text{diam}(f[\{x_{n+1}, x_{n+2}, \cdots\}])\big)_{n \in \omega}$ converges to 0, so that the sequence $\big(f(x_n)\big)$ is Cauchy and thus converges to $g(x) \in Y$. Note that g is well defined, and extends f. In order to see that g is continuous, we need to check that $osc_g(x) = 0$ for each $x \in G$. If O is open in X, then $g[O \cap G] \subseteq \overline{f[O \cap S]}$, so $\text{diam}(g[O \cap G]) \leq \text{diam}(f[O \cap S])$ and $osc_g(x) = osc_f(x) = 0$. $\square$

Theorem 2.2 *A subspace of a Polish space is Polish if and only if it is G_δ.*

Proof. Let X be a Polish space, and $Y \subseteq X$. Assume first that Y, equipped with the induced topology, is Polish. Consider the identity function Id_Y of Y, which is continuous. Theorem 2.1 gives a G_δ subset G of X with $Y \subseteq G \subseteq \overline{Y}$ and $g : G \to Y$ continuous extending Id_Y. As Y is dense in G, $g = Id_G$, so $G = Y$.

Conversely, assume that $Y = \bigcap_{n\in\omega} O_n$, where the O_n's are open subsets of X. We set

$$C_n := Y \setminus O_n.$$

Let d be a metric defining the topology of X. We define a new metric on Y by

$$d'(x,y) := d(x,y) + \Sigma_{n\in\omega} \min\left(2^{-n-1}, \left|\frac{1}{d(x,C_n)} - \frac{1}{d(y,C_n)}\right|\right).$$

Note that d' defines the topology of Y. It remains to see that (Y,d') is complete. So let (y_i) be a Cauchy sequence. It is also Cauchy in (X,d), and thus converges to $y \in X$. If n is fixed, then $\left(\left|\frac{1}{d(y_i,C_n)}\right|\right)_{i\in\omega}$ converges in $\mathbb{R}$. Thus $\big(d(y_i,C_n)\big)_{i\in\omega}$ is bounded away from 0. As it converges to $d(y,C_n)$, $d(y,C_n) \neq 0$ and $y \in O_n$. Thus $y \in Y$. It remains to note that (y_i) converges to y in Y. □

Similarly, the following holds.

Theorem 2.3 *Every Polish space is homeomorphic to a closed subspace of* $\mathbb{R}^\omega$.

Proof. Let X be a Polish space. By Theorems 1.6 and 2.2, we may assume that X is a G_δ subset of of the Hilbert cube. So we can write $X = \bigcap_{n\in\omega} O_n$, where the O_n's are open subsets of $[0,1]^\omega$. We set $C_n := Y \setminus O_n$. Let d be a metric defining the topology of $[0,1]^\omega$. We define $f : X \to \mathbb{R}^\omega$ by $f(x)(2n) := \frac{1}{d(x,C_n)}$ and $f\big((x_i)_{i\in\omega}\big)(2n+1) := x_n$. Note that f is one-to-one and continuous. We now check that $f[X]$ is closed and $f^{-1} : f[X] \to X$ is continuous: if $(y^n) := \big(f(x^n)\big)$ converges to $y \in \mathbb{R}^\omega$, then (x^n) converges to $x \in [0,1]^\omega$ and also $(\frac{1}{d(x^n,C_i)})_{n\in\omega}$ converges for each i, so $\big(d(x^n,C_i)\big)_{n\in\omega}$ is bounded away from 0. Thus $\big(d(x^n,C_i)\big)_{n\in\omega}$ converges to $d(x,C_i) \neq 0$, and $x \in X$. Note that $f(x) = y$. □