Chapter 3-The Cantor and the Baire spaces

1 Trees

The concept of a tree is a basic combinatorial tool in descriptive set theory. What is referred to a tree in this domain is not, however, the same notion as the one used in graph theory or combinatorial set theory, although it is closely related.

Definition 1.1 *Let S be a nonempty set.*

(a) We denote by $S^{<\omega}$ the set of **finite sequences of elements of** S.

(b) If s is an element of $S^{<\omega}$, then the **length** *of s is denoted by $|s|$, so that*

$$s = \big(s(0), \cdots, s(|s|-1)\big).$$

The empty sequence has length 0 and is denoted by $\emptyset$. If $x \in S^{\omega}$, then the length $|x|$ of x is ω.

(c) If $s \in S^{<\omega}$, and $t \in S^{\leq\omega}$ is a finite or infinite sequence of elements of S, then we say that s is an **initial segment** *of t, which we denote by $s \subseteq t$, if $|s| \leq |t|$ and $s(i) = t(i)$ if $i < |s|$.*

(d) A **tree** *on S is a subset T of $S^{<\omega}$ closed under initial segments, i.e., if $t \in T$ and $s \subseteq t$, then $s \in T$.*

(e) If $x \in S^{\leq\omega}$ and $n < |x|$, then $x|n := \big(x(0), \cdots, x(n-1)\big)$ is the initial segment of x of length n. If x is finite, then $x||x| := x$.

(f) An **infinite branch** *of a tree T on S is a sequence $x \in S^{\omega}$ such that $x|n \in T$ for each $n \in \omega$. The set of infinite branches of T is denoted by $[T]$. The tree T is* **well founded** *if $[T]$ is empty.*

(g) A tree T on S is **pruned** *if, for every $s \in T$, there is $t \in T$ such that $s \subsetneqq t$ (i.e., $s \subseteq t$ and $s \neq t$).*

(h) A tree T on S is **finite splitting** *if, for every $s \in T$, there are at most finitely many $a \in S$ with $sa \in T$.*

Exercise. (König's lemma) Prove that a finite splitting tree is infinite if and only if it is not well founded.

Exercise. If T is a well founded tree, then we inductively define, for $s \in T$,

$$\rho_T(s) := \sup\{\rho_T(sa) + 1 \mid sa \in T\},$$

and the **rank** of T by $\rho(T) := \sup\{\rho_T(s) + 1 \mid s \in T\}$. If T' is another tree, then a function $\varphi : T \to T'$ is **strictly monotone** if $\varphi(s) \subsetneqq \varphi(t)$ whenever $s \subsetneqq t$. Prove that if T' is well founded, then T is well founded with $\rho(T) \leq \rho(T')$ if and only if there is $\varphi : T \to T'$ strictly monotone.

Notation. We put the discrete topology on S, so that S is metrizable. We then put the product topology on S^ω, so that S^ω is metrizable again, and the metric defined by

$$d(x,y) := \begin{cases} 2^{-n-1} \text{ if } x \neq y \wedge n := \min\{i \in \omega \mid x(i) \neq y(i)\} \\ 0 \text{ otherwise} \end{cases}$$

defines the topology of S^ω. The **standard basis** for this topology is given by the clopen sets

$$N_s := \{x \in S^\omega \mid s \subseteq x\},$$

for $s \in S^{<\omega}$. In particular, S^ω, equipped with this topology, is zero-dimensional. Note that the Cantor space and the Baire space are particular cases of this construction.

Exercise. Let S be nonempty set. Prove that the function $T \mapsto [T]$ is a bijection from the set of pruned trees on S onto the set of closed subsets of S^ω.

Definition 1.2 *Let X be a topological space, C be a closed subset of X, and $r : X \to C$ be a continuous function. We say that r is a* **retraction** *if r is the identity on C.*

Proposition 1.3 *Let S be a nonempty set, and $C \subseteq F$ be closed nonempty subsets of S^ω. Then there is a retraction from F onto C.*

Proof. Let T, R be pruned trees on S with $C = [T]$ and $F = [R]$. Note that $T \subseteq R$. We define $\varphi : R \to T$. In fact, $\varphi(s)$ is defined by induction on $|s|$. We first set $\varphi(\emptyset) := \emptyset$. Then, if $s \in S^{<\omega}$, $a \in S$ and $sa \in R$, $\varphi(sa) := sa$ if $sa \in T$, $\varphi(s)b$, where $b \in S$ is arbitrary with $\varphi(s)b \in T$, if $sa \notin T$. This is possible since T is pruned. As $\varphi(s) \subsetneqq \varphi(sa)$, we can define $r(x) := \lim_{n\to\infty} \varphi(x|n)$, and r is as desired. □

2 The Cantor space

We first prove the topological continuum hypothesis.

Proposition 2.1 *Let X be a nonempty perfect completely metrizable space. Then the Cantor space $\mathcal{C}$ is homeomorphic to a subset of X.*

Proof. Fix a complete metric $d \leq 1$ defining the topology of X. We construct a family $(O_s)_{s\in 2^{<\omega}}$ of nonempty open subsets of X satisfying the following properties:

$$(1)\ \overline{O_{s\varepsilon}} \subseteq O_s$$
$$(2)\ \text{diam}(O_s) \leq 2^{-|s|}$$
$$(3)\ O_{s0} \cap O_{s1} = \emptyset$$

Assume that this is done. Fix $\alpha \in \mathcal{C}$. Then $(\overline{O_{\alpha|n}})$ is a decreasing sequence of nonempty closed subsets of X with vanishing diameters. As X is complete, its intersection, equal to $\bigcap_{n\in\omega} O_{\alpha|n}$, is a singleton $h(\alpha)$. Note that h is injective and continuous, and thus a homeomorphism onto its range by compactness of $\mathcal{C}$.

We now construct O_s, by induction on $|s|$. We first set $O_\emptyset := X$. Given O_s, we choose $x_{s0} \neq x_{s1}$ in O_s, which is possible since X is perfect. We then choose O_{s0}, O_{s1} small enough balls with centers x_{s0}, x_{s1} respectively. □

Theorem 2.2 *(Cantor-Bendixson) Let X be a Polish space. Then X is the disjoint union of a closed perfect set and a countable set.*

Proof. We set $C := \bigcup\{O \subseteq X \mid O \text{ is open countable}\}$. By Lindelöf's lemma, C is countable open. We set $P := X \setminus C$, so that P is closed and X is the disjoint union of P ans C. If $x \in P$, then every neighborhood N of x is uncountable. Thus $N \setminus C$ is uncountable. In particular, x is not isolated in P for the induced topology. Thus P is perfect. □

Corollary 2.3 *Let X be an uncountable Polish space. Then X contains a homeomorphic copy of $\mathcal{C}$ and thus has size continuum.*

Proof. By Theorem 2.2, X contains a nonempty perfect Polish subspace. By Proposition 2.1, this subspace contains a homeomorphic copy of $\mathcal{C}$. This shows that X has size at least continuum. As X can be seen as a subspace of the Hilbert cube, X has size at most continuum. Thus X has size continuum. □

We now prove that the Cantor space is in some sense universal for metrizable compact spaces.

Theorem 2.4 *Let X be a nonempty metrizable compact space. Then X is a continuous image of $\mathcal{C}$.*

Proof. We first show that the Hilbert cube is a continuous image of $\mathcal{C}$. The function $f : \alpha \mapsto \Sigma_{n\in\omega} \frac{\alpha(n)}{2^{n+1}}$ is a continuous bijection from $\mathcal{C}$ onto $[0,1]$. Thus $(\alpha_n) \mapsto \big(f(\alpha_n)\big)$ maps $\mathcal{C}^\omega$, which is homeomorphic to $\mathcal{C}$, to the Hilbert cube. As every metrizable compact space is homeomorphic to a compact subspace of the Hilbert cube, for every metrizable compact space X there is a closed subset C of $\mathcal{C}$ and a continuous onto function from C onto X. It remains to apply Proposition 1.3. □

We now characterize the Cantor space, up to homeomorphism.

Theorem 2.5 *(Brouwer) The Cantor space $\mathcal{C}$ is the unique, up to homeomorphism, nonempty perfect zero-dimensional metrizable compact space.*

Proof. We saw that $\mathcal{C}$ has these properties. Now let X be such a space, and d be a metric defining its topology. We construct a family $(C_s)_{s\in 2^{<\omega}}$ of nonempty clopen subsets of X satisfying the following conditions:

(1) $C_\emptyset = X$
(2) $C_{s0} \cup C_{s1} = C_s$
(3) $\lim_{n\to\infty} \text{diam}(C_{\alpha|n}) = 0$ if $\alpha \in \mathcal{C}$
(4) $C_{s0} \cap C_{s1} = \emptyset$

Assume that this is done. Fix $\alpha \in \mathcal{C}$. Then $(C_{\alpha|n})$ is a decreasing sequence of nonempty closed subsets of X with vanishing diameters. As X is complete, its intersection is a singleton $h(\alpha)$. Note that h is injective and continuous, and thus a homeomorphism onto its range by compactness of $\mathcal{C}$. By (2), the range of h is X.

We now construct $(C_s)_{s\in 2^{<\omega}}$. We first consider a partition $(X_i)_{0<i\leq n}$ of X into nonempty clopen subsets of X with diameter at most $\frac{1}{2}$. We then set $C_{0^i 1} := X_{i+1}$, $C_{0^i} := \bigcup_{i<j\leq n} X_j$ if $i < n-1$, and $C_{0^{n-1}} := X_n$. Then we repeat this process in each X_i. □

3 The Baire space

We now prove that the Baire space is universal for zero-dimensional metrizable separable spaces.

Theorem 3.1 *(a) Every zero-dimensional metrizable separable space is homeomorphic to a subspace of $\mathcal{N}$, and also $\mathcal{C}$.*

(b) Every zero-dimensional Polish space is homeomorphic to a closed subset of $\mathcal{N}$, and to a G_δ subset of $\mathcal{C}$.

Proof. Note first that $\mathcal{N}$ is homeomorphic to a G_δ subset of $\mathcal{C}$. Indeed, the map $\alpha \to 0^{\alpha(0)}10^{\alpha(1)}1\cdots$ is a homeomorphism from $\mathcal{N}$ onto the G_δ subset $\{\beta \in \mathcal{C} \mid \forall m \in \omega \;\; \exists n \geq m \;\; \alpha(n)=1\}$ of $\mathcal{C}$. So it is enough to prove the assertions about $\mathcal{N}$.

(a) Let X be as in the statement. Fix a metric $d \leq 1$ defining the topology of X. We construct a family $(C_s)_{s\in\omega^{<\omega}}$ of possibly empty clopen subsets of X satisfying the following conditions:

$$\begin{array}{l}
(1)\ C_\emptyset = X\\
(2)\ \bigcup_{n\in\omega} C_{sn} = C_s\\
(3)\ \text{diam}(C_s) \leq 2^{-|s|}\\
(4)\ C_{sm} \cap C_{sn} = \emptyset \text{ if } m \neq n
\end{array}$$

Assume that this is done. We set $D := \{\alpha \in \mathcal{N} \mid \bigcap_{n\in\omega} C_{\alpha|n} \neq \emptyset\}$. Fix $\alpha \in D$. Then $(C_{\alpha|n})$ is a decreasing sequence of subsets of X with vanishing diameters. By definition of D, its intersection is a singleton $h(\alpha)$. Note that $h : D \to X$ is injective and continuous. By (2), h is onto. As $h[N_s \cap D] = C_s$, h is a homeomorphism. It remains to note that we can construct $(C_s)_{s\in 2^{<\omega}}$.

(b) It is enough to see that D is closed if (X, d) is complete. Assume that (α_n) is a sequence of points of D converging to $\alpha \in \mathcal{N}$. Then $\big(f(\alpha_n)\big)$ is Cauchy since, given $\eta > 0$, there are N with $\text{diam}(C_{\alpha|N}) < \eta$ and M such that $\alpha_n|N = \alpha|N$ for all $n \geq M$, so that $d\big(f(\alpha_m), f(\alpha_n)\big) < \eta$ if $m, n \geq M$. Thus $\big(f(\alpha_n)\big)$ converges to $y \in X$. As the C_s's are closed, $y \in C_{\alpha|n}$ for each n, so that $\alpha \in D$ and $f(\alpha) = y$. □

The Baire space is in some other sense universal for Polish spaces.

Theorem 3.2 *Let X be Polish space. Then X is a bijective continuous image of a closed subset of $\mathcal{N}$. If moreover X is nonempty, then X is a continuous image of $\mathcal{N}$.*

Proof. The last assertion is a consequence of the first one and Proposition 1.3. For the first assertion, fix a complete metric $d \leq 1$ defining the topology of X. We construct a family $(F_s)_{s\in\omega^{<\omega}}$ of F_σ subsets of X satisfying the following conditions:

$$\begin{array}{l}
(1)\ F_\emptyset = X\\
(2)\ \bigcup_{n\in\omega} F_{sn} = \bigcup_{n\in\omega} \overline{F_{sn}} = F_s\\
(3)\ \text{diam}(F_s) \leq 2^{-|s|}\\
(4)\ F_{sm} \cap F_{sn} = \emptyset \text{ if } m \neq n
\end{array}$$

Assume that this is done. We set $D := \{\alpha \in \mathcal{N} \mid \bigcap_{n\in\omega} F_{\alpha|n} \neq \emptyset\}$. Fix $\alpha \in D$. Then $(F_{\alpha|n})$ is a decreasing sequence of subsets of X with vanishing diameters. By definition of D, its intersection is a singleton $f(\alpha)$. Note that $f : D \to X$ is injective and continuous. By (2), f is onto. It remains to see that D is closed to see that f is as desired. Assume that (α_n) is a sequence of points of D converging to $\alpha \in \mathcal{N}$. As in the proof of Theorem 3.1, $\big(f(\alpha_n)\big)$ is Cauchy. Thus $\big(f(\alpha_n)\big)$ converges to $y \in X$ and $y \in \bigcap_{n\in\omega} \overline{F_{\alpha|n}} = \bigcap_{n\in\omega} F_{\alpha|n}$, so that $\alpha \in D$ and $f(\alpha) = y$.

In order to construct $(F_s)_{s\in\omega^{<\omega}}$, it is enough to show that for every F_σ set $F \subseteq X$ and every $\eta > 0$, we can write $F = \bigcup_{n\in\omega} F_n$, where the F_n's are pairwise disjoint F_σ sets of diameter $< \eta$ such that $\overline{F_n} \subseteq F$. Let $(C_i)_{i\in\omega}$ be an increasing sequence of closed sets with union F. Note that

$$F = C_0 \cup \bigcup_{i\in\omega} (C_{i+1} \setminus C_i)$$

and this union is disjoint. Now write $C_0 = \bigcup_{j\in\omega} E^0_j$ and $C_{i+1} \setminus C_i = \bigcup_{j\in\omega} E^{i+1}_j$, where the E^i_j's are pairwise disjoint F_σ sets of diameter smaller than η. Then $F = \bigcup_{i,j} E^i_j$ and $\overline{E^0_j} \subseteq C_0 \subseteq F$, $\overline{E^{i+1}_j} \subseteq \overline{C_{i+1} \setminus C_i} \subseteq C_{i+1} \subseteq F$. □

We now characterize the Baire space, up to homeomorphism.

Theorem 3.3 *(Alexandrov-Urysohn) The Baire space $\mathcal{N}$ is the unique, up to homeomorphism, nonempty Polish zero-dimensional space for which all compact subsets have empty interior.*

Proof. Assume that K is a compact subset of $\mathcal{N}$ and n is a natural number. Then the restriction to K of the continuous n'th projection of $\mathcal{N}$ on to ω is bounded. This shows that K has empty interior and $\mathcal{N}$ has the properties in the statement. Assume now that X has these properties. Fix a complete metric $d \leq 1$ defining the topology of X. We construct a family $(C_s)_{s\in\omega^{<\omega}}$ of nonempty clopen subsets of X satisfying the following properties:

(1) $C_\emptyset = X$
(2) $\bigcup_{n\in\omega} C_{sn} = C_s$
(3) $\text{diam}(C_s) \leq 2^{-|s|}$
(4) $C_{sm} \cap C_{sn} = \emptyset$ if $m \neq n$

Assume that this is done. Fix $\alpha \in \mathcal{N}$. Then $(C_{\alpha|n})$ is a decreasing sequence of nonempty closed subsets of X with vanishing diameters. As X is complete, its intersection is a singleton $h(\alpha)$. Note that h is injective and continuous. By (2), h is onto. As $h[N_s] = C_s$, h is a homeomorphism.

In order to construct $(C_s)_{s\in\omega^{<\omega}}$, it is enough to show that for every nonempty open set $O \subseteq X$ and every $\eta > 0$, we can write $O = \bigcup_{n\in\omega} O_n$, where the O_n's are pairwise disjoint nonempty clopen sets of diameter $< \eta$. As $\overline{O}$ is not compact, it is not precompact. This gives $0 < \eta' < \eta$ such that no cover of O by finitely many open sets of diameter smaller than η' exists. If we write $O = \bigcup_{j\in\omega} O_j$, where the O_j's are pairwise disjoint clopen sets of diameter smaller than η', infinitely many O_j's are nonempty, □

Exercise. Prove that the space $\mathbb{R} \setminus \mathbb{Q}$ is homeomorphic to $\mathcal{N}$.