Chapter 4-Baire category

1 Nowhere dense sets and meager sets

We now introduce some notions of topological smallness and bigness.

Definition 1.1 *Let X be a topological space, and $S \subseteq X$. We say that S is*

(a) **nowhere dense** *if its closure $\overline{S}$ has empty interior,*

(b) **meager** *if it is a countable union of nowhere dense sets,*

(c) **comeager** *if its complement $\neg S$ is meager.*

Examples. (a) Any compact subset of the Baire space $\mathcal{N}$ is nowhere dense.
(b) A countable subset of a perfect space is meager.

Definition 1.2 *A topological space X is* **Baire** *if it satisfies one of the following equivalent conditions:*

(a) every nonempty open set is non-meager,

(b) every comeager set is dense,

(c) the intersection of countably many dense open sets is dense.

Example. Every completely metrizable space is Baire, by Baire's theorem.

Exercise. Any open subset of a Baire space, equipped with the induced topology, is also a Baire space.

Exercise. We identify $\mathcal{C}$ with the power set of ω, using characteristic functions. Prove that if $G \subseteq \mathcal{C}$ is comeager, then we can find a partition (A_0, A_1) of ω and, for $\varepsilon \in 2$, $B_i \subseteq A_i$ such that if $A \subseteq \omega$ and $A \cap A_\varepsilon = B_\varepsilon$ for some $\varepsilon \in 2$, then $A \in G$.

2 Baire measurability

We now introduce a notion of regularity, being equal to an open set modulo a meager set.

Definition 2.1 *Let X, Y be topological spaces, $S \subseteq X$, and $f : X \to Y$ be a function. We say that*

(a) S has the **Baire property** *(denoted* **BP***) if there is an open subset O of X such that the* **symmetric difference** *$S \Delta O := (S \setminus O) \cup (O \setminus S)$ is meager,*

(b) f is **Baire measurable** *if the pre-image of any open subset of Y has the Baire property in X.*

Exercise. Prove that S has BP if and only if S is the union of a G_δ set and a meager set.

Definition 2.2 *Let X be a set. A σ-***algebra** *on X is a family of subsets of X containing $\emptyset$ and closed under complements and countable unions (and thus countable intersections).*

Proposition 2.3 *Let X be a topological space. Then the family of the subsets of X having the BP is a σ-algebra on X, contains the open subsets of X and the meager subsets of X.*

Proof. Note first that if O is an open subset of X, then $\overline{O}\setminus O$ is closed nowhere dense and thus meager. Similarly, if C is a closed subset of X, then $C\setminus Int(C)$ is closed nowhere dense and thus meager. Assume now that $S\subseteq X$ has the BP, which gives an open subset O of X such that $S\Delta O$ is meager. Note that $(\neg S)\Delta(\neg O)$ is meager, so that $(\neg S)\Delta Int(\neg O)$ is meager, so $\neg S$ has the BP. Finally, if each S_n has the BP with witness O_n, then $\bigcup_{n\in\omega}\ S_n$ has the BP with witness $\bigcup_{n\in\omega}\ O_n$. □

Proposition 2.4 *Let X be a topological space, and $S\subseteq X$ having the BP. Then S is meager, or there is a nonempty open subset O of X such that $O\setminus S$ is meager.*

Proof. As S has the BP, there is a witness O. If S is not meager, then O is as desired. □

Theorem 2.5 *Let X be a Baire space, Y be a second countable space, and $f : X\rightarrow Y$ be a Baire measurable function. Then we can find a dense G_δ subset G of X such that the restriction $f_{|G}$ of f to G is continuous.*

Proof. Let (O_n) be a countable basis for the topology of Y. As $f^{-1}(O_n)$ has the BP in X, we get an open subset U_n of X and a countable union of closed nowhere dense sets F_n with $f^{-1}(O_n)\Delta U_n\subseteq F_n$. Then $G_n := X\setminus F_n$ is a countable intersection of dense open subsets of X, as well as $G := \bigcap_{n\in\omega}\ G_n$. As X is Baire, G is a dense G_δ subset of X. As $f^{-1}(O_n)\cap G = U_n\cap G$, $f_{|G}$ is continuous. □

3 The Kuratowski-Ulam theorem

We now consider sets in product spaces. We will see a Fubini-like theorem for Baire category.

Lemma 3.1 *Let X be a topological space, Y be a second countable space, $S\subseteq X\times Y$, $x\in X$ and $S_x := \{y\in Y \mid (x,y)\in S\}$ be the vertical section of S at x.*

(a) If S is nowhere dense, then S_x is nowhere dense in Y for comeagerly many $x\in X$.

(b) If S is meager, then S_x is meager in Y for comeagerly many $x\in X$.

Proof. (a) We can assume that Y is not empty and S is closed. Let O be the complement of S. It is enough to show that O_x is dense for comeagerly many $x\in X$. Let (Y_n) be a basis for the topology of Y made of nonempty sets. Then $O_n := \text{proj}_X\big(O\cap(X\times Y_n)\big)$ is dense open in X. If $x\in\bigcap_{n\in\omega}\ O_n$, then $O_x\cap Y_n$ is not empty for all n, i.e., O_x is dense.

(b) This follows from (a). □

Lemma 3.2 *Let X, Y be second countable spaces, $A\subseteq X$ and $B\subseteq Y$. Then $A\times B$ is meager if and only if A is meager or B is meager.*

Proof. If $A \times B$ is meager and A is not meager, then there is $x \in X$ such that $(A \times B)_x = B$ is meager, by Lemma 3.1. Conversely, if A is meager and $A = \bigcup_{n \in \omega} N_n$ with N_n nowhere dense, then $A \times B = \bigcup_{n \in \omega} N_n \times B$, so it is enough to show that $N_n \times B$ is nowhere dense. This comes from the fact that if O is dense open in X, then $O \times Y$ is dense open in $X \times Y$. □

Theorem 3.3 *(Kuratowski-Ulam) Let X, Y be second countable spaces, and $S \subseteq X \times Y$ having the BP.*

(a) S_x has the BP for comeagerly many $x \in X$. Similarly, $S^y := \{x \in X \mid (x, y) \in S\}$ has the BP for comeagerly many $y \in Y$.

(b) S is meager is equivalent to S_x is meager for comeagerly many $x \in X$, and to S^y is meager for comeagerly many $y \in Y$.

(c) S is comeager is equivalent to S_x is comeager for comeagerly many $x \in X$, and to S^y is comeager for comeagerly many $y \in Y$.

Proof. Let O be an open set and M be a meager set with $S \Delta O \subseteq M$.

(a) Note that, for any $x \in X$, $S_x \Delta O_x \subseteq M_x$. By Lemma 3.1, S_x has the BP for comeagerly many $x \in X$.

(b) By Lemma 3.1, if S is meager, then S_x is meager for comeagerly many $x \in X$. Conversely, if S is not meager, then O is not meager, which gives open sets $U \subseteq X$ and $V \subseteq Y$ such that $U \times V \subseteq O$ and $U \times V$ is not meager. By Lemma 3.2, U, V are not meager. This gives $x \in U$ such that S_x and M_x are meager. As $V \setminus M_x \subseteq O_x \setminus M_x \subseteq S_x$, $V \subseteq S_x \cup M_x$ is meager, a contradiction.

(c) This comes from (b). □

4 Meager relations

We now strengthen the perfect set theorem.

Notation. If X is a set, then $\Delta(X) := \{(x, x) \mid x \in X\}$ is the **diagonal** of X.

Theorem 4.1 *(Mycielski-Kuratowski) Let X be a nonempty perfect Polish space, and $R \subseteq X^2$ be meager. Then X contains a copy C of the Cantor space $\mathcal{C}$ such that $(x, y) \notin R$ if $x \neq y \in C$.*

Proof. As R is meager, there is an increasing sequence (C_l) of closed nowhere dense relations on X whose union contains R. We set $U_l := X^2 \setminus C_l$, so that U_l is dense open in X^2. Fix a complete metric $d \leq 1$ defining the topology of X. We construct a family $(O_s)_{s \in 2^{<\omega}}$ of nonempty open subsets of X satisfying the following properties:

(1) $\overline{O_{s\varepsilon}} \subseteq O_s$
(2) $\text{diam}(O_s) \leq 2^{-|s|}$
(3) $O_{s0} \cap O_{s1} = \emptyset$
(4) $O_s \times O_t \subseteq U_l$ if $s \neq t \in 2^l$

Assume that this is done. Fix $\alpha\in\mathcal{C}$. Then $(\overline{O_{\alpha|n}})$ is a decreasing sequence of nonempty closed subsets of X with vanishing diameters. As X is complete, its intersection, equal to $\bigcap_{n\in\omega}\ O_{\alpha|n}$, is a singleton $h(\alpha)$. Note that h is injective and continuous, and thus a homeomorphism onto its range C by compactness of $\mathcal{C}$. If $x\neq y\in C$, then we can write $x=h(\alpha)$ and $y=h(\beta)$, where $\alpha\neq\beta\in\mathcal{C}$. This gives l_0 with $\alpha(l_0)\neq\beta(l_0)$. If $l>l_0$, then $\alpha|l\neq\beta|l$, so that $(x,y)\in O_{\alpha|l}\times O_{\beta|l}\subseteq U_l$ and $(x,y)\notin R$ since (C_l) is increasing.

We now construct O_s, by induction on $|s|$. We first set $O_\emptyset:=X$. We enumerate

$$\{(s,t)\in 2^{l+1}\times 2^{l+1}\mid s\neq t\}$$

by $\{(s_i,t_i)\mid i<N\}$. We construct, inductively on i, a family $(O^i_s)_{s\in 2^{l+1}}$ of nonempty open subsets of X satisfying the following properties:

$$\begin{array}{l}
(1)\ \overline{O^i_{s_i}}\subseteq O_{s_i|l}\cap\bigcap_{j<i}\ O^j_{s_i}\\
(2)\ \text{diam}(O^i_{s_i})\leq 2^{-l-1}\\
(3)\ O^i_{s_i}\times O^i_{t_i}\subseteq U_{l+1}\setminus\Delta(X)\\
(4)\ O^i_s\subseteq O_{s|l}\cap\bigcap_{j<i}\ O^j_s
\end{array}$$

Then we will just have to take $O_s:=O^{N-1}_s$. Note that $(O_{s_i|l}\cap\bigcap_{j<i}\ O^j_{s_i})\times(O_{t_i|l}\cap\bigcap_{j<i}\ O^j_{t_i})$ meets $U_{l+1}\setminus\Delta(X)$ since U_{l+1} is dense open and $\Delta(X)$ is closed nowhere dense since X is perfect. Let (x,y) be in the intersection I. We choose open subsets $O^i_{s_i},O^i_{t_i}$ of X with diameter at most 2^{-l-1} such that $(x,y)\in O^i_{s_i}\times O^i_{t_i}\subseteq\overline{O^i_{s_i}}\times\overline{O^i_{t_i}}\subseteq I$. If $s\in 2^{l+1}\setminus\{s_i,t_i\}$, then we set $O^i_s:=O_{s|l}\cap\bigcap_{j<i}\ O^j_s$. This finishes the proof. $\square$

Exercise. (Galvin) Let X be a nonempty perfect Polish space, and $R\subseteq X^2$ be non meager and having the Baire property. Prove that we can find copies C_0,C_1 of $\mathcal{C}$ in X with $C_0\times C_1\subseteq R$.

5 Choquet and strong Choquet games

Being a Baire space can be expressed in terms of a game.

Definition 5.1 *Let X be a nonempty topological space.*

(a) The **Choquet game** *G_X on X is defined as follows. Players 1 and 2 take turns in playing nonempty open subsets of X*

$$\begin{array}{llllll}
1 & U_0 & & U_1 & & \\
 & & & & & \ldots\\
2 & & V_0 & & V_1 &
\end{array}$$

in such a way that $U_0\supseteq V_0\supseteq U_1\supseteq V_1\cdots$ We say that 2 **wins this run of the game** *if $\bigcap_{n\in\omega}\ V_n$ $(=\bigcap_{n\in\omega}\ U_n)$ is nonempty (1 wins if $\bigcap_{n\in\omega}\ V_n$ is empty).*

(b) Let

$$\mathcal{T}_X:=\{s\in(2^X)^{<\omega}\mid\forall i<|s|\ s(i)\ \textit{is a nonempty open subset of}\ X\wedge\forall i<|s|-1\ s(i)\supseteq s(i+1)\}.$$

A **strategy for 1** *in G_X is a subtree σ of $\mathcal{T}_X$ such that*

(1) $\sigma \neq \emptyset$
(2) $\forall (U_0, V_0, \cdots, U_n) \in \sigma$, $\forall V_n \subseteq U_n$ *nonempty open,* $(U_0, V_0, \cdots, U_n, V_n) \in \sigma$
(3) $\forall (U_0, V_0, \cdots, U_{n-1}, V_{n-1}) \in \sigma$, $\exists! U_n$ *nonempty open such that* $(U_0, V_0, \cdots, U_n) \in \sigma$

Similarly, we define the notion of a **strategy for 2** *in G_X.*

(c) A strategy σ for 1 is a **winning strategy for 1** *if 1 wins every run* $(U_0, V_0, U_1, V_1, \cdots) \in [\sigma]$. *Similarly, we define the notion of a* **winning strategy for 2** *in G_X.*

Theorem 5.2 *(Oxtoby) Let X be a nonempty topological space. Then X is a Baire space if and only if player 1 has no winning strategy in the Choquet game G_X.*

Proof. Assume first that X is not a Baire space, and let U_0 be a nonempty open subset of X and (O_n) be a sequence of dense open subsets of X whose intersection does not meet U_0. Player 1 first plays U_0. Then player 2 plays $V_0 \subseteq U_0$. Note that $U_1 := V_0 \cap O_0$ is nonempty, so that 1 can play it. Then 2 plays $V_1 \subseteq U_1$. Note that $U_2 := V_1 \cap O_1$ is nonempty, so that 1 can play it. And so on. Note that $\bigcap_{n\in\omega} U_n \subseteq \bigcap_{n\in\omega} O_n \cap U_0 = \emptyset$, so that this strategy is winning for 1.

Assume now that player 1 has a winning strategy σ in the Choquet game G_X. Let U_0 be player 1's first move according to σ. We will show that U_0 is not Baire. We construct a nonempty pruned tree $S \subseteq \sigma$ as follows. We determine inductively which sequences from σ of length n we put in S. First $\emptyset \in S$. If $(U_0, V_0, \cdots, U_{n-1}, V_{n-1}) \in S$, then $(U_0, V_0, \cdots, U_n) \in S$, for the unique U_n with $(U_0, V_0, \cdots, U_n) \in \sigma$. If now $p := (U_0, V_0, \cdots, U_n) \in S$ and $V_n \subseteq U_n$ is nonempty open, then let $V_n^* := U_{n+1}$ be what σ requires 1 to play next. Zorn's lemma gives a maximal collection $\mathcal{V}_p$ of nonempty open subsets $V_n \subseteq U_n$ such that $\{V_n^* \mid V_n \in \mathcal{V}_p\}$ is pairwise disjoint. We put in S all $(U_0, V_0, \cdots, U_n, V_n, V_n^*)$ with $V_n \in \mathcal{V}_p$. Then $\mathcal{U}_p := \{V_n^* \mid V_n \in \mathcal{V}_p\}$ is a family of pairwise disjoint sets and $\bigcup \mathcal{U}_p$ is dense in U_n, by maximality. We now set $W_n := \bigcup \{U_n \mid (U_0, V_0, \cdots, U_n) \in S\}$. Note that W_n is open and dense in U_0 for each n. It remains to show that $\bigcap_{n\in\omega} W_n = \emptyset$. We argue by contradicton, which gives $x \in \bigcap_{n\in\omega} W_n$. Then there is a unique $(U_0, V_0, U_1, V_1, \cdots) \in [S]$ such that $x \in \bigcap_{n\in\omega} U_n$, which contradicts the fact that $(U_0, V_0, U_1, V_1, \cdots) \in [\sigma]$ and σ is winning for 1.□

Definition 5.3 *A nonempty topological space X is a* **Choquet space** *if player 2 has a winning strategy in the Choquet game G_X.*

By Theorem 5.2, every Choquet space is Baire. A strong version of the Choquet game will help us to characterize the Polish spaces.

Definition 5.4 *Let X be a nonempty topological space.*

(a) The **strong Choquet game** *G_X^s on X is defined as follows. Players 1 and 2 take turns in playing points of X and nonempty open subsets of X*

1	x_0, U_0		x_1, U_1		$\cdots$
2		V_0		V_1	

in such a way that $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \cdots$ *and* $x_n \in V_n$. *We say that 2* **wins this run of the game** *if* $\bigcap_{n\in\omega} V_n$ *(*$= \bigcap_{n\in\omega} U_n$*) is nonempty (1 wins if* $\bigcap_{n\in\omega} V_n$ *is empty).*

(b) X is a **strong Choquet space** *if player 2 has a winning strategy in the strong Choquet game G^s_X.*

Exercise. (a) Any strong Choquet space is Choquet.

(b) Every nonempty completely metrizable space is strong Choquet.

(c) Products of strong Choquet spaces are strong Choquet.

(d) Nonempty G_δ subspaces of strong Choquet spaces are strong Choquet.

6 Characterization of Polish spaces

Theorem 6.1 *(Choquet) Let X be a nonempty metrizable separable space, and $\hat{X}$ be a Polish space in which X is dense. Then X is strong Choquet if and only if X is G_δ in $\hat{X}$ if and only if X is Polish.*

Proof. The last equivalence is known. If X is Polish, then X is strong Choquet by the last exercise. Conversely, let d be a metric defining the topology of $\hat{X}$, and σ be a winning strategy for 2 in the strong Choquet game G^s_X. Using point-finite refinement, we can construct (as in the proof of Theorem 5.2) a tree S of sequences of the form $\big(x_0,(V_0,\hat{V}_0),x_1,(V_1,\hat{V}_1),\cdots,x_n\big)$ or $\big(x_0,(V_0,\hat{V}_0),x_1,(V_1,\hat{V}_1),\cdots,x_n,(V_n,\hat{V}_n)\big)$, where V_i is open in X, $\hat{V}_i$ is open in $\hat{X}$, $x_i\in\hat{V}_{i-1}\cap X$ (with $\hat{V}_{-1}:=\hat{X}$), $x_i\in V_i$, $\hat{V}_i\cap X\subseteq V_i$, $\hat{V}_0\supseteq\hat{V}_1\supseteq\cdots$, and $\big((x_0,X),V_0,(x_1,\hat{V}_0\cap X),V_1,\cdots\big)$ is compatible with σ, such that S additionally has the following property: for each

$$p=\big(x_0,(V_0,\hat{V}_0),x_1,(V_1,\hat{V}_1),\cdots,x_{n-1},(V_{n-1},\hat{V}_{n-1})\big)\in S$$

(including the empty sequence), if

$$\mathcal{V}_p:=\big\{\hat{V}_n \mid (x_0,(V_0,\hat{V}_0),x_1,\cdots,(V_{n-1},\hat{V}_{n-1}),x_n,(V_n,\hat{V}_n))\in S\big\},$$

then $X\cap\hat{V}_{n-1}\subseteq\bigcup\mathcal{V}_p$, $\mathrm{diam}(\tilde{V}_n)<2^{-n}$ for all $\hat{V}_n\in\mathcal{V}_p$, and for every $\hat{x}\in\hat{X}$ there are at most finitely many $\big(x_n,(V_n,\hat{V}_n)\big)$ with $\big(x_0,(V_0,\hat{V}_0),\cdots,(V_{n-1},\hat{V}_{n-1}),x_n,(V_n,\hat{V}_n)\big)\in S$ and $\hat{x}\in\hat{V}_n$. Let $W_n:=\bigcup\big\{\tilde{V}_n \mid \big(x_0,(V_0,\hat{V}_0),\cdots,x_n,(V_n,\hat{V}_n)\big)\in S\big\}$. Then W_n is open and $X\subseteq W_n$ (as we can see by an easy induction on n). It remains to show that $\bigcap_{n\in\omega}W_n\subseteq X$. Let $\hat{x}\in\bigcap_{n\in\omega}W_n$. Consider the subtree $S_{\hat{x}}$ of S consisting of all initial segments of the sequences

$$\big(x_0,(V_0,\hat{V}_0),\cdots,x_n,(V_n,\hat{V}_n)\big)\in S$$

for which $\hat{x}\in\hat{V}_n$. Since $\hat{x}\in\bigcap_{n\in\omega}W_n$, $S_{\hat{x}}$ is infinite. By the preceding conditions on S, it is also finite splitting. So, by König's Lemma, $[S_{\hat{x}}]\neq\emptyset$. Say $\big(x_0,(V_0,\hat{V}_0),\cdots\big)\in[S_{\hat{x}}]$. Then $\big((x_0,X),V_0,(x_1,\hat{V}_0\cap X),V_1,(x_2,\hat{V}_1\cap X),\cdots\big)$ is a run of G^s_X compatible with σ, so $\bigcap_{n\in\omega}\hat{V}_n\cap X$ is not empty; thus, since $\mathrm{diam}(\hat{V}_n)<2^{-n}$, $\hat{x}\in X$. □

The next characterization will be very important in the applications.

Theorem 6.2 *(Choquet) A nonempty, second countable topological space is Polish if and only if it is T_1, regular, and strong Choquet.*

Proof. We apply Theorem 6.1 and the Urysohn Metrization Theorem. □

Exercise. (Sierpinski) Let X be a Polish space, Y be a metrizable separable space, and $f:X\to Y$ be a function. Prove that if f is continuous, open and onto, then Y is Polish.