Chapter 5-Borel sets and functions

1 The Borel hierarchy

Definition 1.1 *Let X, Y be topological spaces.*

(a) A subset of X is a **Borel set** *if it is in the σ-algebra generated by the open subsets of X.*

(b) A function $f : X \to Y$ is a **Borel function** *if the pre-image by f of any open subset of Y is a Borel subset of X. If f is a Borel bijection with Borel inverse, then we say that f is a* **Borel isomorphism**.

Any continuous function is Borel. Note that the Borel sets have the BP, and every Borel function is Baire-measurable. Also, the class of Borel subsets of X contains, the open, the closed, the F_σ, the G_δ subsets of X.

Definition 1.2 *(1) If $\mathbf{\Gamma}$ is a class of sets, then*

(a) $\check{\mathbf{\Gamma}} := \{\neg S \mid S \in \mathbf{\Gamma}\}$ is the class of the complements of the elements of $\mathbf{\Gamma}$,

(b) $\mathbf{\Gamma}_\sigma$ is the class of countable unions of elements of $\mathbf{\Gamma}$,

(c) $\mathbf{\Gamma}(X)$ is the class of subsets of X which are in $\mathbf{\Gamma}$.

(2) Let ω_1 be the first uncountable ordinal. We define, by induction on $1 \leq \xi < \omega_1$, the following classes of subsets of the metrizable spaces:

	$\mathbf{\Sigma}^0_1 = \textit{open}$		$\mathbf{\Sigma}^0_2 = F_\sigma$		$\mathbf{\Sigma}^0_\xi = (\bigcup_{1 \leq \eta < \xi} \mathbf{\Pi}^0_\eta)_\sigma$	
$\mathbf{\Delta}^0_1 = \textit{clopen}$		$\mathbf{\Delta}^0_2 = \mathbf{\Sigma}^0_2 \cap \mathbf{\Pi}^0_2$	$\cdots$	$\mathbf{\Delta}^0_\xi = \mathbf{\Sigma}^0_\xi \cap \mathbf{\Pi}^0_\xi$		$\cdots$
	$\mathbf{\Pi}^0_1 = \textit{closed}$		$\mathbf{\Pi}^0_2 = G_\delta$		$\mathbf{\Pi}^0_\xi = \check{\mathbf{\Sigma}}^0_\xi$	

In the picture above, the inclusion of classes hold from the left to the right, by transfinite induction, since we saw that in any metrizable space a closed set is G_δ. This gives a ramification of the Borel sets in a hierarchy of at most ω_1 levels, called the **Borel hierarchy**, or hierarchy of the **Borel classes**. This is the most classical hierarchy of topological complexity in descriptive set theory. Note that the class of Borel sets is the σ-algebra $\bigcup_{\xi<\omega_1} \mathbf{\Sigma}^0_\xi = \bigcup_{\xi<\omega_1} \mathbf{\Pi}^0_\xi = \bigcup_{\xi<\omega_1} \mathbf{\Delta}^0_\xi$.

Proposition 1.3 *The Borel classes are closed under finite unions and intersections, and continuous pre-images. Moreover, $\mathbf{\Sigma}^0_\xi$ is closed under countable unions, $\mathbf{\Pi}^0_\xi$ is closed under countable intersections, and $\mathbf{\Delta}^0_\xi$ is closed under complements.*

Proof. We argue by transfinite induction. □

Exercise. Prove that $C_0 := \{(x_n) \in [0,1]^\omega \mid (x_n) \text{ converges to } 0\}$ is $\mathbf{\Pi}^0_3$.

Exercise. (a) Let (f_n) be a sequence of Borel functions from a topological space X into a metrizable space Y. We assume that this sequence converges pointwise to a function $f:X\to Y$, i.e., $\big(f_n(x)\big)_{n\in\omega}$ converges to $f(x)$ for each $x\in X$. Prove that f is Borel.

(b) Let X be a topological space and $f:X\to\mathbb{R}$ be a **lower** (resp., **upper**) **semi-continuous** function, i.e., $\{x\in X\mid f(x)>a\}$ (resp., $\{x\in X\mid f(x)<a\}$) is open for each $a\in\mathbb{R}$. Prove that f is Borel.

Theorem 1.4 *(Lebesgue, Hausdorff) Let X be a metrizable space. Then the class of Borel functions from X into $\mathbb{R}$ is the smallest class of functions from X into $\mathbb{R}$ which contains all the continuous functions and is closed under pointwise limit.*

Proof. Let $\mathcal{S}$ be the smallest class of functions from X into $\mathbb{R}$ which contains all the continuous functions and is closed under pointwise limit. Note that $\mathcal{S}$ is a vector space, i.e., if $a,b\in\mathbb{R}$ and $f,g\in\mathcal{S}$, then $af+bg\in\mathcal{S}$.

Let us prove that the characteristic function χ_B of any Borel subset B of X is in $\mathcal{S}$. Assume first that O is an open subset of X, which gives an increasing sequence (C_n) of closed subsets of X with union O. Urysohn's lemma gives $f_n:X\to\mathbb{R}$ continuous such that $0\leq f_n\leq 1$, $f_n=1$ on C_n, and $f_n=0$ on $\neg O$. Note that (f_n) converges pointwise to χ_O, so that $\chi_O\in\mathcal{S}$. We then note that $\chi_{\neg B}=1-\chi_B$. Finally, if (B_n) is a sequence of pairwise disjoint sets, then the sequence $\big(\chi_{B_0}+\cdots+\chi_{B_p}\big)_{p\in\omega}$ pointwise converges to $\chi_{\bigcup_{n\in\omega}B_n}$. For instance, if $B\in\mathbf{\Sigma}^0_2(X)$, then $B=\bigcup_{n\in\omega}\ C_n$, where the C_n's are closed. Thus B is the disjoint union of the $C_n\setminus(\bigcup_{p<n}\ C_p)$'s. Note that $\neg\big(C_n\setminus(\bigcup_{p<n}\ C_p)\big)$ is the disjoint union of $\neg C_n$ and $\bigcup_{p<n}\ C_p$. Thus $\chi_B\in\mathcal{S}$. We then argue inductively.

Now let $f:X\to\mathbb{R}$ be a Borel function. Note that $f=f^+-f^-$, where $f^+:=\frac{|f|+f}{2}$ and $f^-:=\frac{|f|-f}{2}$. As $|f|$, f^+ and f^- are Borel, we may assume that f is non-negative. We set, for $n\geq 1$ natural number and $1\leq i\leq n2^n$, $A_{n,i}:=f^{-1}\big([\frac{i-1}{2^n},\frac{i}{2^n})\big)$. We then put $f_n:=\Sigma_{1\leq i\leq n2^n}\ \frac{i-1}{2^n}\cdot\chi_{A_{n,i}}$. As $A_{n,i}$ is Borel, $f_n\in\mathcal{S}$. As (f_n) pointwise converges to f, $f\in\mathcal{S}$.

As the class of Borel functions contains all the continuous functions and is closed under pointwise limit, the proof is complete. □

We will give a quantitative version of Theorem 1.4. In order to do this, we first establish some important structural properties of the Borel classes.

Definition 1.5 *Let $\mathbf{\Gamma}$ be a class of sets, X,Y be sets, and $R\subseteq X\times Y$.*

(a) A **uniformization** *of R is a subset R^* of R which is the graph of a partial function defined on the projection $proj_X[R]$ of R on X. Such a function is called a* **uniformizing function** *for R.*

(b) The class $\mathbf{\Gamma}$ has the **number uniformization property** *if, for any $R\subseteq X\times\omega$ in $\mathbf{\Gamma}$, there is a uniformization R^* of R in $\mathbf{\Gamma}$.*

(c) The class $\mathbf{\Gamma}$ has the **reduction property** *if, for any $A,B\subseteq X$ in $\mathbf{\Gamma}$, there are $A^*,B^*\subseteq X$ in $\mathbf{\Gamma}$ disjoint such that $A^*\subseteq A$, $B^*\subseteq B$, and $A^*\cup B^*=A\cup B$. We then say that A^*,B^** **reduce** *A,B.*

(d) The class $\mathbf{\Gamma}$ has the **separation property** *if, for any $A,B\subseteq X$ in $\mathbf{\Gamma}$ disjoint, there is $C\subseteq X$ in $\mathbf{\Gamma}\cap\check{\mathbf{\Gamma}}$ such that $A\subseteq C\subseteq\neg B$.*

Theorem 1.6 *In metrizable spaces, for any countable ordinal $\xi \geq 2$, the class $\mathbf{\Sigma}^0_\xi$ has the number uniformization property and the reduction property, and the class $\mathbf{\Pi}^0_\xi$ has the separation property. This also holds for $\xi = 1$ in zero-dimensional spaces.*

Proof. Let $\xi \geq 2$ be a countable ordinal, and $R \subseteq X \times \omega$ in $\mathbf{\Sigma}^0_\xi$. We can write $R = \bigcup_{i\in\omega} R_i$, where R_i is in $\mathbf{\Pi}^0_{\xi_i}$ and $1 \leq \xi_i < \xi$. Let $k \mapsto \big((k)_0, (k)_1\big)$ be a bijection from ω onto ω^2, with inverse $< .,. >: \omega^2 \rightarrow \omega$. We put $Q(x,k) \Leftrightarrow \big(x, (k)_1\big) \in R_{(k)_0}$, $Q^*(x,k) \Leftrightarrow Q(x,k) \wedge \forall j < k \ \neg Q(x,j)$ and $R^*(x,n) \Leftrightarrow \exists i \in \omega \ Q^*(x, < i, n >)$. Then R^* is a uniformization of R in $\mathbf{\Sigma}^0_\xi$. Thus $\mathbf{\Sigma}^0_\xi$ has the number uniformization property. If moreover X is zero-dimensional and $\xi = 1$, then we can repeat this proof, taking the R_i's clopen.

If $A, B \subseteq X$ are in $\mathbf{\Sigma}^0_\xi$, then we define $R \subseteq X \times \omega$ by

$$R(x,n) \Leftrightarrow (n=0 \wedge x \in A) \vee (n=1 \wedge x \in B).$$

Note that R is in $\mathbf{\Sigma}^0_\xi$. This gives a uniformization R^* of R in $\mathbf{\Sigma}^0_\xi$. We set $A^*(x) \Leftrightarrow R^*(x,0)$ and $B^*(x) \Leftrightarrow R^*(x,1)$. Note that A^*, B^* reduce A, B.

If $A, B \subseteq X$ are in $\mathbf{\Pi}^0_\xi$ disjoint, then $\neg A, \neg B$ are in $\mathbf{\Sigma}^0_\xi$, which gives A^*, B^* reducing $\neg A, \neg B$. We just have to set $C := B^*$. □

Definition 1.7 *Let X be a set, and (S_n) be a sequence of subsets of X.*

(a) $\overline{\mathit{lim}}_{n\in\omega} \ S_n := \{x \in X \mid \forall m \in \omega \ \exists n \geq m \ x \in S_n\}$ is the set of points of X in infinitely many S_n's.

(b) $\underline{\mathit{lim}}_{n\in\omega} \ S_n := \{x \in X \mid \exists m \in \omega \ \forall n \geq m \ x \in S_n\}$ is the set of points of X in all but finitely many S_n's.

(c) If $\overline{\mathit{lim}}_{n\in\omega} \ S_n = \underline{\mathit{lim}}_{n\in\omega} \ S_n$, then this set is denoted by $\mathit{lim}_{n\in\omega} \ S_n$.

Proposition 1.8 *(Kuratowski) Let $\xi \geq 2$ be a countable ordinal, X be a metrizable space, and $S \subseteq X$. Then $S \in \mathbf{\Delta}^0_{\xi+1}$ if and only if there is a sequence (S_n) of subsets of X in $\mathbf{\Delta}^0_\xi$ with $S = \mathit{lim}_{n\in\omega} \ S_n$. This also holds for $\xi = 1$ if X is zero-dimensional. If λ is an infinite limit countable ordinal, then $S \in \mathbf{\Delta}^0_{\lambda+1}$ if and only if there is a sequence (S_n) of subsets of X in $\bigcup_{\eta<\lambda} \ \mathbf{\Delta}^0_\eta$ with $S = \mathit{lim}_{n\in\omega} \ S_n$.*

Proof. Assume first that $S \in \mathbf{\Delta}^0_{\xi+1}$. We can write $S = \bigcup_{n\in\omega} \ A_n$ and $\neg S = \bigcup_{n\in\omega} \ B_n$, where (A_n) and (B_n) are sequences of $\mathbf{\Pi}^0_\xi$ subsets of X. Moreover, replacing A_n with $\bigcup_{p\leq n} \ A_p$ if necessary, we may assume that (A_n) is increasing, and similarly with (B_n). Theorem 1.6 provides $S_n \in \mathbf{\Delta}^0_\xi$ with $A_n \subseteq S_n \subseteq \neg B_n$. Now note that

$$S = \bigcup_{n\in\omega} \ A_n = \underline{\lim}_{n\in\omega} \ A_n \subseteq \underline{\lim}_{n\in\omega} \ S_n \subseteq \overline{\lim}_{n\in\omega} \ S_n \subseteq \overline{\lim}_{n\in\omega} \ \neg B_n = \bigcap_{n\in\omega} \ \neg B_n = S,$$

so that $\lim_{n\in\omega} \ S_n$ is defined and equal to S. Conversely, $S \in \mathbf{\Delta}^0_{\xi+1}$ if (S_n) is a sequence of subsets of X in $\mathbf{\Delta}^0_\xi$ with $S = \lim_{n\in\omega} \ S_n$.

In particular, $S\in\mathbf{\Delta}^0_{\lambda+1}$ if (S_n) is a sequence of subsets of X in $\bigcup_{\eta<\lambda}\ \mathbf{\Delta}^0_\eta$ with $S=\lim_{n\in\omega}\ S_n$. Assume now that $S\in\mathbf{\Delta}^0_{\lambda+1}$. We can write $S=\bigcup_{n\in\omega}\ \bigcap_{m\in\omega}\ B_{n,m}=\bigcap_{m\in\omega}\ \bigcup_{n\in\omega}\ C_{n,m}$, where $(B_{n,m})_{n,m\in\omega}$ and $(C_{n,m})_{n,m\in\omega}$ are sequences of subsets of X in $\bigcup_{\eta<\lambda}\ \bigcup_{\eta<\lambda}\ \mathbf{\Delta}^0_\eta$. As above, we may assume that $\bigcup_{n\in\omega}\ C_{n,m+1}\subseteq\bigcup_{n\in\omega}\ C_{n,m}$. We put $S_n:=\bigcup_{k\leq n}\ \big((\bigcap_{j\leq n}\ B_{k,j})\cap(\bigcup_{l\leq n}\ C_{l,k})\big)$. Note first that $S\subseteq\underline{\lim}_{n\in\omega}\ S_n$. Indeed, let $x\in S$, which gives $k\in\omega$ such that $x\in\bigcap_{j\in\omega}\ B_{k,j}$. Note also that there is $i_k\in\omega$ with $x\in C_{i_k,k}$. Let $N:=\max(k,i_k)$. If $n\geq N$, then $C_{i_k,k}\subseteq\bigcup_{l\leq n}\ C_{l,k}$, so that $x\in(\bigcap_{j\leq n}\ B_{k,j})\cap C_{i_k,k}\subseteq(\bigcap_{j\leq n}\ B_{k,j})\cap(\bigcup_{l\leq n}\ C_{l,k})\subseteq S_n$. Note then that $\neg S\subseteq\underline{\lim}_{n\in\omega}\ (\neg S_n)$. Indeed, let $x\notin S$, which gives $k_0\in\omega$ such that $x\in\bigcap_{l\in\omega}\ \neg C_{l,k_0}$, and thus $x\in\bigcap_{l\in\omega}\ \neg C_{l,k}$ if $k\geq k_0$. Note also that, for each j, there is $i_j\in\omega$ with $x\notin B_{j,i_j}$. Let $M:=\max_{k\leq k_0}\ i_k$, $n\geq M$, and $k\leq n$. If $k\leq k_0$, then $x\in\neg B_{k,i_k}\subseteq\bigcup_{j\leq n}\ \neg B_{k,j}$. If $k\geq k_0$, then $x\in\bigcap_{l\leq n}\ \neg C_{l,k}$. Thus $x\in(\bigcup_{j\leq n}\ \neg B_{k,j})\cup(\bigcap_{l\leq n}\ \neg C_{l,k})$ and thus $x\notin S_n$. Therefore $S=\lim_{n\in\omega}\ S_n$. $\square$

Definition 1.9 *Let $\mathbf{\Gamma}$ be a class of subsets of metrizable spaces, X,Y be metrizable spaces, and $f:X\rightarrow Y$ be a function.*

(a) f is **Baire class one** *if the pre-image by f of any open subset of Y is a subset of X in $\mathbf{\Sigma}^0_2$.*

(b) Inductively, if $\xi\geq 2$ is a countable ordinal, then f is **Baire class** *ξ if f is the pointwise limit of a sequence of functions (f_n), where f_n is Baire class $\xi_n<\xi$.*

(c) f is **$\mathbf{\Gamma}$-measurable** *if the pre-image by f of any open subset of Y is a subset of X in $\mathbf{\Gamma}$.*

The following is an extension and refinement of Theorem 1.4.

Theorem 1.10 *(Lebesgue, Hausdorff, Banach) Let $\xi\geq 1$ be a countable ordinal, X,Y be metrizable spaces with Y separable, and $f:X\rightarrow Y$ be a function. Then f is Baire class ξ if and only if f is $\mathbf{\Sigma}^0_{\xi+1}$-measurable. In particular, f is Borel if and only if f is Baire class ξ for some countable ordinal $\xi\geq 1$.*

Proof. Assume first that f is Baire class ξ. We argue by induction on ξ, the case $\xi=1$ coming from the definitions. Assume that f is the limit of (f_n), where f_n is Baire class $\xi_n<\xi$. Let

$$O=\bigcup_{m\in\omega}\ B_m=\bigcup_{m\in\omega}\ \overline{B_m}$$

be an open subset of Y, where the B_m's are open balls. Note that

$$f^{-1}(O)=\bigcup_{m,N\in\omega}\ \bigcap_{n\geq N}\ f_n^{-1}(\overline{B_m})$$

is in $\mathbf{\Sigma}^0_{\xi+1}$.

Assume now that f is $\mathbf{\Sigma}^0_{\xi+1}$-measurable. We argue by induction on ξ, the case $\xi=1$ coming from the definitions once again. We first solve the case where $Y=2$ and f is the characteristic function χ_S of $S\subseteq X$. Note that $S\in\mathbf{\Delta}^0_{\xi+1}$. If $\xi=\eta+1$ is a successor ordinal, then we can write $S=\lim_{n\rightarrow\infty}\ S_n$, for some sequence (S_n) of subsets of X in $\mathbf{\Delta}^0_{\eta+1}$, by Proposition 1.8. By induction assumption, χ_{S_n} is Baire class η. As χ_S is the pointwise limit of (χ_{S_n}), f is Baire class ξ. If ξ is a limit ordinal, then we can write $S=\lim_{n\rightarrow\infty}\ S_n$, for some sequence (S_n) of subsets of X in $\bigcup_{\eta<\xi}\ \mathbf{\Delta}^0_\eta$, by Proposition 1.8. Say that $S_n\in\mathbf{\Delta}^0_{\eta_n+1}$ with $\eta_n<\xi$. By induction assumption, χ_{S_n} is Baire class η_n. As χ_S is the pointwise limit of (χ_{S_n}), f is Baire class ξ.

The preceding argument can be extended to the case where Y is finite. For this, note that if $(S_i)_{i<k}$ is a partition of X, $S_i=\lim_{n\to\infty} S^i_n$ for $i<k$, and $T^i_n:=S^i_n\setminus(\bigcup_{j<i} S^j_n)$, then $T^0_n,\cdots,T^{k-1}_n$ are pairwise disjoint and still $S_i=\lim_{n\to\infty} T^i_n$.

Note also that if Y is finite with a metric d and if $f,g:X\to Y$ are such that $d\big(f(x),g(x)\big)\leq\delta$ for all x and $(f_n),(g_n)$ are sequences of $\mathbf{\Sigma}^0_\eta$-measurable functions with $f=\lim_{n\to\infty} f_n$, $g=\lim_{n\to\infty} g_n$ pointwise, then we can find a sequence (g'_n) of $\mathbf{\Sigma}^0_\eta$-measurable functions with $g=\lim_{n\to\infty} g'_n$ and $d\big(f_n(x),g'_n(x)\big)\leq\delta$ for all x. For that just define

$$g'_n(x):=\begin{cases} g_n(x) \text{ if } d\big(f_n(x),g_n(x)\big)\leq\delta,\\ f_n(x) \text{ otherwise.}\end{cases}$$

Assume now that Y is an arbitrary metrizable separable space. Considering an embedding of Y into $[0,1]^\omega$ if necessary, we can find a metric d defining the topology of Y such that, for any $\delta>0$, there are finitely many points $y_0,\cdots,y_{n-1}$ of Y such that $Y\subseteq\bigcup_{i<n} B(y_i,\delta)$. For each natural number k, this gives $Y^k:=\{y^k_0,\cdots,y^k_{k-1}\}\subseteq Y$ such that $Y\subseteq\bigcup_{i<n_k} B(y^k_i,2^{-k})$ and $Y^k\subseteq Y^{k+1}$. Note that $f^{-1}\big(B(y^k_i,2^{-k})\big)\in\mathbf{\Sigma}^0_{\xi+1}$. By the reduction property, we get a partition $(A^k_i)_{i<n_k}$ of X into $\mathbf{\Delta}^0_{\xi+1}$ sets with $A^k_i\subseteq f^{-1}\big(B(y^k_i,2^{-k})\big)$. We define $f^k:X\to Y^k$ by $f^k(x)=y^k_i \Leftrightarrow x\in A^k_i$. Note that f^k is $\mathbf{\Sigma}^0_{\xi+1}$-measurable. By the finite case, we get a sequence (f^k_n) of functions with $f^k=\lim_{n\to\infty} f^k_n$ pointwise, as well as $\eta_{k,n}<\xi$ such that f^k_n is Baire class $\eta_{k,n}$. Since $d\big(f(x),f^k(x)\big)\leq 2^{-k}$, so that $d\big(f^k(x),f^{k+1}(x)\big)\leq 2^{1-k}$, we may assume by the preceding remark that $d\big(f^k_n(x),f^{k+1}_n(x)\big)\leq 2^{1-k}$. We now set $f_n:=f^n_n$. Note that f_n is Baire class ξ_n for some $\xi_n<\xi$, and $f=\lim_{n\to\infty} f_n$, so that f is Baire class ξ. □

Our definition of a Baire class ξ function is not uniform. We can make it uniform in some cases. Note that the pointiwse limit of a sequence of continuous functions is Baire class one. The converse is false in general (consider any non constant Baire class one function from $\mathbb{R}$ into 2).

Definition 1.11 *Let (f_n) be a sequence of functions from a set S into a metric space X, and $f:S\to X$ be a function. We say that (f_n)* **converges uniformly** *to f if, for any $\eta>0$, there is $N\in\omega$ such that, for each $n\geq N$ and each $s\in S$, $d\big(f_n(s),f(s)\big)<\eta$.*

Lemma 1.12 *Let X be a metrizable space, and (p_n) be a sequence of pointwise limits of a sequence of continuous functions from X into $\mathbb{R}$ which converges uniformly to p. Then p is also the pointiwse limit of a sequence of continuous functions.*

Proof. It is enough to show that if (q_n) is a sequence of pointwise limits of a sequence of continuous functions from X into $\mathbb{R}$ such that q_n is uniformly bounded by 2^{-n}, then $\Sigma_{n\in\omega}\ q_n$ is the pointwise limit of a sequence of continuous functions. So let $(q^n_i)_{i\in\omega}$ be a sequence of continuous functions from X into $\mathbb{R}$ pointwise converging to q_n. We can assume that q^n_i is uniformly bounded by 2^{-n}. So $r_i:=\Sigma_{n\in\omega}\ q^n_i$ is continuous and it is enough to show that (r_i) pointwise converges to $\Sigma_{n\in\omega}\ q_n$. Fix $x\in X$ and $\eta>0$. Find N so that, for all i, $|\Sigma_{n>N}\ q^n_i(x)|\leq\frac{\eta}{3}$ and $|\Sigma_{n>N}\ q_n(x)|\leq\frac{\eta}{3}$. Then $|r_i(x)-\Sigma_{n\in\omega}\ q_n(x)|\leq\frac{2\eta}{3}+\Sigma_{n\leq N}\ |q^n_i(x)-q_n(x)|$ and we are done. □

Lemma 1.13 *Let X be a separable metrizable space, and $S\subseteq X$ in $\mathbf{\Delta}^0_2$. Then χ_S is the pointiwse limit of a sequence of continuous functions.*

Proof. We can write $S=\bigcup_{n\in\omega}\ C_n$ and $\neg S=\bigcup_{n\in\omega}\ F_n$, where $(C_n),(F_n)$ are increasing sequences of closed subsets of X. Urysohn's lemma provides $f_n : X\rightarrow\mathbb{R}$ continuous such that $f_n(x)=1$ if $x\in C_n$ and $f_n(x)=0$ if $x\in F_n$. It remains to note that (f_n) pointwise converges to χ_S. $\square$

Theorem 1.14 *(Lebesgue, Hausdorff, Banach) Let X,Y be separable metrizable spaces with X zero-dimensional or $Y=\mathbb{R}$, and $f:X\rightarrow Y$ be a Baire class one function. Then f is the pointwise limit of a sequence of continuous functions.*

Proof. If X is zero-dimensional, then we argue as in the proof of Theorem 1.10. So assume that $Y=\mathbb{R}$. Consider a homeomorphism $h:\mathbb{R}\rightarrow(0,1)$. If $f:X\rightarrow\mathbb{R}$ is Baire class one, then so is $h\circ f : X\rightarrow(0,1)$. If the result holds for $g : X\rightarrow\mathbb{R}$ Baire class one with $g[X]\subseteq(0,1)$, then $h\circ f=\lim_{n\rightarrow\infty} g_n$ with $g_n : X\rightarrow\mathbb{R}$ continuous. By replacing g_n with $(g_n\vee\frac{1}{n+1})\wedge(1-\frac{1}{n+1})$, we can assume that $g_n[X]\subseteq(0,1)$. Then $f_n:=h^{-1}\circ g_n$ is as desired. So we may assume that $f[X]\subseteq(0,1)$.

We set, for $N\geq 2$ and $i\leq N$–2, $A^N_i:=f^{-1}\big((\frac{i}{N},\frac{i+2}{N})\big)$. Note that A^N_i is $\mathbf{\Sigma}^0_2$ and $X=\bigcup_{i\leq N-2}\ A^N_i$. The reduction property gives $B^N_i\subseteq A^N_i$ in $\mathbf{\Delta}^0_2$ such that X is the disjoint union $\bigcup_{i\leq N-2}\ B^N_i$. Note that $\chi_{B^N_i}$ is Baire class one and if $g_N:=\Sigma_{i\leq N-2}\ \frac{i}{N}\cdot\chi_{B^N_i}$, then (g_n) converges to f uniformly. It remains to apply Lemmas 1.12 and 1.13. $\square$

Exercise. Prove that semi-continuous functions on metrizable spaces are Baire class one.

Notation. If X,Y are sets and $R\subseteq X\times Y$, then $\exists^Y R:=\{x\in X\mid\exists y\in Y\ (x,y)\in R\}$. If $\mathbf{\Gamma}$ is a class of sets, then $\exists^Y\mathbf{\Gamma}$ is the class of sets of the form $\exists^Y R$ for some $R\in\mathbf{\Gamma}$.

The following fact will be important when we will study effective descriptive set theory.

Proposition 1.15 *Let $n\geq 1$ be a natural number and X be a metrizable space. Then*

$$\mathbf{\Sigma}^0_{n+1}(X)=\exists^\omega\mathbf{\Pi}^0_n(X).$$

Proof. Assume first that $n=1$. If $S\in\mathbf{\Sigma}^0_2(X)$, then we can find a sequence (C_n) of closed subsets of X with union S. We define $R\subseteq X\times\omega$ by $R(x,n)\Leftrightarrow x\in C_n$. Note that R is closed and $S=\exists^\omega R$, so that $S\in\exists^\omega\mathbf{\Pi}^0_1(X)$. Conversely, let $R\subseteq X\times\omega$ be closed with $S=\exists^\omega R$. Note that $S=\bigcup_{n\in\omega}\ R^n$ is the countable union of the horizontal sections $R^n:=\{x\in X\mid(x,n)\in R\}$ of R. As the function $f_n : x\mapsto(x,n)$ is continuous and $R^n=f_n^{-1}(R)$, R^n is closed and $S\in\mathbf{\Sigma}^0_2(X)$. We then argue by induction. $\square$

2 Universal sets

The Borel classes provide for each Polish space X a hierarchy of at most ω_1 levels. We will see that this hierarchy is strict when X is uncountable. This is based on the existence of universal sets for the classes $\mathbf{\Sigma}^0_\xi$ and $\mathbf{\Pi}^0_\xi$.

Definition 2.1 *Let $\mathbf{\Gamma}$ be a class of sets, X,Y be sets. A subset $\mathcal{U}$ of $Y\times X$ is Y-**universal for the subsets of X in $\mathbf{\Gamma}$** if $\mathcal{U}\in\mathbf{\Gamma}$ and, for each $S\in\mathbf{\Gamma}(X)$, there is $y\in Y$ such that*

$$S=\mathcal{U}_y:=\{x\in X\mid(y,x)\in\mathcal{U}\}$$

is the vertical section of $\mathcal{U}$ at y.

Such a universal set provides a coding of the sets in $\mathbf{\Gamma}(X)$.

Theorem 2.2 *Let X be a metrizable separable space, and $\xi \geq 1$ be a countable ordinal. Then there is a $\mathcal{C}$-universal set for the subsets of X in $\mathbf{\Sigma}^0_\xi$, and similarly with $\mathbf{\Pi}^0_\xi$.*

Proof. We proceed by induction on ξ. Let (O_n) be a countable basis for the topology of X. We put

$$\mathcal{U}(\alpha, x) \Leftrightarrow \exists n \in \omega \ \alpha(n) = 1 \wedge x \in O_n.$$

Note that $\mathcal{U}$ is open, and if $O \subseteq X$ is open, then we can find $\alpha \in \mathcal{C}$ such that $O = \bigcup_{\alpha(n)=1} O_n$, so that $O = \mathcal{U}_\alpha$. Thus $\mathcal{U}$ is $\mathcal{C}$-universal for $\mathbf{\Sigma}^0_1(X)$.

Then we note that if $\mathcal{U}$ is $\mathcal{C}$-universal for $\mathbf{\Gamma}(X)$, then $\neg\mathcal{U}$ is $\mathcal{C}$-universal for $\check{\mathbf{\Gamma}}(X)$. In particular, there is a $\mathcal{C}$-universal set for $\mathbf{\Pi}^0_1(X)$, and if there is a $\mathcal{C}$-universal set for $\mathbf{\Sigma}^0_\xi(X)$, then there is a $\mathcal{C}$-universal set for $\mathbf{\Pi}^0_\xi(X)$.

Assume now that there is a $\mathcal{C}$-universal set $\mathcal{U}_\eta$ for $\mathbf{\Pi}^0_\eta(X)$, for each $\eta < \xi$. Let, for $n \in \omega$, $\eta_n < \xi$ such that $\eta_n \leq \eta_{n+1}$ and $\xi = \sup\{\eta_n + 1 \mid n \in \omega\}$. Let $< .,. >: \omega^2 \to \omega$ be a bijection, and, for $\alpha \in \mathcal{C}$ and $n \in \omega$, $(\alpha)_n \in \mathcal{C}$ defined by $(\alpha)_n(p) := \alpha(< n, p >)$. Then $\alpha \mapsto (\alpha)_n$ is continuous and for any $(\alpha_n) \in \mathcal{C}^\omega$ there is $\alpha \in \mathcal{C}$ such that $(\alpha)_n = \alpha_n$ for each n. We put

$$\mathcal{U}(\alpha, x) \Leftrightarrow \exists n \in \omega \ \big((\alpha)_n, x\big) \in \mathcal{U}_{\eta_n}.$$

Then $\mathcal{U}$ is $\mathcal{C}$-universal for $\mathbf{\Sigma}^0_\xi(X)$. $\square$

Theorem 2.3 *Let X be an uncountable Polish space, and $\xi \geq 1$ be a countable ordinal. Then $\mathbf{\Sigma}^0_\xi(X) \neq \mathbf{\Pi}^0_\xi(X)$. Therefore $\mathbf{\Delta}^0_\xi(X) \subsetneqq \mathbf{\Sigma}^0_\xi(X) \subsetneqq \mathbf{\Delta}^0_{\xi+1}(X)$, and similarly for $\mathbf{\Pi}^0_\xi(X)$.*

Proof. As X is uncountable, we may assume that $\mathcal{C} \subseteq X$. So if $\mathbf{\Sigma}^0_\xi(X) = \mathbf{\Pi}^0_\xi(X)$, then

$$\mathbf{\Sigma}^0_\xi(\mathcal{C}) = \mathbf{\Pi}^0_\xi(\mathcal{C}).$$

Let $\mathcal{U}$ be $\mathcal{C}$-universal for $\mathbf{\Sigma}^0_\xi(\mathcal{C})$. We put $S := \{\alpha \in \mathcal{C} \mid (\alpha, \alpha) \notin \mathcal{U}\}$. Then $S \in \mathbf{\Pi}^0_\xi(\mathcal{C}) = \mathbf{\Sigma}^0_\xi(\mathcal{C})$, which gives $\beta \in \mathcal{C}$ such that $S = \mathcal{U}_\beta$. Now $\beta \in S \Leftrightarrow (\beta, \beta) \in \mathcal{U} \Leftrightarrow \beta \notin S$, which is absurd. $\square$

Exercise. Let X be an uncountable Polish space, and $\lambda \geq 1$ be a countable limit ordinal. Prove that $\bigcup_{\xi<\lambda} \mathbf{\Sigma}^0_\xi(X) \subsetneqq \mathbf{\Delta}^0_\lambda(X)$.

Exercise. A class of sets is called **self-dual** if it is closed under complements. Let $\mathbf{\Gamma}$ be a class of subsets of metrizable spaces closed under continuous pre-images and self-dual. Prove that, for any X, there is no X-universal set for $\mathbf{\Gamma}(X)$. Conclude that, for any $1 \leq \xi < \omega_1$, there is no X-universal set for $\mathbf{\Delta}^0_\xi(X)$.

3 Complete sets

We first give a few notions of game theory. More precisely, we will discuss infinite games with perfect information.

Definition 3.1 *Let S be a nonempty set, $T \subseteq S^{<\omega}$ be a nonempty pruned tree on S, and $A \subseteq [T] \subseteq S^\omega$.*

(a) The **game** *$G(T, A)$ on S is defined as follows. Players 1 and 2 take turns in playing*

$$\begin{array}{llllll} 1 & s_0 & & s_2 & & \\ & & & & & \cdots, \\ 2 & & s_1 & & s_3 & \end{array}$$

in such a way that $(s_n) \in [T]$. We say that 1 **wins this run of the game** *if $(s_n) \in A$.*

(b) A **strategy for 1** *in $G(T, A)$ is a subtree σ of T such that*

(1) σ is nonempty pruned,

(2) if $(s_0, s_1, \cdots, s_{2j}) \in \sigma$ and $s_{2j+1} \in S$ satisfies $(s_0, \cdots, s_{2j}, s_{2j+1}) \in T$, then

$$(s_0, \cdots, s_{2j}, s_{2j+1}) \in \sigma,$$

(3) if $(s_0, s_1, \cdots, s_{2j-1}) \in \sigma$, then there is a unique $s_{2j} \in S$ with $(s_0, \cdots, s_{2j-1}, s_{2j}) \in \sigma$.

Intuitively, σ tells 1 what to play, knowing 2's previous moves. Similarly, we define the notion of a **strategy for 2** *in $G(T, A)$.*

(c) A strategy σ for 1 is a **winning strategy for 1** *if $[\sigma] \subseteq A$. Similarly, we define the notion of a* **winning strategy for 2**.

(d) The game $G(T, A)$ is **determined** *if one of the two players has a winning strategy.*

The next theorem is very important, and we will not prove it.

Theorem 3.2 *(Martin) Let S be a nonempty set, and T be a nonempty pruned tree on S. We equip S with the discrete topology, and S^ω with the product topology. Let $A \subseteq [T]$ be Borel. Then the game $G(T, A)$ is determined.*

An important way to compare the topological complexity of sets is to use pre-images by continuous functions.

Definition 3.3 *Let X, Y be sets, $A \subseteq X$ and $B \subseteq Y$.*

(a) A **reduction** *of A to B is a function $f : X \to Y$ such that $A = f^{-1}(B)$.*

(b) If moreover X, Y are topological spaces, then we say that A is **Wadge reducible** *to B, denoted by $(X, A) \leq_W (Y, B)$ or $A \leq_W B$, if there is a continuous reduction of A to B.*

Remarks. (a) Note that $\leq_W$ is a **quasi-order**, i.e., a reflexive and transitive relation. It is called the **Wadge quasi-order**.

(b) Note that the only continuous functions from $\mathbb{R}$ into $\mathcal{N}$ are the constant functions, because the only clopen subsets of $\mathbb{R}$ are the whole space and the empty set. So the Wadge quasi-order is not very interesting in non zero-dimensional spaces. This is the reason why the Wadge quasi order is studied in zero-dimensional spaces, to ensure the existence of enough continuous functions.

Recall that a zero-dimensional Polish space is homeomorphic to a closed subset of $\mathcal{N}$, and that a closed subset of $\mathcal{N}$ is of the form $[T]$, for some pruned tree T on ω.

Lemma 3.4 *(Wadge) Let S,T be nonempty pruned trees on ω, and $A\subseteq[S]$, $B\subseteq[T]$ be Borel. Then $A\leq_W B$ or $B\leq_W \neg A$.*

Proof. Consider the **Wadge game** on ω, defined as follows. Players 1 and 2 take turns in playing

$$\begin{array}{llllll} 1 & \alpha(0) & & \alpha(1) & & \\ & & & & & \ldots, \\ 2 & & \beta(0) & & \beta(1) & \end{array}$$

in such a way that $\alpha\in[S]$ and $\beta\in[T]$. We say that 2 **wins this run of the game** if $\alpha\in A\Leftrightarrow\beta\in B$. This game can be seen as a Borel game $G(U,C)$ for some suitable nonempty pruned tree on ω and some Borel subset C of $[U]$. By Theorem 3.2, this game is determined.

If 2 has a winning strategy, then this strategy can be seen as a map $\varphi:S\rightarrow T$ such that $s\subseteq t\Rightarrow\varphi(s)\subseteq\varphi(t)$ and $|\varphi(s)|=|s|$. It defines a continuous map $f:[S]\rightarrow[T]$ by $f(\alpha):=\lim_{n\rightarrow\infty}\varphi(\alpha|n)$. As φ is winning for 2, $\alpha\in A\Leftrightarrow f(\alpha)\in B$, so that $A\leq_W B$.

Note that 1 wins the run above of the Wadge game if $\alpha\notin A\Leftrightarrow\beta\in B$. As above, if 1 has a winning strategy, then $B\leq_W\neg A$. $\square$

Remark. If B is $\mathbf{\Sigma}^0_\xi$ (resp., $\mathbf{\Pi}^0_\xi$) and $A\leq_W B$, then A is $\mathbf{\Sigma}^0_\xi$ (resp., $\mathbf{\Pi}^0_\xi$). So $\mathbf{\Sigma}^0_\xi$ and $\mathbf{\Pi}^0_\xi$ are initial segments of $\leq_W$. We will see that any set in $\mathbf{\Sigma}^0_\xi\setminus\mathbf{\Pi}^0_\xi$ is maximal for $\leq_W$ in $\mathbf{\Sigma}^0_\xi$, and similarly if we exchange $\mathbf{\Sigma}^0_\xi$ and $\mathbf{\Pi}^0_\xi$.

Definition 3.5 *Let $\mathbf{\Gamma}$ be a class of subsets of Polish spaces, Y be a Polish space and $B\subseteq Y$.*

(a) We say that B is $\mathbf{\Gamma}$-**hard** *if, for any zero-dimensional Polish space X and any $A\subseteq X$ in $\mathbf{\Gamma}$, $A\leq_W B$.*

(b) If moreover B is in $\mathbf{\Gamma}$, then we say that B is $\mathbf{\Gamma}$-**complete**.

Remark. If $\mathbf{\Gamma}$ is not self dual on zero-dimensional spaces and closed under continuous pre-images, then no $\mathbf{\Gamma}$-hard set is in $\check{\mathbf{\Gamma}}$. If A is $\mathbf{\Gamma}$-hard and $A\leq_W B$, then B is $\mathbf{\Gamma}$-hard. This is a very common method for proving that a set is $\mathbf{\Gamma}$-hard: choose a known $\mathbf{\Gamma}$-hard set A, and show that $A\leq_W B$.

Theorem 3.6 *(Wadge) Let X be a zero-dimensional Polish space, and $A\subseteq X$ be a Borel set.*

(a) A is $\mathbf{\Sigma}^0_\xi$-complete if and only if A is in $\mathbf{\Sigma}^0_\xi\setminus\mathbf{\Pi}^0_\xi$.

(b) A is $\mathbf{\Sigma}^0_\xi$-hard if and only if A is not in $\mathbf{\Pi}^0_\xi$.

Moreover, we can exchange $\mathbf{\Sigma}^0_\xi$ and $\mathbf{\Pi}^0_\xi$.

Proof. If A is $\mathbf{\Sigma}^0_\xi$-hard, then A is not in $\mathbf{\Pi}^0_\xi$ since $\mathbf{\Sigma}^0_\xi(\mathcal{N})\neq\mathbf{\Pi}^0_\xi(\mathcal{N})$. If now A is not in $\mathbf{\Pi}^0_\xi$, Y is zero-dimensional and $B\subseteq Y$ is in $\mathbf{\Sigma}^0_\xi$, then, by Lemma 3.4, $B\leq_W A$ since otherwise $A\leq_W\neg B$. Thus A is $\mathbf{\Sigma}^0_\xi$-hard. $\square$

Exercise. Let $\mathcal{U}$ be $\mathcal{C}$-universal for the $\mathbf{\Sigma}^0_\xi$ subsets of $\mathcal{N}$. Prove that $\mathcal{U}$ is $\mathbf{\Sigma}^0_\xi$-complete.

Exercise. We set $\mathbb{P}_f:=\{\alpha\in\mathcal{C}\mid\exists m\in\omega\ \forall n\geq m\ \alpha(n)=0\}$ and $\mathbb{P}_\infty:=\mathcal{C}\setminus\mathbb{P}_f$. Prove that $\mathbb{P}_f$ is $\mathbf{\Sigma}^0_2$-complete and $\mathbb{P}_\infty$ is $\mathbf{\Pi}^0_2$-complete.

Example. We set $V := \{\alpha \in 2^{\omega\times\omega} \mid \exists n \in \omega\ \forall p \in \omega\ \exists q \geq p\ \alpha(n,q) = 1\}$. Note that V is in $\mathbf{\Sigma}^0_3$. In fact V is $\mathbf{\Sigma}^0_3$-complete. Indeed, let X be a zero-dimensional Polish space, and $A \subseteq X$ in $\mathbf{\Sigma}^0_3$. We can write $A = \bigcup_{n\in\omega} A_n$, where $A_n \in \mathbf{\Pi}^0_2$. As $\mathbb{P}_\infty$ is $\mathbf{\Pi}^0_2$-complete, there is $f_n : X \to \mathcal{C}$ continuous such that $A_n = f_n^{-1}(\mathbb{P}_\infty)$. We define $f : X \to 2^{\omega\times\omega}$ by $f(x)(n,q) := f_n(x)(q)$. Note that f is continuous and $x \in A \Leftrightarrow \exists n \in \omega\ x \in A_n \Leftrightarrow \exists n \in \omega\ f_n(x) \in \mathbb{P}_\infty \Leftrightarrow f(x) \in V$.

4 Turning Borel sets into clopen sets

The following theorem is a fundamental fact about Borel subsets of Polish spaces.

Lemma 4.1 *Let (X,τ) be a Polish space, $C \subseteq X$ be closed, and τ_C be the topology generated by $\tau \cup \{C\}$. Then τ_C is Polish, C is clopen in τ_C, and τ_F, τ have the same Borel sets.*

Proof. Note that τ_C is the sum of the relative topologies on C and $\neg C$. □

Lemma 4.2 *Let (X,τ) be a Polish space and (τ_n) be a sequence of Polish topologies on X containing τ. Then the topology τ_∞ generated by $\bigcup_{n\in\omega} \tau_n$ is Polish. If moreover $\bigcup_{n\in\omega} \tau_n \subseteq \mathbf{\Delta}^1_1(X,\tau)$, then τ_∞, τ have the same Borel sets.*

Proof. We set, for $n \in \omega$, $X_n := X$. Consider the function $\varphi : X \to \Pi_{n\in\omega} X_n$ defined by

$$\varphi(x) := (x, x, \cdots).$$

Note that $\varphi[X]$ is closed in $\Pi_{n\in\omega} (X_n, \tau_n)$. Indeed, if $(x_n) \notin \varphi[X]$, then we can find $i < j$ with $x_i \neq x_j$. Let O, U be disjoint τ-open with $x_i \in O$ and $x_j \in U$. Then

$$(x_n) \in X_0 \times \cdots \times X_{i-1} \times O \times X_{i+1} \times \cdots \times X_{j-1} \times U \times X_{j+1} \times \cdots \subseteq \neg\varphi[X].$$

Thus $\varphi[X]$ is Polish. As φ is a homeomorphism from (X, τ_∞) onto $\varphi[X]$, (X, τ_∞) is Polish. □

Theorem 4.3 *Let (X,τ) be a Polish space, and $B \subseteq X$ be Borel. Then there is a Polish topology τ_B on X containing τ such that B is clopen in τ_B, and τ_B, τ have the same Borel sets.*

Proof. Consider the class $\mathcal{A}$ of subsets A of X for which there is a Polish topology τ_A containing τ such that A is clopen in τ_A, and τ_A and τ have the same Borel sets. By Lemma 4.1, $\mathcal{A}$ contains τ. Note that $\mathcal{A}$ is closed under complements. If (A_n) is a sequence of elements of $\mathcal{A}$, then we get $\tau_n := \tau_{A_n}$. Lemma 4.2 provides τ_∞. Then $A := \bigcup_{n\in\omega} A_n$ is τ_∞-open and one more application of Lemma 4.1 shows that $A \in \mathcal{A}$. Thus $\mathcal{A}$ is a σ-algebra, and contains the Borel subsets of (X,τ). □

Exercise. (a) Let (X,τ) be a Polish space, and (S_n) be a sequence of Borel subsets of X. Prove that there is a zero-dimensional Polish topology τ' on X containing τ such that S_n is clopen in τ' for each n, and τ', τ have the same Borel sets.

(b) Let (X,τ) be a Polish space, Y be a second countable space, and $f : (X,\tau) \to Y$ be a Borel function. Prove that there is a zero-dimensional Polish topology τ' on X containing τ such that $f : (X,\tau') \to Y$ is continuous and τ', τ have the same Borel sets.

A first consequence of the previous theorem is the perfect set theorem for Borel sets.

Theorem 4.4 *(Alexandrov, Hausdorff) Let X be a Polish space, and $B \subseteq X$ be Borel. Then either B is countable, or B contains a homeomorphic copy of the Cantor space $\mathcal{C}$. In particular, every uncountable Borel subset of X has size continuum.*

Proof. Theorem 4.3 gives a finer Polish topology τ_B on X such that B is clopen in τ_B, and τ_B has the same Borel sets as the initial topology of X. In particular, B, equipped with the topology induced by τ_B, is Polish. So if B is uncountable, it contains a homeomorphic copy of the Cantor space $\mathcal{C}$. As τ_B is finer than the initial topology, this is also a homeomorphic copy of the Cantor space $\mathcal{C}$ with respect to the initial topology. □

Another consequence is the following representation of Borel sets.

Theorem 4.5 *(Lusin, Souslin) Let X be a Polish space, and $B \subseteq X$ be Borel. Then there is a closed subset C of $\mathcal{N}$ and a continuous bijection $b : C \to B$. In particular, if B is nonempty, then there is a continuous surjection $s : \mathcal{N} \to B$ extending b.*

Proof. Theorem 4.3 gives a finer Polish topology τ_B on X such that B is clopen in τ_B. In particular, B, equipped with the topology induced by τ_B, is Polish. This gives a closed subset C of $\mathcal{N}$ and a bijection $b : C \to B$ continuous for $\tau_{B|B}$. As τ_B is finer than the initial topology, b is also continuous with respect to the initial topology. The last assertion comes from the existence of a retraction from $\mathcal{N}$ onto C. □