Chapter 6-Analytic and co-analytic sets

1 Definition and characterizations

Definition 1.1 *Let X be a Polish space, and $A \subseteq X$. We say that A is* **analytic** *if we can find a Polish space Y and $f : Y \rightarrow X$ continuous with $A = f[Y]$. The class of analytic sets is denoted by $\mathbf{\Sigma}^1_1$.*

Theorem 1.2 *Let X be a Polish space. Then there is a $\mathcal{C}$-universal set for the analytic subsets of X.*

Proof. Let $\mathcal{F}$ be $\mathcal{C}$-universal for the closed subsets of $X \times \mathcal{N}$. We set

$$\mathcal{A} := \{(\alpha, x) \in \mathcal{C} \times X \mid \exists \gamma \in \mathcal{N}\ (\alpha, x, \gamma) \in \mathcal{F}\}.$$

Then $\mathcal{A}$ is $\mathcal{C}$-universal for the analytic subsets of X. Indeed, as the projections are continuous, $\mathcal{A}$ and its vertical sections are analytic. Conversely, let A be an analytic subset of X, C be a closed subset of $\mathcal{N}$, and $f : C \rightarrow X$ continuous with $A = f[C]$. Note that $G := \{(x, \gamma) \in X \times \mathcal{N} \mid x = f(\gamma)\}$ is closed and $x \in A \Leftrightarrow \exists \gamma \in \mathcal{N}\ (x, \gamma) \in G$. This gives $\alpha \in \mathcal{C}$ with $G = \mathcal{F}_\alpha$. Then $A = \mathcal{A}_\alpha$. □

As any nonempty Borel set is a continuous image of $\mathcal{N}$, any Borel set is analytic. This inclusion is strict in uncountable spaces.

Corollary 1.3 *(Souslin) Let X be an uncountable Polish space. Then there is an analytic subset of X which is not Borel.*

Proof. Theorem 1.2 provides $\mathcal{A}$ which is $\mathcal{C}$-universal for the analytic subsets of $\mathcal{C}$. Note that $\mathcal{A}$ is not Borel. We argue by contradiction to see that. Then $\neg\mathcal{A}$ is Borel, as well as $A := \{\beta \in \mathcal{C} \mid (\beta, \beta) \notin \mathcal{A}\}$. This gives $\alpha \in \mathcal{C}$ with $A = \mathcal{A}_\alpha$. Note that $\alpha \in A \Leftrightarrow (\alpha, \alpha) \in \mathcal{A} \Leftrightarrow \alpha \notin A$, which is absurd. It remains to note that any uncountable Polish space contains a homeomorphic copy of $\mathcal{C}$. □

Exercise. Let X be a Polish space, and $A \subseteq X$. Then the following are equivalent:

(a) A is analytic,

(b) we can find a Polish space Y and a Borel subset B of $X \times Y$ with $A = proj_X[B]$,

(c) we can find a closed subset C of $X \times \mathcal{N}$ with $A = proj_X[C]$,

(d) we can find a G_δ subset G of $X \times \mathcal{C}$ with $A = proj_X[G]$.

Proposition 1.4 *The class of analytic sets is closed under countable unions, countable intersections, and direct images and pre-images by Borel functions.*

Proof. Let X be a Polish space, (A_n) be a sequence of analytic subsets of X, (Y_n) be a sequence of Polish spaces, and $f_n : Y_n \rightarrow X$ continuous with $A_n = f_n[Y_n]$. Then the sum $\oplus_{n \in \omega} Y_n$ is Polish, and the function $(n, y) \mapsto f_n(y)$ is continuous with range $\bigcup_{n \in \omega} A_n$ which is therefore analytic.

Now let $Z := \{(y_n) \in \Pi_{n\in\omega} Y_n \mid \forall m,n\in\omega\ f_n(y_n)=f_m(y_m)\}$. Note that Z is a closed subset of $\Pi_{n\in\omega} Y_n$, and thus a Polish space. We define $f: Z\to X$ by $f\big((y_n)\big) := f_0(y_0)$. Then f is continuous and $f[Z]=\bigcap_{n\in\omega} A_n$ which is therefore analytic.

Let Y be a Polish space and $f: X\to Y$ be Borel. Note that

$$y\in f[A_0] \Leftrightarrow \exists x\in X\ x\in A_0 \wedge f(x)=y \Leftrightarrow \exists x\in X\ (y,x)\in A,$$

where $A := \{(y,x)\in Y\times X \mid x\in A_0 \wedge f(x)=y\}$. As the projection is continuous, it is enough to prove that A is analytic. As $\{(y,x)\in Y\times X \mid f(x)=y\}$ is Borel, it is enough to prove that

$$\{(y,x)\in Y\times X \mid x\in A_0\} = Y\times A_0$$

is analytic. We define $f_0^*: Y\times Y_0 \to Y\times X$ by $f_0^*(y,y_0) := \big(y, f_0(y_0)\big)$. Note that f_0^* is continuous and $f_0^*[Y\times Y_0]=Y\times A_0$ which is therefore analytic.

Finally, let B be an analytic subset of Y. Note that $x\in f^{-1}(B) \Leftrightarrow \exists y\in Y\ f(x)=y \wedge y\in B$, so that $f^{-1}(B)$ is analytic as above. □

2 The separation theorem

The following separation theorem is of fundamental importance.

Definition 2.1 *Let X be a Polish space, and $A,B\subseteq X$ be disjoint. We say that A is* **Borel-separable** *from B if there is a Borel subset C of X with $A\subseteq C\subseteq \neg B$.*

Lemma 2.2 *Let X be a Polish space, and $(P_m),(Q_n)$ be sequences of subsets of X such that, for all $m,n\in\omega$, P_m is Borel-separable from Q_n. Then $\bigcup_{m\in\omega} P_m$ is Borel-separable from $\bigcup_{n\in\omega} Q_n$.*

Proof. Let $R_{m,n}$ be a Borel subset of X separating P_m from Q_n. Then $\bigcup_{m\in\omega}\ \bigcap_{n\in\omega} R_{m,n}$ is Borel and separates $\bigcup_{m\in\omega} P_m$ from $\bigcup_{n\in\omega} Q_n$. □

Theorem 2.3 *(Lusin) Let X be a Polish space, and A,B be disjoint analytic subsets of X. Then A is Borel-separable from B.*

Proof. We may assume that A,B are nonempty, which gives continuous surjections $f:\mathcal{N}\to A$ and $g:\mathcal{N}\to B$. We set, for $s\in\omega^{<\omega}$, $A_s := f[N_s]$ and $B_s := g[N_s]$. Note that $A_s=\bigcup_{m\in\omega} A_{sm}$ and $B_s=\bigcup_{n\in\omega} B_{sn}$. We argue by contradiction. By Lemma 2.2, we can inductively construct $\alpha,\beta\in\mathcal{N}$ such that, for each $n\in\omega$, $A_{\alpha|n}$ is not Borel-separable from $B_{\beta|n}$. Note that $f(\alpha)\neq g(\beta)$ since A,B are disjoint. Let O,U be disjoint open sets with $f(\alpha)\in O$ and $g(\beta)\in U$. By continuity, if n is big enough, then $f[N_{\alpha|n}]\subseteq O$ and $g[N_{\beta|n}]\subseteq U$, so O separates $A_{\alpha|n}$ from $B_{\beta|n}$, which is absurd. □

Corollary 2.4 *Let X be a Polish space, and (A_n) be a sequence of pairwise disjoint analytic subsets of X. Then there is a sequence (B_n) of pairwise disjoint Borel ssubsets of X such that $A_n\subseteq B_n$ for each n.*

Corollary 2.5 *(Souslin) Let X be a Polish space, and $B\subseteq X$. Then B is Borel if and only if B and $\neg B$ are analytic. For this reason, we denote by $\mathbf{\Delta}^1_1$ the class of Borel sets.*

Proof. As the class of Borel sets is closed under complements, if B is Borel, then B and $\neg B$ are analytic. Conversely, assume that B and $\neg B$ are analytic. Theorem 2.3 gives a Borel set separating B from $\neg B$, which shows that B is Borel. $\square$

Exercise. Let X, Y be Polish spaces and $f : X \to Y$ be a function. Prove that f is Borel if and only if its graph is Borel if and only if its graph is analytic.

Exercise. Let X be a Polish space and $A \subseteq X$ be analytic. Prove that A is countable or contains a homeomorphic copy of the Cantor space $\mathcal{C}$. In particular, every uncountable analytic subset of X has size continuum. This is the perfect set theorem for analytic sets.

3 Borel injections

Corollary 1.3 shows that a continuous image of a Borel set may not be Borel. We will see that this cannot happen in the injective case.

Theorem 3.1 *(Lusin-Souslin) Let X, Y be Polish spaces, $f : X \to Y$ be a continuous function, and B be a Borel subset of X such that $f_{|B}$ is one-to-one. Then $f[B]$ is Borel.*

Proof. We may assume that $X = \mathcal{N}$ and B is closed. We set, for $s \in \omega^{<\omega}$, $B_s := f[B \cap N_s]$. As $f_{|B}$ is one-to-one, $B_{sm} \cap B_{sn} = \emptyset$ if $m \neq n$. Moreover, $B_\emptyset = f[B]$, $\bigcup_{n\in\omega} B_{sn} = B_s$, and B_s is analytic. By Corollary 2.4, we can find a family $(B'_s)_{s\in\omega^{<\omega}}$ of Borel subsets of Y with $B'_\emptyset = Y$, $B_s \cup B'_{sn} \subseteq B'_s$, and $B'_{sm} \cap B'_{sn} = \emptyset$ if $m \neq n$. We finally define, by induction on $|s|$, a family $(B^*_s)_{s\in\omega^{<\omega}}$ of Borel subsets of Y such that

(1) $B^*_\emptyset = B'_\emptyset$
(2) $B^*_{(n_0)} = B'_{(n_0)} \cap \overline{B_{(n_0)}}$
(3) $B^*_{(n_0,\cdots,n_k)} = B'_{(n_0,\cdots,n_k)} \cap B^*_{(n_0,\cdots,n_{k-1})} \cap \overline{B_{(n_0,\cdots,n_k)}}$

Note that $B_s \subseteq B^*_s \subseteq \overline{B_s}$ if $s \neq \emptyset$. It is enough to prove that $f[B] = \bigcap_{k\in\omega} \bigcup_{s\in\omega^k} B^*_s$. If $y \in f[B]$, then let $\beta \in B$ with $y = f(\beta)$, so that $y \in \bigcap_{k\in\omega} B_{\beta|k}$, and thus $y \in \bigcap_{k\in\omega} B^*_{\beta|k}$. Conversely, if $y \in \bigcap_{k\in\omega} \bigcup_{s\in\omega^k} B^*_s$, then there is a unique $\beta \in \mathcal{N}$ such that $y \in \bigcap_{k\in\omega} B^*_{\beta|k} \subseteq \bigcap_{k\in\omega} \overline{B_{\beta|k}}$. In particular, $B_{\beta|k}$ and $B \cap N_{\beta|k}$ are nonempty for each k. As B is closed, $\beta \in B$. Thus $f(\beta)$ is in $\bigcap_{k\in\omega} B_{\beta|k}$. Note that $y = f(\beta)$. Indeed, we argue by contraction to see that. Let O be an open subset of Y such that $f(\beta) \in O$ and $y \notin \overline{O}$. As f is continuous, we can find a natural number k_0 such that $f[N_{\beta|k_0}] \subseteq O$. Then $y \notin \overline{f[N_{\beta|k_0}]} \supseteq \overline{B_{\beta|k_0}}$, which is absurd. $\square$

Corollary 3.2 *Let X, Y be Polish spaces, $f : X \to Y$ be a Borel function, and B be a Borel subset of X such that $f_{|B}$ is one-to-one. Then $f[B]$ is Borel and f is a Borel isomorphism from B onto $f[B]$.*

Proof. We apply Theorem 3.1 to the projection of $X \times Y$ onto Y and the set $(B \times Y) \cap \text{Graph}(f)$. $\square$

Exercise. Let (X, τ) be a Polish space, and τ' be a Polish topology on X containing τ. Then τ, τ' have the same Borel sets.

We now prove the Borel Schröder-Bernstein theorem.

Theorem 3.3 *Let X, Y be Polish spaces, and $f:X\to Y$, $g:Y\to X$ be Borel injections. Then we can find Borel sets $A\subseteq X$, $B\subseteq Y$ with $f[A]=Y\setminus B$ and $g[B]=X\setminus A$. In particular, X and Y are Borel isomorphic.*

Proof. We first show that there is a Borel set $A\subseteq X$ such that $g^{-1}(X\setminus A)=Y\setminus f[A]$. We define a function $h:2^X\to 2^X$ by $h(S):=X\setminus g\big[Y\setminus f[S]\big]$. We inductively define $X_n\subseteq X$ as follows: $X_0:=\emptyset$, $X_{n+1}:=h(X_n)$. Then we set $A:=\bigcup_{n\in\omega} X_n$. As $h(\bigcup_{n\in\omega} X_n)=\bigcup_{n\in\omega} h(X_n)$, $h(A)=A$. By Corollary 3.2, the X_n's and A are Borel, so we are done. It remains to define $i:X\to Y$ by

$$i(x):=\begin{cases} f(x) \text{ if } x\in A,\\ g^{-1}(x) \text{ otherwise,}\end{cases}$$

and set $B:=Y\setminus f[A]$ to finish the proof, by Corollary 3.2 again. □

A consequence of this is the isomorphism theorem.

Theorem 3.4 *Let X, Y be Polish spaces. Then X, Y are Borel isomorphic if and only if they have the same cardinality.*

Proof. It is enough to prove that any uncountable Polish space X is Borel isomorphic to $\mathcal{C}$. We saw that we can find a closed subset C of $\mathcal{N}$ and a continous bijection $b:C\to X$, and that C is homeomorphic to a G_δ subset of $\mathcal{C}$. By Corollary 3.2, b is a Borel isomorphism. This provides a Borel injection $f:X\to\mathcal{C}$. We also saw that there is a Borel injection $g:\mathcal{C}\to X$. It remains to apply Theorem 3.3. □

4 Well-founded trees

The following is perhaps the archetypical $\mathbf{\Sigma}^1_1$-complete set.

Exercise. Using the characteristic functions, we can view the trees on ω as elements of $2^{\omega^{<\omega}}$. We equip 2 with the discrete topology and $2^{\omega^{<\omega}}$ with the product topology. We call Tr the set of characteristic functions of trees on ω. Prove that Tr is a closed subset of $2^{\omega^{<\omega}}$.

Notation. We set $IF:=\{\chi_T\in Tr \mid T \text{ is not well founded}\}$.

Theorem 4.1 *The set IF is $\mathbf{\Sigma}^1_1$-complete.*

Proof. Note that IF is $\mathbf{\Sigma}^1_1$ since $\alpha\in IF \Leftrightarrow \exists\beta\in\mathcal{N}\ \forall n\in\omega\ \alpha(\beta|n)=1$. If now $A\subseteq\mathcal{N}$ is $\mathbf{\Sigma}^1_1$, then there is a pruned tree T on ω^2 such that $\alpha\in A \Leftrightarrow \exists\beta\in\mathcal{N}\ \forall n\in\omega\ (\alpha,\beta)|n\in T$. The section function $\alpha\mapsto\chi_{T_\alpha}$ is continuous from $\mathcal{N}$ into Tr. Moreover, $\alpha\in A \Leftrightarrow \chi_{T_\alpha}\in IF$, so that IF is $\mathbf{\Sigma}^1_1$-complete.□

Definition 4.2 *Let X be a Polish space, and $C\subseteq X$. We say that C is* **co-analytic** *if $\neg C$ is analytic. The class of co-analytic sets is denoted by $\mathbf{\Pi}^1_1$.*

Remarks. (a) The set $WF:=\{\chi_T\in Tr \mid T \text{ is well founded}\}$ of characteristic functions of well founded trees on ω is $\mathbf{\Pi}^1_1$-complete.

(b) Note that $\mathbf{\Pi}^1_1=\check{\mathbf{\Sigma}}^1_1$. By Corollary 2.5, $\mathbf{\Delta}^1_1=\mathbf{\Sigma}^1_1\cap\mathbf{\Pi}^1_1$. So this notation is coherent with the one used for the Borel classes.

5 Wellorderings

The set of wellorderings is of great importance for the sequel, in classical descriptive set theory as well as in effective descriptive set theory and our applications.

Notation. (a) Fix a bijection $< .,. >: \omega^2 \to \omega$. With each $\alpha \in \mathcal{N}$ we associate the binary relation $\leq_\alpha$ on ω defined by $\leq_\alpha := \{(m,n) \in \omega^2 \mid \alpha(< m,n >) = 1\}$ and we put

$$\begin{array}{rcl} \alpha \in LO & \Leftrightarrow & \leq_\alpha \text{ is a linear ordering} \\ & \Leftrightarrow & \begin{cases} \forall m,n \in \omega\ m \leq_\alpha n \Rightarrow (m \leq_\alpha m \wedge n \leq_\alpha n) \\ \forall m,n \in \omega\ (m \leq_\alpha n \wedge n \leq_\alpha m) \Rightarrow m = n \\ \forall m,n,p \in \omega\ (m \leq_\alpha n \wedge n \leq_\alpha p) \Rightarrow m \leq_\alpha p \\ \forall m,n \in \omega\ (m \leq_\alpha m \wedge n \leq_\alpha n) \Rightarrow (m \leq_\alpha n \vee n \leq_\alpha m) \end{cases} \end{array}$$

$$\begin{array}{rcl} \alpha \in WO & \Leftrightarrow & \leq_\alpha \text{ is a wellordering} \\ & \Leftrightarrow & \alpha \in LO \wedge <_\alpha \text{ has no infinite descending chains} \\ & \Leftrightarrow & \alpha \in LO \wedge \forall \beta \in \mathcal{N}\ \big(\forall n \in \omega\ \beta(n+1) \leq_\alpha \beta(n)\big) \Rightarrow \big(\exists n \in \omega\ \beta(n+1) = \beta(n)\big). \end{array}$$

Note that LO is closed and WO is co-analytic. If $\alpha \in LO$, then we denote by $|\alpha|$ the order type of $\leq_\alpha$. Recall that two ordered sets X, Y are said to have the same **order type** just when they are order isomorphic, i.e., when there exists a bijection $f : X \to Y$ such that both f and its inverse are strictly increasing. Every well-ordered set is order-equivalent to exactly one ordinal. In particular, the map $\alpha \mapsto |\alpha|$ takes WO onto the set of countable ordinals and provides a coding for this set.

(b) For $s, t \in \omega^{<\omega}$, we set

$$s \leq_{BK} t \Leftrightarrow t \subseteq s \text{ or, for the least } k \text{ with } s(k) \neq t(k),\ s(k) < t(k).$$

So $\leq_{BK}$ is the lexicographical ordering, except that a longer word is considered less than a smaller one. $\leq_{BK}$ is a linear ordering with top element $\emptyset$. This ordering is called the **Brouwer-Kleene ordering**.

Proposition 5.1 *Let T be a tree on ω. The following are equivalent.*

(a) T is wellfounded,

(b) $(T, \leq_{BK})$ is a wellordering.

Proof. If α is a branch through T, then the sequence $(\alpha|n)$ is strictly decreasing in $(T, \leq_{BK})$. So (b) implies (a). Suppose conversely that (s_n) is strictly decreasing in $(T, \leq_{BK})$. Note that, except maybe for s_0, $|s_n| \geq 1$ and the sequence $\big(s_n(0)\big)_{n \geq 1}$ must be decreasing in ω, hence is constant from some n_0 on. But then except maybe for s_{n_0} (which could be $\big(s_{n_0}(0)\big)$), the length of s_n is greater or equal to 2 from this point on, and the sequence $\big(s_n(1)\big)_{n \geq n_0}$ must be decreasing in ω, hence stabilizes from some n_1 on, etc. So we can easily construct an increasing sequence $(n_k)_{k \in \omega}$ of natural numbers such that, for $n > n_k$, $|s_n| \geq k+1$ and $s_n(k) = s_{n_k}(k)$. But α defined by $\alpha(k) := s_{n_k}(k)$ is a branch through T. $\square$

Corollary 5.2 *(Lusin-Sierpinski) The set WO is $\mathbf{\Pi}^1_1$-complete, and $LO \setminus WO$ is $\mathbf{\Sigma}^1_1$-complete.*

Proof. As LO is closed and WO is $\mathbf{\Pi}^1_1$, $LO\setminus WO$ is $\mathbf{\Sigma}^1_1$. Fix a bijection $b:\omega\rightarrow\omega^{<\omega}$. We define $f:Tr\rightarrow\mathcal{N}$ by

$$f(\alpha)(<m,n>)=1\Leftrightarrow\Big(\alpha\big(b(m)\big)=\alpha\big(b(n)\big)=1\wedge b(m)\leq_{BK}b(n)\Big)\vee$$

$$\Big(\alpha\big(b(m)\big)=1\neq\alpha\big(b(n)\big)\Big)\ \vee\ \Big(\alpha\big(b(m)\big)=\alpha\big(b(n)\big)=0\wedge m\leq n\Big).$$

Note that $f(\alpha)\in LO$ for each $\alpha\in Tr$. Intuitively, if $\chi_T\in Tr$ is the characteristic function of a tree T on ω, then the ordering $f(\chi_T)$ of $\omega^{<\omega}$ is as follows: we put the elements of T before the elements of $\omega^{<\omega}\setminus T$, the elements of T are ordered by $\leq_{BK}$, and the elements of $\omega^{<\omega}\setminus T$ are ordered by $b^{-1}(s)\leq b^{-1}(t)$. By Proposition 5.1, $\chi_T\in WF$ is equivalent to $f(\chi_T)\in WO$. As f is continuous, WO is $\mathbf{\Pi}^1_1$-hard and thus $\mathbf{\Pi}^1_1$-complete. Similarly, $LO\setminus WO$ is $\mathbf{\Sigma}^1_1$-hard and thus $\mathbf{\Sigma}^1_1$-complete. $\square$

A consequence of this is the following representation theorem for co-analytic sets.

Theorem 5.3 *(Lusin-Sierpinski) Let X be a Polish space and P be a subset of X. Then P is $\mathbf{\Pi}^1_1$ if and only if there is a Borel function $f:X\rightarrow\mathcal{N}$ such that, for all $x\in X$, $f(x)\in LO$ and*

$$(*)\qquad x\in P\Leftrightarrow f(x)\in WO.$$

If in addition X is zero-dimensional, then $(*)$ holds with a continuous f.

Proof. If f exists, then P is $\mathbf{\Pi}^1_1$ by Proposition 1.4. Assume first that X is zero-dimensional. Corollary 5.2 (and its proof) provides $f:X\rightarrow\mathcal{N}$ continuous such that, for all $x\in X$, $f(x)\in LO$ and $x\in P\Leftrightarrow f(x)\in WO$. If X is arbitrary, then we can find a closed subset C of $\mathcal{N}$ and a continuous bijection $b:C\rightarrow X$. By Corollary 3.2, b^{-1} is Borel, and we just have to use the zero-dimensional case and to compose to conclude. $\square$

6 Co-analytic ranks

Definition 6.1 *Let P be a set. A* **norm** *on P is a function from P into the ordinals.*

A key property of the co-analytic sets is that they admit norms with nice definability properties. Roughly speaking, given a $\mathbf{\Pi}^1_1$ set P in a Polish space, there is a norm $\varphi:P\rightarrow\omega_1$ such that the initial segments $P_\xi:=\{x\in P\mid\varphi(x)\leq\xi\}$ are "uniformly" Borel. We now make this more precise.

Theorem 6.2 *There are binary relations on $\mathcal{N}$, $\leq_\Pi$ in $\mathbf{\Pi}^1_1$ and $\leq_\Sigma$ in $\mathbf{\Sigma}^1_1$, such that, for $\beta\in WO$, $\alpha\leq_\Pi\beta\Leftrightarrow\alpha\leq_\Sigma\beta\Leftrightarrow(\alpha\in WO\wedge|\alpha|\leq|\beta|)$.*

Proof. We define $\leq_\Pi$ and $\leq_\Sigma$ as follows:

$$\begin{aligned}\alpha\leq_\Sigma\beta&\Leftrightarrow\alpha\in LO\wedge\exists\gamma\in\mathcal{N}\ \gamma\text{ maps }\leq_\alpha\text{ into }\leq_\beta\text{ in a one-to-one order-preserving manner}\\&\Leftrightarrow\alpha\in LO\wedge\exists\gamma\in\mathcal{N}\ \forall m,n\in\omega\ m<_\alpha n\Rightarrow\gamma(n)<_\beta\gamma(m).\end{aligned}$$

It is immediate that $\leq_\Sigma$ is $\mathbf{\Sigma}^1_1$ and, for $\beta\in WO$, $\alpha\leq_\Sigma\beta\Leftrightarrow(\alpha\in WO\wedge|\alpha|\leq|\beta|)$. For $\leq_\Pi$, take

$$\begin{aligned}\alpha\leq_\Pi\beta&\Leftrightarrow\alpha\in WO\wedge\text{there is no order-preserving map of }\leq_\beta\text{ onto a proper initial segment of }\leq_\alpha\\&\Leftrightarrow\alpha\in WO\wedge\forall\gamma\in\mathcal{N}\ \neg\exists k\in\omega\ \forall m,n\in\omega\ \big(n\leq_\beta m\Leftrightarrow\gamma(n)\leq_\alpha\gamma(m)<_\alpha k\big),\end{aligned}$$

where of course we abbreviate $p<_\alpha q\Leftrightarrow p\leq_\alpha q\wedge p\neq q$. $\square$

Notation. We set, for $\alpha\in\mathcal{N}$, $D_\alpha:=\{n\in\omega \mid \alpha(<n,n>)=1\}$.

Proposition 6.3 *The relations*

(a) $\alpha\in WO \wedge \beta\in LO \wedge (\beta\in WO \Rightarrow |\alpha|\leq|\beta|)$

(b) $\alpha\in WO \wedge \beta\in LO \wedge (\beta\in WO \Rightarrow |\alpha|<|\beta|)$

(c) $\alpha\in WO \wedge \beta\in LO \wedge \big(\beta\in WO \Rightarrow (|\beta|<|\alpha| \vee |\alpha|<|\beta|)\big)$

are $\mathbf{\Pi}^1_1$ *in* $\mathcal{N}^2$.

Proof. We define $R\subseteq\mathcal{N}^3$ by

$$R(\alpha,\beta,\gamma)\Leftrightarrow\begin{cases}\forall n\in D_\alpha\ \gamma(n)\in D_\beta\ \wedge\\ \forall m,n\in\omega\ \big(m\leq_\alpha n\Leftrightarrow\gamma(m)\leq_\beta\gamma(n)\big)\ \wedge\\ \forall m,n\in\omega\ \Big(\big(m\in D_\beta\wedge n\in D_\alpha\wedge m\leq_\beta\gamma(n)\big)\Rightarrow\exists p\in D_\alpha\ m=\gamma(p)\Big).\end{cases}$$

The relation R is $\mathbf{\Pi}^0_2$ and expresses the fact, for α,β in LO, that $\leq_\alpha$ is embedded, via γ, in $\leq_\beta$ as an initial segment. Now (b) is equivalent to $\alpha\in WO\wedge\beta\in LO\wedge\neg\exists\gamma\in\mathcal{N}\ R(\beta,\alpha,\gamma)$. Moreover, (c) is equivalent to $\alpha\in WO\wedge\beta\in LO\wedge\neg\big(\exists\gamma\in\mathcal{N}\ R(\beta,\alpha,\gamma)\wedge\exists\delta\in\mathcal{N}\ R(\alpha,\beta,\delta)\big)$. Similarly, if

$$R'(\alpha,\beta,\gamma)\Leftrightarrow R(\alpha,\beta,\gamma)\wedge\exists n\in D_\beta\ \forall m\in D_\alpha\ \gamma(m)\neq n,$$

R' is $\mathbf{\Delta}^0_3$ and expresses the fact, for α,β in LO, that $\leq_\alpha$ is embedded, via γ, in $\leq_\beta$ as a strict initial segment. And (a) is equivalent to $\alpha\in WO\wedge\beta\in LO\wedge\neg\exists\gamma\in\mathcal{N}\ R'(\beta,\alpha,\gamma)$. Thus (a)-(c) are $\mathbf{\Pi}^1_1$. $\square$

Definition 6.4 *Let X be a Polish space, $P\subseteq X$ in $\mathbf{\Pi}^1_1$, and φ be a norm on P. We say that φ is a* $\mathbf{\Pi}^1_1$**-norm** *if the following relations*

$$x\leq^*_\varphi y\Leftrightarrow x\in P\wedge\big(y\in P\Rightarrow\varphi(x)\leq\varphi(y)\big)$$
$$x<^*_\varphi y\Leftrightarrow x\in P\wedge\big(y\in P\Rightarrow\varphi(x)<\varphi(y)\big)$$

are in $\mathbf{\Pi}^1_1$.

Theorem 6.5 *Let X be a Polish space and $P\subseteq X$ in $\mathbf{\Pi}^1_1$. Then P admits a $\mathbf{\Pi}^1_1$-norm (we say that* $\mathbf{\Pi}^1_1$ *is* **normed***).*

Proof. Theorem 5.3 provides a Borel function $f:X\to\mathcal{N}$ such that, for all $x\in X$, $\leq_{f(x)}$ is a linear ordering and $x\in P\Leftrightarrow f(x)\in WO$. We put $\varphi(x):=|f(x)|$. By Proposition 6.3, $\leq^*_\varphi$ and $<^*_\varphi$ are in $\mathit{\Pi}^1_1$. $\square$

The fact that $\mathit{\Pi}^1_1$ is normed has several important consequences. We now give some of them.

Theorem 6.6 *The class* $\mathbf{\Pi}^1_1$ *has the number uniformization property and the reduction property, and the class* $\mathbf{\Sigma}^1_1$ *has the separation property.*

Proof. Let X be a Polish space, and $P\subseteq X\times\omega$ in $\mathbf{\Pi}^1_1$. Theorem 6.5 provides a $\mathbf{\Pi}^1_1$-norm φ on P. We now put $(x,n)\in P^*\Leftrightarrow(x,n)\in P\wedge\forall m\in\omega\ (x,n)\leq^*_\varphi(x,m)\wedge\big((x,n)<^*_\varphi(x,m)\vee n\leq m\big)$, or in other words

$$(x,n)\in P^*\Leftrightarrow(x,n)\in P\wedge\varphi(x,n)=\inf\{\varphi(x,m)\mid(x,m)\in P\}\ \wedge$$
$$n=\inf\{m\in\omega\mid(x,m)\in P\wedge\varphi(x,m)=\varphi(x,n)\}.$$

Clearly P^* is in $\mathbf{\Pi}^1_1$ and

$$(x,n)\in P^*\wedge(x,n')\in P^*\Rightarrow(x,n)\in P\wedge(x,n')\in P\wedge\varphi(x,n)=\varphi(x,n')\wedge n\leq n'\wedge n'\leq n\Rightarrow n=n',$$

so P^* is the graph of a function. If $x\in\exists^\omega P$, then let $\xi:=\inf\{\varphi(x,n)\mid(x,n)\in P\}$,

$$n:=\inf\{m\in\omega\mid(x,m)\in P\wedge\varphi(x,m)=\xi\},$$

and verify easily that $P^*(x,n)$. Thus P^* uniformizes P and thus $\mathbf{\Pi}^1_1$ has the number uniformization property. As for the class $\mathbf{\Sigma}^0_\xi$, we deduce from this the fact that $\mathbf{\Pi}^1_1$ has the reduction property. As for the class $\mathbf{\Pi}^0_\xi$, we deduce from this the fact that $\mathbf{\Sigma}^1_1$ has the separation property. □

An important consequence of this is the existence of a coding system for $\varDelta^1_1$ sets.

Theorem 6.7 *Let X be a Polish space. Then there are $C\subseteq\mathcal{C}$ and $P^+,P^-\subseteq\mathcal{C}\times X$ in $\mathbf{\Pi}^1_1$ such that*

(a) for any $\alpha\in C$, P^+_α and P^-_α are complements of each other,

(b) for any $A\subseteq X$ in Borel there is $\alpha\in C$ such that $A=P^+_\alpha$.

Proof. Theorem 1.2 provides $\mathcal{U}^X\subseteq\mathcal{C}\times X$ in $\mathbf{\Pi}^1_1$ which is universal for all subsets of X in $\mathbf{\Pi}^1_1$. If $\alpha,\beta\in\mathcal{C}$, then we define $<\alpha,\beta>\in\mathcal{C}$ by $<\alpha,\beta>(2n):=\alpha(n)$ and $<\alpha,\beta>(2n+1):=\beta(n)$. We define $Q^+,Q^-\subseteq\mathcal{C}\times X$ by

$$Q^+(<\alpha,\beta>,x)\Leftrightarrow\mathcal{U}^X(\alpha,x),$$
$$Q^-(<\alpha,\beta>,x)\Leftrightarrow\mathcal{U}^X(\beta,x).$$

Note that Q^+,Q^- are in $\mathbf{\Pi}^1_1$. Theorem 6.6 provides $P^+,P^-\subseteq\mathcal{C}\times X$ disjoint in $\mathbf{\Pi}^1_1$ such that $P^+\subseteq Q^+$, $P^-\subseteq Q^-$, and $P^+\cup P^-=Q^+\cup Q^-$. We set $\alpha\in C\Leftrightarrow Q^+_\alpha\cup Q^-_\alpha=X\Leftrightarrow P^+_\alpha\cup P^-_\alpha=X$. Then C is in $\mathbf{\Pi}^1_1$ and we are done. □

We now prove the boundedness theorem for WO.

Theorem 6.8 *Let $S\subseteq WO$ be a $\mathbf{\Sigma}^1_1$ set. Then sup$\{|\alpha|\mid\alpha\in S\}<\omega_1$.*

Proof. We argue by contradiction. Let C be a $\mathbf{\Pi}^1_1$ subset of $\mathcal{C}$. Theorem 5.3 provides a continuous function $f:\mathcal{C}\to\mathcal{N}$ such that for all $\alpha\in\mathcal{C}$, $f(\alpha)\in LO$ and $\alpha\in C\Leftrightarrow f(\alpha)\in WO$. Now note that, for every α, if $f(\alpha)\in WO$, then $|f(\alpha)|<\omega_1$. So we get $\alpha\in C\Leftrightarrow f(\alpha)\in WO\wedge|f(\alpha)|<\omega_1$. Now note that $\alpha\in C\Leftrightarrow\exists\beta\in S\ \ \beta\notin WO\vee\big(f(\alpha)\in WO\wedge|f(\alpha)|\leq|\beta|\big)$, which gives a $\mathbf{\Sigma}^1_1$ definition of C by Proposition 6.3.(b). As there is in $\mathcal{C}$ a $\mathbf{\Pi}^1_1$ non $\mathbf{\Sigma}^1_1$ set C by Corollary 1.3, we get our contradiction.□

Another important consequence of the fact that $\mathbf{\Pi}^1_1$ is normed is the following reflection theorem.

Definition 6.9 *Let X be a Polish space, and $\Phi\subseteq 2^X$. We say that Φ is* **$\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$** *if, for any Polish space Y and any $A\subseteq Y\times X$ in $\mathbf{\Sigma}^1_1$, the set $A_\Phi:=\{y\in Y\mid A_y\in\Phi\}$ is in $\mathbf{\Pi}^1_1$.*

Theorem 6.10 *Let X be a Polish space, and $\Phi\subseteq 2^X$. We assume that Φ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. Then for any $S\subseteq X$ in $\mathbf{\Sigma}^1_1\cap\Phi$ there is $D\subseteq X$ in $\mathbf{\Delta}^1_1\cap\Phi$ such that $S\subseteq D$.*

Proof. Theorem 6.5 provides a $\mathbf{\Pi}^1_1$-norm φ on $P:=X\setminus S$. We argue by contradiction. Then

$$x\in S\Leftrightarrow\{y\in X\mid y\not<^*_\varphi x\}\in\Phi.$$

Indeed, if $x\in S$, then $\{y\in X\mid y\not<^*_\varphi x\}=S$, while if $x\notin S$, then $D:=\{y\in X\mid y\not<^*_\varphi x\}$ is Borel and $S\subseteq D$. As $<^*_\varphi$ is in $\mathbf{\Pi}^1_1$ and Φ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, $S\in\mathbf{\Pi}^1_1$ and thus $S\in\mathbf{\Delta}^1_1$, a contradiction. □