Chapter 7-Effective descriptive set theory

Effective descriptive set theory is a very powerful tool to prove results of classical type (i.e., results of descriptive set theory for which there is no effective descriptive set theory in their statement). It can sometimes be replaced by some other tools, but sometimes no classical proof is known. Effective descriptive set theory is based on the notion of a recursive function. We refer to [M] for the basic notions of effective descriptive set theory.

1 Recursive functions

Intuitively, the recursive functions are the computable ones. Recall that the set of natural numbers $\mathbb{N}$ is denoted by ω.

Definition 1.1 *(a) The class of* **recursive functions** *is the smallest collection of functions from some ω^k into ω (for some $k\in\omega$)*

(a) containing

- *the* successor function $S:\omega\rightarrow\omega$ *defined by* $S(n):=n+1$,
- *the* constants $C^k_n:\omega^k\rightarrow\omega$ *defined by* $C^k_n(x_0,\cdots,x_{k-1}):=n$,
- *the* projections $P^k_i:\omega^k\rightarrow\omega$ *defined by* $P^k_i(x_0,\cdots,x_{k-1}):=x_i$ *(where $i<k$),*

(b) closed under

- composition *(if $g_1,...,g_m$ and $h:\omega^m\rightarrow\omega$ are recursive, then $x\mapsto h\big(g_1(x),...,g_m(x)\big)$ also),*
- primitive recursion *(if g and $h:\omega^{2+k}\rightarrow\omega$ are recursive, then $f:\omega^{1+k}\rightarrow\omega$ defined by*

$$f(n,x):=\begin{cases} g(x) \text{ if } n=0,\\ h\big(f(n-1,x),n-1,x\big) \text{ if } n\geq 1,\end{cases}$$

is also recursive),

- minimalization *(if $g:\omega^{1+k}\rightarrow\omega$ is recursive and for all x there is n such that $g(n,x)=0$, then $x\mapsto min\{n\in\omega \mid g(n,x)=0\}$ is also recursive).*

(b) A k-ary relation on ω, say $R\subseteq\omega^k$, is a **recursive relation** *if its* **characteristic function** *$\chi_R:\omega^k\rightarrow\omega$, defined by $\chi_R(x):=1$ if $R(x)$ (meaning that $x\in R$), 0 otherwise, is recursive.*

Exercise. Prove that the following functions and relations are recursive.

- The addition $\mathcal{A}:\omega^2\rightarrow\omega$ defined by $\mathcal{A}(n,m):=n+m$.
- The multiplication $\mathcal{M}:\omega^2\rightarrow\omega$ defined by $\mathcal{M}(n,m):=n\cdot m$.

- The predecessor $pd:\omega\to\omega$ defined by $pd(0):=0$ and $pd(n+1):=n$.

- The arithmetic subtraction $\dot{-}:\omega^2\to\omega$ defined by $k\dot{-}n:=k-n$ if $k\geq n$, 0 otherwise.

- $sg:\omega\to\omega$ defined by $sg(0):=0$ and $sg(n):=1$ if $n\geq 1$.

- $\overline{sg}:\omega\to\omega$ defined by $\overline{sg}(0):=1$ and $\overline{sg}(n):=0$ if $n\geq 1$.

- $[./.]:\omega^2\to\omega$ defined by $[n/k]:=$ the unique q such that, for some $r<k$, $n=qk+r$ if $n\geq k>0$, 0 otherwise.

- $rm:\omega^2\to\omega$ defined by $rm(n,k):=$ the unique $r<k$ such that, for some q, $n=qk+r$ if $n,k>0$, 0 otherwise.

- $=(m,n)\Leftrightarrow m=n$,

- $\leq(m,n)\Leftrightarrow m\leq n$,

- $<(m,n)\Leftrightarrow m<n$.

- Prove that the class of recursive relations is closed under the operations $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\exists^\leq$, $\forall^\leq$ and substitution of recursive functions.

- $Divides(m,n)\Leftrightarrow n$ divides m.

- $Prime(m)\Leftrightarrow m$ is a prime number.

- $p:\omega\to\omega$ defined by $p(i):=p_i:=$ the i'th prime number.

- $<.>:\omega^k\to\omega$ defined by $<n_0,\cdots,n_{k-1}>:=p_0^{n_0+1}\cdot\cdots\cdot p_{k-1}^{n_{k-1}+1}$ if $k\geq 1$, 1 otherwise.

- $Seq(m)\Leftrightarrow$ we can find $k\in\omega$ and $(n_0,\cdots,n_{k-1})\in\omega^k$ such that $m=<n_0,\cdots,n_{k-1}>$.

- $lh:\omega\to\omega$ defined by $lh(m):=k$ if we can find $k\geq 1$ and $(n_0,\cdots,n_{k-1})\in\omega^k$ such that $m=<n_0,\cdots,n_{k-1}>$, 0 otherwise.

- $(.)_i:\omega\to\omega$ defined for $i\in\omega$ by $(m)_i:=n_i$ if we can find $k>i$ and $(n_0,\cdots,n_{k-1})\in\omega^k$ such that $m=<n_0,\cdots,n_{k-1}>$, 0 otherwise.

Proposition 1.2 *A function $f:\omega^k\to\omega$ is recursive if and only if Graph(f) is recursive.*

Proof. Note that $(x,n)\in$ Graph$(f)\Leftrightarrow f(x)=n$. If f is recursive, then Graph(f) is recursive since $=$ is recursive and the class of recursive relations is closed under substitution of recursive functions. Conversely, note that $f(x)=\min\big\{n\in\omega\mid\overline{sg}\big(\chi_{\text{Graph}(f)}(x,n)\big)=0\big\}$. This shows that if Graph(f) is recursive, then f is recursive. □

2 Recursive presentations

2.1 Recursively presented Polish spaces

Definition 2.1 *(a) A* **recursive presentation** *of a Polish space X is a pair $\big((x_n),d\big)$ such that*

- *(x_n) is a dense sequence of points of X,*

- *d is a complete distance defining the topology of X such that the following relations are recursive:*

$$P(i,j,m,k)\Leftrightarrow d(x_i,x_j)\leq\tfrac{m}{k+1},$$
$$Q(i,j,m,k)\Leftrightarrow d(x_i,x_j)<\tfrac{m}{k+1}.$$

(b) We say that $\Big(X,\big((x_n),d\big)\Big)$ is a **recursively presented Polish space** *if X is a Polish space and $\big((x_n),d\big)$ is a recursive presentation of X. We will often say that X is a recursively presented Polish space, for short, which means that it is given with a recursive presentation.*

Not every Polish space admits a recursive presentation, but the usual spaces do.

Exercise. Find a recursive presentation of ω, $\mathbb{R}$, the Baire space $\mathcal{N} := \omega^\omega$, and the Cantor space $\mathcal{C} := 2^\omega$.

Exercise. Let $\big(X_i,(x^i_n)_{n\in\omega},d_i\big)_{i<k}$ be a finite sequence of recursively presented Polish spaces. We set, for $n\in\omega$, $x_n := (x^0_{(n)_0},\cdots,x^{k-1}_{(n)_{k-1}})$ and define $d:\Pi_{i<k}\ X_i\rightarrow\mathbb{R}^+$ by

$$d\big((x_0,\cdots,x_{k-1}),(y_0,\cdots,y_{k-1})\big) := \max_{i<k}\ d_i(x_i,y_i).$$

Prove that $\big((x_n),d\big)$ is a recursive presentation of $\Pi_{i<k}\ X_i$, called the **product recursive presentation**.

Definition 2.2 *Let X be a recursively presented Polish space. We say that X is*

(a) **of type 0** *if $X=\omega^k$ for some $k\in\omega$,*

(b) **of type 1** *if $X=\Pi_{i<k}\ X_i$, X_i is either ω or $\mathcal{N}$ for each $i<k$, and X_i is $\mathcal{N}$ for at least one $i<k$.*

2.2 Basic spaces

In product spaces, it is more convenient in practice to work with the natural basis for the topology, rather than the previous recursive presentation. This is why we introduce the following notion.

Definition 2.3 *Let X be a topological space, and $\big(N(X,n)\big)_{n\in\omega}$ be an enumeration (possibly with repetitions) of a basis for the topology of X. We say that $\Big(X,\big(N(X,n)\big)_{n\in\omega}\Big)$ is a* **basic space** *if there is $R\subseteq\omega^3$ recursive such that $x\in N(X,m)\cap N(X,n)\Leftrightarrow\exists p\in\omega\ x\in N(X,p)\wedge R(m,n,p)$. We will often say that X is a basic space, for short, which means that it is given with an enumeration of a basis for its topology witnessing the fact that it has a basic space structure.*

Proposition 2.4 *Let X be a recursively presented Polish space. Then the formula*

$$N(X,n) := B(x_{(n)_0},\frac{(n)_1}{(n)_2+1})$$

defines a basic space structure on X.

Proof. Note that $x\in N(X,m)\cap N(X,n)$ holds exactly when there are $i,k\in\omega$ such that

$$d(x_i,x)<\frac{(k)_1}{(k)_2+1}\wedge d(x_{(m)_0},x_i)<\frac{(m)_1}{(m)_2+1}-\frac{(k)_1}{(k)_2+1}\wedge d(x_{(n)_0},x_i)<\frac{(n)_1}{(n)_2+1}-\frac{(k)_1}{(k)_2+1}.$$

Indeed, the implication from right to left is trivial, while if the left-hand side holds, then

$$A:=\{z\in X \mid \exists k\in\omega\ d(z,x)<\tfrac{(k)_1}{(k)_2+1}\wedge d(x_{(m)_0},z)<\tfrac{(m)_1}{(m)_2+1}-\tfrac{(k)_1}{(k)_2+1}\wedge$$
$$d(x_{(n)_0},z)<\tfrac{(n)_1}{(n)_2+1}-\tfrac{(k)_1}{(k)_2+1}\}$$

is open and nonempty (since $x\in A$), so A must contain one x_i. Using this equivalence and the definition of a recursive presentation, it is easy to see that there is $R\subseteq\omega^3$ recursive as desired. □

From now on, we view the space ω as a basic space by setting $N(\omega,n):=\{n\}$ (the relation defined by $R(m,n,p)\Leftrightarrow m=n=p$ is a witness for the fact that ω is a basic space). In $\mathcal{N}$, we work with the basic space structure given by Proposition 2.4.

Proposition 2.5 *There are recursive functions $g:\omega\to\omega$ and $h:\omega^2\to\omega$ such that*

$$\alpha\in N(\mathcal{N},n)\Leftrightarrow (n)_1\neq 0\wedge\forall i<g(n)\ \alpha(i)=h(n,i),$$

where the $N(\mathcal{N},n)$'s are given by Proposition 2.4.

Proof. If $(n)_1=0$, then $N(\mathcal{N},n)=\emptyset$. If $(n)_1\neq 0$, then $N(\mathcal{N},n)=\{\alpha\in\mathcal{N}\mid\forall i<l\ \alpha(i)=k_i\}$, where l, $k_0,\cdots,k_{l-1}$ are effectively computable from n (if $l=0$, then $N(\mathcal{N},n)=\mathcal{N}$). Write $l:=g(n)$, $k_i:=h(n,i)$ with suitable recursive functions. □

Proposition 2.6 *Let $(X_i)_{i<k}$ be a finite family of basic spaces. Then the formula*

$$N(\Pi_{i<k}\ X_i,n):=\begin{cases}\emptyset \text{ if } \neg Seq(n)\vee lh(n)<k,\\ \Pi_{i<k}\ N\big(X_i,(n)_i\big) \text{ otherwise}\end{cases}$$

defines a basic space structure on $\Pi_{i<k}\ X_i$ called the **product basic structure***.*

Proof. Clearly, $\big(N(\Pi_{i<k}\ X_i,n)\big)_{n\in\omega}$ is a basis for the topology of $\Pi_{i<k}\ X_i$. Let R_i be a witness for the fact that $\Big(X_i,\big(N(X_i,n)\big)_{n\in\omega}\Big)$ is basic. Then

$$\begin{array}{ll} & (x_i)_{i<k}\in N(\Pi_{i<k}\ X_i,m)\cap N(\Pi_{i<k}\ X_i,n)\\ \Leftrightarrow & Seq(m)\wedge Seq(n)\wedge lh(m),lh(n)\geq k\wedge\forall i<k\ \ x_i\in N\big(X_i,(m)_i\big)\cap N\big(X_i,(n)_i\big)\\ \Leftrightarrow & \exists p\in\omega\ \ Seq(p)\wedge lh(p)\geq k\wedge\forall i<k\ \ x_i\in N\big(X,(p)_i\big)\wedge\\ & \qquad Seq(m)\wedge Seq(n)\wedge lh(m),lh(n)\geq k\wedge R_i\big((m)_i,(n)_i,(p)_i\big)\\ \Leftrightarrow & \exists p\in\omega\ \ (x_i)_{i<k}\in N(\Pi_{i<k}\ X_i,p)\wedge\\ & \qquad \forall i<k\ \ Seq(m)\wedge Seq(n)\wedge lh(m),lh(n)\geq k\wedge R_i\big((m)_i,(n)_i,(p)_i\big),\end{array}$$

so that we just have to set

$$T(m,n,p)\Leftrightarrow\forall i<k\ \ Seq(m)\wedge Seq(n)\wedge lh(m),lh(n)\geq k\wedge R_i\big((m)_i,(n)_i,(p)_i\big)$$

to finish the proof. □

In the spaces of type 0 or 1 other than ω and $\mathcal{N}$, we work with the product basic structure.

Proposition 2.7 *Let X,Y be basic spaces, and $\big(N(X\times Y,n)\big)_{n\in\omega}$ given by Proposition 2.6. Then there are recursive functions f,g,h such that $N(X,m)\times N(Y,n)=N\big(X\times Y,f(m,n)\big)$ and*

$$N(X\times Y,n)=N\big(X,g(n)\big)\times N\big(Y,h(n)\big).$$

Proof. We just have to set $f(m,n):=<m,n>$, $g(n):=(n)_0$ and $h(n):=(n)_1$. □

3 The Kleene classes

3.1 Semirecursive sets and functions

We first work in some ω^k.

Notation. If X, Y are sets and $S \subseteq X \times Y$, then we set $\exists^Y S := \{x \in X \mid \exists y \in Y \ \ (x,y) \in S\}$.

Definition 3.1 *Let $R \subseteq \omega^k$. We say that R is* **semirecursive** *if there is a recursive relation $S \subseteq \omega^{k+1}$ such that $R = \exists^\omega S$.*

Exercise. Prove that a set $R \subseteq \omega$ is semirecursive if and only if R is empty or there exists a recursive function $f : \omega \to \omega$ which enumerates R, i.e., $R = \{f(0), f(1), f(2), \cdots\}$.

Proposition 3.2 *A relation $R \subseteq \omega^k$ is recursive if and only if R and $\neg R$ are semirecursive.*

Proof. If R is recursive, then the relation S defined by $S(x,n) \Leftrightarrow R(x)$ is also recursive, so that R is semirecursive. Moreover, $\neg R$ is also recursive, and thus semirecursive. Conversely, if R and $\neg R$ are semirecursive with recursive witnesses S, T, then $S \cup T$ is recursive. Moreover, for any $x \in \omega^k$ there is n with $(x,n) \in S \cup T$ so the formula $f(x) := \min\{n \in \omega \mid (x,n) \in S \cup T\}$ defines a recursive function, and $x \in R \Leftrightarrow S\big(x, f(x)\big)$ so R is recursive. □

Definition 3.3 *Let X be a basic space, and $S \subseteq X$. We say that S is* **semirecursive** *if there is a semirecursive subset S^* of ω such that $S = \bigcup_{n \in S^*} N(X,n)$. We say that S is* **recursive** *if S and $\neg S$ are semirecursive.*

Intuitively, S is semirecursive if it can be written as a recursive union of basic neighborhoods. Note that a subset S of ω is semirecursive in the sense of Definition 3.1 if and only if it is semirecursive in the sense of 3.3, so that our notion is not ambiguous for the space ω. The same remark applies for the product spaces ω^k, viewed as basic spaces. By Proposition 3.2, this remarks also holds for recursive relations.

Definition 3.4 *Let $X_0, \cdots, X_{k-1}, Y$ be basic spaces. We say that $f : \Pi_{i<k} X_i \to Y$ is* **trivial** *if*

$$f(x_0, \cdots, x_{k-1}) = (x_{i_0}, \cdots, x_{i_l}),$$

where $i_0, \cdots, i_l < k$.

Theorem 3.5 *The class of semirecursive sets contains the emptyset, every basic space, every basic neighborhood $N(X,n)$ of a basic space, every recursive relation on some ω^k, and the basic neighborhood relation $\{(x,n) \in X \times \omega \mid x \in N(X,n)\}$ for each basic space X; moreover, it is closed under $\vee$, $\wedge$, $\exists^\omega$, substitution of trivial functions, $\exists^\leq$, and $\forall^\leq$.*

Proof. Clearly, $\emptyset = \bigcup_{n \in \emptyset} N(X,n)$, $X = \bigcup_{n \in \omega} N(X,n)$, and $N(X,n) = \bigcup_{m \in \{n\}} N(X,m)$, so these three sets are semirecursive. Every recursive relation on some ω^k is semirecurive by Proposition 3.2. In order to check that $\{(x,n) \in X \times \omega \mid x \in N(X,n)\}$ is semirecursive, notice that

$$N(X,n) \times \{n\} = N(X,n) \times N(\omega,n) = N\big(X \times \omega, f(n,n)\big)$$

using the recursive function f of Proposition 2.7.

Thus $\{(x,n)\in X\times\omega \mid x\in N(X,n)\}=\bigcup_{n\in\omega} N\big(X\times\omega, f(n,n)\big)$. We are done since the range of a semirecursive subset of some ω^k by a recursive function is semirecursive.

For the closure properties, suppose first that $S=\bigcup_{m\in S^*} N(X,m)$ and $T=\bigcup_{n\in T^*} N(X,n)$, with both S^* and T^* semirecursive. Then $S\cup T=\bigcup_{n\in S^*\cup T^*} N(X,n)$. Similarly,

$$S\cap T=\bigcup_{m\in S^*,n\in T^*} N(X,m)\cap N(X,n)=\bigcup_{m\in S^*,n\in T^*,p\in\omega,R(m,n,p)} N(X,p).$$

We are done since $\{p\in\omega \mid \exists m\in S^*\ \exists n\in T^*\ R(m,n,p)\}$ is semirecursive. This establishes closure under $\vee$ and $\wedge$. To prove closure under $\exists^\omega$, suppose that $S=\exists^\omega T$ and $T=\bigcup_{n\in T^*} N(X\times\omega,n)$. Then

$$S(x)\Leftrightarrow\exists m\in\omega\ \exists n\in T^*\ (x,m)\in N(X\times\omega,n)\Leftrightarrow\exists m\in\omega\ \exists n\in T^*\ (x,m)\in N\big(X,g(n)\big)\times N\big(\omega,h(n)\big),$$

where g,h are recursive and given by Proposition 2.7. The relation defined by

$$R(m,n)\Leftrightarrow m\in N\big(\omega,h(n)\big)$$

is easily proved recursive, so that $S^*:=\{p\in\omega \mid \exists n\in T^*\ p=g(n)\wedge\exists m\in\omega R(m,n)\}$ is semirecursive, as well as $S=\bigcup_{p\in S^*} N(X,p)$.

Suppose that $f:\Pi_{i<k} X_i\to Y$ is trivial and defined by

$$f(x_0,\cdots,x_{k-1}):=(x_{i_0},\cdots,x_{i_l}),$$

where $i_0,\cdots,i_l<k$. If $S=\bigcup_{n\in S^*} N(Y,n)$ and $T(x)\Leftrightarrow S\big(f(x)\big)$, then

$$\begin{aligned}T(x_0,\cdots,x_{k-1})&\Leftrightarrow\exists n\in S^*\ (x_{i_0},\cdots,x_{i_l})\in N(Y,n)\\&\Leftrightarrow\exists n\in S^*\ x_{i_0}\in N\big(X_{i_0},(n)_0\big)\wedge\cdots\wedge x_{i_l}\in N\big(X_{i_l},(n)_l\big)\end{aligned}$$

For a fixed j, $x_j\in N\big(X_j,m\big)$ is equivalent to

$$\exists p\in\omega\ x_0\in N\big(X_0,(p)_0\big)\wedge\cdots\wedge x_j\in N\big(X_j,m\big)\wedge\cdots\wedge x_{k-1}\in N\big(X_{k-1},(p)_{k-1}\big)$$

and to $\exists p\in\omega\ (x_0,\cdots,x_{k-1})\in N\big(\Pi_{i<k} X_i, g_j(m,p)\big)$, where g_j is recursive. Using the argument which established that $\{(x,n)\in X\times\omega \mid x\in N(X,n)\}$ is semirecursive, it is easy to verify that each relation $R_j(x_0,\cdots,x_{k-1},m,p)\Leftrightarrow(x_0,\cdots,x_{k-1})\in N\big(\Pi_{i<k} X_i, g_j(m,p)\big)$ is semirecursive, so by closure under $\exists^\omega$ we get

$$T(x_0,\cdots,x_{k-1})\Leftrightarrow\exists n\in\omega\ \ R^*_{i_0}(x_0,\cdots,x_{k-1},n)\wedge\cdots\wedge R^*_{i_l}(x_0,\cdots,x_{k-1},n),$$

with suitable semirecursive $R^*_{i_0},\cdots,R^*_{i_l}$, and T is semirecursive by closure under $\wedge$ and $\exists^\omega$. If $T(x,n)\Leftrightarrow\exists i\leq n\ S(x,i)$ with S semirecursive, then

$$T(x,n)\Leftrightarrow\exists i\in\omega\ \ i\leq n\wedge S(x,i)\Leftrightarrow\exists i\in\omega\ \ R(x,n,i)\wedge U(x,n,i),$$

where $R(x,n,i)\Leftrightarrow i\leq n$, $U(x,n,i)\Leftrightarrow S(x,i)$ are both semirecursive by closure under the trivial substitutions $(x,n,i)\mapsto(i,n)$, $(x,n,i)\mapsto(x,i)$ and the semirecursiveness of $\leq$ and S. Now use closure under $\wedge$ and $\exists^\omega$.

Similarly, if $T(x,n) \Leftrightarrow \forall i \le n\ S(x,i)$ with $S = \bigcup_{m\in S^*} N(X\times\omega, m)$, then let $V \subseteq \omega^2$ be recursive with $S^* = \exists^\omega V$. Note that

$$\begin{aligned}T(x,n) &\Leftrightarrow \forall i\le n\ \exists m\in S^*\ (x,i)\in N(X\times\omega, m)\\ &\Leftrightarrow \forall i\le n\ \exists q\in\omega\ \big((q)_0,(q)_1\big)\in V \wedge (x,i)\in N\big(X\times\omega,(q)_0\big)\\ &\Leftrightarrow \exists p\in\omega\ \forall i\le n\ \Big(\big((p)_i\big)_0,\big((p)_i\big)_1\Big)\in V\wedge (x,i)\in N\Big(X\times\omega,\big((p)_i\big)_0\Big)\\ &\Leftrightarrow \exists p\in\omega\ \forall i\le n\ \Big(\big((p)_i\big)_0,\big((p)_i\big)_1\Big)\in V\wedge x\in N\big(X,f_1(p,i)\big)\wedge i\in N\big(\omega,f_2(p,i)\big)\end{aligned}$$

with f_1, f_2 recursive by Proposition 2.7. Thus

$$T(x,n)\Leftrightarrow \exists p,u\in\omega\ \ \forall i\le n\ \Big(\big((p)_i\big)_0,\big((p)_i\big)_1\Big)\in V\ \wedge\ \forall i\le n\ f_1(p,i)=(u)_i\ \wedge\ \forall i\le n\ x\in N\big(X,(u)_i\big)\ \wedge\ \forall i\le n\ i\in N\big(\omega,f_2(p,i)\big).$$

Now using the definition of a basic space and rearranging,

$$T(x,n)\Leftrightarrow \exists u,p,v\in\omega\ \ x\in N\big(X,g(u,n,v)\big)\wedge R(n,p,u)$$

with a recursive function g and a recursive R, i.e.,

$$T(x,n)\Leftrightarrow \exists u,p,v,m\in\omega\ \ m=g(u,n,v)\wedge x\in N(X,m)\wedge R(n,p,u).$$

So T is semirecursive by the closure properties we have established already. □

Theorem 3.6 *The class of recursive sets contains the emptyset, every basic space, every recursive relation on some ω^k, the set $\{(\alpha,n,w)\in\mathcal{N}\times\omega^2 \mid \alpha(n)=w\}$, and for each recursively presented Polish space of type 0 or 1, every basic neighborhood $N(X,n)$, and the basic neighborhood relation*

$$\{(x,n)\in X\times\omega \mid x\in N(X,n)\};$$

moreover, it is closed under $\neg$, $\vee$, $\wedge$, substitution of trivial functions, $\exists^\le$, and $\forall^\le$.

Proof. The closure properties are immediate from Theorem 3.5 and so are the facts that $\emptyset$, each basic space and each recursive relation on some ω^k are recursive. Recall from Proposition 2.5 that there are recursive functions g and h such that

$$\alpha\in N(\mathcal{N},p)\Leftrightarrow (p)_1\ne 0\wedge\forall n<g(p)\ \alpha(n)=h(p,n),$$

where the $N(\mathcal{N},n)$'s are given by Proposition 2.4. This implies that

$$\alpha(n)=w\Leftrightarrow \exists p\in\omega\ \alpha\in N(\mathcal{N},p)\wedge n<g(p)\wedge h(p,n)=w$$

because the implication from right-to-left is trivial, and that from left-to-right is easy to check if we choose p such that $\alpha\in N(\mathcal{N},p)\wedge\forall\beta\in N(\mathcal{N},p)\ \beta(n)=w$. It follows that

$$\{(\alpha,n,w)\in\mathcal{N}\times\omega^2 \mid \alpha(n)=w\}$$

is semirecursive by Theorem 3.5, and it is also recursive, since

$$\alpha(n)\ne w\Leftrightarrow \exists m\in\omega\ m\ne w\wedge\alpha(n)=m.$$

Using again Proposition 2.5,

$$\alpha \notin N(\mathcal{N}, p) \Leftrightarrow (p)_1 = 0 \vee \exists n < g(p)\ \exists w \in \omega\ \alpha(n) = w \wedge w \neq h(p, n),$$

so $\{(\alpha, p) \in \mathcal{N} \times \omega \mid \alpha \notin N(\mathcal{N}, p)\}$ is semirecursive and hence recursive by Theorem 3.5. The corresponding set for ω is trivially recursive, and then by Proposition 2.7 and closure under $\wedge$, $\{(x, p) \in X \times \omega \mid x \in N(X, p)\}$ is recursive for every space X of type 0 or 1. $\square$

Theorem 3.7 *Let X, Y be basic spaces, and $S \subseteq X \times Y$ ($X \times Y$ being equipped with the product basic structure). Then S is semirecursive if and only if there is $S^* \subseteq \omega^2$ semirecursive such that $S(x, y) \Leftrightarrow \exists p, q \in \omega\ x \in N(X, p) \wedge y \in N(Y, q) \wedge S^*(p, q)$. More specifically, $S \subseteq \omega \times X$ is semirecursive if and only if there is $S^* \subseteq \omega^2$ semirecursive such that*

$$S(n, x) \Leftrightarrow \exists p \in \omega\ x \in N(X, p) \wedge S^*(n, p).$$

Proof. By definition, $S \subseteq X \times Y$ is semirecursive if and only if there is $T^* \subseteq \omega$ semirecursive such that $S(x, y) \Leftrightarrow \exists n \in T^*\ (x, y) \in N(X \times Y, n)$. Proposition 2.7 provides recursive functions g, h such that $S(x, y) \Leftrightarrow \exists n \in T^*\ (x, y) \in N\big(X, g(n)\big) \times N\big(Y, h(n)\big)$. It remains to set

$$S^*(p, q) \Leftrightarrow \exists n \in T^*\ p = g(n) \wedge q = h(n).$$

If now $S \subseteq \omega \times X$, then the previous point provides $U^* \subseteq \omega^2$ semirecursive such that

$$S(n, x) \Leftrightarrow \exists p, q \in \omega\ n \in N(\omega, q) \wedge x \in N(X, p) \wedge U^*(q, p).$$

It remains to set $S^*(n, p) \Leftrightarrow \exists q \in \omega\ n \in N(\omega, q) \wedge U^*(q, p)$. $\square$

Theorem 3.8 *Let X be a recursively presented Polish space of type 0 or 1, and $S \subseteq X$. Then S is semirecursive if and only if there is $R \subseteq X \times \omega$ recursive such that $S = \exists^\omega R$.*

Proof. One way is immediate by Theorem 3.5. For the converse, let S^* be a semirecursive subset of ω with $P(x) \Leftrightarrow \exists n \in \omega\ n \in S^* \wedge x \in N(X, n)$, and $R^* \subseteq \omega^2$ recursive with $S^* = \exists^\omega R^*$. Then $P(x) \Leftrightarrow \exists n, p \in \omega\ R^*(n, p) \wedge x \in N(X, n) \Leftrightarrow \exists q \in \omega\ R^*\big((q)_0, (q)_1\big) \wedge x \in N\big(X, (q)_0\big)$. Thus it is enough to show that the relation $S(x, q) \Leftrightarrow x \in N\big(X, (q)_0\big)$ is recursive when X is of type 0 or 1. It is by Theorem 3.6, since $S(x, q) \Leftrightarrow \exists m \in \omega\ (q)_0 = m \wedge x \in N(X, m)$ and $\neg S(x, q) \Leftrightarrow\Leftrightarrow \exists m \in \omega\ (q)_0 = m \wedge x \notin N(X, m)$. $\square$

Proposition 3.9 *A function $f : \omega^k \to \omega$ is recursive if and only if Graph(f) is semirecursive.*

Proof. If f is recursive then Graph(f) is semirecursive, by Propositions 1.2 and 3.2. Conversely, assume that Graph(f) is semirecursive, which gives $R \subseteq \omega^{k+2}$ such that Graph$(f) = \exists^\omega R$. Note that $f(x) = \Big(\min\big\{n \in \omega \mid R\big(x, (n)_0, (n)_1\big)\big\}\Big)_0$, so that f is recursive. $\square$

Definition 3.10 *Let X, Y be basic spaces. We say that a function $f : X \to Y$ is*

(a) Σ^0_1**-recursive** *if $\{(x, n) \in X \times \omega \mid f(x) \in N(Y, n)\}$ is semirecursive in $X \times \omega$, equipped with the product basic structure,*

(b) a **recursive isomorphism** *if f is a bijection such that both f and f^{-1} are Σ^0_1-recursive.*

Proposition 3.11 *A function $f:\omega^k\rightarrow\omega$ is recursive if and only if f is Σ^0_1-recursive. So in the sequel we will say that f is a* **recursive function** *if f is Σ^0_1-recursive.*

Proof. By Proposition 3.9 it is enough to prove that f is Σ^0_1-recursive if and only if Graph(f) is semirecursive. We just have to apply the definition of $N(\omega,n)$. □

Along similar lines, the following holds.

Proposition 3.12 *Let X be a recursively presented Polish space of type 0 or 1, Y be a recursively presented Polish space of type 0, and $S\subseteq X\times Y$ semirecursive. Then there is $S^*\subseteq S$ semirecursive which the graph of a function defined on $\exists^Y S$. If moreover $X=\exists^Y S$, then there is a recursive function $f:X\rightarrow Y$ such that $S\big(x,f(x)\big)$ for each $x\in X$.*

Proof. Theorem 3.8 provides $R\subseteq X\times Y\times\omega$ recursive such that $S=\exists^\omega R$. We set, if $Y=\omega^k$,

$$S^*(x,y)\Leftrightarrow\exists n\in\omega\ R(x,y,n)\wedge\forall m<<y_0,\cdots,y_{k-1},n>\ \neg R\big(x,(m)_0,\cdots,(m)_k\big).$$

Then S^* is semirecursive by Theorem 3.5, and contained in S. If $S^*(x,y)$ and $S^*(x,y')$ both hold, then we can find natural numbers n and n' with $R(x,y,n)$, $R(x,y',n')$, and $\neg R\big(x,(m)_0,\cdots,(m)_k\big)$ if $m<<y_0,\cdots,y_{k-1},n>$ or $m<<y'_0,\cdots,y'_{k-1},n'>$. This shows that $y=y'$ and S^* is the graph of a partial function f.

If $x\in\exists^Y S$, then we can find $y\in Y$ and $n\in\omega$ with $R(x,y,n)$, and if we choose them in such a way that $<y_0,\cdots,y_{k-1},n>$ is minimal, then $f(x)$ is defined and equal to y. If now $X=\exists^Y S$, then note that

$$f(x)\in N(Y,n)\Leftrightarrow\exists m\in\omega\ \Big(x,\big((m)_0,\cdots,(m)_{k-1}\big)\Big)\in S^*\wedge\big((m)_0,\cdots,(m)_{k-1}\big)\in N(Y,n),$$

so f is recursive by Theorem 3.5. □

Exercise. Prove that the following functions are recursive.

- $f:\mathcal{N}\times\omega\rightarrow\omega$ defined by $f(\alpha,n):=\overline{\alpha}(n):=<\alpha(0),\cdots,\alpha(n-1)>$.
- $<.,.>:\mathcal{N}\times\mathcal{N}\rightarrow\mathcal{N}$ defined by $<\alpha,\beta>(2n):=\alpha(n)$ and $<\alpha,\beta>(2n+1):=\beta(n)$.
- $(.).:\mathcal{N}\times\omega\rightarrow\mathcal{N}$ defined by $(\alpha)_i(n):=\alpha(<i,n>)$.
- $.^*:\mathcal{N}\rightarrow\mathcal{N}$ defined by $\gamma^*=<\gamma(1),\gamma(2),\cdots>$.

Proposition 3.13 *The class of semirecursive sets is closed under recursive substitution.*

Proof. Let X,Y be basic spaces, $f:X\rightarrow Y$ be recursive, $S\subseteq Y$ be semirecursive, and S^* be a semicursive subset of ω with $S=\bigcup_{n\in S^*}\ N(Y,n)$. Note that

$$S\big(f(x)\big)\Leftrightarrow\exists n\in S^*\ f(x)\in N(Y,n).$$

It remains to apply Theorem 3.5. □

We met two basic structures on a product $X\times Y$ of two recursively presented Polish spaces. We now check that these two structures are recursively equivalent, and thus can be identified.

Proposition 3.14 *Let X, Y be recursively presented Polish spaces, P_0 be $X \times Y$ equipped with the basic structure defined by the product recursive presentation, and P_1 be $X \times Y$ equipped with the product of the basic structures given by Proposition 2.4. Then P_0 and P_1 are recursively isomorphic.*

Proof. We will check that the identity function is a recursive isomorphism from P_0 onto P_1. Let, for $\varepsilon \in 2$, $\big(N_\varepsilon(X \times Y, n)\big)_{n\in\omega}$ be given by the basic structure on P_ε. Note that

$$\begin{aligned}
(x,y) \in N_0(X \times Y, n) &\Leftrightarrow (x,y) \in B\big((x,y)_{(n)_0}, \tfrac{(n)_1}{(n)_2+1}\big)\\
&\Leftrightarrow (x,y) \in B\big((x_{((n)_0)_0}, y_{((n)_0)_1}), \tfrac{(n)_1}{(n)_2+1}\big)\\
&\Leftrightarrow x \in B(x_{((n)_0)_0}, \tfrac{(n)_1}{(n)_2+1}) \wedge y \in B(y_{((n)_0)_1}, \tfrac{(n)_1}{(n)_2+1})\\
&\Leftrightarrow x \in N(X, <((n)_0)_0, (n)_1, (n)_2>) \wedge y \in N(X, <((n)_0)_1, (n)_1, (n)_2>)\\
&\Leftrightarrow (x,y) \in N_1\big(X \times Y, \big\langle <((n)_0)_0, (n)_1, (n)_2>, <((n)_0)_1, (n)_1, (n)_2> \big\rangle\big)\\
&\Leftrightarrow \exists p \in \omega\ (x,y) \in N_1(X \times Y, p) \wedge\\
&\qquad\qquad p = \big\langle <((n)_0)_0, (n)_1, (n)_2>, <((n)_0)_1, (n)_1, (n)_2> \big\rangle
\end{aligned}$$

Using Theorem 3.7, this shows that the identity function from P_1 into P_0 is recursive. Conversely, the previous computation shows that

$$x \in N(X, n) \Leftrightarrow \exists p \in \omega\ (x,y) \in N_0(X \times Y, p) \wedge n = <((p)_0)_0, (p)_1, (p)_2>,$$

so that the projection from P_0 into X is recursive. Similarly, the projection into Y is recursive. As

$$(x,y) \in N_1(X \times Y, n) \Leftrightarrow x \in N\big(X, (n)_0\big) \wedge y \in N\big(Y, (n)_1\big),$$

we are done by Theorem 3.5 and Proposition 3.13. □

3.2 Polish recursive spaces

Definition 3.15 *A basic space X is* **Polish recursive** *if it is recursively isomorphic to a basic space defined by a recursively presented Polish space.*

In the sequel, we will work in Polish recursive spaces.

Theorem 3.16 *Let X be a Polish recursive space. Then there is $\pi : \mathcal{N} \to X$ recursive and onto.*

Proof. We may assume that X is a basic space defined by a recursively presented Polish space. Let (x_n) be the dense sequence coming from the recursive presentation of X. To each $\alpha \in \mathcal{N}$ we assign the sequence $(x^\alpha_n)_{n\in\omega}$ by the recursion $x^\alpha_0 := x_{\alpha(0)}$ and

$$x^\alpha_{n+1} := \begin{cases} x_{\alpha(n+1)} \text{ if } d(x^\alpha_n, x_{\alpha(n+1)}) < 2^{-n},\\ x^\alpha_n \text{ if } d(x^\alpha_n, x_{\alpha(n+1)}) \geq 2^{-n}.\end{cases}$$

For each n, $d(x^\alpha_n, x^\alpha_{n+1}) < 2^{-n}$, so that $(x^\alpha_n)_{n\in\omega}$ is Cauchy and we can set $\pi(\alpha) := \lim_{n\to\infty} x^\alpha_n$. Note that π is recursive. If $x \in X$, let $\alpha(n) := \min\{k \in \omega \mid d(x, x_k) < 2^{-n-1}\}$ and check that $\pi(\alpha) = \lim_{n\to\infty} x_{\alpha(n)} = x$. □

3.3 The Kleene classes

Notation. If Γ is a class of sets, then $\exists^Y\Gamma := \{\exists^Y S \mid S\in\Gamma\}$. Recall that $\check{\Gamma} := \{\neg S \mid S\in\Gamma\}$. We now introduce the **Kleene classes**, which are classes of subsets of Polish recursive spaces. We first set, for $n\in\omega$,

$$\begin{array}{l} \varSigma^0_1 := \text{the class of semirecursive sets} \\ \varPi^0_n := \check{\varSigma}^0_n \\ \varSigma^0_{n+1} := \exists^\omega \varPi^0_n \\ \varDelta^0_n := \varSigma^0_n \cap \varPi^0_n \end{array}$$

The sets in $\bigcup_{n\geq 1}\ \varSigma^0_n$ are called **arithmetical**. They are the effective versions of the Borel sets of finite rank. Similarly,

$$\begin{array}{l} \varSigma^1_1 := \exists^{\mathcal{N}} \varPi^0_1 \\ \varPi^1_n := \check{\varSigma}^1_n \\ \varSigma^1_{n+1} := \exists^{\mathcal{N}} \varPi^1_n \\ \varDelta^1_n := \varSigma^1_n \cap \varPi^1_n \end{array}$$

The sets in $\bigcup_{n\geq 1}\ \varSigma^1_n$ are the effective versions of the projective sets. We can also define the **relativized Kleene classes** $\varSigma^0_n(x)$, $\varPi^0_n(x)$, $\varSigma^1_n(x)$, $\varPi^1_n(x)$ by the general process as follows.

Let Γ be a class of subsets of Polish recursive spaces, X be a Polish recursive space, and $x\in X$. We say that a subset P of a Polish recursive space Y is in the **relativization** $\Gamma(x)$ of Γ to x if there is $Q\subseteq X\times Y$ in Γ such that $P(y) \Leftrightarrow Q(x,y)$.

We next define $\varDelta^0_n(x) := \varSigma^0_n(x)\cap\varPi^0_n(x)$ and $\varDelta^1_n(x) := \varSigma^1_n(x)\cap\varPi^1_n(x)$. One should be careful with this notation, since it is not the case that $\varDelta^0_n(x)$ is the relativization of $\varDelta^0_n$ to x. We will not always bother to state explicitly results about these relativized classes since they are similar to those about the non-relativized classes, and they are obtained (usually) by the same arguments.

Definition 3.17 *A class of subsets of Polish recursive spaces is called* **adequate** *if it contains the recursive subsets and is closed under recursive substitution, $\vee$, $\wedge$, $\exists^{\leq}$, and $\forall^{\leq}$.*

Theorem 3.18 *Let Γ be an adequate class. Then $\check{\Gamma}$, $\exists^\omega\Gamma$, $\forall^\omega\Gamma$, $\exists^{\mathcal{N}}\Gamma$, and $\forall^{\mathcal{N}}\Gamma$ are also adequate. Moreover, $\exists^\omega\Gamma$ is closed under $\exists^\omega$, $\forall^\omega\Gamma$ is closed under $\forall^\omega$, $\exists^{\mathcal{N}}\Gamma$ is closed under $\exists^Y$ for each Polish recursive space Y, and $\forall^{\mathcal{N}}\Gamma$ is closed under $\forall^Y$ for each Polish recursive space Y.*

Proof. The results for $\forall^\omega\Gamma$, $\forall^{\mathcal{N}}\Gamma$ and $\neg\Gamma$ follow from those for $\exists^\omega\Gamma$ and $\exists^{\mathcal{N}}\Gamma$. If $R\subseteq X$ is recursive, then we set $P(x,n)\Leftrightarrow R(x)$, so that P is recursive by Theorem 3.6 and thus in Γ, and $R=\exists^\omega P$ is in $\exists^\omega\Gamma$. If $f:X\rightarrow Y$ is recursive and $P\subseteq Y\times\omega$ is in Γ, then $f(x)\in\exists^\omega P \Leftrightarrow \exists p\in\omega\ P\big(f(x),p\big)$, so that $\exists^\omega\Gamma$ is closed under recursive substitution. If $P,Q\subseteq X\times\omega$ are in Γ, then

$$x\in\exists^\omega P\cup\exists^\omega Q \Leftrightarrow \exists p\in\omega\ P(x,p)\vee Q(x,p),$$

so that $\exists^\omega\Gamma$ is closed under $\vee$. Similarly, $\exists^\omega\Gamma$ is closed under $\exists^\omega$. Moreover,

$$x\in\exists^\omega P\cap\exists^\omega Q \Leftrightarrow \exists p,q\in\omega\ P(x,p)\wedge Q(x,q) \Leftrightarrow \exists n\in\omega\ P\big(x,(n)_0\big)\wedge Q\big(x,(n)_1\big),$$

so that $\exists^\omega\Gamma$ is closed under $\wedge$.

If now $P\subseteq X\times\omega^2$ is in Γ, then

$$\begin{aligned}\exists m\leq n\ (x,m)\in\exists^\omega P &\Leftrightarrow \exists p\in\omega\ \exists m\leq n\ P(x,m,p),\\ \forall m\leq n\ (x,m)\in\exists^\omega P &\Leftrightarrow \exists p\in\omega\ \forall m\leq n\ P\big(x,m,(p)_m\big),\end{aligned}$$

so that $\exists^\omega\Gamma$ is closed under $\exists^\leq$ and $\forall^\leq$. Similarly, $\exists^\mathcal{N}\Gamma$ contains the recursive subsets and is closed under recursive substitution. In order to prove the closure of $\exists^\mathcal{N}\Gamma$ under $\vee$, $\wedge$, $\exists^\leq$, $\forall^\leq$ and $\exists^\mathcal{N}$, we use quantifier contractions. For example, to prove closure under $\exists^\mathcal{N}$, assume that

$$Q(x,\alpha)\Leftrightarrow \exists\beta\in\mathcal{N}\ P(x,\alpha,\beta)$$

with P in Γ. Then $\exists\alpha\in\mathcal{N}\ Q(x,\alpha)\Leftrightarrow\exists\alpha,\beta\in\mathcal{N}\ P(x,\alpha,\beta)\Leftrightarrow\exists\gamma\in\mathcal{N}\ P\big(x,(\gamma)_0,(\gamma)_1\big)$ and $\exists^\mathcal{N}Q$ is in $\exists^\mathcal{N}\Gamma$ by closure of Γ under recursive substitution. To take one more example, suppose that $Q(x,m)\Leftrightarrow\exists\beta\in\mathcal{N}\ P(x,m,\beta)$. Then

$$\forall m\leq n\ Q(x,m)\Leftrightarrow\forall m\leq n\ \exists\beta\in\mathcal{N}\ P(x,m,\beta)\Leftrightarrow\exists\gamma\in\mathcal{N}\ \forall m\leq n\ P\big(x,m,(\gamma)_m\big)$$

and again $\forall^\leq Q$ is in $\exists^\mathcal{N}\Gamma$ by closure of Γ under recursive substitution and $\forall^\leq$. If Y is a Polish recursive space and $Q\subseteq X\times Y$ is in $\exists^\mathcal{N}\Gamma$, then Theorem 3.16 provides $\pi:\mathcal{N}\to Y$ recursive and onto. Then $\exists y\in Y\ Q(x,y)\Leftrightarrow\exists\alpha\in\mathcal{N}\ Q\big(x,\pi(\alpha)\big)$ and $\exists^Y Q$ is in $\exists^\mathcal{N}\Gamma$ by closure of $\exists^\mathcal{N}\Gamma$ under recursive substitution and $\exists^\mathcal{N}$. □

Corollary 3.19 *The Kleene classes are adequate. Moreover, Σ^0_n is closed under $\exists^\omega$, Π^0_n is closed under $\forall^\omega$, Σ^1_n is closed under $\forall^\omega$ and $\exists^Y$ for each Polish recursive space Y, and Π^1_n is closed under $\exists^\omega$ and $\forall^Y$ for each Polish recursive space Y. The relativizations share these properties.*

Proof. By Theorem 3.5 and Proposition 3.13, Σ^0_1 is adequate and closed under $\exists^\omega$. By Theorem 3.18 and induction, the Kleene classes are adequate. The proof of Theorem 3.18 shows that Σ^1_n is closed under $\forall^\omega$. Thus Π^1_n is closed under $\exists^\omega$. □

Proposition 3.20 *Let X be a Polish recursive space. Then $\Sigma^0_1(X)\subseteq\Sigma^0_2(X)$.*

Proof. Assume first that X is a basic space defined by a recursively presented Polish space. We define a relation P on $X\times\omega^3$ by $P(x,i,m,k)\Leftrightarrow d(x_i,x)<\frac{m}{k+1}$. Note that

$$P(x,i,m,k)\Leftrightarrow\exists n\in\omega\ x\in N(X,n)\wedge(n)_0=i\wedge\frac{(n)_1}{(n)_2+1}<\frac{m}{k+1},$$

so that P is in $\Sigma^0_1(X\times\omega^3)$. Similarly, we define a relation Q on $X\times\omega^3$ by

$$Q(x,i,m,k)\Leftrightarrow d(x_i,x)>\frac{m}{k+1}.$$

Note that $Q(x,i,m,k)\Leftrightarrow\exists n\in\omega\ x\in N(X,n)\wedge\frac{m}{k+1}+\frac{(n)_1}{(n)_2+1}<d(x_i,x_{(n)_0})$, so that Q is in $\Sigma^0_1(X\times\omega^3)$. Moreover,

$$\begin{aligned}P(x,i,m,k)&\Leftrightarrow\exists m',k'\in\omega\ \tfrac{m'}{k'+1}<\tfrac{m}{k+1}\wedge\neg\tfrac{m'}{k'+1}<d(x_i,x)\\ &\Leftrightarrow\exists m',k'\in\omega\ \tfrac{m'}{k'+1}<\tfrac{m}{k+1}\wedge\neg Q(x,i,m',k').\end{aligned}$$

Assume now that $S \in \Sigma^0_1(X)$. This gives $S^* \subseteq \omega$ semirecursive such that $S = \bigcup_{n\in S^*} N(X,n)$, and $R \subseteq \omega^2$ recursive with $S^* = \exists^\omega R$. Now

$$\begin{aligned} x\in S \Leftrightarrow \exists n\in S^*\ d(x_{(n)_0},x) < \tfrac{(n)_1}{(n)_2+1} &\Leftrightarrow \exists n,p\in\omega\ R(n,p)\wedge P\big(x,(n)_0,(n)_1,(n)_2\big)\\ &\Leftrightarrow \exists n,p,m',k'\in\omega\ R(n,p)\wedge \tfrac{m'}{k'+1} < \tfrac{(n)_1}{(n)_2+1}\wedge \neg Q\big(x,(n)_0,m',k'\big) \end{aligned}$$

Thus S is in $\Sigma^0_2(X)$.

If now X is an arbitrary Polish recursive space, then it is recursively isomorphic to a basic space defined by a recursively presented Polish space. We just have to use the closure of Σ^0_1 and Σ^0_2 under recursive substitution. $\square$

Theorem 3.21 *The inclusions hold from left to right in the following picture:*

$$\begin{array}{cccccccccccc} & \Sigma^0_1 & & \Sigma^0_2 & & & & \Sigma^1_1 & & \Sigma^1_2 & & \\ \Delta^0_1 & & \Delta^0_2 & & \cdots & \Delta^1_1 & & & \Delta^1_2 & & & \cdots \\ & \Pi^0_1 & & \Pi^0_2 & & & & \Pi^1_1 & & \Pi^1_2 & & \end{array}$$

In particular, every arithmetical set is Δ^1_1.

Proof. The inclusion $\Sigma^0_1 \subseteq \Pi^0_2$ is proved by vacuous quantifier. If $S \in \Sigma^0_1(X)$, then we define $T \subseteq X\times\omega$ by $T(x,n) \Leftrightarrow S(x)$. Then T is in Σ^0_1 and $S = \forall^\omega T$, so that S is in Π^0_2. Proposition 3.20 shows that $\Sigma^0_1 \subseteq \Sigma^0_2$ and thus $\Sigma^0_1 \subseteq \Delta^0_2$ and $\Pi^0_1 \subseteq \Delta^0_2$. By induction, we get our inclusions for the arithmetical hierarchy.

A vacuous quantifier argument shows that Π^0_1 is contained in Σ^1_1. Thus Σ^1_1 contains Σ^0_2, and also Σ^0_1 by Proposition 3.20. This implies that Π^1_1 and Δ^1_1 also contain Σ^0_1 and Π^0_1. By the closure properties, every arithmetical set is Δ^1_1. The inclusion $\Sigma^1_1 \subseteq \Pi^1_2$ is proved by vacuous quantifier. As $\Pi^0_1 \subseteq \Pi^1_1$, $\Sigma^1_1 \subseteq \Sigma^1_2$. By induction, we get all the remaining inclusions. $\square$

Definition 3.22 *Let X,Y be Polish recursive spaces, $f: X\to Y$ and Γ be a Kleene class. We say that f is* **Γ-recursive** *if the $\{(x,n)\in X\times\omega \mid f(x)\in N(Y,n)\}$ is in Γ. In particular, f is recursive if f is Σ^0_1-recursive.*

Theorem 3.23 *The classes Σ^1_1, Π^1_1, Δ^1_1 are closed under Δ^1_1-recursive substitution.*

Proof. Let $f: X\to Y$ be Δ^1_1-recursive. If $A\subseteq Y$ is Σ^1_1, pick a Π^0_1 subset of $Y\times\mathcal{N}$ with $A = \exists^{\mathcal{N}} B$, and write $A\big(f(x)\big) \Leftrightarrow \exists\alpha\in\mathcal{N}\ B\big(f(x),\alpha\big)$. As $\neg B$ is Σ^0_1, there is a Σ^0_1 subset P^* of ω^2 such that $(y,\alpha)\notin B \Leftrightarrow \exists p,n\in\omega\ y\in N(Y,p)\wedge\alpha\in N(\mathcal{N},n)\wedge P^*(p,n)$. Finally,

$$A\big(f(x)\big) \Leftrightarrow \exists\alpha\in\mathcal{N}\ \forall p,n\in\omega\ f(x)\notin N(Y,p)\vee\alpha\notin N(\mathcal{N},n)\vee\neg P^*(p,n),$$

which is Σ^1_1 by the closure properties of this class which contains Π^0_1 and Δ^1_1. Thus Σ^1_1 is closed under Δ^1_1-recursive substitution. Thus Π^1_1, Δ^1_1 are also closed under Δ^1_1-recursive substitution. $\square$

Theorem 3.24 *Let X,Y be Polish recursive spaces, and $f:X\to Y$. The following are equivalent.*

(a) f is Δ^1_1-recursive,

(b) f is Σ^1_1-recursive,

(c) Graph$(f):=\{(x,y)\in X\times Y \mid f(x)=y\}$ is Σ^1_1,

(d) Graph(f) is Δ^1_1.

Proof. (a) $\Rightarrow$ (b) is immediate and (b) $\Rightarrow$ (c) follows from the equivalence

$$f(x)=y \Leftrightarrow \forall n\in\omega\ \big(y\in N(Y,n) \Rightarrow f(x)\in N(Y,n)\big).$$

In order to prove (c) $\Rightarrow$ (d), note that $f(x)\neq y \Leftrightarrow \exists z\in Y\ f(x)=z \wedge z\neq y$. For (d) $\Rightarrow$ (a), we use

$$\begin{aligned} f(x)\in N(Y,n) &\Leftrightarrow \exists y\in Y\ f(x)=y \wedge y\in N(Y,n)\\ &\Leftrightarrow \forall y\in Y\ f(x)\neq y \vee y\in N(Y,n). \end{aligned}$$

and Corollary 3.19. $\square$

3.4 Partial functions

Definition 3.25 *Let Γ be a class of subsets of Polish recursive spaces.*

(a) We say that Γ is a Σ-class if it contains Σ^0_1, and is closed under trivial substitutions, $\vee$, $\wedge$, $\exists^{\leq}$, $\forall^{\leq}$ and $\exists^{\omega}$.

(b) Let X,Y be Polish recursive spaces, and $f:X\to Y$ be a partial function. We say that f is Γ**-recursive on its domain** *if there is $P\in X\times\omega$ in Γ such that, for each x in the domain of f and each $n\in\omega$, $f(x)\in N(Y,n) \Leftrightarrow P(x,n)$. If f is Γ-recursive on its domain and the domain of f is in Γ, then we say that f is a Γ-recursive partial function.*

(c) We say that Γ has the **substitution property** *if for each Polish recursive spaces X,Y, each partial function $f:X\to Y$ which is Γ-recursive on its domain, and each $Q\subseteq Y$ in Γ, there is $Q^*\subseteq X$ in Γ such that $Q^*(x)\Leftrightarrow Q\big(f(x)\big)$ if $f(x)$ is defined.*

Theorem 3.26 *(a) Σ^0_1 has the substitution property.*

(b) If Γ is a Σ-class with the substitution property, then so is each relativization $\Gamma(z)$.

(c) If Γ is a Σ-class closed under $\forall^{\omega}$ and either $\exists^Y$ or $\forall^Y$, then Γ has the substitution property; in particular, Σ^1_1, Π^1_1 do.

Proof. (a) Suppose that $Q\subseteq Y$ is semirecursive, so that $Q(y) \Leftrightarrow \exists n\in\omega\ y\in N(Y,n)\wedge Q^*(n)$, with a semirecursive Q^*. If $f:X\to Y$ is partial and computed on its domain by some semirecursive $P\subseteq X\times\omega$, put $Q'(x)\Leftrightarrow \exists n\in\omega\ P(x,n)\wedge Q^*(n)$. If $f(x)$ is defined, then $f(x)\in N(Y,n)\Leftrightarrow P(x,n)$, so that $Q'(x)\Leftrightarrow \exists n\in\omega\ f(x)\in N(Y,n)\wedge Q^*(n)\Leftrightarrow Q\big(f(x)\big)$.

(b) Suppose that $Q\subseteq Y$ is in $\Gamma(z)$, so that $Q(y)\Leftrightarrow Q'(z,y)$ for some Q' in Γ and suppose that $f:X\to Y$ is computed on its domain by some $P\subseteq X\times\omega$ in $\Gamma(z)$. Again $P(x,n)\Leftrightarrow P'(z,x,n)$ for some P' in Γ. Now P' computes on its domain the partial function $f':Z\times X\to Y$ defined as follows. $f'(z',x)$ is defined exactly when, for some $y\in Y$, $y\in N(X,n)\Leftrightarrow P'(z',x,n)$, and $f'(z',x)\in N(Y,n)\Leftrightarrow P'(z',x,n)$. Notice that, for the specific fixed z, $f(x)$ is defined exactly when $f'(z,x)$ is defined, and $f(x)$ is $f'(z,x)$ in this case.

The partial function $g(z',x)=\big(z',f'(z',x)\big)$ is Γ-recursive on its domain, so by the substitution property for Γ, there is some $Q''\subseteq Z\times X$ in Γ so that $Q''(z',x)\Leftrightarrow Q'\big(z',f'(z',x)\big)$ if $g(z',x)$ is defined. Setting $z':=z$, we get $Q''(z,x)\Leftrightarrow Q'\big(z,f'(z,x)\big)$ and $Q''(z,x)\Leftrightarrow Q\big(f(x)\big)$ if $f(x)$ is defined, and we can take $Q^*(x)\Leftrightarrow Q''(z,x)$.

(c) Suppose that the partial function $f:X\to Y$ is computed on its domain by $P\subseteq X\times\omega$ in Γ, $Q\subseteq Y$ is in Γ and Γ is closed under $\forall^\omega$ and $\exists^Y$. Take

$$Q^*(x)\Leftrightarrow \exists y\in Y\ Q(y)\wedge\forall n\in\omega\ \big(y\in N(Y,n)\Rightarrow P(x,n)\big).$$

This is easily in Γ and if $f(x)$ is defined, then for any y,

$$\forall n\in\omega\ \big(y\in N(Y,n)\Rightarrow P(x,n)\big)\Rightarrow\forall n\in\omega\ \big(y\in N(Y,n)\Rightarrow f(x)\in N(Y,n)\big)\Rightarrow y=f(x),$$

so that $Q^*(x)\Leftrightarrow Q\big(f(x)\big)$. Similarly, if Γ is closed under $\forall^Y$, take

$$Q^*(x)\Leftrightarrow\forall y\in Y\ Q(y)\vee\exists n\in\omega\ \big(P(x,n)\wedge y\notin N(Y,n)\big).$$

This finishes the proof. $\square$

We now prove a tranfer result.

Proposition 3.27 *Let X be a Polish recursive space. Then there exists $i_X:X\to\mathcal{N}$ one-to-one and Σ^0_2-recursive, with Π^0_2 range, whose inverse is recursive on its domain.*

Proof. Let Y be a basic space defined by a recursively presented Polish space and $r:X\to Y$ be a recursive isomorphism. We define $i_Y:Y\to\mathcal{N}$ by

$$i_Y(y)(n):=\begin{cases}1\text{ if }y\in N(Y,n),\\0\text{ otherwise.}\end{cases}$$

Note that i_Y is one-to-one. It is Σ^0_2-recursive since, using Proposition 2.5,

$$\begin{aligned}i_Y(y)\in N(\mathcal{N},n)&\Leftrightarrow (n)_1\neq 0\wedge\forall i<g(n)\ i_Y(y)(i)=h(n,i)\\&\Leftrightarrow (n)_1\neq 0\wedge\forall i<g(n)\ \Big(\big(h(n,i)=1\wedge y\in N(Y,i)\big)\vee\\&\qquad\qquad\big(h(n,i)=0\wedge y\notin N(Y,i)\big)\Big).\end{aligned}$$

We now use an idea in the proof of Proposition 2.4. We define $S\subseteq\omega^2$ by

$$S(k,n)\Leftrightarrow d(y_{(k)_0},y_{(n)_0})<\frac{(n)_1}{(n)_2+1}-\frac{(k)_1}{(k)_2+1},$$

which implies that $\overline{N(Y,k)}\subseteq N(Y,n)$. The proof of Proposition 2.4 shows that the relation $R\subseteq\omega^3$ defined by $R(m,n,k)\Leftrightarrow S(k,m)\wedge S(k,n)$ is a witness for the fact that $\Big(Y,\big(N(Y,n)\big)_{n\in\omega}\Big)$ is a basic space. Now the equivalence

$$\alpha\in i_Y[Y]\Leftrightarrow\alpha\in\mathcal{C}\wedge\begin{cases}\forall n\in\omega\ \ \alpha(n)=1\Rightarrow(n)_1\neq 0\\ \forall m,n,p\in\omega\ \ \alpha(m)=\alpha(n)=1\Rightarrow\exists k\in\omega\ \ \alpha(k)=1\wedge R(m,n,k)\wedge\\ \qquad\qquad 0\neq\frac{(k)_1}{(k)_2+1}\leq 2^{-p}\\ \forall n\in\omega\ \ \big(\exists k\in\omega\ \ S(k,n)\wedge\alpha(k)=1\Rightarrow\alpha(n)=1\big)\end{cases}$$

shows that $i_Y[Y]$ is in Π^0_2. The inverse $j_Y:i_Y[Y]\to Y$ of i_Y is recursive on its domain since $j_Y(\alpha)\in N(Y,n)\Leftrightarrow\alpha(n)=1$. It remains to set $i_X:=i_Y\circ r$, by Theorem 3.26.(a). $\square$

3.5 Universal sets

Theorem 3.28 *Let Γ be a Kleene class of the form Σ^0_n, Π^0_n, Σ^1_n or Π^1_n, $\mathbf{\Gamma}$ be its corresponding boldface class, and X be a Polish recursive space. Then there exists $\mathcal{U}^X \subseteq \mathcal{N} \times X$ in Γ which is universal for all subsets of X in $\mathbf{\Gamma}$, i.e., for every $P \subseteq X$ in Γ there is $\alpha \in \mathcal{N}$ such that $P = \mathcal{U}^X_\alpha$.*

Proof. Any open set is the union of a subfamily of the basis $\big(N(X,n)\big)_{n\in\omega}$ of X. But as it may be the case that no $N(X,n)$ is empty, we need the empty union to get the empty set, so we define $\mathcal{U}^X$ by $(\alpha, x) \in \mathcal{U}^X \Leftrightarrow \alpha(0) \neq 0 \wedge \exists p \geq 1\ \exists q \in \omega\ \alpha(p) = q \wedge x \in N(X,q)$. By Theorem 3.5, $\mathcal{U}^X$ is a Σ^0_1 subset of $\mathcal{N} \times X$. If $\alpha(0) = 0$, then $\mathcal{U}^X_\alpha$ is empty. So the empty set is coded. If now P is a nonempty open set, pick $\alpha \in \mathcal{N}$ enumerating the non empty set of ns with $N(X,n) \subseteq P$. Then clearly $P = \mathcal{U}^X_\alpha$. The result follows by a trivial induction. □

Corollary 3.29 $\mathbf{\Sigma}^0_1 = \bigcup_{\alpha\in\mathcal{N}} \Sigma^0_1(\alpha)$. *Similarly,* $\mathbf{\Pi}^0_1 = \bigcup_{\alpha\in\mathcal{N}} \Pi^0_1(\alpha)$, $\mathbf{\Sigma}^1_1 = \bigcup_{\alpha\in\mathcal{N}} \Sigma^1_1(\alpha)$, *and* $\mathbf{\Pi}^1_1 = \bigcup_{\alpha\in\mathcal{N}} \Pi^1_1(\alpha)$. *We can also say that* $\mathbf{\Delta}^0_1 = \bigcup_{\alpha\in\mathcal{N}} \Delta^0_1(\alpha)$ *and* $\mathbf{\Delta}^1_1 = \bigcup_{\alpha\in\mathcal{N}} \Delta^1_1(\alpha)$.

Proof. Let us check the left to right inclusion in the last assertion, the other ones being immediate consequences of Theorem 3.28. For example, let $B \in \mathbf{\Delta}^0_1(X)$. We can find $\alpha, \beta \in \mathcal{N}$ such that $B \in \Sigma^0_1(\alpha) \cap \Pi^0_1(\beta)$. Note that $B \in \Sigma^0_1(<\alpha,\beta>) \cap \Pi^0_1(<\alpha,\beta>) \subseteq \Delta^0_1(<\alpha,\beta>)$. □

Corollary 3.30 *The inclusions in Theorem 3.21 are strict in $\mathcal{N}$.*

Proof. We apply Cantor's diagonal method. Theorem 3.28 provides $\mathcal{U}^\mathcal{N} \subseteq \mathcal{N} \times \mathcal{N}$ in Γ which is universal for all subsets of $\mathcal{N}$ in $\mathbf{\Gamma}$. We set $\alpha \in H \Leftrightarrow (\alpha, \alpha) \in \mathcal{U}^\mathcal{N}$. Clearly $H \subseteq \mathcal{N}$ is in Γ. Now H is not in $\check{\mathbf{\Gamma}}$, otherwise we could find $\alpha \in \mathcal{N}$ such that $\neg H = \mathcal{U}^\mathcal{N}_\alpha$. In particular,

$$\alpha \notin H \Leftrightarrow \alpha \in \mathcal{U}^\mathcal{N}_\alpha \Leftrightarrow \alpha \in H,$$

a contradiction. □

If X is an arbitrary Polish space and (x_n) is any dense sequence in X, we can always pick $\alpha \in \mathcal{N}$ such that the associated relations P and Q become recursive in α. Then $\Big(X, \big((x_n), d\big)\Big)$ becomes a recursively-in-α presented Polish space. The slogan behind Theorem 3.28 and Corollary 3.29 is "boldface=topological". It explains why the classical theory, concerned with the topological notions, and the modern theory of the effective notions, can be put in a unified theory. In fact the effective (or also called lightface) results, once relativized, automatically give results for their boldface counterparts. We will not write explicitly the relativized-to-α results, although we will often use them: adding everywhere the symbols (α) would not help understanding the ideas, and would be notationally awkward. But the reader must consider these relativized-to-α statements as part of this course, for they play a fundamental role: they are the bridge between the classical and the effective approaches to descriptive set theory.

Theorem 3.28 provides $\mathcal{N}$-parametrizations of the elements of some Kleene classes. We can also find ω-parametrizations of the elements of these Kleene classes. This is a much deeper result, based on the following result, called the **enumeration theorem for semirecursive relations on** ω, that we will not prove here.

Theorem 3.31 *(Kleene) Let* $k \geq 1$.

(a) There is a Σ^0_1 *subset* $\mathcal{S}^{\omega^k}$ *of* $\omega \times \omega^k$ *such that for every* Σ^0_1 *subset* S *of* ω^k *there is* $n \in \omega$ *such that* $S = \mathcal{S}^{\omega^k}_n$.

(b) There is a Σ^0_1 *subset* $\mathcal{T}^{\omega^k}$ *of* $\mathcal{N} \times \omega \times \omega^k$ *such that, for every* $\alpha \in \mathcal{N}$*, the section* $\mathcal{T}^X_\alpha \subseteq \omega \times \omega^k$ *is in* $\Sigma^0_1(\alpha)$ *and universal for all subsets of* ω^k *in* $\Gamma(\alpha)$.

Corollary 3.32 *Let* Γ *be a Kleene class of the form* Σ^0_n, Π^0_n, Σ^1_n *or* Π^1_n*, and* X *be a Polish recursive space.*

(a) There exists $\mathcal{U}^X \subseteq \omega \times X$ *in* Γ *which is universal for all subsets of* X *in* Γ*, i.e., for every* $P \subseteq X$ *in* Γ *there is* $n \in \omega$ *such that* $P = \mathcal{U}^X_n$.

(b) The relativized result also holds, in fact uniformly. There exists $\mathcal{V}^X \subseteq \mathcal{N} \times \omega \times X$ *in* Γ *such that, for every* $\alpha \in \mathcal{N}$*, the section* $\mathcal{V}^X_\alpha \subseteq \omega \times X$ *is in* $\Gamma(\alpha)$ *and universal for all subsets of* X *in* $\Gamma(\alpha)$.

Proof. (a) Theorem 3.31 provides a Σ^0_1 subset $\mathcal{S}^\omega$ of $\omega \times \omega$ such that for every Σ^0_1 subset S of ω there is $p \in \omega$ such that $S = \mathcal{S}^\omega_p$. We put $\mathcal{U}^X(n, x) \Leftrightarrow \exists p \in \omega \ x \in N(X, p) \wedge \mathcal{S}^\omega(n, p)$. By 3.7, $\mathcal{U}^X$ is universal for all open subsets of X. The result follows by a trivial induction.

(b) We argue as in (a). □

4 The basic representation theorem for Π^1_1 sets

4.1 The representation

Theorem 4.1 *Let* X *be a Polish recursive space.*

(a) A set $S \subseteq X \times \mathcal{N}^l$ $(l \geq 1)$ *is in* Σ^0_1 *if and only if there is a set* $Q \subseteq X \times \omega^l$ *in* Σ^0_1 *such that* $S(x, \alpha_0, \cdots, \alpha_{l-1}) \Leftrightarrow \exists m \in \omega \ Q\big(x, \overline{\alpha_0}(m), \cdots, \overline{\alpha_{l-1}}(m)\big)$ *and, for each* $n \in \omega$,

$$\Big(Q\big(x, \overline{\alpha_0}(m), \cdots, \overline{\alpha_{l-1}}(m)\big) \wedge m < n\Big) \Rightarrow Q\big(x, \overline{\alpha_0}(n), \cdots, \overline{\alpha_{l-1}}(n)\big).$$

Moreover, if X *is of type 0 or 1, then* Q *may be chosen to be recursive.*

(b) A set $P \subseteq X$ *is in* Π^1_1 *if and only if there is a set* $Q \subseteq X \times \omega$ *in* Σ^0_1 *such that*

$$P(x) \Leftrightarrow \forall \alpha \in \mathcal{N} \ \exists m \in \omega \ Q\big(x, \overline{\alpha}(m)\big)$$

and, for each $n \in \omega$, $\Big(Q\big(x, \overline{\alpha}(m)\big) \wedge m < n\Big) \Rightarrow Q\big(x, \overline{\alpha}(n)\big)$*. Moreover, if* X *is of type 0 or 1, then* Q *may be chosen to be recursive.*

Proof. (b) follows immediately from (a). In order to prove (a), we take $l = 1$ for simplicity of notation. Suppose by Theorem 3.7 that $S(x, \alpha) \Leftrightarrow \exists p, q \in \omega \ x \in N(X, p) \wedge \alpha \in N(\mathcal{N}, q) \wedge S^*(p, q)$ with S^* semirecursive, so there is a recursive R such that

$$S(x, \alpha) \Leftrightarrow \exists p, q, n \in \omega \ x \in N(X, p) \wedge \alpha \in N(\mathcal{N}, q) \wedge R(p, q, n).$$

By Proposition 2.5, there are recursive functions g, h such that

$$\alpha \in N(\mathcal{N}, q) \Leftrightarrow (q)_1 \neq 0 \wedge \forall i < g(q)\ \alpha(i) = h(q, i),$$

so that whenever $m \geq g(q)$, we easily have $\alpha \in N(\mathcal{N}, q) \Leftrightarrow (q)_1 \neq 0 \wedge \forall i < g(q)\ \big(\overline{\alpha}(m)\big)_i = h(q, i)$. Now put

$$Q(x, w) \Leftrightarrow Seq(w) \wedge \exists p, q, n \leq lh(w)\ x \in N(X, p) \wedge (q)_1 \neq 0 \wedge g(q) \leq lh(w) \wedge \forall i < g(q)\ (w)_i = h(q, i) \wedge R(p, q, n)$$

and verify easily that $S(x, \alpha) \Leftrightarrow \exists m \in \omega\ Q\big(x, \overline{\alpha}(m)\big)$. If X is of type 0 or 1, then Q is recursive since $\{(x, p) \in X \times \omega \mid x \in N(X, p)\}$ is recursive by Theorem 3.6. □

We can now state the basic representation theorem for Π^1_1 sets.

Theorem 4.2 *(Lusin-Sierpinski, Kleene) Let X be a Polish recursive space and P be a subset of X. Then P is Π^1_1 if and only if there is a Δ^1_1-recursive function $f : X \to \mathcal{N}$ such that for all $x \in X$, $f(x) \in LO$ and*

$$(*) \qquad P(x) \Leftrightarrow f(x) \in WO.$$

In fact, if P is Π^1_1, then we can choose $f : X \to \mathcal{N}$ so that for all $x \in X$, $\leq_{f(x)}$ is a non-empty linear ordering, $(*)$ holds, and the relation $R(x, m, n) \Leftrightarrow f(x)(m) = n$ is arithmetical; if in addition X is of type 0 or 1, then $(*)$ holds with a recursive f.

Proof. Theorem 4.1 provides a set $Q \subseteq X \times \omega$ semirecursive (or recursive if X is of type 0 or 1) such that

$$P(x) \Leftrightarrow \forall \alpha \in \mathcal{N}\ \exists m \in \omega\ Q\big(x, \overline{\alpha}(m)\big)$$

and, for each $n \in \omega$, $\Big(Q\big(x, \overline{\alpha}(m)\big) \wedge m < n\Big) \Rightarrow Q\big(x, \overline{\alpha}(n)\big)$.

For each $x \in X$, put $T(x) := \{(u_0, \cdots, u_{l-1}) \in \omega^{<\omega} \mid \neg Q(x, < u_0, \cdots, u_{l-1} >)\}$, so that $T(x)$ is a tree on ω and clearly $P(x) \Leftrightarrow T(x)$ is wellfounded. What we must do is replace $T(x)$ by a linear ordering on ω which will be wellfounded precisely when $T(x)$ is. Put

$$(v_0, \cdots, v_{k-1}) >^x (u_0, \cdots, u_{l-1}) \Leftrightarrow (v_0, \cdots, v_{k-1}), (u_0, \cdots, u_{l-1}) \in T(x) \wedge \big(v_0 > u_0 \vee (v_0 = u_0 \wedge v_1 > u_1) \vee (v_0 = u_0 \wedge v_1 = u_1 \wedge v_2 > u_2) \vee \cdots \vee (v_0 = u_0 \wedge v_1 = u_1 \wedge \cdots \wedge v_{k-1} = u_{k-1} \wedge k < l)\big),$$

where $>$ on the right is the usual "greater than" in ω.

It is immediate that if $(v_0, \cdots, v_{k-1}), (u_0, \cdots, u_{l-1})$ are both in $T(x)$ and $(v_0, \cdots, v_{k-1})$ is a proper initial segment of $(u_0, \cdots, u_{l-1})$, then $(v_0, \cdots, v_{k-1}) >^x (u_0, \cdots, u_{l-1})$; thus if $T(x)$ has an infinite branch, then $>^x$ has an infinite descending chain. Assume now that $>^x$ has an infinite descending chain, say $v^0 >^x v^1 >^x v^2 >^x \cdots$, where $v^i = (v^i_0, v^i_1, \cdots, v^i_{l_i - 1})$, and consider the following array.

$$\begin{array}{l} v^0 = (v_0^0, v_1^0, \cdots, v_{l_0-1}^0) \\ v^1 = (v_0^1, v_1^1, \cdots, v_{l_1-1}^1) \\ \cdots \\ v^i = (v_0^i, v_1^i, \cdots, v_{l_i-1}^i) \\ \cdots \end{array}$$

The definition of $>^x$ implies immediately that $v_0^0 \geq v_0^1 \geq v_0^2 \geq \cdots$, i.e., the first column is a nonincreasing sequence of integers. Hence after a while they all are the same, say $v_0^i = k_0$ for $i \geq i_0$. Now the second column is nonincreasing below the level i_0, so that, for some i_1, k_1, $v_1^i = k_1$ for $i \geq i_1$. Proceeding in the same way we find an infinite sequence $k_0, k_1, \cdots$ such that for each l,

$$(k_0, \cdots, k_{l-1}) \in T(x),$$

so $T(x)$ is not wellfounded. Thus we have shown that

$$P(x) \Leftrightarrow T(x) \text{ is wellfounded} \Leftrightarrow >^x \text{ has no infinite descending chains.}$$

Finally put

$$u \leq^x v \Leftrightarrow \exists l \leq u\ \exists k \leq v\ Seq(u) \wedge lh(u) = l \wedge Seq(v) \wedge lh(v) = k \wedge \\ u = v \vee \big((v)_0, \cdots, (v)_{k-1}\big) >^x \big((u)_0, \cdots, (u)_{l-1}\big)$$

and notice that $\leq^x$ is always a linear ordering, it is not empty (because the code 1 of the empty sequence is in its field), and $P(x) \Leftrightarrow \leq^x$ is a wellordering. Moreover, the relation

$$P(x, u, v) \Leftrightarrow u \leq^x v$$

is easily arithmetical for arbitrary X and recursive if X is of type 0 or 1. It remains to take

$$f(x)(n) := \begin{cases} 1 \text{ if } (n)_0 \leq^x (n)_1, \\ 0 \text{ otherwise.} \end{cases}$$

This finishes the proof. □

4.2 Π_1^1-norms

Theorem 4.3 *The set WO is Π_1^1. Moreover, there are relations, $\leq_\Pi$ in Π_1^1 and $\leq_\Sigma$ in Σ_1^1, on $\mathcal{N}$ such that $\alpha \leq_\Pi \beta \Leftrightarrow \alpha \leq_\Sigma \beta \Leftrightarrow (\alpha \in WO \wedge |\alpha| \leq |\beta|)$ if $\beta \in WO$.*

Proof. The definition of WO shows that it is Π_1^1. Then copy the proof of the classical version of this result. Simply note that we can replace the boldface classes by the lightface ones. □

Proposition 4.4 *The relations*

(a) $\alpha \in WO \wedge \beta \in LO \wedge (\beta \in WO \Rightarrow |\alpha| \leq |\beta|)$

(b) $\alpha \in WO \wedge \beta \in LO \wedge (\beta \in WO \Rightarrow |\alpha| < |\beta|)$

(c) $\alpha \in WO \wedge \beta \in LO \wedge \big(\beta \in WO \Rightarrow (|\beta| < |\alpha| \vee |\alpha| < |\beta|)\big)$

are Π_1^1 in $\mathcal{N}^2$.

Proof. We copy the proof of the classical version of this result. Simply note that we can replace the boldface classes by the lightface ones. □

Definition 4.5 *Let X be a Polish recursive space, $P \subseteq X$ in Π^1_1, and φ be a function from P into the ordinals. We say that φ is a Π^1_1-**norm** if the following relations*

$$x \leq^*_\varphi y \Leftrightarrow P(x) \wedge \big(\neg P(y) \vee \varphi(x) \leq \varphi(y)\big)$$
$$x <^*_\varphi y \Leftrightarrow P(x) \wedge \big(\neg P(y) \vee \varphi(x) < \varphi(y)\big)$$

are in Π^1_1.

Theorem 4.6 *Let X be a Polish recursive space and $P \subseteq X$ in Π^1_1. Then P admits a Π^1_1-norm (we say that Π^1_1 is **normed**).*

Proof. Theorem 4.2 provides a Δ^1_1-recursive function $f: X \rightarrow \mathcal{N}$ such that, for all $x \in X$, $\leq_{f(x)}$ is a non-empty linear ordering,

$$(*) \qquad P(x) \Leftrightarrow f(x) \in WO$$

and the relation $R(x, n, p) \Leftrightarrow f(x)(n) = p$ is arithmetical. We put $\varphi(x) := |f(x)|$. By Proposition 4.4, $\leq^*_\varphi$ and $<^*_\varphi$ are in Π^1_1. □

The fact that Π^1_1 is normed has several important consequences. We now give some of them. We first prove the easy uniformization theorem.

Theorem 4.7 *(Kreisel) Let X be a Polish recursive space, and $P \subseteq X \times \omega$ in Π^1_1. Then P can be uniformized by some P^* in Π^1_1.*

Proof. We copy the proof of the classical version of this result. Simply note that we can replace the boldface classes by the lightface ones. □

Definition 4.8 *Let Γ be a class of subsets of Polish recursive spaces.*

*(a) We say that Γ has the **reduction property** if for any Polish recursive space X and any $A, B \subseteq X$ in Γ, there are $A^*, B^* \subseteq X$ disjoint in Γ such that $A^* \subseteq A$, $B^* \subseteq B$, and $A^* \cup B^* = A \cup B$.*

*(b) We say that Γ has the **separation property** if for any Polish recursive space X and any disjoint $A, B \subseteq X$ in Γ, there is $D \subseteq X$ in $\Gamma \cap \check{\Gamma}$ such that $A \subseteq D \subseteq \neg B$.*

Theorem 4.9 *The class Π^1_1 has the reduction property, and Σ^1_1 has the separation property.*

Proof. We copy the proof of the classical version of this result. Simply note that we can replace the boldface classes by the lightface ones. □

Exercise. (Novikov, Kleene, Addison) Prove that Π^1_1 does not have the separation property.

An important consequence of this is the existence of a coding system for Δ^1_1 sets.

Theorem 4.10 *Let X be a Polish recursive space. Then there are $C \subseteq \omega$ and $P^+, P^- \subseteq \omega \times X$ in Π^1_1 such that*

(a) for any $n \in C$, P^+_n and P^-_n are complements of each other,

(b) for any $A \subseteq X$ in Δ^1_1 there is $n \in C$ such that $A = P^+_n$.

Proof. Corollary 3.32 provides $\mathcal{U}^X \subseteq \omega \times X$ in Π^1_1 which is universal for all subsets of X in Π^1_1. We copy the proof of the classical version of this result, replacing $\mathcal{C}$ with ω. Simply note that we can replace the boldface classes by the lightface ones. $\square$

Another important consequence of the fact that Π^1_1 is normed is the following reflection theorem.

Definition 4.11 *Let X be a Polish recursive space, and $\Phi \subseteq 2^X$. We say that Φ is Π^1_1* **on** *Σ^1_1 if, for any Polish recursive space Y and any $A \subseteq Y \times X$ in Σ^1_1, the set $A_\Phi := \{y \in Y \mid A_y \in \Phi\}$ is in Π^1_1.*

Theorem 4.12 *Let X be a Polish recursive space, and $\Phi \subseteq 2^X$. We assume that Φ is Π^1_1 on Σ^1_1. Then for any $S \subseteq X$ in $\Sigma^1_1 \cap \Phi$ there is $D \subseteq X$ in $\Delta^1_1 \cap \Phi$ such that $S \subseteq D$.*

Proof. We copy the proof of the classical version of this result. Simply note that we can replace the boldface classes by the lightface ones. $\square$

4.3 The parametrization of Δ^1_1 points

Definition 4.13 *(a) Let X be a Polish recursive space, $x \in X$ and Γ be a Kleene class. We say that x* **is a Γ-recursive point** *if the set of codes of neighborhoods of x is in Γ, i.e., if*

$$\{n \in \omega \mid x \in N(X, n)\}$$

is in Γ. We will also say that x **is in** *Γ. We say that x* **is recursive** *if x is in Σ^0_1.*

(b) A countable ordinal ξ is a **recursive ordinal** *if there is $\alpha \in WO \cap \Sigma^0_1$ such that $|\alpha| = \xi$. Similarly, for each Polish recursive space and each $x \in X$, a countable ordinal ξ is a recursive in x ordinal if there is $\alpha \in WO \cap \Sigma^0_1(x)$ such that $|\alpha| = \xi$.*

(c) The ordinal ω_1^{CK}, called the **Church-Kleene** *ω_1, is the first non recursive ordinal. Similarly, for each Polish recursive space and each $x \in X$, the ordinal ω_1^x is the first non recursive in x ordinal.*

Remark. As there are only countably many recursive functions from some ω^k into ω, $\Sigma^0_1(X)$ is countable for each Polish recursive space X, as well as the set of Σ^0_1-recursive points of X. In particular, ω_1^{CK} is well defined and countable. Similarly, ω_1^x is well defined and countable.

Proposition 4.14 *The set $\{\alpha \in \mathcal{N} \mid \alpha \in \Sigma^0_1\}$ is Σ^0_3.*

Proof. Theorem 3.31 provides a Σ^0_1 subset $\mathcal{S}^\omega$ of ω^2 such that for every Σ^0_1 subset S of ω there is $p \in \omega$ such that $S = \mathcal{S}^\omega_p$. Thus

$$\alpha \in \Sigma^0_1 \Leftrightarrow \{n \in \omega \mid \alpha \in N(\mathcal{N}, n)\} \in \Sigma^0_1 \Leftrightarrow \exists p \in \omega\ \forall n \in \omega\ \big(\alpha \in N(\mathcal{N}, n) \Leftrightarrow (p, n) \in \mathcal{S}^\omega\big),$$

which provides a Σ^0_3 definition of our set. $\square$

Notation. If $(Z, \leq)$ is a wellordering, then we can define, by induction on $\leq$, the **rank function** ρ of $\leq$, from Z into the ordinals, by $\rho(z) := \{\rho(y) \mid y \in Z \wedge y < z\}$. Note that ρ maps Z onto some ordinal α. This is because, if α is the least ordinal not in the range of ρ, then, by induction on $\leq$, $\rho(z) < \alpha$ if $z \in Z$. We denote this ordinal by $\rho(Z)$, so that $\rho(Z) = \{\rho(z) \mid z \in Z\}$. The ordinal $\rho(Z)$ is called the **rank** of the wellordering.

It is the unique ordinal isomorphic to the wellordering, and ρ is the unique isomorphism from the wellordering onto it. If f is an order preserving function from Z into the ordinals, then $\rho(z) \leq f(z)$ for each $z \in Z$. Note that the order type of $(Z, \leq)$ is $\rho(Z)$. One also denotes by $\rho(z, Z)$ the rank $\rho(z)$ of z in Z. Note that if $Z_{|z} := \{y \in Z \mid y < z\}$ is ordered by the restriction of $\leq$, then $Z_{|z}$ is a wellordering, and $\rho(Z_{|z}) = \rho(z, Z)$.

Proposition 4.15 *There is a recursive function* $.|.:\mathcal{N}\times\omega\rightarrow\mathcal{N}$*, sending* $LO\times\omega$ *into* LO *and* $WO\times\omega$ *into* WO*, and such that, for* $\alpha\in WO$*,*

$$|\alpha|n| = \begin{cases} 0 \text{ if } n \notin D_\alpha, \\ \rho\big(n, (D_\alpha, \leq_\alpha)\big) \text{ if } n \in D_\alpha. \end{cases}$$

Proof. We set

$$(\alpha|n)(< m, p >) = \begin{cases} \alpha(< m, p >) \text{ if } n, m, p \in D_\alpha \wedge m, p <_\alpha n, \\ 0 \text{ otherwise.} \end{cases}$$

This function is as desired. □

Proposition 4.16 *The equality* $\omega_1^{CK} = \{|\alpha| \mid \alpha \in WO \cap \Sigma_1^0\}$ *holds.*

Proof. If $\alpha \in WO \cap \Sigma_1^0$ and $|\alpha| = \xi$, then, by Proposition 4.15, $\alpha|n$ is in WO for each n and the $|\alpha|n| < \xi$. As $\alpha|n$ is recursive, the set of recursive ordinals is a (countable) ordinal. □

We now prove the boundedness theorem for WO.

Theorem 4.17 *Let* $S \subseteq WO$ *be a* Σ_1^1 *set. Then* $sup\{|\alpha| \mid \alpha \in S\} < \omega_1^{CK}$.

Proof. We argue by contradiction. Let C be a Π_1^1 subset of ω. Theorem 4.2 provides a recursive function $f : \omega \rightarrow \mathcal{N}$ such that for all $n \in \omega$, $f(n) \in LO$ and $C(n) \Leftrightarrow f(n) \in WO$. Now note that, for every n, $f(n)$ is a recursive element of $\mathcal{N}$, hence if $f(n) \in WO$, then $|f(n)| < \omega_1^{CK}$. So we get $n \in C \Leftrightarrow f(n) \in WO \wedge |f(n)| < \omega_1^{CK}$. Now note that

$$n \in C \Leftrightarrow \exists \beta \in S \;\; \beta \notin WO \vee \big(f(n) \in WO \wedge |f(n)| \leq |\beta|\big),$$

which gives a Σ_1^1 definition of C by Proposition 4.4.(b). As there is in ω a Π_1^1 non Σ_1^1 set C by Corollary 3.30, we get our contradiction. □

Corollary 4.18 *(Spector) The equality* $\omega_1^{CK} = sup\{|\alpha| \mid \alpha \in WO \cap \Delta_1^1\}$ *holds.*

Proof. Let $\alpha \in WO$ be Δ_1^1. Then $\{\alpha\}$ is Δ_1^1 since $\beta \in \{\alpha\} \Leftrightarrow \forall n \in \omega \; \big(\alpha \in N(X, n) \Leftrightarrow \beta \in N(X, n)\big)$. By Theorem 4.17, $|\alpha| < \omega_1^{CK}$. □

This result is rather surprising, as one might expect to get longer wellorderings in the complicated pointclass Δ_1^1 than one gets in Σ_1^0.

Theorem 4.19 *Let* X, Y *be Polish recursive spaces,* $f : X \rightarrow Y$ *be a* Π_1^1*-recursive partial function,* $Q \subseteq Y$ *in* Π_1^1 *and* $R \subseteq X$ *defined by* $R(x) \Leftrightarrow f(x)$ *is defined* $\wedge\, Q\big(f(x)\big)$*. Then* R *is in* Π_1^1*.*

Proof. Choose $Q^* \subseteq X$ in Π^1_1 by Theorem 3.26 and the substitution property, so that, if $f(x)$ is defined, $Q^*(x) \Leftrightarrow Q\big(f(x)\big)$. Notice that $R(x) \Leftrightarrow f(x)$ is defined $\wedge\, Q^*(x)$. $\square$

We now prove the parametrization theorem for the points in Δ^1_1.

Theorem 4.20 *There is a Π^1_1-recursive partial function $\mathbf{d}:\omega\to\mathcal{N}$ such that, for every $\alpha\in\mathcal{N}$,*

$$\alpha\in\Delta^1_1 \Leftrightarrow \exists i\in\omega\ \ \mathbf{d}(i) \text{ is defined} \wedge \mathbf{d}(i)=\alpha.$$

Similarly, for any Polish recursive space X, there is a Π^1_1-recursive partial function $\mathbf{d}:\omega\times X\to\mathcal{N}$ such that, for every $(x,\alpha)\in X\times\mathcal{N}$,

$$\alpha\in\Delta^1_1(x) \Leftrightarrow \exists i\in\omega\ \ \mathbf{d}(i,x) \text{ is defined} \wedge \mathbf{d}(i,x)=\alpha.$$

Proof. We prove the second assertion, the first being simpler. Corollary 3.32 provides

$$\mathcal{U}^{X\times\omega^2}\subseteq\omega\times X\times\omega^2$$

in Π^1_1 which is universal for all subsets of $X\times\omega^2$ in Π^1_1. Theorem 4.7 provides $\mathcal{U}^*\subseteq\mathcal{U}^{X\times\omega^2}$ in Π^1_1 uniformizing $\mathcal{U}^{X\times\omega^2}$. Here we are thinking of $\mathcal{U}^{X\times\omega^2}$ as a subset of $(\omega\times X\times\omega)\times\omega$, i.e., we uniformize only on the last variable.

Now $\mathbf{d}(i,x)$ is defined exactly when $\forall n\in\omega\ \exists m\in\omega\ \mathcal{U}^*(i,x,n,m)$, and, if this is the case, we set $\mathbf{d}(i,x):=\alpha$, where for all n,m, $\alpha(n)=m \Leftrightarrow \mathcal{U}^*(i,x,n,m)$. We omit the trivial computation which establishes that $\mathbf{d}$ is Π^1_1-recursive partial.

From this it follows that $\mathbf{d}(i,x)\in N(\mathcal{N},n) \Leftrightarrow (n)_1\neq 0 \wedge \forall j<g(n)\ \mathcal{U}^*\big(i,x,j,h(n,j)\big)$, where $g:\omega\to\omega$ and $h:\omega^2\to\omega$ are the recursive functions given by Proposition 2.5. This shows that $\mathbf{d}(i,x)$ is in $\Pi^1_1(x)$. Now the relation $Q(\alpha,n) \Leftrightarrow \alpha\notin N(\mathcal{N},n)$ is in Π^0_1 and thus in Π^1_1. Since the partial function $(i,x,n)\mapsto\big(\mathbf{d}(i,x),n\big)$ is Π^1_1-recursive on its domain, the substitution property established in Theorem 3.26 gives Q^* in Π^1_1 so that $\mathbf{d}(i,x)\notin N(\mathcal{N},n) \Leftrightarrow Q^*(i,x,n)$ if $\mathbf{d}(i,x)$ is defined, so that $\mathbf{d}(i,x)$ is also $\Sigma^1_1(x)$, and hence $\Delta^1_1(x)$.

Conversely, if $\alpha\in\Delta^1_1(x)$, choose i so that $\alpha(n)=m \Leftrightarrow \mathcal{U}^{X\times\omega^2}(i,x,n,m)$ so that

$$\alpha(n)=m \Leftrightarrow \mathcal{U}^*(i,x,n,m)$$

and hence $\mathbf{d}(i,x)$ is defined and $\mathbf{d}(i,x)=\alpha$. $\square$

These last results allow us to prove the theorem on restricted quantification, which is as follows.

Theorem 4.21 *(Kleene) Let X be a Polish recursive space, $Q\subseteq X\times\mathcal{N}$ in Π^1_1 and put*

$$P(x) \Leftrightarrow \exists\alpha\in\Delta^1_1\ Q(x,\alpha).$$

Then P is in Π^1_1. Similarly, if Z is a Polish recursive space, $Q\subseteq X\times Z\times\mathcal{N}$ is in Π^1_1 and

$$P(x,z) \Leftrightarrow \exists\alpha\in\Delta^1_1(z)\ Q(x,z,\alpha),$$

then P is in Π^1_1.

Proof. Taking the second case, $P(x,z) \Leftrightarrow \exists i\in\omega\ \ \mathbf{d}(i,z)$ is defined $\wedge\, Q\big(x,z,\mathbf{d}(i,z)\big)$, so P is in Π^1_1 by Theorems 4.19 and 4.20. $\square$

Exercise. Prove that the collection of Π^1_1-recursive partial functions is closed under composition.

5 Gandy's basis theorem

We now introduce Kleene's $\mathcal{O}$.

Theorem 5.1 *(1) There is a Π^1_1 relation $\mathcal{O}\subseteq\omega^2$ such that*

(a) $\mathcal{O}$ is a wellordering of domain$(\mathcal{O}):=\{n\in\omega \mid (n,n)\in\mathcal{O}\}$ of order type ω_1^{CK},

(b) $\mathcal{O}$ is Δ^1_1 in domain$(\mathcal{O})\times\omega$.

(2) Similarly, there is a Π^1_1 relation $\boldsymbol{\mathcal{O}}\subseteq\mathcal{N}\times\omega^2$ such that, for each $\alpha\in\mathcal{N}$,

$$\mathcal{O}_\alpha:=\{(m,n)\in\omega^2 \mid (\alpha,m,n)\in\boldsymbol{\mathcal{O}}\}$$

satisfies

(a) $\mathcal{O}_\alpha$ is a wellordering of domain$(\mathcal{O}_\alpha):=\{n\in\omega \mid (n,n)\in\mathcal{O}_\alpha\}$ of order type ω_1^α,

(b) $\mathcal{O}_\alpha$ is $\Delta^1_1(\alpha)$ in domain$(\mathcal{O}_\alpha)\times\omega$.

Proof. (1) Corollary 3.30 provides $C\subseteq\omega$ in Π^1_1 but not in Σ^1_1. Theorem 4.2 provides $f:\omega\to\mathcal{N}$ recursive such that, for all $n\in\omega$, $\leq_{f(n)}$ is a non-empty linear ordering and

$$C(n)\Leftrightarrow f(n)\in WO\Leftrightarrow f(n)\in WO\wedge|f(n)|<\omega_1^{CK}.$$

Note that $\sup\{|f(n)| \mid C(n)\}=\omega_1^{CK}$. Indeed, we argue by contradiction, which gives $\alpha\in WO$ recursive such that $|f(n)|\leq|\alpha|$ if $C(n)$. Theorem 4.3 provides a Σ^1_1 relation $\leq_\Sigma$ on $\mathcal{N}$ such that $f(n)\leq_\Sigma\alpha\Leftrightarrow(f(n)\in WO\wedge|f(n)|\leq|\alpha|)$. Then $C(n)\Leftrightarrow f(n)\leq_\Sigma\alpha$. This gives a Σ^1_1 definition of C, which is absurd. We set

$$C^*:=\{n\in C \mid \forall m<n\ \ m\notin C\vee|f(m)|<|f(n)|\vee|f(n)|<|f(m)|\}.$$

By Proposition 4.4.(c), C^* is Π^1_1 and $n\mapsto|f(n)|$ is one-to-one on it. As

$$\sup\{|f(n)| \mid C^*(n)\}=\sup\{|f(n)| \mid C(n)\},$$

$\sup\{|f(n)| \mid C^*(n)\}=\omega_1^{CK}$. Note then that the relation $m\in C^*\wedge\big(n\in C^*\Rightarrow|f(m)|\leq|f(n)|\big)$ is Π^1_1 in ω^2. Indeed, it is equivalent to

$$m\in C^*\wedge\Big(n\notin C\vee\big(n\in C\wedge\exists p<n\ \ p\in C\wedge|f(p)|=|f(n)|\big)\vee|f(m)|\leq|f(n)|\Big),$$

which is Π^1_1 by Theorem 4.3 and Proposition 4.4.(a).

Note also that the relation $m\in C^*\wedge\big(n\in C^*\Rightarrow|f(m)|<|f(n)|\big)$ is Π^1_1 in ω^2. We set

$$\mathcal{O}:=\{(m,n)\in\omega^2 \mid m,n\in C^*\wedge|f(m)|\leq|f(n)|\},$$

so that $\mathcal{O}$ is a wellordering of domain$(\mathcal{O})=C^*$ in Π^1_1 which is Δ^1_1 in domain$(\mathcal{O})\times\omega$.

It remains to see that $\rho(\mathcal{O})=\omega_1^{CK}$. Note that $\rho(\mathcal{O})\leq\omega_1^{CK}$ since $\rho(m,\mathcal{O})\leq|f(m)|$ if $m\in C^*$. Assume that $\rho(\mathcal{O})<\omega_1^{CK}$, which gives $\alpha\in WO$ recursive with $\rho(\mathcal{O})=|\alpha|$. we set

$$\beta(n):=\begin{cases}0\text{ if }n\notin D_\alpha,\\ \text{the unique }m\in C^*\text{ with }\rho(m,\mathcal{O})=|\alpha|n|\text{ if }n\in D_\alpha.\end{cases}$$

Note that $\beta\in\Delta^1_1$. Indeed, it is enough to see that the relation R on ω^2 defined by

$$R(n,m)\Leftrightarrow n\in D_\alpha\wedge m\in C^*\wedge\rho(m,\mathcal{O})=|\alpha|n|$$

is Π^1_1 since $\beta(n)=m\ \Leftrightarrow\ \forall p\neq m\ \ \beta(n)\neq p$. Note first that

$$\rho(m,\mathcal{O})<|\alpha|n|\ \Leftrightarrow\ \exists\gamma\in\mathcal{N}\ \ \exists p\in\omega\ \ p<_\alpha n\ \wedge\ \forall q\neq r\neq m\in\omega \\ \big((q,r)\notin\mathcal{O}\vee(r,m)\notin\mathcal{O}\vee\gamma(q)<_\alpha\gamma(r)<_\alpha p\big),$$

so that the relation $n\in D_\alpha\wedge m\in C^*\wedge\rho(m,\mathcal{O})\geq|\alpha|n|$ is Π^1_1. Now

$$\rho(m,\mathcal{O})>|\alpha|n|\ \Leftrightarrow\ \exists\gamma\in\mathcal{N}\ \ \exists p\in\omega\ \ f(p)\leq_\Sigma f(m)\ \wedge\ f(m)\not\leq_\Pi f(p)\ \wedge\forall q\neq r\in\omega \\ \Big(q\not<_{\alpha|n}r\ \vee\ \Big(f\big(\gamma(r)\big)\leq_\Sigma f(p)\ \wedge\ f(p)\not\leq_\Pi f\big(\gamma(r)\big)\ \wedge \\ f\big(\gamma(q)\big)\leq_\Sigma f\big(\gamma(r)\big)\ \wedge\ f\big(\gamma(r)\big)\not\leq_\Pi f\big(\gamma(q)\big)\Big)\Big),$$

where $\leq_\Sigma$ and $\leq_\Pi$ are given by Theorem 4.3, so that the relation $n\in D_\alpha\wedge m\in C^*\wedge\rho(m,\mathcal{O})\leq|\alpha|n|$ is Π^1_1, as well as R. As $\beta\in\Delta^1_1$, $C^*=\{m\in\omega\mid\exists n\in D_\alpha\ m=\beta(n)\}$ is Δ^1_1 too. But this contradicts the fact that $\sup\{|f(n)|\mid C^*(n)\}=\omega_1^{CK}$, by Theorem 4.17.

(2) We argue as in (1), starting with $C\subseteq\mathcal{N}\times\omega$ such that, for every $\beta\in\mathcal{N}$, the section C_β is in $\Pi^1_1(\beta)$ but not in $\Sigma^1_1(\beta)$. This is possible by Corollary 3.32. □

We are now ready to prove Spector's criterion.

Theorem 5.2 *Let $\alpha,\beta\in\mathcal{N}$ with $\alpha\in\Delta^1_1(\beta)$. Then $\omega_1^\alpha<\omega_1^\beta\Leftrightarrow\mathcal{O}_\alpha\in\Delta^1_1(\beta)$.*

Proof. If $\mathcal{O}_\alpha$ is $\Delta^1_1(\beta)$, then $\rho(\mathcal{O}_\alpha)=\omega_1^\alpha$ is a $\Delta^1_1(\beta)$-recursive ordinal, hence recursive in β by the relativized version of Corollary 4.18, and so $\omega_1^\alpha<\omega_1^\beta$.

Conversely suppose that $\alpha\in\Delta^1_1(\beta)$ and $\omega_1^\alpha<\omega_1^\beta$. Let f be recursive in α such that

$$\begin{aligned}(m,n)\in\mathcal{O}_\alpha&\Leftrightarrow f(m,n)\in WO\\&\Leftrightarrow f(m,n)\in WO\wedge|f(m,n)|<\omega_1^\alpha.\end{aligned}$$

Let $\gamma\in WO$ be recursive in β such that $|\gamma|=\omega_1^\alpha$. Then

$$(m,n)\in\mathcal{O}_\alpha\Leftrightarrow f(m,n)\in WO\wedge|f(m,n)|<|\gamma|.$$

As $\alpha\in\Delta^1_1(\beta)$, f is $\Delta^1_1(\beta)$-recursive, so the above equivalence gives a $\Delta^1_1(\beta)$ definition of $\mathcal{O}_\alpha$. □

Notation (1) We define a coding $\mathbf{s}:\omega\to\omega^{<\omega}$ by $\mathbf{s}(n):=\mathbf{s}_n:=\Big(\big((n)_1\big)_0,\cdots,\big((n)_1\big)_{(n)_0-1}\Big)$, i.e., by considering first n as a pair $\big((n)_0,(n)_1\big)$ and then $(n)_1$ as a $(n)_0$-tuple, using the appropriate brackets. So any $n=<0,k>$ codes the empty sequence, $<1,k>$ codes (k), and, for $p\geq 2$, $<p,k>$ codes $\big((k)_0,\cdots,(k)_{p-1}\big)$.

(2) We define $\beta_{\mathcal{O}}\in\mathcal{N}$ by

$$\beta_{\mathcal{O}}(n):=\begin{cases} 1 \text{ if } \big((n)_0,(n)_1\big)\in\mathcal{O} \\ 0 \text{ otherwise,}\end{cases}$$

and similarly for $\mathcal{O}_\alpha$ for each $\alpha\in\mathcal{N}$.

(3) We set, for any ordinal ξ, $WO_\xi:=\{\alpha\in WO \mid |\alpha|<\xi\}$.

Theorem 5.3 *Let X be a Polish recursive space and A be a nonempty Σ^1_1 (resp., $\Sigma^1_1(\alpha)$) subset of X. Then A contains a point x which is $\Delta^1_1(\beta_{\mathcal{O}})$ (resp., $\Delta^1_1(\beta_{\mathcal{O}_\alpha})$).*

Proof. Proposition 3.27 provides $i_X:X\to\mathcal{N}$ one-to-one and Σ^0_2-recursive, with Π^0_2 range, whose inverse $j_X:i_X[X]\to X$ is recursive on its domain. We define $A'\subseteq\mathcal{N}$ by

$$A'(\beta)\Leftrightarrow\beta\in i_X[X]\wedge j_X(\beta)\in A.$$

By Corollary 3.19 and Theorems 3.21, 3.26.(c), $A'=i_X[A]$ is a nonempty Σ^1_1 set. Assume that $\beta\in A'$ is $\Delta^1_1(\beta_{\mathcal{O}})$, which gives $S\subseteq\omega\times\mathcal{N}$ in Σ^1_1 and $P\subseteq\omega\times\mathcal{N}$ in Π^1_1 such that

$$\beta\in N(\mathcal{N},p)\Leftrightarrow(p,\beta_{\mathcal{O}})\in S\Leftrightarrow(p,\beta_{\mathcal{O}})\in P.$$

Let $R\subseteq\mathcal{N}\times\omega$ be semirecursive such that $j_X(\gamma)\in N(X,n)\Leftrightarrow R(\gamma,n)$ if $\gamma\in i_X[X]$. Note that

$$\begin{aligned}j_X(\beta)\in N(X,n) &\Leftrightarrow \exists\gamma\in\mathcal{N}\ \forall p\in\omega\ \big(\gamma\notin N(\mathcal{N},p)\vee\beta\in N(\mathcal{N},p)\big)\wedge j_X(\gamma)\in N(X,n)\\ &\Leftrightarrow \forall\gamma\in\mathcal{N}\ \exists p\in\omega\ \big(\gamma\notin N(\mathcal{N},p)\vee\beta\in N(\mathcal{N},p)\big)\vee j_X(\gamma)\in N(X,n)\\ &\Leftrightarrow \exists\gamma\in\mathcal{N}\ \forall p\in\omega\ \big(\gamma\notin N(\mathcal{N},p)\vee(p,\beta_{\mathcal{O}})\in S\big)\wedge R(\gamma,n)\\ &\Leftrightarrow \forall\gamma\in\mathcal{N}\ \exists p\in\omega\ \big(\gamma\notin N(\mathcal{N},p)\vee(p,\beta_{\mathcal{O}})\in P\big)\vee R(\gamma,n),\end{aligned}$$

so that $j_X(\beta)\in A$ is $\Delta^1_1(\beta_{\mathcal{O}})$. So we may assume that $X=\mathcal{N}$.

We can assume that A is Π^0_1, as the projection of a $\Delta^1_1(\beta_{\mathcal{O}})$ point in $\mathcal{N}^2$ is $\Delta^1_1(\beta_{\mathcal{O}})$. Theorem 4.1 provides $Q\subseteq\omega$ recursive (coding a tree T on ω) such that $A(\alpha)\Leftrightarrow\forall m\in\omega\ Q\big(\overline{\alpha}(m)\big)$ and, for each $n\in\omega$, $Q\big(\overline{\alpha}(n)\big)\Rightarrow\forall m<n\ Q\big(\overline{\alpha}(m)\big)$. We inductively define

$$\alpha(n):=\min\big\{p\in\omega\mid\exists\gamma\in\mathcal{N}\ \ (\alpha|n)p\subseteq\gamma\wedge\forall m\in\omega\ Q\big(\overline{\gamma}(m)\big)\big\}.$$

Intuitively, α is the left-most branch of the tree T coded by Q. As A is nonempty, α is well defined and in A. So it is enough to show that $\alpha\in\Delta^1_1(\beta_{\mathcal{O}})$. We set, for each $s\in\omega^{<\omega}$,

$$T_s:=\{t\in\omega^{<\omega}\mid<s(0),\cdots,s(|s|-1),t(0),\cdots,t(|t|-1)>\in Q\},$$

and define a recursive function $f:\omega\to\mathcal{N}$ by

$$f(n)(<p,q>):=\begin{cases} 1 \text{ if } \mathbf{s}_p\in T_{\mathbf{s}_n}\wedge\mathbf{s}_q\in T_{\mathbf{s}_n}\wedge\mathbf{s}_p\leq_{BK}\mathbf{s}_q \\ 0 \text{ otherwise,}\end{cases}$$

so that $f(n)\in LO$ for each n, and $f(n)\in WO$ if and only if $T_{\mathbf{s}_n}$ is wellfounded.

By definition of α, γ is the left-most branch if and only if

$$\forall i\in\omega\ \forall k<\gamma(i)\ f(<\gamma(0),\cdots,\gamma(i-1),k>)\in WO_{\omega_1^{CK}}\wedge f(<\gamma(0),\cdots,\gamma(i)>)\notin WO_{\omega_1^{CK}}.$$

So it is enough to prove that $WO_{\omega_1^{CK}}$ is $\Delta^1_1(\beta_{\mathcal{O}})$. But as $\beta_{\mathcal{O}}\in WO$ and $|\beta_{\mathcal{O}}|=\omega_1^{CK}$, we get $WO_{\omega_1^{CK}}=\{\alpha\in WO\mid |\alpha|<|\beta_{\mathcal{O}}|\}$ so $WO_{\omega_1^{CK}}$ is $\Delta^1_1(\beta_{\mathcal{O}})$ and we are done. □

Notation Let X be a Polish recursive space. We define $X_{low}:=\{x\in X\mid \omega_1^x=\omega_1^{CK}\}$, and similarly $X^\alpha_{low}:=\{x\in X\mid \omega_1^x=\omega_1^\alpha\}$ if $\alpha\in\mathcal{N}$.

Theorem 5.4 *(Gandy) Let X be a Polish recursive space and A be a nonempty Σ^1_1 (resp., $\Sigma^1_1(\alpha)$) subset of X. Then A meets X_{low} (resp., X^α_{low}).*

Proof. Proposition 3.27 provides $i_X:X\to\mathcal{N}$ one-to-one and Σ^0_2-recursive, with Π^0_2 range, whose inverse $j_X:i_X[X]\to X$ is recursive on its domain. We define $A'\subseteq\mathcal{N}$ by

$$A'(\beta)\Leftrightarrow\beta\in i_X[X]\wedge j_X(\beta)\in A.$$

By Corollary 3.19 and Theorems 3.21, 3.26.(c), $A'=i_X[A]$ is a nonempty Σ^1_1 set. Assume that $\beta\in A'\cap\mathcal{N}_{low}$. Then $j_X(\beta)\in A$ and $\omega_1^{CK}=\omega_1^\beta=\sup\{|\alpha|\mid\alpha\in WO\cap\Delta^1_1(\beta)\}$. Assume that $\alpha\in\Delta^1_1(\beta)$. Then there are $S\subseteq\omega\times\mathcal{N}$ in Σ^1_1 and $P\subseteq\omega\times\mathcal{N}$ in Π^1_1 such that, for each $n\in\omega$, $\alpha\in N(\mathcal{N},n)\Leftrightarrow S(n,\beta)\Leftrightarrow P(n,\beta)$. We define $S',P'\subseteq\omega\times X$ by $S'(n,x)\Leftrightarrow S\big(n,i_X(x)\big)$ and $P'(n,x)\Leftrightarrow P\big(n,i_X(x)\big)$. By Theorem 3.23, S' is in Σ^1_1 and P' is in Π^1_1. Moreover,

$$(*)\qquad \alpha\in N(\mathcal{N},n)\Leftrightarrow S'\big(n,j_X(\beta)\big)\Leftrightarrow P'\big(n,j_X(\beta)\big),$$

so that $\alpha\in\Delta^1_1\big(j_X(\beta)\big)$. Conversely, assume that $\alpha\in\Delta^1_1\big(j_X(\beta)\big)$, which gives $S'\subseteq\omega\times X$ in Σ^1_1 and $P'\subseteq\omega\times X$ in Π^1_1 such that, for each $n\in\omega$, $(*)$ holds. we define $S,P\subseteq\omega\times\mathcal{N}$ by $S(n,\gamma)\Leftrightarrow\gamma\in i_X[X]\wedge S'\big(n,j_X(\gamma)\big)$ and $P(n,\gamma)\Leftrightarrow\gamma\in i_X[X]\wedge P'\big(n,j_X(\gamma)\big)$. By Corollary 3.19 and Theorems 3.21, 3.26.(c), S is in Σ^1_1 and P is in Π^1_1. Moreover,

$$\alpha\in N(\mathcal{N},n)\Leftrightarrow S(n,\beta)\Leftrightarrow P(n,\beta),$$

so that $\alpha\in\Delta^1_1(\beta)$. Thus $j_X(\beta)\in X_{low}$. So we may assume that $X=\mathcal{N}$.

We define $B\subseteq\mathcal{N}$ by $\gamma\in B\Leftrightarrow\forall\beta\in A\ \gamma\in\Delta^1_1(\beta)$. By Theorem 4.20, B is Π^1_1. And as A is non empty, pick $\beta_0\in A$. Then $B\subseteq\{\gamma\in\mathcal{N}\mid\gamma\in\Delta^1_1(\beta_0)\}$, so B is countable. Note also that if $\gamma\in B$ and $\beta\in\Delta^1_1(\gamma)$, then $\beta\in B$ by transitivity. Consider $C:=\mathcal{N}\setminus B$. The set C is Σ^1_1 and non empty in $\mathcal{N}$, hence by Theorem 5.3 it contains a point $\beta\in\Delta^1_1(\beta_{\mathcal{O}})$. But then, by the preceding remark, $\beta_{\mathcal{O}}\in C$ (for if $\beta_{\mathcal{O}}\in B$, any $\Delta^1_1(\beta_{\mathcal{O}})$ point would be in B too). By definition of C this means that there is $\beta\in A$ such that $O\notin\Delta^1_1(\beta)$. By Theorem 5.2, this implies that $\omega_1^\beta=\omega_1^{CK}$, as desired. □

6 The Gandy-Harrington topology

Definition 6.1 *Let X be a Polish recursive space. The* **Gandy-Harrington topology** *on X is generated by the Σ^1_1 subsets of X. We denote it by τ_{GH}.*

Theorem 6.2 *Let X be a nonempty Polish recursive space. The Gandy-Harrington topology has the following properties:*

(a) it is second countable,

(b) it is finer than the initial topology of X, and is in particular T_1,

(c) it is not regular (and thus not metrizable) in general,

(d) it is strong Choquet,

(e) the set X_{low} is Σ^1_1, and thus τ_{GH}-open, and dense,

(f) if $S \subseteq X$ is Σ^1_1, then $S \cap X_{low}$ is τ_{GH}-clopen in X_{low},

(g) the set X_{low}, equipped with τ_{GH}, is a zero-dimensional Polish space.

Proof. (a) This comes from the fact that the set of Σ^1_1 subsets of X is countable.

(b) Any basic open set $N(X,n)$ is semirecursive, and thus Σ^1_1. Thus τ_{GH} is finer than the initial topology of X. As X is Polish, its topology is Hausdorff, as well as τ_{GH} which is therefore T_1.

(c) We will check that in $\mathcal{N}$, τ_{GH} is not regular. By Theorem 3.28 provides $\mathcal{U}^{\mathcal{N}} \subseteq \mathcal{N}^2$ in Π^1_1 which is universal for all subsets of $\mathcal{N}$ in $\mathbf{\Pi}^1_1$. We set $P := \{\alpha \in \mathcal{N} \mid (\alpha, \alpha) \in \mathcal{U}^{\mathcal{N}}\}$. Note that P is in Π^1_1. But it is not in $\mathbf{\Sigma}^1_1$. Indeed, we argue by contradiction. This gives $\beta \in \mathcal{N}$ with $\neg P = \mathcal{U}^{\mathcal{N}}_\beta$. Now $\beta \notin P \Leftrightarrow (\beta, \beta) \in \mathcal{U}^{\mathcal{N}} \Leftrightarrow \beta \in P$, which is absurd. This implies that P is τ_{GH}-closed but not τ_{GH}-$\mathbf{\Pi}^0_2$. Thus τ_{GH} is not metrizable, and not regular by (a).

(d) We first prove the result in $\mathcal{N}$. We describe a strategy τ for Player 2. Player 1 first plays $\sigma_0 \in \mathcal{N}$ and a τ_{GH}-open neighborhood U_0 of σ_0. Let L_0 in Σ^1_1 with $\sigma_0 \in L_0 \subseteq U_0$. Let $C_0 \subseteq \mathcal{N}^2$ be Π^0_1 with $L_0 = \exists^{\mathcal{N}} C_0$. This gives $\alpha_0 \in \mathcal{N}$ such that $(\sigma_0, \alpha_0) \in C_0$. We set $w_0 := \sigma_0|1$, $s^0_0 := \alpha_0|1$ and $V_0 := \pi_0[C_0 \cap (N_{w_0} \times N_{s^0_0})]$. Note that V_0 is in Σ^1_1 and thus τ_{GH}-open. Moreover, $\sigma_0 \in V_0 \subseteq L_0 \subseteq U_0$, so that Player 2 respects the rules of the game if he plays V_0.

Now Player 1 plays $\sigma_1 \in V_0$ and a τ_{GH}-open neighborhood U_1 of σ_1 contained in V_0. Let L_1 in Σ^1_1 with $\sigma_1 \in L_1 \subseteq U_1$. Let $C_1 \subseteq \mathcal{N}^2$ be Π^0_1 with $L_1 = \exists^{\mathcal{N}} C_1$. This gives $\alpha_1 \in \mathcal{N}$ such that $(\sigma_1, \alpha_1) \in C_1$. As $\sigma_1 \in V_0$, there is $\alpha'_0 \in \mathcal{N}$ such that $(\sigma_1, \alpha'_0) \in C_0 \cap (N_{w_0} \times N_{s^0_0})$. We set $w_1 := \sigma_1|2$, $s^0_1 := \alpha'_0|2$, $s^1_0 := \alpha_1|1$ and $V_1 := \pi_0[C_0 \cap (N_{w_1} \times N_{s^0_1})] \cap \pi_0[C_1 \cap (N_{w_0} \times N_{s^1_0})]$. Here again, V_1 is τ_{GH}-open. Moreover, $\sigma_1 \in V_1 \subseteq U_1$ and Player 2 can play V_1.

Next, Player 1 plays $\sigma_2 \in V_1$ and a τ_{GH}-open neighborhood U_2 of σ_2 contained in V_1. Let L_2 in Σ^1_1 with $\sigma_2 \in L_2 \subseteq U_2$. Let $C_2 \subseteq \mathcal{N}^2$ be Π^0_1 with $L_2 = \exists^{\mathcal{N}} C_2$. This gives $\alpha_2 \in \mathcal{N}$ such that $(\sigma_2, \alpha_2) \in C_2$. As $\sigma_2 \in V_1$, there is $\alpha'_1 \in \mathcal{N}$ such that $(\sigma_2, \alpha'_1) \in C_1 \cap (N_{w_0} \times N_{s^1_0})$. As $\sigma_2 \in V_1$, there is $\alpha''_0 \in \mathcal{N}$ such that $(\sigma_2, \alpha''_0) \in C_0 \cap (N_{w_1} \times N_{s^0_1})$. We set $w_2 := \sigma_2|3$, $s^0_2 := \alpha''_0|3$, $s^1_1 := \alpha'_1|2$, $s^2_0 := \alpha_2|1$ and $V_2 := \pi_0[C_0 \cap (N_{w_2} \times N_{s^0_2})] \cap \pi_0[C_1 \cap (N_{w_1} \times N_{s^1_1})] \cap \pi_0[C_2 \cap (N_{w_0} \times N_{s^2_0})]$. Here again, V_2 is τ_{GH}-open. Moreover, $\sigma_2 \in V_2 \subseteq U_2$ and Player 2 can play V_2.

If we go on like this, we build $w_l \in \omega^{l+1}$ and $s^n_l \in \omega^{<\omega}$ such that $w_0 \subseteq w_1 \subseteq \cdots$ and $s^n_0 \subsetneqq s^n_1 \subsetneqq \cdots$ This allows us to define $\sigma := \lim_{l\to\infty} w_l \in \mathcal{N}$ and, for each $n \in \omega$, $\alpha_n := \lim_{l\to\infty} s^n_l \in \mathcal{N}$. As (σ, α_n) is the limit of (w_l, s^n_l) as l goes to infinity and $N_{w_l} \times N_{s^n_l}$ meets C_n (which is closed in $\mathcal{N} \times \mathcal{N}$), $(\sigma, \alpha_n) \in C_n$. Thus $\sigma \in \bigcap_{n\in\omega} \pi_0[C_n] = \bigcap_{n\in\omega} L_n \subseteq \bigcap_{n\in\omega} U_n \subseteq \bigcap_{n\in\omega} V_n$, so that τ is winning for Player 2.

By Proposition 3.27, the result also holds in X.

(e) By Corollary 4.18, $\omega^x_1 \leq \omega^{CK}_1$ is equivalent to

$$\forall\alpha \in \Delta^1_1(x)\ \big(\alpha \in WO \Rightarrow \exists\beta, \gamma \in \mathcal{N}\ \beta \text{ is recursive and } \gamma \text{ is an order-preserving bijection from } (\omega, \leq_\alpha) \text{ onto } (\omega, \leq_\beta)\big),$$

which is Σ^1_1 by Proposition 4.14 and Theorem 4.21. This shows that X_{low} is Σ^1_1. By Theorem 5.4, X_{low} is τ_{GH}-dense.

(f) By definition, $S \cap X_{low}$ is τ_{GH}-open in X_{low}. Theorem 4.2 provides a Δ^1_1-recursive map $f : X \to \mathcal{N}$ such that $X \setminus (S \cap X_{low}) = f^{-1}(WO)$. We get

$$x \in X_{low} \setminus (S \cap X_{low}) \Leftrightarrow x \in X_{low} \wedge \exists\xi < \omega^{CK}_1\ f(x) \in WO \wedge |f(x)| \leq \xi.$$

This proves that $S \cap X_{low}$ is τ_{GH}-closed in X_{low}, by Theorem 4.3.

(g) By (f), our space is zero-dimensional, and thus regular. By (a), (b), (d) and Choquet's theorem, it is Polish. $\square$