Chapter 8-Applications: some dichotomy results

1 The Hurewicz dichotomy

We first prove the level one version of a general Louveau result, which is a strong form of the effective separation theorem. The general result is as follows.

Theorem 1.1 *(Louveau) Let $\xi \geq 1$ be a recursive ordinal, X be a Polish recursive space, and $A, B \subseteq X$ in Σ^1_1 be disjoint. We assume that A is separable from B by a $\mathbf{\Sigma}^0_\xi$ set. Then A is separable from B by a set which is $\mathbf{\Sigma}^0_\xi$ and Δ^1_1.*

Theorem 1.2 *(Louveau) Let X be a Polish recursive space, and $A, B \subseteq X$ in Σ^1_1 be disjoint. We assume that A is separable from B by an open set. Then there is $D \subseteq \omega$ in Δ^1_1 such that $\bigcup_{n \in D} N(X, n)$ separates A from B. In particular, A is separable from B by a set which is open and Δ^1_1.*

Proof. We define $P \subseteq X \times \omega$ by $P(x, n) \Leftrightarrow x \notin A \vee x \in N(X, n) \subseteq \neg B$. Note that P is in Π^1_1. Kreisel's easy uniformization theorem gives P^* in Π^1_1 uniformizing P. In particular, P^* is the graph of a partial function $f : X \to \omega$. As A is separable from B by an open set, f is in fact defined on X. Note that $f(x) = n \Leftrightarrow P^*(x, n) \Leftrightarrow \forall m \in \omega \ \ m = n \vee \neg P^*(x, m)$, so that f is Δ^1_1-recursive. Note that the Σ^1_1 set $f[X]$ is contained in the Π^1_1 set $\{n \in \omega \mid N(X, n) \subseteq \neg B\}$. As Σ^1_1 has the separation property, there is $D \subseteq \omega$ in Δ^1_1 such that $f[X] \subseteq D \subseteq \{n \in \omega \mid N(X, n) \subseteq \neg B\}$. The set D is as desired. $\square$

A generalization of the Hurewicz dichotomy is as follows.

Theorem 1.3 *(Louveau-Saint Raymond) Let $\xi \geq 1$ be a countable ordinal, $\mathbb{A}$ be a $\mathbf{\Sigma}^0_\xi$ subset of $\mathcal{C}$, $\mathbb{B} := \mathcal{C} \setminus \mathbb{A}$, X be a Polish space, and A, B be disjoint analytic subsets of X. Then one of the following holds:*

(a) A is separable from B by a $\mathbf{\Pi}^0_\xi$ set,

(b) there is $f : \mathcal{C} \to X$ continuous such that $\mathbb{A} \subseteq f^{-1}(A)$ and $\mathbb{B} \subseteq f^{-1}(B)$.

These authors obtained a further generalization.

Definition 1.4 *Let $\mathbf{\Gamma}$ be a class of subsets of zero-dimensional Polish spaces. We say that $\mathbf{\Gamma}$ is a* **Wadge class** *if there is $A \subseteq \omega^\omega$ which is $\mathbf{\Gamma}$-complete.*

The **Wadge hierarchy** is obtained by the inclusion of the Wadge classes. The Wadge hierarchy of Wadge classes of Borel sets is much finer than the hierarchy obtained by the inclusion of the non self-dual Borel classes $\mathbf{\Sigma}^0_\xi$ and $\mathbf{\Pi}^0_\xi$. It is the finest hierarchy of topological complexity considered in descriptive set theory.

Theorem 1.5 *(Louveau-Saint Raymond) Let $\mathbf{\Gamma}$ be a non self-dual Wadge class of Borel sets, $\mathbb{A}\in\mathbf{\Gamma}(\mathcal{C})$, $\mathbb{B}:=\mathcal{C}\setminus\mathbb{A}$, X be a zero-dimensional Polish space, and A,B be disjoint analytic subsets of X. Then one of the following holds:*

(a) A is separable from B by a $\check{\mathbf{\Gamma}}$ set,

(b) there is $f:\mathcal{C}\rightarrow X$ continuous such that $\mathbb{A}\subseteq f^{-1}(A)$ and $\mathbb{B}\subseteq f^{-1}(B)$.

Notation. Let $\mathbb{P}_f:=\{\alpha\in\mathcal{C}\mid\exists m\in\omega\ \ \forall n\geq m\ \ \alpha(n)=0\}$, and $\mathbb{P}_\infty:=\mathcal{C}\setminus\mathbb{P}_f$. The following is proved in [L-SR].

Theorem 1.6 *(Hurewicz) Let X be a Polish space, and A,B be disjoint analytic subsets of X. Then exactly one of the following holds:*

(a) A is separable from B by a $\mathbf{\Pi}^0_2$ set,

(b) there is $f:\mathcal{C}\rightarrow X$ one-to-one and continuous such that $\mathbb{P}_f\subseteq f^{-1}(A)$ and $\mathbb{P}_\infty\subseteq f^{-1}(B)$.

Proof. By Baire's theorem, $\mathbb{P}_f$ is not in $\mathbf{\Pi}^0_2$, so that (a) and (b) cannot hold simultaneously. In order to simplify the notation, by relativization, we may assume that X is recursively presented and that $A,B\in\Sigma^1_1$. Let τ_2 be the topology on X generated by the $\mathbf{\Pi}^0_1\cap\Sigma^1_1$ subsets of X, and $N:=\overline{A}^{\tau_2}\cap B$. Note that $\overline{A}^{\tau_2}\in\mathbf{\Pi}^0_2\cap\Sigma^1_1$. Indeed, by Theorem 1.2,

$$\begin{aligned} x\notin\overline{A}^{\tau_2} &\Leftrightarrow \exists C\in\mathbf{\Pi}^0_1\cap\Sigma^1_1\ \ x\in C \text{ et } C\cap A=\emptyset\\ &\Leftrightarrow \exists D\subseteq\omega \text{ in } \Delta^1_1\ \ x\notin\textstyle\bigcup_{n\in D}\ N(X,n)\wedge\forall y\in X\ \big(y\notin A\vee y\in\textstyle\bigcup_{n\in D}\ N(X,n)\big). \end{aligned}$$

and we are done, using the coding system for Δ^1_1 sets. Thus $N\in\Sigma^1_1$.

Case 1. $N=\emptyset$.

The set $\overline{A}^{\tau_2}$ is $\mathbf{\Pi}^0_2$ and separates A from B and (a) holds.

Case 2. $N\neq\emptyset$.

We set $D:=\big\{s\in 2^{<\omega}\mid s=\emptyset \text{ or } \big(s\neq\emptyset \text{ and } s(|s|-1)=1\big)\big\}$. Fix $s\in 2^{<\omega}$. We set $s^-:=s|(|s|-1)$ if $s\neq\emptyset$, and

$$s^0:=\begin{cases} s^- \text{ if } s\neq\emptyset \text{ and } s^-\notin D,\\ s \text{ otherwise,}\end{cases}$$

$$s^1:=\begin{cases} s \text{ if } s=\emptyset,\\ s|\max\{n<|s|\mid s|n\in D\} \text{ otherwise.}\end{cases}$$

We construct sequences

- $(x_s)_{s\in 2^{<\omega}}$ of points of X,
- $(O_s)_{s\in 2^{<\omega}}$ of Σ^0_1 subsets of X,
- $(U_s)_{s\in 2^{<\omega}}$ of Σ^1_1 subsets of X.

We want these objects to satisfy the following conditions:

(1) $\overline{O_{s\varepsilon}} \subseteq O_s$
(2) $x_s \in O_s \cap U_s$ et $U_s \subseteq X_{low}$
(3) $\text{diam}(O_s), \text{diam}_{GH}(U_s) \leq 2^{-|s|}$
(4) $O_{s0} \cap O_{s1} = \emptyset$
(5) $U_s \subseteq N \cap U_{s^1}$ if $s \in D$
(6) $U_s \subseteq A \cap U_{s^0} \cap \overline{U_{s^1}}$ if $s \notin D$

Assume that this is done. Fix $\alpha \in \mathcal{C}$. Note that $(\overline{O_{\alpha|n}})$ is a decreasing sequence of nonempty closed subsets of X with vanishing diameters, and thus defines $f(\alpha) \in X$ by

$$\{f(\alpha)\} := \bigcap_{n\in\omega} \overline{O_{\alpha|n}} = \bigcap_{n\in\omega} O_{\alpha|n}.$$

This defines a function $f : \mathcal{C} \to X$ such that $f(\alpha) = \lim_{n\to\infty} x_{\alpha|n}$. Note that f is one-to-one and continuous.

If $\alpha \in \mathbb{P}_f$, then $\alpha|n \notin D$ if $n \geq n_0$. Note that $(U_{\alpha|n})_{n\geq n_0}$ is a decreasing sequence of nonempty clopen subsets of (X_{low}, τ_{GH}) with vanishing diameters, and thus defines $F(\alpha) \in A$ by

$$\{F(\alpha)\} := \bigcap_{n\geq n_0} U_{\alpha|n}.$$

Moreover, $F(\alpha)$ is the limit, for τ_{GH} and thus for the initial topology of X, of $(x_{\alpha|n})_{n\in\omega}$. Therefore, $f(\alpha) = F(\alpha) \in A$.

If $\alpha \in \mathbb{P}_\infty$, then the sequence $(n_k)_{k\in\omega}$ of natural numbers for which $\alpha|n_k \in D$ is infinite. Note that $(U_{\alpha|n_k})_{k\in\omega}$ is a decreasing sequence of nonempty clopen subsets of (X_{low}, τ_{GH}) with vanishing diameters, and thus defines $F(\alpha) \in N$ by $\{F(\alpha)\} := \bigcap_{k\in\omega} U_{\alpha|n_k}$. Moreover, $F(\alpha)$ is the limit, for τ_{GH} and thus for the initial topology of X, of $(x_{\alpha|n_k})_{k\in\omega}$. Therefore, $f(\alpha) = F(\alpha) \in N \subseteq B$.

Let us show that the construction is possible. Let $x_\emptyset \in N \cap X_{low}$, which is nonempty since N is a nonempty Σ^1_1 subset of X. We choose $O_\emptyset \in \Sigma^0_1$ and $U_\emptyset \in \Sigma^1_1$ with small diameter containing $x_\emptyset$ with $U_\emptyset \subseteq N \cap X_{low}$. Suppose that $(x_s)_{|s|\leq l}$, $(O_s)_{|s|\leq l}$ and $(U_s)_{|s|\leq l}$ satisfying (1)-(5) have been constructed, which is the case for $l = 0$.

Let $s \in 2^l$. If $s \in D$, then we set $x_{s1} := x_s$. Note that $x_s \in \overline{A}^{\tau_2} \cap \overline{U_{s^1}} \cap O_s$. As $\overline{U_{s^1}} \in \Sigma^1_1$, $\overline{U_{s^1}} \cap O_s$ is τ_2-open. Therefore, there is $x_{s0} \in A \cap \overline{U_{s^1}} \cap O_s \cap X_{low}$. If $s \notin D$, then we set $x_{s0} := x_s$. As $x_s \in \overline{U_{s^1}} \cap O_s$, there is $x_{s1} \in U_{s^1} \cap O_s$. Note that $x_{s0} \in A$ and $x_{s1} \in N$, so that $x_{s0} \neq x_{s1}$.

We choose $O_{s0}, O_{s1} \in \Sigma^0_1$ disjoint with small diameter such that $x_{s\varepsilon} \in \overline{O_{s\varepsilon}} \subseteq O_s$, and $U_{s\varepsilon} \in \Sigma^1_1$ with small diameter containing $x_{s\varepsilon}$ such that $U_{s\varepsilon} \subseteq U_s$ if $s\varepsilon \in D \Leftrightarrow s \in D$, $U_{s0} \subseteq A \cap \overline{U_{s^1}} \cap X_{low}$ if $s \in D$, and $U_{s1} \subseteq U_{s^1}$ if $s \notin D$. Thus (b) holds. □

Exercise. (Kechris-Saint Raymond) Let X be a Polish space and A be an analytic subset of X. Prove that exactly one of the following holds: either there is a closed subset of X homeomorphic to $\mathcal{N}$ contained in A, or A is contained in a K_σ subset of X.

2 The Silver dichotomy

One of the main subject of research in descriptive set theory is the study of the complexity of classification problems in mathematics. A classification problem is given by a collection of objects X and an equivalence relation E on X. A **complete classification** of X up to E consists of a set of invariants I and a function $c : X \to I$ such that $xEy \Leftrightarrow c(x) = c(y)$. The theory of equivalence relations studies the set-theoretic nature of possible (complete) invariants and develops a mathematical framework for measuring the complexity of classification problems. In order to compare equivalence relations, we use the notion of reducibility. If E, F are equivalence relations on X, Y respectively, then a **reduction** of (X, E) to (Y, F) is a function $f : X \to Y$ such that $xEy \Leftrightarrow f(x)Ff(y)$. Intuitively this means that the classification problem represented by E is at most as complicated as that of F, and that the F-classes are complete invariants for E. Note that the reduction function f induces an injection from the quotient space X/E into Y/F. For this to be of any interest, I, c and f must be as explicit and concrete as possible. This is the reason why we are particularily interested in the case where I, X, Y are Polish, c, f are Borel, and E, F are Borel (or analytic). In this case, if there is a Borel reduction of (X, E) to (Y, F), then we write $(X, E) \leq_B (Y, F)$. If moreover f can be one-to-one and continuous, then we write $(X, E) \sqsubseteq_c (Y, F)$.

The proof of the following perfect set theorem for equivalence relations can be found in [G].

Theorem 2.1 *(Silver) Let X a Polish space, and E be a Borel equivalence relation on X. Then exactly one of the following holds:*

(a) E has countably many equivalence classes (i.e., $(X, E) \leq_B (\omega, =)$),

(b) there is $f : \mathcal{C} \to X$ one-to-one and continuous such that $\big(f(\alpha), f(\beta)\big) \notin E$ if $\alpha \neq \beta$ (i.e., $(\mathcal{C}, =) \sqsubseteq_c (X, E)$).

Proposition 2.2 *Let X be a Polish space, and E be an equivalence relation on X. If there is a nonempty open subset O of X such that E is meager on O^2, then there is $f : \mathcal{C} \to X$ one-to-one and continuous such that $\big(f(\alpha), f(\beta)\big) \notin E$ if $\alpha \neq \beta$.*

Proof. Note that O is a nonempty Polish space. Moreover, O is perfect since otherwise it has an isolated point x, and $E \cap O^2 \supseteq \{(x, x)\}$ cannot be meager. It remains to apply the Mycielski-Kuratowski theorem. $\square$

Corollary 2.3 *Let τ be the Gandy-Harrington topology on $\mathcal{N}$, and E be an equivalence relation on $\mathcal{N}$. If there is a nonempty Σ^1_1 subset V of $\mathcal{N}$ such that E is τ^2-meager on V^2, then there is $f : \mathcal{C} \to \mathcal{N}$ one-to-one and continuous such that $\big(f(\alpha), f(\beta)\big) \notin E$ if $\alpha \neq \beta$.*

Proof. By Gandy's basis theorem, $O := X_{low} \cap V$ is a nonempty τ-open subset of X_{low}. Now E is τ^2-meager on O^2, and we saw that (X_{low}, τ) is Polish. It remains to apply Proposition 2.2 since τ is finer than the usual topology on $\mathcal{N}$. $\square$

Proof of Theorem 2.1. We will see that Silver's theorem in fact holds for any co-analytic equivalence relation. Note first that (a) and (b) cannot hold simultaneously. In order to simplify the notation, by relativization, we may assume that X is recursively presented and that $E \in \Pi^1_1$.

We saw that there is $\pi:\mathcal{N}\rightarrow X$ recursive and onto. We set $E':=(\pi\times\pi)^{-1}(E)$. Note that E' is a Π^1_1 equivalence relation on $\mathcal{N}$. If E' has countably many equivalence classes, then E too. Assume that there is $f\!:\!\mathcal{C}\!\rightarrow\!\mathcal{N}$ one-to-one and continuous such that $\big(f(\alpha),f(\beta)\big)\!\notin\! E'$ if $\alpha\!\neq\!\beta$. Then $\pi\circ f\!:\!\mathcal{C}\!\rightarrow\! X$ is one-to-one since E is reflexive, continuous, and as desired. This shows that we may assume that $X\!=\!\mathcal{N}$ and $E\!\in\!\Pi^1_1$.

Let τ be the Gandy-Harrington topology on $\mathcal{N}$. We set

$$V:=\mathcal{N}\setminus\{\alpha\!\in\!\mathcal{N}\mid\exists U\!\in\!\Delta^1_1(\mathcal{N})\ \alpha\!\in\! U\!\subseteq\![\alpha]_E\}.$$

Case 1. $V\!=\!\emptyset$.

Every E-equivalence class contains a nonempty Δ^1_1 subset of $\mathcal{N}$. As there are only countably many Δ^1_1 subsets of $\mathcal{N}$, (a) holds.

Case 2. $V\!\neq\!\emptyset$.

Note that V is Σ^1_1. Indeed, $\alpha\!\in\! V\Leftrightarrow\forall U\!\in\!\Delta^1_1(\mathcal{N})\ \alpha\!\notin\! U\vee\exists\beta\!\in\! U\ (\alpha,\beta)\!\notin\! E$. Using the coding system for Δ^1_1 sets, we get

$$\alpha\!\in\! V\Leftrightarrow\forall n\!\in\!\omega\ \big((n\!\in\! C\wedge\alpha\!\in\! P^+_n)\Rightarrow\exists\beta\!\in\!\mathcal{N}\ (\beta\!\notin\! P^-_n\wedge(\alpha,\beta)\!\notin\! E)\big),$$

which shows that V is in Σ^1_1. In order to see that (b) holds, it is enough to see that E is τ^2-meager in V^2, by Corollary 2.3. We proceed in several steps.

We first check that for every $\alpha\in V$ there is no Σ^1_1 set U such that $\alpha\in U\subseteq[\alpha]_E$. We argue by contradiction, which gives α and U. Then note that $[\alpha]_E$ is Π^1_1 since

$$\beta\!\in\![\alpha]_E\Leftrightarrow\forall\gamma\!\in\!\mathcal{N}\ \gamma\!\notin\! U\vee(\beta,\gamma)\!\in\! E.$$

By the reduction property of Π^1_1, we can find $W,W'\subseteq\mathcal{N}$ disjoint in Π^1_1 such that $W\subseteq[\alpha]_E$, $W'\subseteq\neg U$, and $W\cup W'=[\alpha]_E\cup\neg U=\mathcal{N}$. This implies that W is in Δ^1_1 and $U\subseteq W$, and thus $\alpha\!\notin\! V$, a contradiction. From this it follows immediately that every nonempty Σ^1_1 set U contained in V meets more than one E-equivalence class.

We then note that E has the Baire property for τ^2, and each E-equivalence class has the Baire property for τ. By the Kuratowski-Ulam theorem, it is enough to see that, for all $\alpha\in V$, $[\alpha]_E$ is meager in V. Thus it suffices to show that, for all $\alpha\in V$, $[\alpha]_E$ is not τ-comeager in any $U\subseteq V$ which is Σ^1_1 and nonempty. We argue by contradiction, which gives α and U. It is enough to check that $[\alpha]^2_E$ is comeager in U^2 for the Gandy-Harrington topology of $\mathcal{N}^2$. Indeed, assume this. Since $U^2\!\setminus\! E$ is nonempty and Σ^1_1, $[\alpha]^2_E\cap U^2\!\setminus\! E$ is nonempty, which is absurd. It remains to note that the projections from U^2, equipped with the Gandy-Harrington topology of $\mathcal{N}^2$, onto U, equipped with the Gandy-Harrington topology of $\mathcal{N}$, are continuous and open. $\square$

We now consider a natural invariant for $\leq_B$, the notion of potential complexity.

Definition 2.4 *(Louveau) Let $\mathbf{\Gamma}$ be a Borel class or a Wadge class of Borel sets, X,Y be Polish spaces, and B be a Borel subset of $X\times Y$. We say that B is* **potentially in $\mathbf{\Gamma}$** *if we can find finer zero-dimensional Polish topologies σ and τ on X and Y respectively such that $B\in\mathbf{\Gamma}\big((X,\sigma)\times(Y,\tau)\big)$.*

One should emphasize the fact that the point of this definition is to consider product topologies. Indeed, if B is a Borel subset of a Polish space X, then there is a finer Polish topology τ on X such that B is a clopen subset of (X,τ). This is not the case in products: if for example $\mathbf{\Gamma}$ is a non self-dual Wadge class of Borel sets, then there are sets in $\mathbf{\Gamma}(\mathcal{N}^2)$ that are not in pot$(\check{\mathbf{\Gamma}})$. For example, the equality on $\mathcal{C}$ is not potentially open, since the potentially open sets are the countable unions of Borel rectangles. The notion of potential complexity is an invariant for $\leq_B$ in the sense that if $(X,E)\leq_B(Y,F)$ and $F\in$pot$(\mathbf{\Gamma})$, then $E\in$pot$(\mathbf{\Gamma})$ too.

Corollary 2.5 *Let $\mathbf{\Gamma}\in\{\mathbf{\Delta}^0_1,\mathbf{\Sigma}^0_1\}$, X be a Polish space, and E be a Borel equivalence relation on X. Then exactly one of the following holds.*

(a) E is potentially in $\mathbf{\Gamma}$,

(b) $(\mathcal{C},=)\sqsubseteq_c(X,E)$.

Proof. As the equality on $\mathcal{C}$ is not potentially open, (a) and (b) cannot hold simultaneously. If E has countably many Borel equivalence classes $(C_n)_{n\in I}$, then $E=\bigcup_{n\in I}\ C_n^2$ is a countable union of Borel rectangles, as well as $\neg E=\bigcup_{n\in I}\ C_n\times(\neg C_n)$, so that E is potentially clopen. It remains to apply Silver's theorem 2.1. $\square$

3 The $\mathbb{E}_0$-dichotomy

This next dichotomy characterizes when a Borel equivalence relation is potentially closed. The Borel equivalence relation just after the equality on $\mathcal{C}$ in $\leq_B$ is the relation on $\mathcal{C}$ defined by

$$\mathbb{E}_0:=\{(\alpha,\beta)\in\mathcal{C}^2\mid\exists m\in\omega\ \ \forall n\geq m\ \ \alpha(n)=\beta(n)\},$$

which is the version of the Vitali equivalence relation, on $\mathcal{C}$.

Theorem 3.1 *(Harrington-Kechris-Louveau) Let X a Polish space, and E be a Borel equivalence relation on X. Then exactly one of the following holds:*

(a) $(X,E)\leq_B(\mathcal{C},=)$,

(b) $(\mathcal{C},\mathbb{E}_0)\sqsubseteq_c(X,E)$.

Proof. For further use, we first prove that $\mathbb{E}_0$ is not potentially G_δ. We argue by contradiction, which gives a finer Polish topology τ on X such that $\mathbb{E}_0\in\mathbf{\Pi}^0_2\big((X,\tau)^2\big)$. The identity map from (X,τ) into X is continuous, so that its inverse is Borel. This gives a dense G_δ subset G of X on which the two topologies coincide. We define, for $s\in 2^{<\omega}$, $f_s:\mathcal{C}\rightarrow\mathcal{C}$ by $f(\alpha)(n):=1-\alpha(n)$ if $n<|s|$ and $s(n)=1$, $f(\alpha)(n):=\alpha(n)$ otherwise. Note that $(\alpha,\beta)\in\mathbb{E}_0$ if and only if there is $s\in 2^{<\omega}$ such that $\beta=f_s(\alpha)$. Moreover, f_s is a homeomorphism. In particular, $\bigcap_{s\in 2^{<\omega}}\ f_s^{-1}(G)$ is a dense G_δ subset of $\mathcal{C}$ contained in G and $\mathbb{E}_0$-invariant. So we may assume that G is $\mathbb{E}_0$-invariant. Note that $\mathbb{E}_0\cap G^2$ is a $\mathbf{\Pi}^0_2$ subset of G (for both topologies). Pick $\alpha\in G$. Then the equivalence class C of α is a $\mathbf{\Pi}^0_2$ subset of G and thus $\mathcal{C}$, and C is dense in $\mathcal{C}$. As C is countable, it is comeager in $\mathcal{C}$. But there is no comeager dense G_δ subset of $\mathcal{C}$, by Baire's theorem.

If (a) holds, then E is potentially closed and thus potentially G_δ. This and the previous point show that (a) and (b) cannot hold simultaneously.

In order to simplify the notation, by relativization, we may assume that X is recursively presented and that $E\in\Delta^1_1$. Let τ be the Gandy-Harrington topology on X. We set $\overline{E}:=\overline{E}^{\tau^2}$. If $A\subseteq X$, then $[A]_E:=\{x\in X \mid \exists y\in A\ \ (x,y)\in E\}$. We say that A is E-**invariant** if $A=[A]_E$.

Case 1. $\overline{E}=E$.

Claim 1. *Let $A,B\in\Sigma^1_1(X)$ with $[A]_E\cap[B]_E=\emptyset$. Then there is $C\in\Delta^1_1(X)$ which is E-invariant and separates A from B.*

Indeed, note that $[A]_E,[B]_E$ are Σ^1_1. The separation theorem gives $C_0\in\Delta^1_1(X)$ separating $[A]_E$ from $[B]_E$. As $[C_0]_E$ is Σ^1_1 and disjoint from $[B]_E$, the separation theorem gives $C_1\in\Delta^1_1(X)$ separating $[C_0]_E$ from $[B]_E$. Continuing like this, we get a sequence (C_n) of Δ^1_1 subsets of X with $[C_n]_E\subseteq C_{n+1}\subseteq\neg[B]_E$. Note that $C:=\bigcup_{n\in\omega}C_n$ is E-invariant and separates A from B. The problem is that C is not necessarily Δ^1_1 since this class is not closed under countable unions. So we have to make the construction uniformly to solve this problem.

Let $\mathcal{U}^{\omega^2\times X}\in\Sigma^1_1(\omega^3\times X)$ be ω-universal for $\Sigma^1_1(\omega^2\times X)$. Note that

$$\mathcal{U}^{\omega\times X}:=\big\{(e,n,x)\in\omega^2\times X \mid \big((e)_0,(e)_1,n,x\big)\in\mathcal{U}^{\omega^2\times X}\big\}$$

is ω-universal for $\Sigma^1_1(\omega\times X)$. Similarly, $\mathcal{U}^X:=\big\{(e,x)\in\omega\times X \mid \big((e)_0,(e)_1,x\big)\in\mathcal{U}^{\omega\times X}\big\}$ is ω-universal for $\Sigma^1_1(X)$. We define subsets of $\omega^2\times X$, P_0 and P_1, by $P_0(m,n,x)\Leftrightarrow(m,x)\notin\mathcal{U}^X$ and $P_1(m,n,x)\Leftrightarrow(n,x)\notin\mathcal{U}^X$. Note that P_0,P_1 are Π^1_1. The reduction property of Π^1_1 provides $P^*_0,P^*_1\in\Pi^1_1(\omega^2\times X)$ disjoint with $P^*_\varepsilon\subseteq P_\varepsilon$ and $P^*_0\cup P^*_1=P_0\cup P_1$. Let $e_\varepsilon\in\omega$ with $\neg P^*_\varepsilon=\mathcal{U}^{\omega^2\times X}_{e_\varepsilon}$. We define $f,g:\omega^2\rightarrow\omega$ by $f(m,n):=\big\langle <e_0,m>,n\big\rangle$ and $g(m,n):=\big\langle <e_1,m>,n\big\rangle$. Note that f,g are recursive. Assume that $\mathcal{U}^X_m$ and $\mathcal{U}^X_n$ are disjoint. Then $(\neg\mathcal{U}^X_m)\cup(\neg\mathcal{U}^X_n)=X$,

$$(P^*_0)_{m,n}\cup(P^*_1)_{m,n}=(P_0)_{m,n}\cup(P_1)_{m,n}=X,$$

$(\mathcal{U}^{\omega^2\times X}_{e_0})_{m,n}\cap(\mathcal{U}^{\omega^2\times X}_{e_1})_{m,n}=(\neg P^*_0)_{m,n}\cap(\neg P^*_1)_{m,n}=\emptyset$. Note that

$$x\in(\mathcal{U}^{\omega^2\times X}_{e_0})_{m,n}\Leftrightarrow(e_0,m,n,x)\in\mathcal{U}^{\omega^2\times X}\Leftrightarrow(<e_0,m>,n,x)\in\mathcal{U}^{\omega\times X}\Leftrightarrow\big(f(m,n),x\big)\in\mathcal{U}^X$$

and, similarly, $x\in(\mathcal{U}^{\omega^2\times X}_{e_1})_{m,n}\Leftrightarrow\big(g(m,n),x\big)\in\mathcal{U}^X$, so that $\mathcal{U}^X_{f(m,n)},\mathcal{U}^X_{g(m,n)}$ are disjoint. Moreover, $\mathcal{U}^X_m\subseteq\mathcal{U}^X_{f(m,n)}$, $\mathcal{U}^X_n\subseteq\mathcal{U}^X_{g(m,n)}$, and $\mathcal{U}^X_{g(m,n)}=\neg\mathcal{U}^X_{f(m,n)}$. This shows that $\mathcal{U}^X_{f(m,n)}\in\Delta^1_1(X)$ separates $\mathcal{U}^X_m$ from $\mathcal{U}^X_n$. In other words, the separation theorem is uniform.

We now check that the map $A\mapsto[A]_E$ is uniform, for Σ^1_1 sets A. In order to see this, consider $B:=\{(n,x)\in\omega\times X \mid \exists y\in\mathcal{U}^X_n\ \ (x,y)\in E\}$. As B is Σ^1_1, there is $e\in\omega$ with $B=\mathcal{U}^{\omega\times X}_e$. Thus $[\mathcal{U}^X_n]_E=\mathcal{U}^X_{h(n)}$, where h is recursive defined by $h(n):=<e,n>$.

Fix now a Σ^1_1 code p_0 for $[B]_E$, and pick $C_0\in\Delta^1_1(X)$ separating $[A]_E$ from $[B]_E$, and Σ^1_1 codes m_0 and n_0 for C_0 and $\neg C_0$ respectively. We inductively define $k\omega\rightarrow\omega$ by $k(0):=m_0$ and $k(n+1):=f\Big(h\big(k(n)\big),p_0\Big)$.

We also set $k'(0):=n_0$ and $k'(n+1):=g\Big(h\big(k(n)\big),p_0\Big)$. Then $k(n),k'(n)$ are Σ^1_1 codes of C_n and $\neg C_n$, for a sequence (C_n) as above. Finally, note that

$$x\in C \Leftrightarrow \exists n\in\omega\ \ x\in\mathcal{U}^X_{k(n)} \Leftrightarrow \forall n\in\omega\ \ x\notin\mathcal{U}^X_{k'(n)},$$

so that $C\in\Delta^1_1$ as desired. $\diamond$

If now $(x,y)\notin\overline{E}$, then there are $A,B\in\Sigma^1_1$ with $x\in A$, $y\in B$, and $(A\times B)\cap E=\emptyset$, so that $[A]_E,[B]_E$ are disjoint. Claim 1 provides $C\in\Delta^1_1(X)$ which is E-invariant and separates A from B. In particular, $x\in C$ and $y\notin C$. Let (I_n) be an enumeration of the E-invariant Δ^1_1 subsets of X. We define $f:X\to\mathcal{C}$ by $f(x)(n):=1$ if $x\in I_n$, 0 otherwise. Then f is Borel and

$$(x,y)\in E \Leftrightarrow f(x)=f(y).$$

Case 2. $\overline{E}\neq E$.

Let us check that $\overline{E}$ is Σ^1_1. In order to do this, we use the coding of Δ^1_1 sets we met. Recall that there are $C\subseteq\omega$ and $P^+,P^-\subseteq\omega\times X$ in Π^1_1 such that

(a) for any $n\in C$, P^+_n and P^-_n are complements of each other,
(b) for any $A\subseteq X$ in Δ^1_1 there is $n\in C$ such that $A=P^+_n$.

The previous arguments show that

$$\begin{aligned}(x,y)\notin\overline{E} &\Leftrightarrow \exists A\in\Delta^1_1(X)\ \ A \text{ is } E\text{-invariant} \wedge x\in A\wedge y\notin A\\ &\Leftrightarrow \exists n\in C\ \ P^+_n \text{ is } E\text{-invariant} \wedge x\in P^+_n\wedge y\in P^-_n\\ &\Leftrightarrow \exists n\in C\ \ \Big(\forall z,t\in X\ \ \big(z\notin P^-_n\wedge (z,t)\in E\big)\Rightarrow t\in P^+_n\Big)\wedge x\in P^+_n\wedge y\in P^-_n,\end{aligned}$$

so we are done.

We set $Y:=\{x\in X\mid E_x\neq\overline{E}_x\}$. As $\overline{E}$ is Σ^1_1 and $\overline{E}\neq E$, Y is a nonempty Σ^1_1 subset of X.

Claim 2. *We equip $\overline{E}$ with the topology induced by τ^2. Then $E\cap Y^2$ is dense and meager in $\overline{E}\cap Y^2$.*

Indeed, the density comes from the fact that Y is τ-open. As E is Borel for the usual topology, it is also Borel for τ^2. Thus $E\cap Y^2$ is Borel in $(\overline{E}\cap Y^2,\tau^2)$. In this space, $E\cap Y^2$ has the Baire property. We argue by contradiction, which gives $A,B\in\Sigma^1_1(X)$ such that $A,B\subseteq Y$, $\overline{E}\cap(A\times B)\neq\emptyset$, and $E\cap(A\times B)$ is comeager in $\overline{E}\cap(A\times B)$. By considering if necessary the two projections of $\overline{E}\cap(A\times B)$, we may assume that $\forall x\in A\ \ \exists y\in B\ \ (x,y)\in\overline{E}$ and $\forall x\in B\ \ \exists y\in A\ \ (x,y)\in\overline{E}$.

Let us prove that $\overline{E}\cap A^2\subseteq E$. We argue by contradiction. We set

$$\overline{E}^3:=\{(x,y,z)\in X^3\mid (x,z),(y,z)\in\overline{E}\}.$$

We equip $\overline{E}^3$ with the topology induced by $\tau_2\times\tau$, where τ_2 is the Gandy-Harrington topology on X^2. By Claim 1, $\overline{E}=\bigcap\{C^2\cup(\neg C)^2\mid C\in\Delta^1_1(X) \text{ is } E\text{-invariant}\}$. In particular, $\overline{E}$ is a G_δ equivalence relation on X, for τ^2. As the projections $(X^3,\tau_2\times\tau)\to(X^2,\tau)$ are continuous, $\overline{E}^3$ is a nonempty G_δ subset of $(X^3,\tau_2\times\tau)$. In particular, $\overline{E}^3$ is a strong Choquet space, and in particular a Baire space.

We set $Z:=\{(x,y,z)\in\overline{E}^3 \mid x,y\in A\wedge z\in B\}$. Note that Z is a nonempty open subset of $\overline{E}^3$. The two projections previously considered are also open. As $E\cap(A\times B)$ is comeager in $\overline{E}\cap(A\times B)$, $Z_1:=\{(x,y,z)\in Z \mid (x,z)\in E\}$ and $Z_2:=\{(x,y,z)\in Z \mid (y,z)\in E\}$ are comeager in Z for $\tau_2\times\tau$. Now note that $Z_3:=\{(x,y,z)\in Z \mid (x,y)\notin E\}$ is nonempty, and open for $\tau_2\times\tau$. By Baire's theorem, Z_3 has to meet $Z_1\cap Z_2$, which contradicts the transitivity of E.

Note now that $\overline{E}\cap[A]^2_E\subseteq E$. Indeed, if $(x,y)\in\overline{E}\cap[A]^2_E$, then pick $z,t\in A$ with $(x,z)\in E$ and $(y,t)\in E$. Note that $(z,t)\in\overline{E}$ which is an equivalence relation, and $(z,t)\in$since $\overline{E}\cap A^2\subseteq E$. Thus $(x,y)\in E$, by transitivity of E. This implies that $[A]_E=[A]_{\overline{E}}$. Indeed, we argue by contradiction to see that. We set $A':=\{x\in X \mid \exists y\in[A]_E\ \ (x,y)\in\overline{E}\wedge(x,y)\notin E\}$. Then A' is a nonempty Σ^1_1 set, and $\overline{E}\cap(A'\times[A]_E)$ is not empty. By density, $E\cap(A'\times[A]_E)$ is not empty. This gives $x\in A'$ and $z\in[A]_E$ with $(x,z)\in E$. As $x\in A'$, there is $y\in[A]_E$ with $(x,y)\in\overline{E}$ and $(x,y)\notin E$. But then $y,z\in[A]_E$ and $(y,z)\in\overline{E}$, so that $(y,z)\in E$, which contradicts the transitivity of E. If now $x\in A$ and $(x,y)\in\overline{E}$, then $y\in[A]_E$, and as $\overline{E}\cap[A]^2_E\subseteq E$, $(x,y)\in E$. Thus $E_x=\overline{E}_x$, which contradicts the fact that $A\subseteq Y$. $\diamond$

By Claim 2, there is a decreasing sequence (W_n) of τ^2-open subsets of X^2 such that $W_n\subseteq Y^2$, $\overline{E}\cap W_n$ is dense in $\overline{E}\cap Y^2$, and $E\cap(\bigcap_{n\in\omega}\ W_n)=\emptyset$. Moreover, since the diagonal

$$\Delta(X):=\{(x,x) \mid x\in X\}$$

is contained in E and τ^2-closed, we may assume that $\Delta(X)$ does not meet W_0. We construct sequences

- $(x_s)_{s\in 2^{<\omega}\setminus\{\emptyset\}}$ of points of X,
- $(U_s)_{s\in 2^{<\omega}\setminus\{\emptyset\}}$ of Σ^1_1 subsets of X,
- $(E_{k,s})_{k\in\omega,s\in 2^{<\omega}}$ of Σ^1_1 subsets of X^2.

We want these objects to satisfy the following conditions:

(1) $E_{k,\emptyset}=E\cap(X^2)_{low}$
(2) $U_{si}\subseteq U_s\subseteq Y\cap X_{low}\wedge E_{k,si}\subseteq E_{k,s}$
(3) $x_s\in U_s\wedge(x_{0^k0s},x_{0^k1s})\in E_{k,s}$
(4) $\text{diam}_{GH}(U_s),\text{diam}_{GH}(E_{k,s})\leq 2^{-|s|}$
(5) $(x_s,x_t)\in\overline{E}$ if $|s|=|t|$
(6) $U_s\times U_t\subseteq W_{|s|}$ if $|s|=|t|\wedge s(|s|-1)<t(|s|-1)$

Assume that this is done. Fix $\alpha\in\mathcal{C}$. Note that $(\overline{U_{\alpha|n}})$ is a decreasing sequence of nonempty clopen subsets of X_{low} with vanishing GH-diameters, and thus defines $f(\alpha)\in X$ by $\{f(\alpha)\}:=\bigcap_{n\in\omega}\ U_{\alpha|n}$. This defines a function $f:\mathcal{C}\to X$ such that $f(\alpha)=\lim_{n\to\infty}x_{\alpha|n}$. Note that f is continuous (for (X,τ), and thus for X). The map f is also injective since if $\alpha\neq\beta$, then there is n with $\alpha(n)\neq\beta(n)$, and $\big(f(\alpha),f(\beta)\big)\in U_{\alpha|(n+1)}\times U_{\beta|(n+1)}\subseteq W_{n+1}\subseteq W_0\subseteq\neg\Delta(X)$. If $(\alpha,\beta)\notin\mathbb{E}_0$, then there is $(n_k)_{k\in\omega}$ strictly increasing such that $\alpha(n_k)\neq\beta(n_k)$ for each k. Again, $\big(f(\alpha),f(\beta)\big)\in\bigcap_{k\in\omega}\ W_{n_k+1}$, so that $\big(f(\alpha),f(\beta)\big)\notin E$. If now $(\alpha,\beta)\in\mathbb{E}_0$, then we can find $k\in\omega$, $s,t\in 2^k$, and $\gamma\in 2^\omega$ with $(\alpha,\beta)=(s\gamma,t\gamma)$. We prove that $\big(f(s\gamma),f(t\gamma)\big)\in E$ by induction on k, the case $k=0$ being clear.

So assume that $s_1,t_1\in 2^k$, $i,j\in 2$ and $(s,t)=(s_1i,t_1j)$. If $i=j$, then we are done by induction assumption. Assume for example that $i<j$. The induction assumption ensures that $\big(f(s_10\gamma),f(0^k0\gamma)\big)\in E$ and $\big(f(t_11\gamma),f(0^k1\gamma)\big)\in E$. So it is enough to show that

$$\big(f(0^k0\gamma),f(0^k1\gamma)\big)\in E.$$

Note that $(E_{k,\gamma|l})_{l\in\omega}$ is a decreasing sequence of nonempty clopen subsets of $(X^2)_{low}$ with vanishing GH-diameters, hence $\bigcap_{l\in\omega}\ E_{k,\gamma|l}$ consists of a single point, which must be $\big(f(0^k0\gamma),f(0^k1\gamma)\big)$. As $E_{k,\emptyset}=E\cap (X^2)_{low}$, $\big(f(0^k0\gamma),f(0^k1\gamma)\big)\in E$.

Let us show that the construction is possible. As $\overline{E}\cap W_0$ is dense open in $\overline{E}\cap Y^2$ and $E\cap Y^2$ is dense in $\overline{E}\cap Y^2$, $E\cap W_0\cap Y^2$ is not empty, as well as $E\cap W_0\cap Y^2\cap (X^2)_{low}$. We choose $(x_0,x_1)\in E\cap W_0\cap Y^2\cap (X^2)_{low}$. As $(X^2)_{low}\subseteq X^2_{low}$, $x_0,x_1\in Y\cap X_{low}$. We choose Σ^1_1 sets U_0,U_1 with GH-diameter at most 2^{-1} such that $x_\varepsilon\in U_\varepsilon\subseteq Y\cap X_{low}$ and $U_0\times U_1\subseteq W_1$. Assume that $(x_s)_{|s|\leq l}$, $(U_s)_{|s|\leq l}$ and $(E_{k,s})_{k+1+|s|\leq l}$ satisfying (1)-(6) have been constructed, which is the case for $l=1$.

We set $\overline{E}^{2^l}:=\{(y_s)_{s\in 2^l}\mid \forall s,t\in 2^l\ \ (y_s,y_t)\in\overline{E}\}$. Consider the space $\overline{E}^{2^l}\times\overline{E}^{2^l}$, with the product topology $\tau_{2^l}\times\tau_{2^l}$ of the Gandy-Harrington topology τ_{2^l} on X^{2^l}. A typical element of $\overline{E}^{2^l}\times\overline{E}^{2^l}$ is denoted by $\overline{y}=\big((y_{s0})_{s\in 2^l},(y_{s1})_{s\in 2^l}\big)$. We set

$$\begin{aligned}Y_l:=\{\overline{y}\in\overline{E}^{2^l}\times\overline{E}^{2^l}\mid \forall s\in 2^l\ y_{s0},y_{s1}\in U_s\ \wedge\ \forall k,m\leq l\ \forall s\in 2^m\\ k+1+m=l\Rightarrow (y_{0^k0u0},y_{0^k1u0}),(y_{0^k0u1},y_{0^k1u1})\in E_{k,s}\}.\end{aligned}$$

Note that Y_l is an open subset of $\overline{E}^{2^l}\times\overline{E}^{2^l}$ containing $\big((x_s)_{s\in 2^l},(x_s)_{s\in 2^l}\big)$, by the induction assumption. Note then that, for each $s,t\in 2^l$, the projection map $\overline{y}\mapsto (y_{s0},y_{t1})$ from

$$\{\overline{y}\in Y_l\mid (y_{0^l0},y_{0^l1})\in\overline{E}\}$$

into $\overline{E}\cap Y^2$ is continuous and open. As $\overline{E}\cap W_{l+1}$ is dense in $\overline{E}\cap Y^2$, the set

$$V_l:=\{\overline{y}\in Y_l\mid \forall s,t\in 2^l\ \ (y_{s0},y_{t1})\in W_{l+1}\}$$

is open and nonempty. But then the set $V_l^*:=\{(y_{0^l0},y_{0^l1})\in\overline{E}\cap Y^2\mid \exists (y_{s0},y_{s1})_{s\in 2^l\setminus\{0^l\}}\ \ \overline{y}\in V_l\}$ is open and nonempty, hence meets E. In other words, we can find a family $(x_{si})_{s\in 2^l,i\in 2}$ such that

(a) $\forall s,t\in 2^l\ \ \forall i,j\in 2\ \ (x_{si},x_{tj})\in\overline{E}$
(b) $\forall s\in 2^l\ \ \forall i\in 2\ \ x_{si}\in U_s$
(c) $\forall k,m\leq l\ \ \forall s\in 2^m\ \ \forall i\in 2\ \ (x_{0^k0si},x_{0^k1si})\in E_{k,s}$ if $k+1+m=l$
(d) $\forall s,t\in 2^l\ \ (x_{s0},x_{t1})\in W_{l+1}$
(e) $(x_{0^l0},x_{0^l1})\in E$

As W_{l+1} is τ^2-open, we can find neighborhoods of small diameter U_{si} of x_{si} such that $U_{si}\subseteq U_s$, and $U_{s0}\times U_{t1}\subseteq W_{l+1}$ for all $s,t\in 2^l$. We can choose, for $k,m\leq l$ and $s\in 2^m$ with $k+1+m=l$, $E_{k,si}\in\Sigma^1_1$ of small GH-diameter with $(x_{0^k0si},x_{0^k1si})\in E_{k,si}\subseteq E_{k,s}$. Finally, we set $E_{l,\emptyset}:=E\cap (X^2)_{low}$. $\square$

Corollary 3.2 *Let $\mathbf{\Gamma}\in\{\mathbf{\Pi}^0_1,\mathbf{\Delta}^0_2,\mathbf{\Pi}^0_2\}$, X be a Polish space, and E be a Borel equivalence relation on X. Then exactly one of the following holds.*

(a) E is potentially in $\mathbf{\Gamma}$,

(b) $(\mathcal{C},\mathbb{E}_0)\sqsubseteq_c(X,E)$.

Proof. As $\mathbb{E}_0$ is not potentially G_δ, (a) and (b) cannot hold simultaneously. If $(X,E)\leq_B(\mathcal{C},=)$, then E is potentially closed. It remains to apply the Harrington-Kechris-Louveau theorem 3.1. $\square$

This kind of result cannot be extended to higher classes.

Theorem 3.3 *(Clemens-Lecomte-Miller) Let $\mathbf{\Gamma}$ be a Borel class containing $\mathbf{\Sigma}^0_2$. Then there is no Borel equivalence relation $\mathbb{E}$ on a Polish space $\mathbb{X}$ such that, for any Borel equivalence relation E on a Polish space X, exactly one of the following holds:*

(a) E is in potentially in $\mathbf{\Gamma}$,

(b) $(\mathbb{X},\mathbb{E})\sqsubseteq_c(X,E)$ (or even $(\mathbb{X},\mathbb{E})\leq_B(X,E)$).

4 The $\mathbb{G}_0$-dichotomy

Another important subject of research in descriptive set theory is the study of the analytic digraphs on Polish spaces. Recall that if X is a set, then the **diagonal** of X is $\Delta(X):=\{(x,x)\mid x\in X\}$. A binary relation on X is a **digraph** if it does not meet $\Delta(X)$. If A,B are digraphs on X,Y respectively, then a **homomorphism** of (X,A) to (Y,B) is a function $f:X\to Y$ such that $xAy\Rightarrow f(x)Bf(y)$. A **coloring** from (X,A) into some set Y is a map $c:X\to Y$ such that $c(x)\neq c(x')$ if $(x,x')\in A$, i.e., a homomorphism from (X,A) into $(Y,\neq)$. The study of definable colorings of analytic graphs was initiated in [K-S-T]. The Borel chromatic number of a digraph A on a Polish space X is the smallest cardinality of a Polish space Y for which there is a Borel coloring from (X,A) into Y. If there is a Borel homomorphism from (X,A) into (Y,B), then we write $(X,A)\preceq_B(Y,B)$. If moreover f can be continuous, then we write $(X,A)\preceq_c(Y,B)$.

Notation. Let $\psi:\omega\to 2^{<\omega}$ be a natural bijection ($\psi(0)=\emptyset$, $\psi(1)=0$, $\psi(2)=1$, $\psi(3)=0^2$, $\psi(4)=01$, $\psi(5)=10$, $\psi(6)=1^2$, $\ldots$). Note that $|\psi(n)|\leq n$, so that we can define $s_n:=\psi(n)0^{n-|\psi(n)|}$. Some crucial properties of (s_n) are that it is **dense** (for each $s\in 2^{<\omega}$, there is n such that $s\subseteq s_n$), and that $|s_n|=n$. We set $\mathbb{G}_0:=\{(s_n0\gamma,s_n1\gamma)\mid n\in\omega\wedge\gamma\in\mathcal{C}\}$, which was introduced in [K-S-T] where the following is proved.

Theorem 4.1 *(Kechris, Solecki, Todorčević) Let X be a Polish space and $A\subseteq X^2$ be analytic. Then exactly one of the following holds:*

(a) there is $c:X\to\omega$ Borel such that $c(x)\neq c(y)$ if $(x,y)\in A$ (i.e., $(X,A)\preceq_B(\omega,\neq)$),

(b) there is $f:\mathcal{C}\to X$ continuous such that $\big(f(\alpha),f(\beta)\big)\in A$ if $(\alpha,\beta)\in\mathbb{G}_0$ (which means that $(\mathcal{C},\mathbb{G}_0)\preceq_c(X,A)$).

Proof. Note first that we cannot have (a) and (b) simultaneously. Indeed, we argue by contradiction. This gives $g:\mathcal{C}\to\omega$ Borel such that $g(\alpha)\neq g(\beta)$ if $(\alpha,\beta)\in\mathbb{G}_0$. Let i_0 be a natural number such that $G:=g^{-1}(\{i_0\})$ is not meager, and $s\in 2^{<\omega}$ such that $N_s\setminus G$ is meager.

Let H be a dense G_δ subset of $\mathcal{C}$ such that $H\cap N_s\subseteq G$. We choose $n\in\omega$ with $s\subseteq s_n$. Note that $h_n:N_{s_n0}\to N_{s_n1}$ defined by $h_n(s_n0\gamma):=s_n1\gamma$ is a homeomorphism. This implies that $H\cap h_n^{-1}(H)$ is a dense G_δ subset of N_{s_n0}. We choose $s_n0\gamma\in H\cap h_n^{-1}(H)$. We get

$$(s_n0\gamma, s_n1\gamma)\in\mathbb{G}_0\cap(H\cap N_s)^2\subseteq G^2,$$

which contradicts the definition of g.

In order to simplify the notation, by relativization, we may assume that X is recursively presented and that $A\in\Sigma^1_1$. We say that $S\subseteq X$ is A**-discrete** if $A\cap S^2=\emptyset$. We put

$$U:=\{D\in\Delta^1_1(X)\mid D\text{ is }A\text{-discrete}\}.$$

Note that $U\subseteq X$ is in Π^1_1 since, using the coding system for Δ^1_1 sets,

$$U(x)\Leftrightarrow\exists n\in\omega\ \ n\in C\wedge P^+_n(x)\wedge\forall(y,z)\in X^2\ \big((y,z)\notin A\vee y\in P^-_n\vee z\in P^-_n\big).$$

Case 1. $U=X$.

There is a partition (D_n) of X into A-discrete Δ^1_1 sets. We define a function $c:X\to\omega$ by $c(x)=n\Leftrightarrow x\in D_n$, so that c is Borel. If $(x,y)\in A$, then we cannot have $c(x)=c(y)$ since the D_n's are A-discrete.

Case 2. $U\neq X$.

We set $Y:=X\setminus U$, so that Y is a nonempty Σ^1_1 subset of X. We set

$$\Phi:=\{S\subseteq X\mid S\text{ is }A\text{-discrete}\}.$$

As Φ is Π^1_1 on Σ^1_1, the reflection theorem ensures that if $S\subseteq X$ is in Σ^1_1 and A-discrete, then there is $D\subseteq X$ in Δ^1_1 which is A-discrete and contains S. This gives the following key property:

$$\forall S\in\Sigma^1_1(X)\quad(\emptyset\neq S\subseteq Y\ \Rightarrow\ A\cap S^2\neq\emptyset).$$

We construct sequences

- $(x_s)_{s\in 2^{<\omega}}$ of points of Y,
- $(V_s)_{s\in 2^{<\omega}}$ of Σ^1_1 subsets of X,
- $(U_{n,t})_{(n,t)\in\omega\times 2^{<\omega}}$ of Σ^1_1 subsets of X^2.

We want these objects to satisfy the following conditions:

(1) $x_s\in V_s\subseteq Y\cap X_{low}$ and $(x_{s_n0t},x_{s_n1t})\in U_{n,t}\subseteq A\cap Y^2\cap(X^2)_{low}$,

(2) $V_{sm}\subseteq V_s$ and $U_{n,tm}\subseteq U_{n,t}$,

(3) $\text{diam}_{GH}(V_s)\leq 2^{-|s|}$ and $\text{diam}_{GH}(U_{n,t})\leq 2^{-n-1-|t|}$.

Assume that this is done. Fix $\alpha \in \mathcal{C}$. Then $(V_{\alpha|p})$ is a decreasing sequence of nonempty clopen subsets of (X_{low}, τ_{GH}) with vanishing diameters, so there is $f(\alpha)$ in their intersection. This defines $f:\mathcal{C}\rightarrow X$. Note that $d_{GH}\big(x_{\alpha|p}, f(\alpha)\big) \leq \text{diam}_{GH}(V_{\alpha|p}) \leq 2^{-p}$, so that f is continuous and $(x_{\alpha|p})$ tends to $f(\alpha)$ for τ_{GH}.

If $(s_n0\gamma, s_n1\gamma) \in \mathbb{G}_0$, then $(U_{n,\gamma|p})_{p\in\omega}$ is a decreasing sequence of nonempty clopen subsets of $\big((X^2)_{low}, \tau_{GH}\big)$ with vanishing diameters, so there is (x,y) in their intersection. Note that (x,y) is in A. Moreover, the sequence $\big((x_{s_n0(\gamma|p)}, x_{s_n1(\gamma|p)})\big)_{p\in\omega}$ tends to (x,y) for τ_{GH}, and for τ_{GH}^2 too. As $(x_{s_n\varepsilon(\gamma|p)})_{p\in\omega}$ tends to $f(s_n\varepsilon\gamma)$ for each $\varepsilon \in 2$, we get $f(s_n0\gamma) = x$ and $f(s_n1\gamma) = y$. Thus $\big(f(s_n0\gamma), f(s_n1\gamma)\big) \in A$.

So it is enough to see that the construction is possible. As Y is a nonempty Σ_1^1 subset of X, we can choose $x_\emptyset \in Y \cap X_{low}$, and $V_\emptyset \subseteq X$ in Σ_1^1 such that $x_\emptyset \in V_\emptyset \subseteq Y \cap X_{low}$ and $\text{diam}_{GH}(V_\emptyset) \leq 1$. Assume that $(x_s)_{|s|\leq l}$, $(V_s)_{|s|\leq l}$ and $(U_{n,t})_{n+1+|t|\leq l}$ satisfying (1)-(3) have been constructed, which is the case for $l=0$. Let S be the following set:

$$\{x\in X \mid \exists (\underline{x_s})_{s\in 2^l} \in X^{2^l}\ \ \underline{x_{s_l}} = x \wedge \forall s\in 2^l\ \underline{x_s} \in V_s \wedge \forall n<l\ \ \forall t\in 2^{l-n-1}\ (\underline{x_{s_n0t}}, \underline{x_{s_n1t}}) \in U_{n,t}\}.$$

Then $S\in \Sigma_1^1(X)$, $x_{s_l} \in S \subseteq Y$ by induction assumption. So there is (x_{s_l0}, x_{s_l1}) in $A\cap S^2 \cap (X^2)_{low}$, by the key property. As $x_{s_l\varepsilon} \in S$, we get $(x_{s\varepsilon})_{s\in 2^l\setminus\{s_l\}}$. It remains to choose

- $V_{s\varepsilon} \subseteq X$ in Σ_1^1 with $x_{s\varepsilon} \in V_{s\varepsilon} \subseteq V_s$ and $\text{diam}_{GH}(V_{s\varepsilon}) \leq 2^{-l-1}$, for $s\in 2^l$ and $\varepsilon \in 2$.

- $U_{l,\emptyset} \subseteq X^2$ in Σ_1^1 with $(x_{s_l0}, x_{s_l1}) \in U_{l,\emptyset} \subseteq A\cap Y^2 \cap (X^2)_{low}$ and $\text{diam}_{GH}(U_{l,\emptyset}) \leq 2^{-l-1}$.

- $U_{n,t\varepsilon} \in \Sigma_1^1(X^d)$ with $(x_{s_n0t\varepsilon}, x_{s_n1t\varepsilon}) \in U_{n,t\varepsilon} \subseteq U_{n,t}$ and $\text{diam}_{GH}(U_{n,t\varepsilon}) \leq 2^{-l-1}$, for (n,t) in $\omega \times 2^{<\omega}$ with $n+1+|t| = l$ and $\varepsilon \in 2$. $\square$

Problem. (Miller) Use the $\mathbb{G}_0$-dichotomy to prove Silver's theorem.

Theorem 4.1 can be used to characterize the potentially closed sets.

Theorem 4.2 *(Lecomte) Let X, Y be Polish spaces, and A, B be disjoint analytic subsets of $X\times Y$. Then exactly one of the following holds:*

(a) A is separable from B by a potentially closed set,

(b) there are $f:\mathcal{C}\rightarrow X$, $g:\mathcal{C}\rightarrow Y$ continuous such that the inclusions $\mathbb{G}_0 \subseteq (f\times g)^{-1}(A)$ and $\Delta(\mathcal{C}) \subseteq (f\times g)^{-1}(B)$ hold.

Proof. If (a) and (b) hold simultaneously, then $\mathbb{G}_0$ can be separated from $\Delta(\mathcal{C})$ by a potentially closed set. In other words, $\Delta(\mathcal{C})$ can be separated from $\mathbb{G}_0$ by a potentially open set, which has to be a countable union of Borel rectangles $A_n \times B_n$. We set $C_n := A_n \cap B_n$, so that $\bigcup_{n\in\omega}\ C_n^2$ separates $\Delta(\mathcal{C})$ from $\mathbb{G}_0$. We then set $D_n := C_n \setminus (\bigcup_{p<n}\ C_p)$, so that (D_n) is a partition of $\mathcal{C}$ into Borel sets and $\bigcup_{n\in\omega}\ D_n^2$ separates $\Delta(\mathcal{C})$ from $\mathbb{G}_0$. In other words, the map $c:\mathcal{C}\rightarrow\omega$ defined by $c(\alpha) := n$ if $\alpha \in D_n$ contradicts Theorem 4.1. So (a) and (b) cannot hold simultaneously.

If B is empty, then (a) holds. So assume that B is not empty, which gives $s:\omega^\omega\rightarrow X\times Y$ continuous such that $s[\omega^\omega]=B$. We set $s(\alpha):=\big(s_0(\alpha),s_1(\alpha)\big)$, so that $(s_0\times s_1)[\Delta(\omega^\omega)]=B$. We set $R:=(s_0\times s_1)^{-1}(A)$, so that R is an analytic relation on ω^ω. So we can apply Theorem 4.1.

If there is $c:\omega^\omega\rightarrow\omega$ Borel such that $c(x)\neq c(y)$ if $(x,y)\in R$, then we set $C_n:=c^{-1}(\{n\})$. Note that $\Delta(\omega^\omega)\subseteq\bigcup_{n\in\omega}\ C_n^2\subseteq\neg R$, so that $B\subseteq\bigcup_{n\in\omega}\ (s_0[C_n]\times s_1[C_n])\subseteq\neg A$. The reflection theorem gives sequences $(X_n),(Y_n)$ of Borel sets with $\bigcup_{n\in\omega}\ (s_0[C_n]\times s_1[C_n])\subseteq\bigcup_{n\in\omega}\ (X_n\times Y_n)\subseteq\neg A$. As $\bigcup_{n\in\omega}\ (X_n\times Y_n)$ is potentially open, (a) holds.

If there is $h:\mathcal{C}\rightarrow\omega^\omega$ continuous such that $\big(h(\alpha),h(\beta)\big)\in R$ if $(\alpha,\beta)\in\mathbb{G}_0$, then we set $f:=s_0\circ h$ and $g:=s_1\circ h$. $\square$

One can check that Theorem 4.1 is also a consequence of Theorem 4.2. Theorem 4.2 can be extended to any Borel class and any Wadge class of Borel sets.

Theorem 4.3 *(Lecomte) Let $\mathbf{\Gamma}$ be a Wadge class of Borel sets, or the class $\mathbf{\Delta}^0_\xi$ for some $1\leq\xi<\omega_1$. Then there are Borel binary relations $\mathbb{S}_0$, $\mathbb{S}_1$ on $\mathcal{C}$ such that for any Polish spaces X,Y, and for any disjoint analytic subsets A,B of $X\times Y$, exactly one of the following holds:*

(a) A is separable from B by a set potentially in $\mathbf{\Gamma}$,

(b) there are $f:\mathcal{C}\rightarrow X$, $g:\mathcal{C}\rightarrow Y$ continuous such that the inclusions $\mathbb{S}_0\subseteq(f\times g)^{-1}(A)$ and $\mathbb{S}_1\subseteq(f\times g)^{-1}(B)$ hold.

The proof of this result provides a new proof of the Louveau-Saint Raymond Theorems 1.3 and 1.5. These proofs involve games, which is not the case in higher dimensions. Theorem 4.1 can be extended to any countable dimension. This is straightforward in finite dimension. This is not the case in countably infinite dimension, and we now prove this extension.

Notation. Let, for $2\leq\kappa\leq\omega$, $\psi_\kappa:\omega\rightarrow\kappa^{<\omega}$ be a bijection. More precisely,

- If $\kappa<\omega$, then $\psi_\kappa(0):=\emptyset$ is the sequence of length 0, $\psi_\kappa(1):=0$, ..., $\psi_\kappa(\kappa):=\kappa-1$ are the sequences of length 1, and so on.

- If $\kappa=\omega$, then let $(p_n)_{n\in\omega}$ be the sequence of prime numbers, and $I:\omega^{<\omega}\rightarrow\omega$ defined by $I(\emptyset):=1$, and $I(s):=p_0^{s(0)+1}...p_{|s|-1}^{s(|s|-1)+1}$ if $s\neq\emptyset$. Note that I is one-to-one, so that there is an increasing bijection $\varphi:Seq:=I[\omega^{<\omega}]\rightarrow\omega$. If $t\in Seq$, then we will denote by $\overline{t}:=I^{-1}(t)$ the finite sequence of natural numbers coded by the natural number t. We set $\psi_\omega:=(\varphi\circ I)^{-1}:\omega\rightarrow\omega^{<\omega}$.

Note that $|\psi_\kappa(n)|\leq n$ if $n\in\omega$. Indeed, this is clear if $\kappa<\omega$. If $\kappa=\omega$, then

$$I[\psi_\omega(n)|0]<I[\psi_\omega(n)|1]<...<I[\psi_\omega(n)],$$

so that $(\varphi\circ I)[\psi_\omega(n)|0]<(\varphi\circ I)[\psi_\omega(n)|1]<...<(\varphi\circ I)[\psi_\omega(n)]=n$. This implies that $|\psi_\omega(n)|\leq n$. Fix $n\in\omega$. As $|\psi_\kappa(n)|\leq n$, we can define $s_{n,\kappa}:=\psi_\kappa(n)0^{n-|\psi_\kappa(n)|}$.

The sequence $(s_{n,\kappa})_{n\in\omega}$ satisfies the following properties:

- $(s_{n,\kappa})_{n\in\omega}$ is **dense** in $\kappa^{<\omega}$ (i.e., any element of $\kappa^{<\omega}$ can be extended by one of the $s_{n,\kappa}$'s),
- the length of $s_{n,\kappa}$ is n.

We put $\mathbb{G}_{0,\kappa} := \{(s_{n,\kappa}\ i\ \gamma)_{i\in\kappa} \mid n\in\omega\ \wedge\ \gamma\in\kappa^\omega\} \subseteq (\kappa^\omega)^\kappa$. In particular, $\mathbb{G}_{0,2} = \mathbb{G}_0$.

Theorem 4.4 *(Kechris, Solecki, Todorčević) Let $2\leq\kappa<\omega$, X be a Polish space, and $A\subseteq X^\kappa$ be analytic. Then exactly one of the following holds:*

(a) there is $c : X\rightarrow\omega$ Borel such that $\big(c(x_i)\big)_{i\in\kappa} \notin \{(n_i)_{i\in\kappa}\in\omega^\kappa \mid \forall i\in\kappa\ \ n_i = n_0\}$ if $(x_i)_{i\in\kappa}\in A$,

(b) there is $f : \kappa^\omega\rightarrow X$ continuous such that $\big(f(\alpha_i)\big)_{i\in\kappa}\in A$ if $(\alpha_i)_{i\in\kappa}\in\mathbb{G}_{0,\kappa}$.

We cannot directly extend Theorem 4.1 to the case $\kappa=\omega$. In order to get a positive result in the case of the infinite dimension, we put $\mathbb{G} := \{\alpha\in\omega^\omega \mid \exists^\infty n\in\omega\ \ s_{n,\omega}\ 0\subseteq\alpha\}$. Note that $\mathbb{G}$ is a G_δ subset of ω^ω, and thus a Polish space.

Theorem 4.5 *(Lecomte) Let X be a Polish space, and $A\subseteq X^\omega$ be analytic. Then exactly one of the following holds:*

(a) there is $c : X\rightarrow\omega$ Borel such that $\big(c(x_i)\big)_{i\in\omega} \notin \{(n_i)_{i\in\omega}\in\omega^\omega \mid \forall i\in\omega\ \ n_i = n_0\}$ if $(x_i)_{i\in\omega}\in A$,

(b) there is $f : \mathbb{G}\rightarrow X$ continuous such that $\big(f(\alpha_i)\big)_{i\in\omega}\in A$ if $(\alpha_i)_{i\in\omega}\in\mathbb{G}_{0,\omega}\cap\mathbb{G}^\omega$.

Proof. As in the proof of Theorem 4.1, we see that (a) and (b) cannot hold simultaneously.

Note that there is a recursive map $\tilde{s} : \omega\rightarrow\omega$ such that $\tilde{s}(l)$ codes $s_{l,\omega}$, i.e., $\tilde{s}(l) = I(s_{l,\omega})$. Indeed, there is a recursive map $\tilde{\varphi} : \omega\rightarrow\omega$ whose restriction to Seq is an increasing bijection from Seq onto ω. Now $(\tilde{\varphi}|_{Seq})^{-1}$ defines a recursive map $\tilde{\psi} : \omega\rightarrow\omega$. It remains to note that $\tilde{s}(l) = t$ is equivalent to

$$t\in Seq\ \wedge\ lh(t)=l\ \wedge\ \forall i<l\ \ \Big(i<lh\big(\tilde{\psi}(l)\big)\ \wedge\ (t)_i = (\tilde{\psi}(l))_i\Big)\ \vee\ \Big(i\geq \mathrm{lh}\big(\tilde{\psi}(l)\big)\ \wedge\ (t)_i = 0\Big).$$

We may assume that

- the X^{ω^l}'s are recursively presented Polish spaces, for $l\in\omega$,

- the projections are recursive,

- the maps $\Pi_l : \omega\times X^{\omega^l}\rightarrow X$ defined by

$$\Pi_l\big(t, (x_s)_{s\in\omega^l}\big) = x\ \ \Leftrightarrow\ \ t\in Seq\ \text{ and }\ lh(t) = l\ \text{ and }\ x = x_{\bar{t}}$$

are partial recursive functions on $\{t\in\omega \mid t\in Seq\ \text{ and }\ lh(t)=l\}\times X^{\omega^l}$, for $l\in\omega$,

- the maps $\Pi'_l : \omega^2\times X^{\omega^l}\rightarrow X^\omega$ defined by

$$\Pi'_l\big(n, t, (x_s)_{s\in\omega^l}\big) = (y_i)_{i\in\omega}\ \ \Leftrightarrow\ \ t\in Seq\ \text{ and }\ n+1+lh(t) = l\ \text{ and }\ \forall i\in\omega\ \ y_i = x_{s_{n,\omega} i \bar{t}}$$

are partial recursive functions on $\{(n,t)\in\omega^2 \mid t\in Seq\ \text{ and }\ n+1+lh(t)=l\}\times X^{\omega^l}$, for $l\in\omega$,

- $A\in\Sigma^1_1(X^\omega)$.

We set $\Phi := \{C\subseteq X \mid A\cap C^\omega = \emptyset\}$. As Φ is Π^1_1 on Σ^1_1, the first reflection theorem ensures that if $C\in\Sigma^1_1(X)$ is in Φ, then there is $D\in\Delta^1_1(X)$ which is in Φ and contains C. As in the proof of Theorem 4.1 we may assume that $U\neq X$, so that $Y := X\setminus U$ is a nonempty Σ^1_1 subset of X.

The previous point gives the following key property:

$$\forall C\in\Sigma^1_1(X)\ \ (\emptyset\neq C\subseteq Y\ \Rightarrow\ A\cap C^\omega\neq\emptyset).$$

We construct $(x_s)_{s\in\omega^{<\omega}}\subseteq Y$, $(V_s)_{s\in\omega^{<\omega}}\subseteq\Sigma^1_1(X)$, and $(U_{n,t})_{(n,t)\in\omega\times\omega^{<\omega}}\subseteq\Sigma^1_1(X^\omega)$ satisfying the following conditions:

(1) $x_s\in V_s\subseteq Y\cap X_{low}$ and $(x_{s_{n,\omega}it})_{i\in\omega}\in U_{n,t}\subseteq A\cap Y^\omega\cap(X^\omega)_{low}$,

(2) $V_{sm}\subseteq V_s$ and $U_{n,tm}\subseteq U_{n,t}$,

(3) $\text{diam}_{d_X}(V_{s_{l,\omega}0})\leq 2^{-l}$ and $\big(s_{n,\omega}0t=s_{l,\omega}0\ \Rightarrow\ \text{diam}_{d_{X^\omega}}(U_{n,t})\leq 2^{-l}\big)$,

(4) For any fixed $|s|$, the relation "$x\in V_s$" is a Σ^1_1 condition in (x,s),

(5) For any fixed n and fixed $|t|$, the relation "$(x_i)_{i\in\omega}\in U_{n,t}$" is a Σ^1_1 condition in $\big((x_i)_{i\in\omega},t\big)$.

Assume that this is done. Fix $\alpha\in\mathbb{G}$. Then $(V_{\alpha|p})_{p\in\omega}$ is a decreasing sequence of nonempty clopen subsets of (X_{low},τ_{GH}) whose d_X-diameters tend to zero, so there is $f(\alpha)$ in their intersection. This defines $f:\mathbb{G}\to X$. Note that $d_X\big(x_{\alpha|p},f(\alpha)\big)\leq\text{diam}_{d_X}(V_{\alpha|p})$, so that f is continuous and $(x_{\alpha|p})_{p\in\omega}$ tends to $f(\alpha)$ in (X,τ_{GH}).

If $(s_{n,\omega}i\gamma)_{i\in\omega}\in\mathbb{G}_{0,\omega}\cap\mathbb{G}^\omega$, then $(U_{n,\gamma|p})_{p\in\omega}$ is a decreasing sequence of nonempty clopen subsets of $\big((X^\omega)_{low},\tau_{GH}\big)$ whose d_{X^ω}-diameters tend to zero, so there is $(\alpha_i)_{i\in\omega}$ in their intersection. Note that $(\alpha_i)_{i\in\omega}\in A$. Moreover, the sequence $\big((x_{s_{n,\omega}i(\gamma|p)})_{i\in\omega}\big)_{p\in\omega}$ tends to $(\alpha_i)_{i\in\omega}$ in (X^ω,τ_{GH}), and in $(X,\tau_{GH})^\omega$ too. As $(x_{s_{n,\omega}i(\gamma|p)})_{p\in\omega}$ tends to $f(s_{n,\omega}i\gamma)$ in (X,τ_{GH}), we get $f(s_{n,\omega}i\gamma)=\alpha_i$, for each $i\in\omega$. Thus $\big(f(s_{n,\omega}i\gamma)\big)_{i\in\omega}\in A$.

So it is enough to see that the construction is possible. If $V_\emptyset$ is any Σ^1_1 set, then clearly (4) holds for s of length 0. Now suppose that V_s has been defined for all $s\in\omega^{\leq l}$ and that (4) holds. Then in order to define V_r for $r\in\omega^{l+1}$, while ensuring (4), we will let $V_{s_{l,\omega}0}\subseteq V_{s_{l,\omega}}$ be some chosen Σ^1_1 set of diameter at most 2^{-l} (to be determined later on) and $V_{sm}:=V_s$ for all $sm\neq s_{l,\omega}0$. Then for $r\in\omega^{l+1}$

$$x\in V_r\Leftrightarrow(r=s_{l,\omega}0\ \text{ and }\ x\in V_{s_{l,\omega}0})\ \text{ or }\ (r=sm\neq s_{l,\omega}0\ \text{ and }\ x\in V_s),$$

which is Σ^1_1 in (x,r) by the induction hypothesis.

Similarly, if $U_{n,\emptyset}$ is any Σ^1_1 set, then clearly (5) holds for t of length 0. Now suppose that $U_{n,t}$ has been defined for all $t\in\omega^{\leq k}$ and that (5) holds. Then in order to define $U_{n,r}$ for $r\in\omega^{k+1}$, while ensuring (5), we again split into two cases. If $s_{n,\omega}0r=s_{n,\omega}0t0=s_{l,\omega}0$, then $U_{n,r}\subseteq U_{n,t}$ will be some chosen Σ^1_1 set of diameter at most 2^{-l} (to be determined later on). On the other hand, if $s_{n,\omega}0r=s_{n,\omega}0tm\neq s_{l,\omega}0$, then we set $U_{n,r}:=U_{n,t}$. Then for $r\in\omega^{k+1}$

$$(x_i)_{i\in\omega}\in U_{n,r}\Leftrightarrow\begin{cases}(s_{n,\omega}0r=s_{n,\omega}0t0=s_{l,\omega}0\ \text{ and }\ (x_i)_{i\in\omega}\in U_{n,r})\\ \text{or}\\ (s_{n,\omega}0r=s_{n,\omega}0tm\neq s_{l,\omega}0\ \text{ and }\ (x_i)_{i\in\omega}\in U_{n,t}),\end{cases}$$

which is Σ^1_1 in $\big((x_i)_{i\in\omega},r\big)$ by the induction hypothesis, since $s_{n,\omega}0r=s_{l,\omega}0$ can hold for only finitely many $(n,r)\in\omega\times\omega^{<\omega}$.

Notice that in this way (2) and (3) are also satisfied, so it remains to define $V_{s_{l,\omega}0}$, $U_{n,\emptyset}$ and $U_{n,r}$ for $s_{n,\omega}0r = s_{l,\omega}0$ of diameter small enough such that (1) also holds.

- As Y is a nonempty Σ^1_1 subset of X, we can choose $x_\emptyset \in Y \cap X_{low}$, and set $V_\emptyset := Y \cap X_{low}$.

- The key property applied to $V_\emptyset$ gives $(x_i)_{i\in\omega} \in A \cap V_\emptyset^\omega \cap (X^\omega)_{low}$. We choose $U_{0,\emptyset} \in \Sigma^1_1(X^\omega)$ such that $(x_i)_{i\in\omega} \in U_{0,\emptyset} \subseteq A \cap V_\emptyset^\omega \cap (X^\omega)_{low}$ and $\mathrm{diam}_{d_{X^\omega}}(U_{0,\emptyset}) \leq 1$. Then we choose $V_0 \in \Sigma^1_1(X)$ such that $x_0 \in V_0 \subseteq V_\emptyset$ and $\mathrm{diam}_{d_X}(V_0) \leq 1$. Assume that $(x_s)_{|s|\leq l}$, $(V_s)_{|s|\leq l}$, and $(U_{n,t})_{n+1+|t|\leq l}$ satisfying (1)-(5) have been constructed, which is the case for $l \leq 1$.

- We put

$$C := \Big\{ x \in X \mid \exists (y_s)_{s\in\omega^l} \in X^{\omega^l} \;\; y_{s^\omega_l} = x \;\text{ and }\; \forall s \in \omega^l \;\; y_s \in V_s \;\text{ and }\; \forall n < l \;\; \forall t \in \omega^{l-n-1} \;\; (y_{s_{n,\omega}it})_{i\in\omega} \in U_{n,t} \Big\}.$$

Then $x_{s_{l,\omega}} \in C$, by induction assumption. Moreover, $C \in \Sigma^1_1$, by conditions (4) and (5) since Σ^1_1 is closed under $\forall^\omega$. The key property applied to C gives $(x_{s_{l,\omega}i})_{i\in\omega} \in A \cap C^\omega \cap (X^\omega)_{low}$. As $x_{s_{l,\omega}m} \in C$, there is $(x_{sm})_{s\in\omega^l\setminus\{s_{l,\omega}\}} \subseteq X$ such that $x_{sm} \in V_s$ for each $s \in \omega^l$ and $(x_{s_{n,\omega}itm})_{i\in\omega} \in U_{n,t}$ for each $n < l$ and each $t \in \omega^{l-n-1}$. This defines $(x_s)_{s\in\omega^{l+1}}$.

We choose $U_{l,\emptyset} \in \Sigma^1_1(X^\omega)$ such that $(x_{s_{l,\omega}i})_{i\in\omega} \in U_{l,\emptyset} \subseteq A \cap V^\omega_{s_{l,\omega}} \cap (X^\omega)_{low}$ and

$$\mathrm{diam}_{d_{X^\omega}}(U_{l,\emptyset}) \leq 2^{-l},$$

and $V_{s_{l,\omega}0} \in \Sigma^1_1(X)$ such that $x_{s_{l,\omega}0} \in V_{s_{l,\omega}0} \subseteq V_{s_{l,\omega}}$ and $\mathrm{diam}_{d_X}(V_{s_{l,\omega}0}) \leq 2^{-l}$. If

$$s_{n,\omega}0r = s_{n,\omega}0t0 = s_{l,\omega}0,$$

then we choose $U_{n,r} \in \Sigma^1_1(X^\omega)$ such that $\mathrm{diam}_{d_{X^\omega}}(U_{n,r}) \leq 2^{-l}$ and $(x_{s_{n,\omega}ir})_{i\in\omega} \in U_{n,r} \subseteq U_{n,t}$. $\square$

Passing to complements, Theorem 4.2 characterizes when two disjoint analytic binary relations can be separated by a potentially open set. We saw that the potentially open sets are the countable unions of Borel rectangles. It is natural to ask about a level by level version of this. We will prove such a version at the level two. The problem at the level three is still open.

Notation. Let $b : \omega \to 3^{<\omega}$ be the following bijection: $b(0) := \emptyset$ is the sequence of length 0, $b(1) := 2$, $b(2) := 1$, $b(3) := 0$ are the sequences of length 1, and so on. Note that $|b(n)| \leq n$ if $n \in \omega$. Let $n \in \omega$. As $|b(n)| \leq n$, we can define $t_n := b(n)2^{n-|b(n)|}$. Note that $(t_n)_{n\in\omega}$ is dense in $3^{<\omega}$ and $|t_n| = n$. We then put $\mathbb{X} := 3^\omega \setminus \{t_n1^\infty \mid n \in \omega\}$, $\mathbb{Y} := 3^\omega \setminus \{t_n0^\infty \mid n \in \omega\}$, $\mathbb{A} := \Delta(3^\omega \setminus \{t_n\varepsilon^\infty \mid n \in \omega \wedge \varepsilon \in 2\})$ and $\mathbb{B} := \{(t_n0^\infty, t_n1^\infty) \mid n \in \omega\}$.

Theorem 4.6 *(Lecomte-Zelený) Let X, Y be Polish spaces, and A, B be disjoint analytic subsets of $X \times Y$. Then exactly one of the following holds:*

(a) A is separable from B by a $(\mathbf{\Sigma}^0_2 \times \mathbf{\Sigma}^0_2)_\sigma$ set,

(b) there are $f : \mathbb{X} \to X$, $g : \mathbb{Y} \to Y$ continuous such that the inclusions $\mathbb{A} \subseteq (f \times g)^{-1}(A)$ and $\mathbb{B} \subseteq (f \times g)^{-1}(B)$ hold.

Proof. We argue by contradiction for the exactly part, which gives $C_n \in \mathbf{\Pi}^0_1(\mathbb{X})$ and $D_n \in \mathbf{\Pi}^0_1(\mathbb{Y})$ with $\mathbb{A} \subseteq \bigcup_{n\in\omega}\ (C_n \times D_n) \subseteq \neg\mathbb{B}$. In particular, $3^\omega \setminus \{t_n\varepsilon^\infty \mid n\in\omega \wedge \varepsilon\in 2\} \subseteq \bigcup_{n\in\omega}\ C_n \cap D_n$, and Baire's theorem gives n and $s\in 3^{<\omega}$ such that $N_s \setminus \{t_n\varepsilon^\infty \mid n\in\omega \wedge \varepsilon\in 2\} \subseteq C_n \cap D_n$. Note that $N_s \cap \mathbb{X} \subseteq C_n$ and $N_s \cap \mathbb{Y} \subseteq D_n$. Choose p with $s \subseteq t_p$. Then $(t_p 0^\infty, t_p 1^\infty) \in \mathbb{B} \cap (C_n \times D_n)$, which is absurd.

In order to simplify the notation, by relativization, we may assume that X, Y are recursively presented and that $A, B \in \Sigma^1_1$. Let τ^X_2, τ^Y_2 be the topology on X, Y generated by the $\mathbf{\Pi}^0_1 \cap \Sigma^1_1$ subsets of X, Y respectively, and $N := A \cap \overline{B}^{\tau^X_2 \times \tau^Y_2}$. Note first that

$$(x,y) \notin \overline{B}^{\tau^X_2 \times \tau^Y_2} \Leftrightarrow \exists C_X \in \mathbf{\Pi}^0_1 \cap \Sigma^1_1(X)\ \ \exists C_Y \in \mathbf{\Pi}^0_1 \cap \Sigma^1_1(Y)\ \ (x,y) \in C_X \times C_Y \subseteq \neg B,$$

so that $\neg\overline{B}^{\tau^X_2 \times \tau^Y_2} = \bigcup_{C_X \in \mathbf{\Pi}^0_1 \cap \Sigma^1_1(X), C_Y \in \mathbf{\Pi}^0_1 \cap \Sigma^1_1(Y), C_X \times C_Y \subseteq \neg B}\ (C_X \times C_Y) \in (\mathbf{\Sigma}^0_2 \times \mathbf{\Sigma}^0_2)_\sigma$ since τ^X_2, τ^Y_2 have a countable basis. Moreover, $\overline{B}^{\tau^X_2 \times \tau^Y_2} \in \Sigma^1_1(X \times Y)$. Indeed,

$$\begin{aligned}(x,y) \notin \overline{B}^{\tau^X_2 \times \tau^Y_2} \Leftrightarrow\ & \exists D_X, D_Y \in \Delta^1_1(\omega)\ \ x \notin \textstyle\bigcup_{n\in D_X}\ N(X,n)\ \wedge\ y \notin \bigcup_{n\in D_Y}\ N(Y,n)\ \wedge \\ & \forall (z,t) \in X \times Y\ \ \big((z,t) \notin B \vee z \in \textstyle\bigcup_{n\in D_X}\ N(X,n) \vee t \in \bigcup_{n\in D_Y}\ N(Y,n)\big).\end{aligned}$$

In order to se this, assume first that $(x,y) \notin \overline{B}^{\tau^X_2 \times \tau^Y_2}$, which gives C_X, C_Y as above. Then

$$C_X \subseteq P_X := \{x \in X \mid \forall y \in Y\ \ y \notin C_Y \vee (x,y) \notin B\}.$$

The set P_X is Π^1_1. In particular, the Σ^1_1 set $\neg P_X$ is separable from the Σ^1_1 set C_X by the open set $\neg C_X$. Theorem 1.2 provides $D_X \in \Delta^1_1(\omega)$ such that $\bigcup_{n\in D_X}\ N(X,n)$ separates $\neg P_X$ from C_X. Note then that $C_Y \subseteq P_Y := \{y \in Y \mid \forall x \in X\ \ x \in \bigcup_{n\in D_X}\ N(X,n) \vee (x,y) \notin B\}$. The set P_Y is Π^1_1. In particular, the Σ^1_1 set $\neg P_Y$ is separable from the Σ^1_1 set C_Y by the open set $\neg C_Y$. Theorem 1.2 provides $D_Y \in \Delta^1_1(\omega)$ such that $\bigcup_{n\in D_Y}\ N(Y,n)$ separates $\neg P_Y$ from C_Y. We are done, using the coding system for Δ^1_1 sets. Thus $N \in \Sigma^1_1(X \times Y)$.

Case 1. $N = \emptyset$.

The set $\neg\overline{B}^{\tau^X_2 \times \tau^Y_2}$ is $(\mathbf{\Sigma}^0_2 \times \mathbf{\Sigma}^0_2)_\sigma$ and separates A from B and (a) holds.

Case 2. $N \neq \emptyset$.

We say that $s \in 3^{<\omega}$ is **suitable** if there is no triple $(n,\varepsilon,k) \in \omega \times 2 \times \omega$ such that $s = t_n\varepsilon^{k+1}$. Note that if s is not suitable, then the triple (n,ε,k) is unique, by the third crucial property of $(t_n)_{n\in\omega}$: $\forall \varepsilon \in 2\ \ \forall p < n\ \ \ t_n \not\subseteq t_p\varepsilon^\infty$. If $\emptyset \neq s$ is suitable, then we set $s^- := s|\max\{l < |s| \mid s|l \text{ is suitable}\}$. We construct

- a sequence $(x_s)_{s\in 3^{<\omega}}$ of points of X,
- a sequence $(y_s)_{s\in 3^{<\omega}}$ of points of Y,
- a sequence $(X_s)_{s\in 3^{<\omega}}$ of Σ^0_1 subsets of X,
- a sequence $(Y_s)_{s\in 3^{<\omega}}$ of Σ^0_1 subsets of Y,
- a sequence $(S_s)_{s\in 3^{<\omega}\text{ suitable}}$ of Σ^1_1 subsets of $X \times Y$.

We want these objects to satisfy the following conditions:

(1) $(x_s,y_s)\in X_s\times Y_s$
(2) $(x_s,y_s)\in S_s\subseteq N\cap (X\times Y)_{low}$ if s is suitable
(3) $\overline{X_{s\varepsilon}}\subseteq X_s$ if s is suitable or $s=t_n0^{k+1}$, and $\overline{X_{t_n1^{k+1}\varepsilon}}\subseteq X_{t_n}$
(4) $\overline{Y_{s\varepsilon}}\subseteq Y_s$ if s is suitable or $s=t_n1^{k+1}$, and $\overline{Y_{t_n0^{k+1}\varepsilon}}\subseteq Y_{t_n}$
(5) $S_s\subseteq S_{s^-}$ if $\emptyset\neq s$ is suitable
(6) $\text{diam}(X_s),\text{diam}(Y_s)\leq 2^{-|s|}$
(7) $\text{diam}_{\text{GH}}(S_s)\leq 2^{-|s|}$ if s is suitable
(8) $(x_{t_n0},y_{t_n1})\in\big(\overline{\Pi_0[(X_{t_n}\times Y_{t_n})\cap S_{t_n}]}\times\overline{\Pi_1[(X_{t_n}\times Y_{t_n})\cap S_{t_n}]}\big)\cap B$
(9) $(x_{t_n0^{k+1}},y_{t_n1^{k+1}})=(x_{t_n0},y_{t_n1})$

Assume that this is done. Let $\alpha\in\mathbb{X}$. Then the sequence (p_k) of integers such that $\alpha|p_k$ is suitable or of the form t_n0^{k+1} is infinite, by the third crucial property of $(t_n)_{n\in\omega}$. Condition (3) implies that $(\overline{X_{\alpha|p_k}})_{k\in\omega}$ is decreasing. Moreover, $(\overline{X_{\alpha|p_k}})_{k\in\omega}$ is a sequence of nonempty closed subsets of X whose diameters tend to 0, so that we can define $f(\alpha)$ by $\{f(\alpha)\}:=\bigcap_{k\in\omega}\ \overline{X_{\alpha|p_k}}=\bigcap_{k\in\omega}\ X_{\alpha|p_k}$. This defines a continuous map $f:\mathbb{X}\rightarrow X$ with $f(\alpha)=\lim_{k\rightarrow\infty}\ x_{\alpha|p_k}$. Similarly, we define $g:\mathbb{Y}\rightarrow Y$ continuous with $g(\beta)=\lim_{k\rightarrow\infty}\ y_{\beta|q_k}$.

If $\alpha\notin\{t_n\varepsilon^\infty\mid n\in\omega\wedge\varepsilon\in 2\}$, then the sequence (k_j) of integers such that $\alpha|p_{k_j}$ is suitable is infinite. Note that $(S_{\alpha|p_{k_j}})_{j\in\omega}$ is a decreasing sequence of nonempty closed subsets of $(X\times Y)_{low}$ whose GH-diameters tend to 0, so that we can define $F(\alpha)$ by $\{F(\alpha)\}:=\bigcap_{j\in\omega}\ S_{\alpha|p_{k_j}}\subseteq N\subseteq A$. As $F(\alpha)$ is the limit (in $(X\times Y,\text{GH})$, and thus in $X\times Y$) of $(x_{\alpha|p_{k_j}},y_{\alpha|p_{k_j}})_{j\in\omega}$, we get the equality $F(\alpha)=\big(f(\alpha),g(\alpha)\big)$. Thus $\mathbb{A}\subseteq(f\times g)^{-1}(A)$.

Note that $x_{t_n0}=x_{t_n0^2}=...=x_{t_n0^{q+1}}$ for each n. Thus $f(t_n0^\infty)=\lim_{q\rightarrow\infty}\ x_{t_n0^q}=x_{t_n0}$. Similarly, $g(t_n1^\infty)=y_{t_n1}$ and $\big(f(t_n0^\infty),g(t_n1^\infty)\big)=(x_{t_n0},y_{t_n1})\in B$. Thus $\mathbb{B}\subseteq(f\times g)^{-1}(B)$.

Let us prove that the construction is possible. As N is not empty, we can choose $(x_\emptyset,y_\emptyset)$ in $N\cap (X\times Y)_{low}$, a Σ^1_1 subset $S_\emptyset$ of $X\times Y$ with $(x_\emptyset,y_\emptyset)\in S_\emptyset\subseteq N\cap (X\times Y)_{low}$ of GH-diameter at most 1, and a Σ^0_1 neighborhood $X_\emptyset$ (resp., $Y_\emptyset$) of $x_\emptyset$ (resp., $y_\emptyset$) of diameter at most 1. Assume that $(x_s)_{s\in 3^{\leq l}}$, $(y_s)_{s\in 3^{\leq l}}$, $(X_s)_{s\in 3^{\leq l}}$, $(Y_s)_{s\in 3^{\leq l}}$ and $(S_s)_{s\in 3^{\leq l}}$ satisfying (1)-(9) have been constructed, which is the case for $l=0$.

Note that $(x_{t_l},y_{t_l})\in(X_{t_l}\times Y_{t_l})\cap S_{t_l}\subseteq\overline{B}^{\tau_2^X\times\tau_2^Y}$ since t_l is suitable. As $\Pi_\varepsilon[(X_{t_l}\times Y_{t_l})\cap S_{t_l}]$ is Σ^1_1, $\overline{\Pi_\varepsilon[(S_{t_l}\times Y_{t_l})\cap S_{t_l}]}\in\Sigma^1_1\cap\mathbf{\Pi}^0_1$. In particular, $\overline{\Pi_\varepsilon[(X_{t_l}\times Y_{t_l})\cap S_{t_l}]}$ is τ_2-open. This shows the existence of $(x_{t_l0},y_{t_l1})\in\Big(\big(X_{t_l}\cap\overline{\Pi_0[(X_{t_l}\times Y_{t_l})\cap S_{t_l}]}\big)\times\big(Y_{t_l}\cap\overline{\Pi_1[(X_{t_l}\times Y_{t_l})\cap S_{t_l}]}\big)\Big)\cap B$. We set $x_{t_l1}:=x_{t_l}$, $y_{t_l0}:=y_{t_l}$. We defined x_s,y_s when $s\in 3^{l+1}$ is not suitable but $s|l$ is suitable.

Assume now that s is suitable, but not $s|l$. This gives $(n,\varepsilon,k,\varepsilon')\in\omega\times 2\times\omega\times 3$ such that $s=t_n\varepsilon^{k+1}\varepsilon'$, with $\varepsilon'\neq\varepsilon$. Assume first that $\varepsilon=0$. Note that

$$x_{t_n0^{k+1}}=x_{t_n0}\in X_{t_n0^{k+1}}\cap\overline{\Pi_0[(X_{t_n}\times Y_{t_n})\cap S_{t_n}]}.$$

This gives $x_s\in X_{t_n0^{k+1}}\cap\Pi_0[(X_{t_n}\times Y_{t_n})\cap S_{t_n}]$, and also y_s with

$$(x_s,y_s)\in\big((X_{t_n}\cap X_{t_n0^{k+1}})\times Y_{t_n}\big)\cap S_{t_n}=(X_{t_n0^{k+1}}\times Y_{t_n})\cap S_{t_n}.$$

If $\varepsilon = 1$, then similarly we get $(x_s, y_s) \in (X_{t_n} \times Y_{t_n 1^{k+1}}) \cap S_{t_n}$.

If s and $s|l$ are both suitable, or both non suitable, then we set $(x_s, y_s) := (x_{s|l}, y_{s|l})$. So we defined x_s, y_s in any case. Note that Conditions (8) and (9) are fullfilled, and that $(x_s, y_s) \in S_{s^-}$ if s is suitable. Moreover, $x_s \in X_{s|l}$ if $s|l$ is suitable or $s|l = t_n 0^{k+1}$, and $x_s \in X_{t_n}$ if $s = t_n 1^{k+1}\varepsilon$, and similarly in Y. We choose Σ^0_1 sets X_s, Y_s of diameter at most 2^{-l-1} with

$$(x_s, y_s) \in X_s \times Y_s \subseteq \overline{X_s} \times \overline{Y_s} \subseteq \begin{cases} X_{s|l} \times Y_{s|l} \text{ if } s \text{ is not suitable or } s|l \text{ is suitable,} \\ \\ X_{s|l} \times Y_{t_n} \text{ if } s = t_n 0^{k+1}\varepsilon' \wedge \varepsilon' \neq 0, \\ \\ X_{t_n} \times Y_{s|l} \text{ if } s = t_n 1^{k+1}\varepsilon' \wedge \varepsilon' \neq 1. \end{cases}$$

It remains to choose, when s is suitable, $S_s \in \Sigma^1_1(X \times Y)$ of GH-diameter at most 2^{-l-1} such that $(x_s, y_s) \in S_s \subseteq S_{s^-}$. □

5 References

[C-L-Mi] J. D. Clemens, D. Lecomte and B. D. Miller, Dichotomy theorems for families of non-cofinal essential complexity, *Adv. Math.* 304 (2017), 285-299

[G] S. Gao, *Invariant descriptive set theory,* Pure and Applied Mathematics, A Series of Monographs and Textbooks, 293, Taylor and Francis Group, 2009

[H-K-Lo] L. A. Harrington, A. S. Kechris and A. Louveau, A Glimm-Effros dichotomy for Borel equivalence relations, *J. Amer. Math. Soc.* 3 (1990), 903-928

[K] A. S. Kechris, *Classical descriptive set theory,* Springer-Verlag, 1995

[K-S-T] A. S. Kechris, S. Solecki and S. Todorčević, *Borel chromatic numbers,* Adv. Math. 141 (1999), 1-44

[Ku] K. Kuratowski, *Topology, Vol. 1,* Academic Press, New-York, 1966

[LM] F. Le Maître, *Notes de cours de théorie descriptive des ensembles,* Handwritten notes, 2018

[L1] D. Lecomte, On minimal non potentially closed subsets of the plane, *Topology Appl.* 154, 1 (2007), 241-262

[L2] D. Lecomte, A dichotomy characterizing analytic digraphs of uncountable Borel chromatic number in any dimension, *Trans. Amer. Math. Soc.* 361 (2009), 4181-4193

[L3] D. Lecomte, *Potential Wadge classes,* Mem. Amer. Math. Soc. 221, 1038 (2013)

[L-Z] D. Lecomte and M. Zeleny, Baire-class xi colorings: the first three levels, *Trans. Amer. Math. Soc.* 366, 5 (2014), 2345-2373

[Lo-SR1] A. Louveau and J. Saint Raymond, Borel classes and closed games: Wadge-type and Hurewicz-type results, *Trans. Amer. Math. Soc.* 304 (1987), 431-467

[Lo-SR2] A. Louveau and J. Saint Raymond, The strength of Borel Wadge determinacy, *Cabal Seminar 81-85, Lecture Notes in Math.* 1333 (1988), 1-30

[Lo] A. Louveau, *The descriptive theory of Borel sets,* Handwritten notes, 1990

[M] Y. N. Moschovakis, *Descriptive set theory,* North-Holland, 1980

[Sr] S. M. Srivastava, *A course on Borel sets,* Graduate Texts in Mathematics, 180, Springer-Verlag, New-York, 1998