How can we recover Baire class one functions?

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Abstract. Let X and Y be separable metrizable spaces, and $f : X \rightarrow Y$ be a function. We want to recover f from its values on a small set via a simple algorithm. We show that this is possible if f is Baire class one, and in fact we get a characterization. This leads us to the study of sets of Baire class one functions and to a characterization of the separability of the dual space of an arbitrary Banach space.

1 Introduction.

This paper is the continuation of a study by U. B. Darji and M. J. Evans in [DE]. We specify the term "simple algorithm" used in the abstract. We work in separable metrizable spaces X and Y, and f is a function from X into Y. Recall that f is Baire class one if the inverse image of each open set is F_σ. Assume that we only know the values of f on a countable dense set $D \subseteq X$. We want to recover, in a simple way, all the values of f. For each point x of X, we extract a subsequence of D which tends to x. Let $(s_n[x,D])_n$ be this sequence. We will say that f is $recoverable\ with\ respect\ to\ D$ if, for each x in X, the sequence $\big(f(s_n[x,D])\big)_n$ tends to $f(x)$. The function f is $recoverable$ if there exists D such that f is recoverable with respect to D. Therefore, continuous functions are recoverable with respect to any countable dense sequence in X. We will show that results concerning recoverability depend on the way of extracting the subsequence. We let $D := (x_p)$.

Definition 1 *Let X be a topological space. We say that a basis (W_m) for the topology of X is a $good\ basis$ if for each open subset U of X and each point x of U, there exists an integer m_0 such that, for each $m \geq m_0$, $W_m \subseteq U$ if $x \in W_m$.*

We show that every separable metrizable space has a good basis, using the embedding into the compact space $[0,1]^\omega$. In the sequel, (W_m) will be a good basis of X, except where indicated.

Definition 2 *Let $x \in X$. The $path\ to\ x\ based\ on\ D$ is the sequence $(s_n[x,D])_{n\in\omega}$, denoted $\mathcal{R}(x,D)$, defined by induction as follows:*

$$\begin{cases} s_0[x,D] \quad := x_0, \\ \\ s_{n+1}[x,D] := \begin{cases} s_n[x,D] \ \text{ if } \ x = s_n[x,D], \\ \\ x_{\text{min}\left\{p\ /\ \exists m\in\omega\ \{x,x_p\}\subseteq W_m\subseteq X\setminus\left\{s_0[x,D],\ldots,s_n[x,D]\right\}\right\}} \ \text{ otherwise.} \end{cases} \end{cases}$$

Now the definition of a $recoverable$ function is complete.

In Section 2, we show the

Theorem 4 *A function f is recoverable if and only if f is Baire class one.*

In Section 3, we study the limits of U. B. Darji and M. J. Evans's result, using their way of extracting the subsequence. We give some possible extensions, and we show that we cannot extend it to any Polish space.

In Section 4, we study the question of the uniformity of sequence (x_p) for a set of Baire class one functions. We consider $A \subseteq \mathcal{B}_1(X, Y)$, equipped with the pointwise convergence topology. We study the existence of a dense sequence (x_p) of X such that each function of A is recoverable with respect to (x_p) (if this happens, we say that A is $uniformly\ recoverable$).

In the first part, we give some necessary conditions for uniform recoverability. We deduce among other things from this an example of a metrizable compact space $A \subseteq \mathcal{B}_1(2^\omega, 2)$ which is not uniformly recoverable.

In the second part, we study the link between the uniform recoverability of A and the fact that J. Bourgain's ordinal rank is bounded on A. J. Bourgain wondered whether his rank was bounded on a separable compact space A when X is a metrizable compact space. We show among other things that, if X and A are Polish spaces, then this rank is bounded (this is a partial answer to J. Bourgain's question).

In the third part, we give some sufficient conditions for uniform recoverability. We study among other things the link between uniform recoverability and F_σ subsets with open vertical sections of a product of two spaces.

In the fourth part, we give a characterization of the separability of the dual space of an arbitrary Banach space:

Theorem 30 *Let E be a Banach space, $X := [B_{E^*}, w^*]$, $A := \{G\lceil X / G \in B_{E^{**}}\}$, and $Y := \mathbb{R}$. The following statements are equivalent:*
(a) E^* is separable.
(b) A is metrizable.
(c) Every singleton of A is G_δ.
(d) A is uniformly recoverable.

In the fifth part, we introduce a notion similar to that of equicontinuity, the notion of an $equi$-$Baire\ class\ one$ set of functions. We give several characterizations of it, and we use it to study similar versions of Ascoli's theorems for Baire class one functions. Finally, the study of the link between the notion of an equi-Baire class one set of functions and uniform recoverability is made.

2 A characterization of Baire class one functions.

As mentionned in Section 1, we show the

Proposition 3 *Every separable metrizable space has a good basis.*

Proof. Let X be a separable metrizable space. Then X embeds into the compact metric space $[0,1]^\omega$, by ϕ. So let, for r integer, n_r be an integer and $(U^r_j)_{j\leq n_r}$ be a covering of $[0,1]^\omega$ made of open subsets of $[0,1]^\omega$ whose diameter is at most 2^{-r}. To get (W_m), it is enough to enumerate the sequence $(\phi^{-1}(U^r_j))_{r\in\omega,\ j\leq n_r}$. □

Theorem 4 *A function f is recoverable if and only if f is Baire class one.*

In order to prove this, we first give a lemma. It is essentially identical to U. B. Darji and M. J. Evans's proof of the "only if" direction. But we will use it later. So we give the details. Notice that it does not really depend of the way of extracting the subsequence.

Lemma 5 *Assume that, for $q\in\omega$, $\{x\in X\ /\ \exists n\ s_n[x,D]=x_q\}$ is an open subset of X. If f is recoverable with respect to D, then f is Baire class one.*

Proof. Let F be a closed subset of Y. We let, for k integer, $O_k:=\{y\in Y\ /\ d(y,F)<2^{-k}\}$. This defines an open subset of Y containing F. Let us fix an integer k. Let $(x_{p_j})_j$ be the subsequence of D made of the elements of $f^{-1}(O_k)$ (we may assume that it is infinite and enumerated in a 1-1 way). We let, for j integer, $U_j:=\{x\in X\ /\ \exists n\ s_n[x,D]=x_{p_j}\}$. This set is an open subset of X by hypothesis. Let $H_k:=\bigcap_{i\in\omega}[(\bigcup_{j\geq i}U_j)\cup\{x_{p_0},...,x_{p_{i-1}}\}]$. This set is a G_δ subset of X.

Let $x\in f^{-1}(O_k)$ and i be an integer. Then $s_n[x,D]\in f^{-1}(O_k)$ if n is bigger than n_0 and there exists $j(n)$ such that $s_n[x,D]=x_{p_{j(n)}}$; thus $x\in U_{j(n)}$. Either there exists $n\geq n_0$ such that $j(n)\geq i$ and $x\in\bigcup_{j\geq i}U_j$, or $x_{p_{j(n)}}$ is x_{p_q} if n is big enough, with $q<i$, and $x=x_{p_q}$. In both cases, $x\in H_k$.

If $x\in H_k$, either there exists an integer q such that $x=x_{p_q}$ and $f(x)\in O_k$, or for each integer i, there exists $j\geq i$ such that $x\in U_j$, and $\exists n\ s_n[x,D]=x_{p_j}$, and thus $f(x)\in\overline{O_k}$.

Therefore $f^{-1}(F)\subseteq\bigcap_{k\in\omega}f^{-1}(O_k)\subseteq\bigcap_{k\in\omega}H_k\subseteq\bigcap_{k\in\omega}f^{-1}(\overline{O_k})\subseteq f^{-1}(F)$. We deduce that

$$f^{-1}(F)=\bigcap_{k\in\omega}H_k$$

is a G_δ subset of X. □

Proof of Theorem 4. In order to show the "only if" direction, let us show that Lemma 5 applies. Set

$$\mathcal{O}(x,D,n):=\begin{cases}\emptyset\ \text{ if }\ x=s_n[x,D],\\ \\ W_{\min\left\{m\ /\ \{x,s_{n+1}[x,D]\}\subseteq W_m\subseteq X\setminus\left\{s_0[x,D],...,s_n[x,D]\right\}\right\}}\ \text{otherwise.}\end{cases}$$

Note that $\mathcal{O}(x,D,n)\neq\emptyset$ if and only if $x\neq s_n[x,D]$. In this case $\mathcal{O}(x,D,n)$ is an open neighborhood of x. If $n<n'$ and $\mathcal{O}(x,D,n),\ \mathcal{O}(x,D,n')\neq\emptyset$, $s_{n+1}[x,D]\in\mathcal{O}(x,D,n)\setminus\mathcal{O}(x,D,n')$, so $\mathcal{O}(x,D,n)$ is distinct from $\mathcal{O}(x,D,n')$. As (W_m) is a good basis, for each open neighborhood V of x there exists an integer n_0 such that $\mathcal{O}(x,D,n)\subseteq V$ if $n\geq n_0$, and therefore $s_{n+1}[x,D]\in V$. So path to x based on D tends to x.

To show that $\{x \in X \,/\, x_q \in \mathcal{R}(x, D)\}$ is an open subset of X, we may assume that $q > 0$ and that $x_r \neq x_q$ if $r < q$. So let $t_0 \in X$ and n be a minimal integer such that $s_{n+1}[t_0, D] = x_q$. Let m be a minimal integer such that $\{t_0, x_q\} \subseteq W_m \subseteq X \setminus \{s_0[t_0, D], ..., s_n[t_0, D]\}$. By definition of the path, q is minimal such that $x_q \in W_m$. Let us show that if $x \in W_m$, then $x_q \in \mathcal{R}(x, D)$; this will be enough since $t_0 \in W_m$. We notice that if we let $p_n(x) := \text{min}\{p \in \omega \,/\, x_p = s_n[x, D]\}$, then the sequence $(p_n(x))_n$ increases, strictly until it may be eventually constant. We have $x \in W_m$, which is a subset of $X \setminus \{x_0, ..., x_{q-1}\}$. Thus, as the path to x based on D tends to x, there exists a minimal integer n' such that $p_{n'+1}(x) \geq q$. Then we have $x_q = s_{n'+1}[x, D] \in \mathcal{R}(x, D)$.

Let us show the "if" direction. The proof looks like C. Freiling and R. W. Vallin's ones in [FV]. The main difference is the choice of the dense sequence, which has to be valid in any separable metrizable space.

We say that D $approximates$ $F \subseteq X$ if for all $x \in F \setminus D$, $\mathcal{R}(x, D) \setminus F$ is finite. Let us show that if (F_i) is a sequence of closed subsets of X, then there is $D \subseteq X$ which approximates each F_i.

Consider a countable dense sequence of X, and also a countable dense sequence of each finite intersection of the F_i's. Put this together, to get a countable dense sequence (q_i) of X. This countable dense set is the set D we are looking for. But we've got to describe how to order the elements of this sequence.

We will construct D in stages, called D_i, for each integer i. If F is a finite intersection of the F_i's and G is a finite subset of D, we set

$$A^F(G) := \bigcup_{m \in \omega, x \in G \setminus F, x \in W_m \not\subseteq X \setminus F} \{q_{\text{min}\{i/q_i \in W_m \cap F\}}\}.$$

Put on $2^i = \{\sigma_1, \ldots, \sigma_{2^i}\}$ the lexicographic ordering, and let $F^\sigma := \bigcap_{j \in \sigma} F_j$ for each finite subset σ of ω. We set

$$G_0 := \{q_i\},\, G_{k+1} := G_k \cup A^{F^{\sigma_{k+1}}}(G_k) \quad (\text{for } k < 2^i),$$
$$D_i := (\bigcup_{k \leq 2^i} G_k) \setminus (\bigcup_{l < i} D_l).$$

We order the elements of D_i as follows. Let $\sigma^i(x) := \{k < i / x \in F_k\}$. Put the elements of D_i whose σ^i is σ_{2^i} first (in any order). Then put the elements of D_i whose σ^i is σ_{2^i-1}. And so on, until elements of D_i whose σ^i is σ_1.

Now let us suppose that F_i is not approximated by D, with x as a witness and i minimal. Let $y \in \mathcal{R}(x, D) \setminus F_i$ such that y is put into D at some stage $j > i$ and satisfying $x \in F_k \Leftrightarrow y \in F_k$ for each $k < i$. Let $m \in \omega$ such that $x, y \in W_m$. We have $y \notin F^{\sigma^j(x)}$, and $W_m \not\subseteq X \setminus F^{\sigma^j(x)}$ because $x \in F^{\sigma^j(x)}$. So we can define $z := q_{\text{min}\{i/q_i \in W_m \cap F^{\sigma^j(x)}\}}$. Then $\sigma^j(z) > \sigma^j(y)$ in the lexicographic order. We have $z \in A^{F^{\sigma^j(x)}}(\{y\})$. We conclude that z is put before y and that $y \notin \mathcal{R}(x, D)$. This is the contradiction we were looking for.

Now let (Y_p) be a basis for the topology of Y. Consider the inverse images of the Y_p's by f. Express each of these sets as a countable union of closed sets. This gives D which approximates each of these closed sets. It is now clear that the set D is what we were looking for. $\square$

3 About the limits of U. B. Darji and M. J. Evans's method.

Let us recall the original way of extracting the subsequence. Fix a compatible distance d on X.

Definition 6 *Let $x \in X$. The $route\ to\ x\ based\ on\ D$ is the sequence $(s'_n[x,D])_{n\in\omega}$, denoted $\mathcal{R}'(x,D)$, defined by induction as follows:*

$$\begin{cases} s'_0[x,D] \quad := x_0, \\ s'_{n+1}[x,D] := \begin{cases} s'_n[x,D] & \textit{if } x = s'_n[x,D], \\ x_{min\{p\ /\ d(x,x_p)<d(x,s'_n[x,D])\}} & \textit{otherwise.} \end{cases} \end{cases}$$

If f is recoverable in the sense of Definition 6, we say that f is $first\ return\ recoverable$. U. B. Darji and M. J. Evans showed the following:

Theorem *If f is first return recoverable, then f is Baire class one. Conversely, if f is Baire class one and X is a compact space, then f is first return recoverable.*

Definition 7 *We will say that an ultrametric space (X,d) is $discrete$ if the following condition is satisfied:* $\forall\ (d_n)_{n\in\omega} \subseteq d[X\times X] \quad [(\forall\ n\in\omega\ \ d_{n+1}<d_n) \ \Rightarrow\ (\lim_{n\to\infty} d_n = 0)]$.

We can show the following extensions:

Theorem 8 *Assume that f is Baire one. Then f is first return recoverable in the following cases:*
(a) X is a metric space countable union of totally bounded subspaces.
(b) X is a discrete ultrametric space.

Corollary 9 *Let X be a metrizable separable space. Then there exists a compatible distance d on X such that for each $f : X \to Y$, f is Baire class one if and only if f is first return recoverable relatively to d.*

This corollary comes from the fact that we can find a compatible distance on X making X totally bounded. Now we will show that the notion of a first return recoverable function is a metric notion and not a topological one. More precisely, we will show that the hypothesis "X is discrete" in Theorem 8 is useful. In fact, we will give an example of an ultrametric space homeomorphic to ω^ω in which there exists a closed subset whose characteristic function is not first return recoverable (notice that ω^ω, equipped with its usual metric, is a discrete ultrametric space). So the equivalence between "f is Baire class one " and "f is first return recoverable" depends on the choice of the distance. And the equivalence in Theorem 4 does not depend on the choice of the good basis, and is true without any restriction on X. The algorithm given in Definition 2 is given in topological terms only, as the notion of a Baire class one function. Furthermore, Definition 2 uses only countably many open subsets of X, namely the W_m's.

Lemma 10 *Let X be an ultrametric space, $t \in X$, $x, y \in X \setminus \{t\}$. Then the open balls $B(x, d(x,t)[$ and $B(y, d(y,t)[$ are equal or disjoint.*

Proof. Let us show that $d(x,t) = d(y,t)$ or $B(x, d(x,t)[\cap B(y, d(y,t)[= \emptyset$. Let

$$z \in B(x, d(x,t)[\cap B(y, d(y,t)[.$$

If for example $d(x,t) < d(y,t)$, let r be in $]d(x,t), d(y,t)[$. As $z \in B(x, r[$,

$$B(x, r[= B(z, r[\subseteq B(z, d(y,t)[.$$

As $z \in B(y, d(y,t)[$, we can write $B(z, d(y,t)[= B(y, d(y,t)[\subseteq X \setminus \{t\}$. But this contradicts the fact that $t \in B(x, r[$.

If $B(x, d(x,t)[\cap B(y, d(y,t)[\neq \emptyset$, let z be in the intersection. Then we have the sequence of equalities $B(x, d(x,t)[= B(z, d(x,t)[= B(z, d(y,t)[= B(y, d(y,t)[$. $\square$

Now we introduce the counterexample. We set

$$Z := \{Q = (q_n)_{n\in\omega} \in \mathbb{Q}_+^{\omega} \ / \ \forall\, n \in \omega \ \ q_n < q_{n+1} \ \text{ and } \lim_{n\to\infty} q_n = +\infty\}.$$

This space is equipped with

$$d : \begin{cases} Z \times Z & \to \mathbb{R}_+ \\ (Q, Q') & \mapsto \begin{cases} 2^{-\min(q_{\min\{n\in\omega / q_n \neq q'_n\}}, q'_{\min\{n\in\omega / q_n \neq q'_n\}})} & \text{if } \ Q \neq Q', \\ 0 \ \text{ otherwise.} \end{cases} \end{cases}$$

Proposition 11 *The space (Z, d) is an ultrametric space homeomorphic to ω^ω and is not discrete.*

Proof. We set $W := \{f \in 2^{\mathbb{R}_+} \ / \ \exists\, Q \in Z \ \ f = \mathbb{I}_{\cup_{p\in\omega}[q_{2p}, q_{2p+1}]}\}$; this space is equipped with the ultrametric on $2^{\mathbb{R}_+}$ defined by $\tilde{d}(f, g) := 2^{-\inf\{x\in\mathbb{R}_+ / f(x) \neq g(x)\}}$ if $f \neq g$. Then the function from Z into W which associates $\mathbb{I}_{\cup_{p\in\omega}[q_{2p}, q_{2p+1}]}$ to Q is a bijective isometry. Thus, it is enough to show the desired properties for W.

We set

$$D := \{f \in 2^{\mathbb{R}_+} \ / \ \exists\, Q \in Z \ \ \exists\, k \in \omega \ \ f = \mathbb{I}_{\cup_{p<k}[q_{2p}, q_{2p+1}]} \ \text{ or } \ f = \mathbb{I}_{\cup_{p<k}[q_{2p}, q_{2p+1}] \cup [q_{2k}, +\infty[}\},$$

$$V := W \cup D.$$

Then W and V are ultrametric, viewed as subspaces of $2^{\mathbb{R}_+}$. Set D is countable and dense in V, so V and W are separable.

Let $(f_p)_{p\in\omega}$ be a Cauchy sequence in V, and m in ω. There exists a minimal integer $N(m)$ such that, for $p, q \geq N(m)$, we have $d(f_p, f_q) \leq 2^{-m}$; that is to say $f_p(t) = f_q(t)$ for each $t < m$. We let, if $E(t)$ is the biggest integer less than or equal to t,

$$f : \begin{cases} \mathbb{R}_+ \to 2 \\ t \quad \mapsto f_{N(E(t)+1)}(t) \end{cases}$$

If $p \geq N(m)$ and $t < m$, $N(E(t)+1) \leq N(m)$ and we have

$$f(t) = f_{N(E(t)+1)}(t) = f_{N(m)}(t) = f_p(t).$$

Thus the sequence $(f_p)_{p\in\omega}$ tends to f in $2^{\mathbb{R}_+}$. We will check that $f \in V$; this will show that V is complete, thus Polish. As W is a G_δ subset of V, W will also be Polish.

Case 1. $\exists\, r \in \mathbb{R}_+ \ \forall\, t \geq r \ \ f(t) = 0$.

If $p \geq N(E(r)+1)$ and $t < E(r)+1$, $f_p(t) = f(t)$; thus, the restriction of f to $[0, E(r)+1[$ is the restriction of $\mathbb{I}_{\cup_{p<k}[q_{2p}, q_{2p+1}]}$ to this interval, and we may assume that $q_{2k-1} < E(r)+1$. Therefore, we have $f = \mathbb{I}_{\cup_{p<k}[q_{2p}, q_{2p+1}]}$ and $f \in D \subseteq V$.

Case 2. $\exists\, r \in \mathbb{R}_+ \ \forall\, t \geq r \ \ f(t) = 1$.

If $p \geq N(E(r)+1)$ and $t < E(r)+1$, then $f_p(t) = f(t)$; thus, the restriction of f to $[0, E(r)+1[$ is the restriction of $\mathbb{I}_{\cup_{p\leq k}[q_{2p}, q_{2p+1}]}$ to this interval, and we may assume that $q_{2k} < E(r)+1$. Therefore, we have $f = \mathbb{I}_{\cup_{p<k}[q_{2p}, q_{2p+1}]\cup[q_{2k}, +\infty[}$ and $f \in D \subseteq V$.

Case 3. $\forall\, r \in \mathbb{R}_+ \ \exists\, t, u \geq r \ \ f(t) = 0$ and $f(u) = 1$.

Let $(r_n)_{n\in\omega} \subseteq \mathbb{R}_+$ be a strictly increasing sequence such that $\lim_{n\to\infty} r_n = +\infty$ and $f(r_n) = 0$ for each integer n. If $t < E(r_n)+1$, then we have $f(t) = f_{N(E(r_n)+1)}(t)$. Thus, the restriction of f to $[0, r_n]$ is the restriction of $\mathbb{I}_{\cup_{p<k_n}[q_{2p}, q_{2p+1}]}$ to this interval, and we may assume that $q_{2k_n-1} < r_n$. The sequence $(k_n)_{n\in\omega}$ is increasing, and $\lim_{n\to\infty} k_n = +\infty$ because f is not ultimately constant. For the same reason, $\lim_{n\to\infty} q_n = +\infty$. Thus $f = \mathbb{I}_{\cup_{p\in\omega}[q_{2p}, q_{2p+1}]} \in W \subseteq V$.

Let $f \in V$ and m in ω. There exists $\varepsilon \in \mathbb{Q} \cap]0, 1[$ and $q \in \mathbb{Q}_+ \cap]m+1, +\infty[$ such that, for each $t \in]q-\varepsilon, q+\varepsilon[$, we have $f(t) = 0$, or, for each $t > q-\varepsilon$, we have $f(t) = 1$. In the first case we set

$$g : \begin{cases} \mathbb{R}_+ \to 2 \\ t \quad \mapsto \begin{cases} f(t) \ \text{ if } \ t \notin [q-\varepsilon/2, q+\varepsilon/2], \\ 1 \text{ otherwise.} \end{cases} \end{cases}$$

In the second case, we set

$$g : \begin{cases} \mathbb{R}_+ \to 2 \\ t \quad \mapsto \begin{cases} f(t) \ \text{ if } \ t \leq q, \\ 0 \text{ otherwise.} \end{cases} \end{cases}$$

In both cases we have $f \neq g$, $d(f,g) \leq 2^{-m}$ and $g \in V$; this shows that V is perfect. Moreover, as D is countable and dense in V, W is locally not compact. Finally, W is a 0-dimensional Polish space, and each of its compact subsets have empty interior; thus it is homeomorphic to ω^ω (see Theorem 7.7 page 37 of chapter 1 in [Ke]).

To finish the proof, we set $f_n := \mathbb{1}_{[0,1-2^{-n-1}]\cup\bigcup_{p>0}[2p,2p+1]}$. We have $f_n \in W$ and $d(f_n, f_{n+1})$ is $2^{-1+2^{-n-1}}$, which strictly decreases to $1/2$. Thus, space W is not a discrete ultrametric space. $\square$

Theorem 12 *There exists a $\mathbf{\Pi}^0_1(Z)$ whose characteristic function is not first return recoverable.*

Proof. Let $F := \{Q \in Z \;/\; \forall\, n \in \omega \;\; n < q_n < n+1\}$, $D := (x_p)$ be a dense sequence of Z. Then F is closed since fixing a finite number of coordinates is a clopen condition. We will show that there exists $x \in Z$ such that the sequence $(\mathbb{1}_F(s_n[x,D]))_{n\in\omega}$ does not tend to $\mathbb{1}_F(x)$. Let us assume that this is not the case.

• We set $n_\emptyset := 0$, $B_\emptyset := Z$. We have $Z \setminus \{x_0\} = \bigcup^{\text{disj.}}_{j\in\omega} \; B(y_j, d(y_j, x_0)[$. Let

$$n_j := \min\{n \in \omega \;/\; x_n \in B(y_j, d(y_j, x_0)[\}.$$

For each x in $B(y_j, d(y_j, x_0)[$ we have $B(y_j, d(y_j, x_0)[= B(x_{n_j}, d(x_{n_j}, x_0)[= B(x, d(x, x_0)[$ and $s_1[x,D] = x_{n_j}$. Then we do this construction again. For $s \in \omega^{<\omega} \setminus \{\emptyset\}$, we set

$$B_s := B(x_{n_s}, d(x_{n_s}, x_{n_{s\lceil |s|-1}})[.$$

We have $B_s \setminus \{x_{n_s}\} = \bigcup^{\text{disj.}}_{j\in\omega} \; B(y_{s^\frown j}, d(y_{s^\frown j}, x_{n_s})[$. Let

$$n_{s^\frown j} := \min\{n \in \omega \;/\; x_n \in B(y_{s^\frown j}, d(y_{s^\frown j}, x_{n_s})[\}.$$

For each x in $B(y_{s^\frown j}, d(y_{s^\frown j}, x_{n_s})[$, we have

$$B(y_{s^\frown j}, d(y_{s^\frown j}, x_{n_s})[= B(x_{n_{s^\frown j}}, d(x_{n_{s^\frown j}}, x_{n_s})[= B(x, d(x, x_{n_s})[\subseteq B_s,$$

and also $s_{|s|+1}[x,D] = x_{n_{s^\frown j}}$.

• For each x in $Z \setminus \{x_n \;/\; n \in \omega\}$, there is α in ω^ω with $x \in \bigcap_{m\in\omega} B_{\alpha\lceil m}$ and $s_m[x,D] = x_{n_{\alpha\lceil m}}$ for each m in ω. Moreover, if $x \in F$, then there exists m_0 in ω such that $x_{n_{\alpha\lceil m}} \in F$ for each $m \geq m_0$.

Case 1. $\forall\, s \in \omega^{<\omega} \;\; B_s \cap F = \emptyset$ or $\exists\, t \succ_{\neq} s \;\; B_t \cap F \neq \emptyset$ and $x_{n_t} \notin F$.

As $B_\emptyset = Z$ meets F which is not empty, there exists $\alpha, \beta \in \omega^\omega$ such that $0 < \beta(n) < \beta(n+1)$, $B_{\alpha\lceil\beta(n)} \cap F \neq \emptyset$ and $x_{n_{\alpha\lceil\beta(n)}} \notin F$ for each n in ω. It is enough to show the existence of x in $\bigcap_{m\in\omega} B_{\alpha\lceil m}$. Indeed, if we have this, we will have $s_m[x,D] = x_{n_{\alpha\lceil m}}$ for each $m \in \omega$. But the diameter of $B_{\alpha\lceil m}$ will be at most $2d(s_m[x,D], s_{m-1}[x,D])$, thus will tend to 0. As $B_{\alpha\lceil\beta(n)}$ meets F, we will deduce that $x \in F$. Thus, the sequence $(\mathbb{1}_F(s_n[x,D]))_{n\in\omega}$ will not tend to $\mathbb{1}_F(x)$ since $s_{\beta(n)}[x,D] \notin F$.

As $x_{n_{\alpha\lceil m+1}} \in B_{\alpha\lceil m+1} \subseteq B_{\alpha\lceil m}$, the sequence $(d(x_{n_{\alpha\lceil m+1}}, x_{n_{\alpha\lceil m}}))_{m\in\omega}$ is strictly decreasing; let l be its inferior bound.

Case 1.1. $l = 0$.

In this case, sequence $(x_{n_{\alpha\lceil m}})_{m\in\omega}$ is a Cauchy sequence. Let Φ be the bijective isometry that we used at the beginning of the proof of Proposition 12. We set $f_m := \Phi(x_{n_{\alpha\lceil m}})$. Then the sequence $(f_m)_{m\in\omega}$ is a Cauchy sequence in $W \subseteq V$, thus tends to $f \in V$ which is complete.

Case 1.1.1. $\exists\, r \in \mathbb{R}_+ \;\; \forall\, t \geq r \;\; f(t) = 0$.

We have $f = \mathbb{1}_{\cup_{p<k}[q_{2p}, q_{2p+1}]}$ and, if m is big enough, then the restriction of f_m to $[0, E(r)+1[$ is the restriction of f to this same interval, and we have $q_{2k-1} < E(r)+1$. Thus, $x_{n_{\alpha\lceil m}}$ starts with $< q_0, q_1, ..., q_{2k-1}, q^m_{2k} >$ and, if m is greater than $p_0 \geq m_0$, then $q^m_{2k} \geq 2k+1$. Let n_0 in ω be such that $\beta(n_0) > p_0$. Then $B_{\alpha\lceil\beta(n_0)}$ is disjoint from F because, if y is in F, then $y \notin B_{\alpha\lceil\beta(n_0)}$ since $d(y, x_{n_{\alpha\lceil\beta(n_0)}}) \geq 2^{-y_{2k}} > 2^{-2k-1} \geq d(x_{n_{\alpha\lceil\beta(n_0)}}, x_{n_{\alpha\lceil\beta(n_0)-1}})$. Thus, this case is not possible.

Case 1.1.2. $\exists\, r \in \mathbb{R}_+ \;\; \forall\, t \geq r \;\; f(t) = 1$.

This case is similar to case 1.1.1.

Case 1.1.3. $\forall\, r \in \mathbb{R}_+ \;\; \exists\, t, u \geq r \;\; f(t) = 0$ and $f(u) = 1$.

In this case, $f \in W$, thus there exists $x \in Z$ such that the sequence $(x_{n_{\alpha\lceil m}})_{m\in\omega}$ tends to x. We have $x \in \bigcap_{m\in\omega} B_{\alpha\lceil m}$, since otherwise we can find an integer m'_0 such that $x \notin B_{\alpha\lceil m}$ for each $m \geq m'_0$; but, as $x_{n_{\alpha\lceil m}} \in B_{\alpha\lceil m}$, x is in $B_{\alpha\lceil m'_0}$ which is closed.

Case 1.2. $l > 0$.

Let $r' \in \mathbb{R}$ be such that $l = 2^{-r'}$.

Case 1.2.1. $E(r') < r'$.

We will show that there exists $x \in \bigcap_{m\in\omega} B_{\alpha\lceil m}$. This will be enough. If m is big enough, $d(x_{n_{\alpha\lceil m}}, x_{n_{\alpha\lceil m-1}}) < 2^{-E(r')}$. As $B_{\alpha\lceil m}$ meets F, let y be in the intersection; y is of the form

$$(n + 1 - \varepsilon_n)_{n\in\omega},$$

where $\varepsilon_n \in]0, 1[$. If m is big enough, then $x_{n_{\alpha\lceil m}}$ starts with $< 1 - \varepsilon_0, ..., E(r') - \varepsilon_{E(r')-1} >$. Then the term number $E(r') + 1$ of sequence $x_{n_{\alpha\lceil m}}$ is called $x^{E(r')}_{n_{\alpha\lceil m}}$.

Case 1.2.1.1. $\exists\, m \in \omega \;\; x^{E(r')}_{n_{\alpha\lceil m}} = x^{E(r')}_{n_{\alpha\lceil m+1}}$.

In this case, as $B_{\alpha\lceil m+1}$ meets F, $x^{E(r')}_{n_{\alpha\lceil p}}$ is of the form $E(r') + 1 - \varepsilon_{E(r')}$ for each $p \geq m$. This shows that if p is big enough, then $x^{E(r')+1}_{n_{\alpha\lceil p}} \neq x^{E(r')+1}_{n_{\alpha\lceil p+1}}$. Thus we are reduced to the following case.

Case 1.2.1.2. $\forall\, m\in\omega\;\; x^{E(r')}_{n_{\alpha\lceil m}}\neq x^{E(r')}_{n_{\alpha\lceil m+1}}$.

The sequence $(x^{E(r')}_{n_{\alpha\lceil m}})_{m\in\omega}$ is strictly increasing. Indeed, assume that $x^{E(r')}_{n_{\alpha\lceil m}}>x^{E(r')}_{n_{\alpha\lceil m+1}}$. Then we have $d(x_{n_{\alpha\lceil m}},x_{n_{\alpha\lceil m+1}})\leq d(x_{n_{\alpha\lceil m+1}},x_{n_{\alpha\lceil m+2}})$, since $x^{E(r')}_{n_{\alpha\lceil m+1}}\neq x^{E(r')}_{n_{\alpha\lceil m+2}}$; but this is absurd. Thus the sequence $(x^{E(r')}_{n_{\alpha\lceil m}})_{m\in\omega}$ is strictly increasing, and $\lim_{m\to\infty} x^{E(r')}_{n_{\alpha\lceil m}}=r'$. But if the point x starts with sequence $<1-\varepsilon_0,...,E(r')-\varepsilon_{E(r')-1},q>$, where $q\in\mathbb{Q}\cap]r',+\infty[$, then $x\in\bigcap_{m\in\omega}B_{\alpha\lceil m}$ since

$$d(x,x_{n_{\alpha\lceil m}})=2^{-x^{E(r')}_{n_{\alpha\lceil m}}}<d(x_{n_{\alpha\lceil m}},x_{n_{\alpha\lceil m-1}})=2^{-x^{E(r')}_{n_{\alpha\lceil m-1}}}.$$

Case 1.2.2. $E(r')=r'$.

This case is similar to case 1.2.1; $r'-1$ plays the role that $E(r')$ played in the preceding case.

Case 2. $\exists\, s\in\omega^{<\omega}\;\; B_s\cap F\neq\emptyset$ and $\forall\, t\succ_{\neq} s\;\; B_t\cap F\neq\emptyset\;\Rightarrow\; x_{n_t}\in F$.

Note that, for each x in Z and each q in $\mathbb{Q}_+$, there exists Q in Z such that $d(Q,x)=2^{-q}$. Indeed, there exists a minimal integer n such that $q<x_n$, and we take Q beginning with $<x_0,...,x_{n-1},q>$ if $x_{n-1}\neq q$; otherwise, we take Q beginning with $<x_0,...,x_{n-2},x_n>$.

We may assume, by shifting s if necessary, that $x_{n_s}\in F$ and $s\neq\emptyset$. Thus we have

$$x_{n_s}=\;<1-\varepsilon^0_0,2-\varepsilon^0_1,...>,$$

where $0<\varepsilon^0_i<1$. Let j_0 be a minimal integer such that $2^{\varepsilon^0_{j_0}-j_0-1}<d(x_{n_s},x_{n_{s\lceil|s|-1}})$, and

$$s_0:=\;<1-\varepsilon^0_0,...,j_0-\varepsilon^0_{j_0-1}>.$$

If $t\succ s$, then x_{n_t} begins with s_0.

There are p_0 in ω and $Q_{n_s\frown p_0}$ in F such that $d(Q_{n_s\frown p_0},x_{n_s})=2^{\varepsilon^0_{j_0}-j_0-1}$. Then $Q_{n_s\frown p_0}$ is of the form $s_0^\frown<1+j_0-\varepsilon^1,2+j_0-\varepsilon^0_{j_0+1},...>$, where $0<\varepsilon^1<\varepsilon^0_{j_0}$. There exists an unique integer n_0 such that

$$Q_{n_s\frown p_0}\in B_{s\frown n_0}=B(Q_{n_s\frown p_0},2^{\varepsilon^0_{j_0}-j_0-1}[.$$

As $B_{s\frown n_0}$ meets F, $x_{n_{s\frown n_0}}\in F$. Thus the point $x_{n_{s\frown n_0}}$ is of the form

$$s_0^\frown<1+j_0-\varepsilon^1_{j_0},2+j_0-\varepsilon^1_{j_0+1},...>,$$

where $0<\varepsilon^1_{j_0}<\varepsilon^0_{j_0}$. More generally, there exists p_k in ω and $Q_{n_{s\frown n_0^\frown...\frown n_{k-1}\frown p_k}}$ in F such that $d(Q_{n_{s\frown n_0^\frown...\frown n_{k-1}\frown p_k}},x_{n_s^\frown n_0^\frown...\frown n_{k-1}})=2^{\varepsilon^k_{j_0}-j_0-1}$. Then $Q_{n_{s\frown n_0^\frown...\frown n_{k-1}\frown p_k}}$ is of the form

$$s_0^\frown<1+j_0-\varepsilon^{k+1},2+j_0-\varepsilon^k_{j_0+1},...>,$$

where $0<\varepsilon^{k+1}<\varepsilon^k_{j_0}$.

There exists an unique integer n_k such that $Q_{n_s\frown n_0^\frown\ldots\frown n_{k-1}\frown p_k}$ is in $B_{s\frown n_0^\frown\ldots\frown n_k}$, which is $B(Q_{n_s\frown n_0^\frown\ldots\frown n_{k-1}\frown p_k}, 2^{\varepsilon_{j_0}^k - j_0 - 1}[$. As $B_{s\frown n_0^\frown\ldots\frown n_k}$ meets F, $x_{n_s\frown n_0^\frown\ldots\frown n_k}$ is in F. Thus the point $x_{n_s\frown n_0^\frown\ldots\frown n_k}$ is of the form $s_0^\frown < 1 + j_0 - \varepsilon_{j_0}^{k+1}, 2 + j_0 - \varepsilon_{j_0+1}^{k+1}, \ldots >$, where $0 < \varepsilon_{j_0}^{k+1} < \varepsilon_{j_0}^k$.

We set $\gamma := < n_0, n_1, \ldots >$ and $x := s_0^\frown (j_0 + 1 + k + \eta_k)_{k\in\omega}$, where $\eta_k \in \mathbb{Q}_+$ are chosen so that $\eta_0 := 0$ and $x \notin \{x_n \ / \ n \in \omega\}$. Then $d(x, x_{n_s\frown\gamma\lceil m}) = 2^{\varepsilon_{j_0}^m - j_0 - 1}$ decreases to $r > 0$, and the sequence $(x_{n_s\frown\gamma\lceil m})_{m\in\omega}$ does not tend to x. But $x \in \bigcap_{m\in\omega} B_{s\frown\gamma\lceil m}$; thus $s_{m+|s|}[x, D] = x_{n_s\frown\gamma\lceil m}$ and the sequence $(s_m[x, D])_{m\in\omega}$ does not tend to x. But this is absurd. $\square$

4 Study of the uniformity of the dense sequence.

(A) Necessary conditions for uniform recoverability.

It is natural to wonder whether there exists a dense sequence (x_p) of X such that every Baire class one function from X into Y is first return recoverable with respect to (x_p). The answer is no when X is uncountable. Indeed, if we choose $x \in X \setminus \{x_p \ / \ p \in \omega\}$, then $\mathbb{I}_{\{x\}}$ is not first return recoverable with respect to (x_p). We can wonder whether (x_p) exists for a set of Baire class one functions.

Notation $\mathcal{B}_1(X, Y)$ is the set of Baire class one functions from X into Y, and is equipped with the pointwise convergence topology.

If A is a subset of $\mathcal{B}_1(X, Y)$, then the map

$$\phi : \begin{cases} X \times A \to Y \\ (x, f) \ \mapsto f(x) \end{cases}$$

has its partial functions $\phi(x, .)$ (respectively $\phi(., f)$) continuous (respectively Baire class one). Therefore ϕ is Baire class two if A is a metrizable separable space (see p 378 in [Ku]).

Definition 13 *We will say that $A \subseteq \mathcal{B}_1(X, Y)$ is $uniformly\ recoverable$ if there exists a dense sequence (x_p) of X such that every function of A is recoverable with respect to (x_p).*

Proposition 14 *If A is uniformly recoverable and compact, then A is metrizable.*

Proof. Let $D := (x_p)$ be a dense sequence of X such that every function of A is recoverable with respect to D. Let $I : A \to Y^\omega$ defined by $I(f) := \big(f(x_p)\big)_p$. This map is continuous by definition of the pointwise convergence topology. It is one-to-one because, if $f \neq g$ are in A, then there is $p \in \omega$ such that $f(x_p) \neq g(x_p)$. Indeed, if this were not the case, then we would have, for each x in X,

$$f(x) = \lim_{n\to\infty} f(s_n[x, D]) = \lim_{n\to\infty} g(s_n[x, D]) = g(x)$$

(because f and g are recoverable with respect to (x_p)). As A is compact, I is a homeomorphism from A onto a subset of Y^ω. Therefore, A is metrizable. $\square$

Example. There are some separable compact spaces which are not metrizable, and whose points are G_δ. For example, "split interval" $A := \{f : [0,1] \to 2 \ / \ f$ is increasing$\}$, viewed as a subset of $\mathcal{B}_1([0,1],2)$, is one of them (see [T]). A is compact because it is a closed subset of $2^{[0,1]}$:

$$f \in A \Leftrightarrow \forall\, x \leq y \ \ f(x) = 0 \text{ or } f(y) = 1.$$

A is separable because $\{\mathbf{1}_{[q,1]} \ / \ q \in [0,1] \cap \mathbb{Q}\} \cup \{\mathbf{1}_{]q,1]} \ / \ q \in [0,1] \cap \mathbb{Q}\}$ is a countable dense subset of A. The family of continuous functions $\phi_x : f \mapsto f(x)$ separates points, and for every sequence $(x_n) \subseteq [0,1]$, $(\phi_{x_n})_n$ does not separates points. Thus A is not analytic and not metrizable (see Corollary 1 page 77 in Chapter 9 of [Bo2]). Finally, every point of A is G_δ; for example, $\{\mathbf{1}_{[x,1]}\} = \bigcap_{n\in\omega, x\geq 2^{-n}} \{f \in A \ / \ f(x - 2^{-n}) \neq 1\} \cap \{f \in A \ / \ f(x) \neq 0\}$. By Proposition 14, "split interval" is not uniformly recoverable.

Proposition 15 *If A is uniformly recoverable and Y is a 0-dimensional space, then ϕ is Baire class one.*

Proof. Let F be a closed subset of Y. We have $\phi(x,f) \in F \Leftrightarrow x \in f^{-1}(F)$. Remember the proof of Lemma 5. We replace the O_k's by a sequence of clopen subsets of Y whose intersection is F (it exists because Y is a 0-dimensional space). The sequence $(x_{p_j})_j$ is finite or infinite and enumerates in a one-to-one way the elements of $(x_p) \cap f^{-1}(O_k)$. We have $U_j := \{t \in X \ / \ x_{p_j} \in \mathcal{R}(t,D)\}$ if x_{p_j} exists ($U_j := \emptyset$ otherwise), and $H_k := \bigcap_{i\in\omega} [(\bigcup_{j\geq i} U_j) \cup \{x_{p_0}, ..., x_{p_{i-1}}\}]$ (in fact, between braces we have the x_{p_j} that exist, for $j < i$). So that $f^{-1}(F) = \bigcap_{k\in\omega} H_k$. The sequence $(x_{p_j})_j$ can be defined as follows, by induction on integer j:

$$q = p_0 \ \ \Leftrightarrow f(x_q) \in O_k \text{ and } \forall\, l < q \ \ f(x_l) \notin O_k$$

$$q = p_{j+1} \Leftrightarrow \forall\, l < q\ x_l \neq x_q \text{ and } \exists\, r < q \ \ (r = p_j \text{ and } f(x_q) \in O_k \text{ and } \forall\, l \in]r,q[\cap\omega \ f(x_l) \notin O_k)$$

We notice that the relation "$q = p_j$" is clopen in f. Then we notice that

$$x \in (\bigcup_{j\geq i} U_j) \cup \{x_{p_0}, ..., x_{p_{i-1}}\}$$

if and only if $[\exists j \geq i \ \ \exists q \in \omega \ \ q = p_j$ and $x_q \in \mathcal{R}(x,D)]$ or $\exists r \leq i$ such that $[(\forall m < r \ \ \exists q \in \omega \ \ q = p_m)$ and $(\forall m \in [r,i[\cap\omega \ \ \forall q \in \omega \ \ q \neq p_m)$ and $(\exists m < r \ \ \forall q \in \omega \ \ q \neq p_m$ or $x = x_q)]$. We can deduce from this that the relation "$x \in (\bigcup_{j\geq i} U_j) \cup \{x_{p_0}, ..., x_{p_{i-1}}\}$" is G_δ in (x,f); thus the relation "$x \in H_k$" is too. □

Corollary 16 *(a) There exists a continuous injection $I : 2^\omega \to \mathcal{B}_1(2^\omega, 2)$ such that $I[2^\omega]$ is not uniformly recoverable (and in fact such that $\phi \notin \mathcal{B}_1(2^\omega \times I[2^\omega], 2)$).*
(b) There exists $A \subseteq \mathcal{B}_1(2^\omega, 2)$, $A \approx \omega^\omega$, which is not uniformly recoverable and such that ϕ is in $\mathcal{B}_1(2^\omega \times A, 2)$.

Proof. (a) Let $\mathcal{S} := \{s \in 2^{<\omega} / s = \emptyset$ or $[s \neq \emptyset$ and $s(|s|-1) = 1]\}$ and

$$I(\alpha) := \begin{cases} 2^\omega \to 2 \\ \beta \ \ \mapsto \begin{cases} 1 \text{ if } \ \exists s \in \mathcal{S} \ \ [s \prec \alpha \text{ and } \beta = s^\frown 0^\omega], \\ 0 \text{ otherwise.} \end{cases} \end{cases}$$

If $\alpha\lceil n = \alpha'\lceil n$ and $\alpha(n) = 1 - \alpha'(n) = 0$, then $I(\alpha)(\alpha\lceil n^\frown 10^\omega) = 0 = 1 - I(\alpha')(\alpha\lceil n^\frown 10^\omega)$. Thus I is one-to-one.

It is continuous because

$$I(\alpha)(\beta)=1\ \Leftrightarrow\ \begin{cases}\alpha\in\emptyset\ \text{ if }\ \beta\in P_\infty:=\{\alpha\in 2^\omega\ /\ \forall n\ \exists m\geq n\ \ \alpha(m)=1\},\\ \\ \alpha\in N_s\ \text{ if }\ \beta=s^\frown 0^\omega\ \text{ and }\ s\in\mathcal{S}.\end{cases}$$

Moreover, $\{\beta\in 2^\omega\ /\ I(\alpha)(\beta)=1\}=\{0^\omega\}\cup\bigcup_{n/\alpha(n)=1}\ \{\alpha\lceil(n+1)^\frown 0^\omega\}\in D_2(\mathbf{\Sigma}^0_1)(2^\omega)$, thus $I[2^\omega]\subseteq\mathcal{B}_1(2^\omega,2)$. Let us argue by contradiction. We have

$$\phi^{-1}(\{0\})\equiv(P_\infty\times 2^\omega)\cup(\bigcup_{s\in\mathcal{S}}\ \{s^\frown 0^\omega\}\times\check{N}_s)=\bigcup_n F_n\in\mathbf{\Sigma}^0_2(2^\omega\times 2^\omega).$$

The diagonal of P_∞ is a subset of $\phi^{-1}(\{0\})$, so there exists an integer n such that $\Delta(P_\infty)\cap F_n$ is not meager in $\Delta(P_\infty)$. Therefore there exists a sequence s in $\mathcal{S}\setminus\{\emptyset\}$ such that $\Delta(N_s\cap P_\infty)\subseteq F_n$. Thus $\Delta(N_s)\subseteq F_n$ and $(s^\frown 0^\omega,s^\frown 0^\omega)\in\phi^{-1}(\{0\})$, which is absurd.

(b) Let $A:=I[P_\infty]$. As I is a homeomorphism from 2^ω onto its range and $P_\infty\approx\omega^\omega$, we have $A\approx\omega^\omega$. We have $F:=\phi^{-1}(\{1\})\cap(2^\omega\times A)\equiv\bigcup_{s\in\mathcal{S}}\ \{s^\frown 0^\omega\}\times(N_s\cap P_\infty)$. Let us show that $\overline{F}^{2^\omega\times P_\infty}\subseteq F\cup\Delta(P_\infty)$. Then $F=\overline{F}^{2^\omega\times P_\infty}\setminus\Delta(P_\infty)$ will be $D_2(\mathbf{\Sigma}^0_1)(2^\omega\times A)\subseteq\mathbf{\Delta}^0_2(2^\omega\times A)$. As $\phi^{-1}(\{0\})\cap(2^\omega\times A)=(2^\omega\times A)\setminus\phi^{-1}(\{1\})$, we will have $\phi\in\mathcal{B}_1(2^\omega\times A,2)$. If $(s_n^\frown 0^\omega,s_n^\frown\gamma_n)\in F$ tends to $(\beta,\alpha)\in(2^\omega\times P_\infty)\setminus F$, we may assume that $|s_n|$ increases strictly. So for each integer p and for n big enough we have $\beta(p)=s_n(p)=\alpha(p)$. Thus $\alpha=\beta$.

If A were uniformly recoverable, we could find a dense sequence $D:=(x_p)$ of 2^ω such that every function of A is recoverable with respect to (x_p). Let $s\in\mathcal{S}$. Then $I(s^\frown 1^\omega)$ is in A, and it is the characteristic function of the following set:

$$\{s\lceil n^\frown 0^\omega\ /\ n=0\ \text{ or }\ (0<n\leq|s|\ \text{ and }\ s(n-1)=1)\}\cup\{s^\frown 1^{p+1}0^\omega\ /\ p\in\omega\}.$$

For n big enough, $s_n[s^\frown 0^\omega,D]$ is in this set, thus $s^\frown 0^\omega\in D$ and $P_f:=2^\omega\setminus P_\infty\subseteq D$. So the functions of $I[2^\omega]$ are all recoverable with respect to D. But this contradicts the previous point. $\square$

(B) Study of the link between recoverability and ranks on Baire class one functions.

So there exists a metrizable compact set of characteristic functions of $D_2(\mathbf{\Sigma}^0_1)$ sets which is not uniformly recoverable. So the boundedness of the complexity of functions of A does not insure that A is uniformly recoverable. Notice that the example of the "split interval" is another proof of this, in the case where the compact space is not metrizable. Indeed, functions of the "split interval" are characteristic functions of open or closed subsets of $[0,1]$ (of the form $]a,1]$ or $[a,1]$, with $a\in[0,1[$).

In [B2], the author introduces a rank which measures the complexity of numeric Baire class one functions defined on a metrizable compact space. Let us recall this definition, which makes sense for functions defined on a Polish space X which is not necessarily compact.

• Let $\mathcal{A}$ and $\mathcal{B}$ be two disjoint G_δ subsets of X, and $R(\mathcal{A},\mathcal{B})$ be the set of increasing sequences $(G_\alpha)_{\alpha\leq\beta}$ of open subsets of X, with $\beta<\omega_1$, which satisfy

1. $G_{\alpha+1}\setminus G_\alpha$ is disjoint from $\mathcal{A}$ or from $\mathcal{B}$ if $\alpha<\beta$.
2. $G_\gamma=\cup_{\alpha<\gamma}\, G_\alpha$ if $0<\gamma\leq\beta$ is a limit ordinal.
3. $G_0=\emptyset$ and $G_\beta=X$.

Then $R(\mathcal{A},\mathcal{B})$ is not empty, because $\mathcal{A}$ and $\mathcal{B}$ can be separated by a $\mathbf{\Delta}^0_2$ set, which is of the form

$$D_\xi((U_\alpha)_{\alpha<\xi}):=\bigcup_{\alpha<\xi \text{ with parity opposite to that of } \xi} U_\alpha\setminus(\,\cup_{\theta<\alpha}\, U_\theta),$$

where $(U_\alpha)_{\alpha<\xi}$ is an increasing sequence of open subsets of X and $1\leq\xi<\omega_1$ (see [Ke]). Then we check that $(G_\alpha)_{\alpha\leq\xi+1}\in R(\mathcal{A},\mathcal{B})$, where $G_{\alpha+1}:=U_\alpha$ if $\alpha<\xi$.

• We set $L(\mathcal{A},\mathcal{B}):=\min\{\beta<\omega_1\ /\ \exists\ (G_\alpha)_{\alpha\leq\beta}\in R(\mathcal{A},\mathcal{B})\}$. If $f\in\mathcal{B}_1(X,\mathbb{R})$ and $a<b$ are real numbers, we let $L(f,a,b):=L(\{f\leq a\},\{f\geq b\})$. Finally,

$$L(f):=\sup\{L(f,q_1,q_2)\ /\ q_1<q_2\in\mathbb{Q}\ \}.$$

In [B2], the author shows that, if $A\subseteq\mathcal{C}(X,\mathbb{R})$ is relatively compact in $\mathcal{B}_1(X,\mathbb{R})$, then

$$\sup\{L(f,a,b)\ /\ f\in\overline{A}^{\text{p.c.}}\}<\omega_1$$

if X is a compact space and if $a<b$ are real numbers. He wonders whether his result remains true for a separable compact subspace A of $\mathcal{B}_1(X,\mathbb{R})$.

We can ask the question of the link between uniform recoverability of A and the fact that

$$\sup\{L(f)/f\in A\}<\omega_1.$$

If $D_\xi(\mathbf{\Sigma}^0_1)(X):=\{D_\xi((U_\eta)_{\eta<\xi})/(U_\eta)_{\eta<\xi}\subseteq\mathbf{\Sigma}^0_1(X)\text{ increasing}\}$ and $A\in D_\xi(\mathbf{\Sigma}^0_1)(X)$, one has $\check{\mathcal{A}}\in D_{\xi+1}(\mathbf{\Sigma}^0_1)(X)$ and $L(\check{\mathcal{A}},\mathcal{A})\leq\xi+2$ by the previous facts. So the rank of the characteristic function of $\mathcal{A}$ is at most $\xi+2$. In the case of the example in Corollary 16 and of the "split interval", one has $\sup\{L(f)\ /\ f\in A\}\leq 4<\omega_1$. Therefore, the fact that L is bounded on A does not imply uniform recoverability of A, does not imply that ϕ is Baire class one, and does not imply that A is metrizable. But we have the following result. It is a partial answer to J. Bourgain's question.

Proposition 17 *If X is a Polish space, $Y\subseteq\mathbb{R}$ and $A\subseteq\mathcal{B}_1(X,Y)$ is a Polish space, then we have sup$\{L(f)\ /\ f\in A\}<\omega_1$.*

Proof. Let $a<b$ be real numbers, $\mathcal{A}:=\{(x,f)\in X\times A\ /\ f(x)\leq a\}$ and

$$\mathcal{B}:=\{(x,f)\in X\times A\ /\ f(x)\geq b\}.$$

As ϕ is Baire class two, $\mathcal{A}$ and $\mathcal{B}$ are $\mathbf{\Pi}^0_3(X\times A)$ with horizontal sections in $\mathbf{\Pi}^0_2(X)$.

So there exists a finer Polish topology $\tau_{\mathcal{A}}$ on A such that $\mathcal{A} \in \mathbf{\Pi}^0_2(X \times [A, \tau_{\mathcal{A}}])$ (see [L1]). The same thing is true for $\mathcal{B}$. Let τ be a Polish topology on A, finer than $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$ (see Lemma 13.3 in [Ke]). As $\mathcal{A}$ and $\mathcal{B}$ are disjoint, there exists $\Delta_{a,b} \in \mathbf{\Delta}^0_2(X \times [A, \tau])$ which separates $\mathcal{A}$ from $\mathcal{B}$. Let $\xi_{a,b} < \omega_1$ be such that $\Delta_{a,b} \in D_{\xi_{a,b}}(\mathbf{\Sigma}^0_1)(X \times [A, \tau])$. For each function f of A, the set $\Delta^f_{a,b}$ is a $D_{\xi_{a,b}}(\mathbf{\Sigma}^0_1)(X)$ which separates $\{f \leq a\}$ from $\{f \geq b\}$. Thus $L(\{f \leq a\}, \{f \geq b\}) \leq \xi_{a,b} + 1$. Therefore $\sup\{L(f) \ / \ f \in A\} \leq \sup\{L(f, a, b) \ / \ a < b \in \mathbb{Q}\} < \omega_1$. □

Corollary 18 *If X is a Polish space, $Y \subseteq \mathbb{R}$ and if $A \subseteq \mathcal{B}_1(X, Y)$ is uniformly recoverable and compact, then sup$\{L(f) \ / \ f \in A\} < \omega_1$.*

We can wonder whether this result is true for the set of recoverable functions with respect to a dense sequence of X. We will see that it is not the case.

Proposition 19 *Let (x_p) be a dense sequence of a nonempty perfect Polish space X, and $Y := 2$. Then sup$\{L(f) \ / \ f$ is recoverable with respect to $(x_p)\} = \omega_1$.*

Proof. Set D of the elements of the dense sequence is countable, metrizable, nonempty and perfect. Indeed, if x_p is an isolated point of D, then it is also isolated in X, which is absurd. Thus D is homeomorphic to $\mathbb{Q}$ (see 7.12 in [Ke]). For $1 \leq \xi < \omega_1$, there exists a countable metrizable compact space K_ξ and $\mathcal{A}_\xi \in D_\xi(\mathbf{\Sigma}^0_1)(K_\xi) \setminus \check{D}_\xi(\mathbf{\Sigma}^0_1)(K_\xi)$ (see [LSR]). So we may assume that $K_\xi \subseteq D$ (see 7.12 in [Ke]). Thus we have $\mathcal{A}_\xi \notin \check{D}_\xi(\mathbf{\Sigma}^0_1)(X)$. We will deduce from this the fact that $L(\mathbb{1}_{\mathcal{A}_{\xi+1}}) > \xi$.

To see this, let us show that, if $L(\mathbb{1}_{\mathcal{A}}) = L(\check{\mathcal{A}}, \mathcal{A}) \leq \xi$, then $\mathcal{A} \in \check{D}_{\xi+1}(\mathbf{\Sigma}^0_1)(X)$. Let $(G_\alpha)_{\alpha \leq \xi'}$ be in $R(\check{\mathcal{A}}, \mathcal{A})$, where $\xi' \in \{\xi, \xi + 1\}$ is odd. We let, for $\alpha < \xi'$,

$$U_\alpha := \begin{cases} \bigcup_{\theta<\alpha} U_\theta \cup \bigcup_{\theta \leq \alpha / \mathcal{A} \cap G_{\theta+1} \setminus G_\theta = \emptyset} G_{\theta+1} & \text{if } \alpha \text{ is even,} \\ \\ \bigcup_{\theta<\alpha} U_\theta \cup \bigcup_{\theta \leq \alpha / \check{\mathcal{A}} \cap G_{\theta+1} \setminus G_\theta = \emptyset} G_{\theta+1} & \text{if } \alpha \text{ is odd.} \end{cases}$$

Then $D_{\xi'}((U_\alpha)_{\alpha<\xi'})$ separates $\check{\mathcal{A}}$ from $\mathcal{A}$. Indeed, if $x \notin \mathcal{A}$, let $\alpha' \leq \xi'$ be minimal such that $x \in G_{\alpha'}$. Then α' is the successor of $\alpha < \xi'$, and $x \in \check{\mathcal{A}} \cap G_{\alpha+1} \setminus G_\alpha$. So $\mathcal{A} \cap G_{\alpha+1} \setminus G_\alpha = \emptyset$, by condition 1. If α is even, then $x \in U_\alpha \setminus (\cup_{\theta<\alpha} U_\theta)$ because $U_\theta \subseteq G_{\theta+1}$ if $\theta < \xi'$. If α is odd, then $x \in U_{\alpha+1} \setminus U_\alpha$. In both cases, $x \in D_{\xi'}((U_\alpha)_{\alpha<\xi'})$. If $x \in U_\alpha \setminus (\cup_{\theta<\alpha} U_\theta)$ with $\alpha < \xi'$ even, there exists $\theta \leq \alpha$ such that $x \in G_{\theta+1}$ and $\mathcal{A} \cap G_{\theta+1} \setminus G_\theta = \emptyset$. Let $\eta' \leq \xi'$ be minimal such that $x \in G_{\eta'}$. As before, η' is the successor of $\eta < \xi'$. Let us argue by contradiction: we assume that $x \in \mathcal{A}$. Then $x \in \mathcal{A} \cap G_{\eta+1} \setminus G_\eta \neq \emptyset$, so $\check{\mathcal{A}} \cap G_{\eta+1} \setminus G_\eta = \emptyset$. If η is odd, then $x \in U_\eta$, thus $\eta = \alpha$. This contradicts the parity of α. If η is even, then $x \in U_{\eta+1}$ and $\eta = \alpha = \theta$. So $x \in G_{\theta+1} \setminus G_\theta \subseteq \check{\mathcal{A}}$. This is the contradiction we were looking for.

It remains to check that $\mathbb{1}_{\mathcal{A}_{\xi+1}}$ is recoverable. If $x \in D$, then $s_n[x, D] = x$ for almost all integer n. Thus $\mathbb{1}_{\mathcal{A}_{\xi+1}}(s_n[x, D])$ tends to $\mathbb{1}_{\mathcal{A}_{\xi+1}}(x)$. If x is not in D, then $x \notin \mathcal{A}_{\xi+1} \subseteq K_{\xi+1} \subseteq D$. So, from some point on, $s_n[x, D] \notin K_{\xi+1}$, and $\mathbb{1}_{\mathcal{A}_{\xi+1}}(s_n[x, D])$ is ultimately constant and tends to $\mathbb{1}_{\mathcal{A}_{\xi+1}}(x)$.□

Remark. We can find in [KL] the study of some other ranks on Baire class one functions. The rank L is essentially the separation rank defined in this paper. In the case where X is a metrizable compact space and where the Baire class one functions considered are bounded, Propositions 17, 19 and Corollary 18 remain valid for these other ranks.

(C) Sufficient conditions for uniform recoverability.

Theorem 20 *Assume that Y is a metric space, and that A, equipped with the compact open topology, is a separable subset of $\mathcal{B}_1(X,Y)$. Then A is uniformly recoverable.*

Proof. Let (l_q) be a dense sequence of A for the compact open topology. By the lemma showed in [Ku], page 388, for each integer q there exists a sequence $(h^q_n)_n \subseteq \mathcal{B}_1(X,Y)$ which uniformly tends to l_q, functions h^q_n having a discrete range. Enumerating the sequence $(h^q_n)_{n,q}$, we get $(h_n)_n$. Every function of A is in the closure of this sequence for the compact open topology. For each integer n, one can get a countable partition $(B^n_p)_p$ of X into $\mathbf{\Delta}^0_2$ sets on which h_n is constant. Express each of these sets as a countable union of closed sets. Putting all these closed sets together gives a countable sequence of closed subsets of X. As in the proof of Theorem 4, this gives D which approximates each of these closed sets. Now let $f \in A$, $x \in X$ and $\varepsilon > 0$. Consider the compact subset $K := \mathcal{R}(x,D) \cup \{x\}$ of X. By uniform convergence on K, there exists $N \in \omega$ such that, for each t in K, we have $d_Y(f(t), h_N(t)) < \varepsilon/2$. Let p be an integer such that $x \in B^N_p$. Now $K \setminus B^N_p$ is finite and we have $h_N(s_n[x,D]) = h_N(x)$ for each $n \in \omega$, except maybe a finite number of them. So we have the following inequality, for all but finitely many n:

$$d_Y(f(x), f(s_n[x,D])) \leq \\ d_Y(f(x), h_N(x)) + d_Y(h_N(x), h_N(s_n[x,D])) + d_Y(h_N(s_n[x,D]), f(s_n[x,D])) < \varepsilon$$

(this last argument is essentially in [DE]). □

The following corollary has been showed in [FV] when $X = \mathbb{R}$ and with another way of extracting the subsequence.

Corollary 21 *Let $A \subseteq \mathcal{B}_1(X,Y)$ be countable. Then A is uniformly recoverable.*

Proof. Put a compatible distance on Y. □

Proposition 22 *Let (Y_p) be a basis for the topology of Y, and*

(1) For each integer p, $\phi^{-1}(Y_p) \in (\mathbf{\Pi}^0_1(X) \times \mathcal{P}(A))_\sigma$.

(2) There exists a finer metrizable separable topology on X, made of $\mathbf{\Sigma}^0_2(X)$, and making functions of A continuous.

(3) A is uniformly recoverable.

Then (1) ⇔ (2) ⇒ (3).

Proof. (1) ⇒ (2) We have $\phi^{-1}(Y_p) = \bigcup_{n\in\omega} F^p_n \times B^p_n$, where F^p_n is a closed subset of X and $B^p_n \subseteq A$. If $f \in A$, then $f^{-1}(Y_p) = \phi^{-1}(Y_p)^f = \bigcup_{n/f\in B^p_n} F^p_n$. Therefore, it is enough to find a finer metrizable separable topology on X, made of $\mathbf{\Sigma}^0_2(X)$, and making the F^p_n's open. Let (X_n) be a basis for the topology of X, closed under finite intersections, and (G_q) be the sequence of finite intersections of F^p_n's. Then set τ of unions of sets of the form X_n or $X_n \cap G_q$ is a topology, with a countable basis, made of $\mathbf{\Sigma}^0_2(X)$, finer than the initial topology on X (thus Hausdorff), and makes the F^p_n's open. It remains to check that it is regular.

So let $x \in X$ and $F \in \mathbf{\Pi}^0_1(X, \tau)$, with $x \notin F$. We have $X \setminus F = \bigcup_p X_{n_p} \cup \bigcup_k X_{m_k} \cap G_{q_k}$. Either there exists p such that $x \in X_{n_p}$; in this case, by regularity of initial topology on X we can find two disjoint open sets V_1 and V_2 with $x \in V_1$ and $X \setminus X_{n_p} \subseteq V_2$. But these two open sets are τ-open and $F \subseteq X \setminus X_{n_p} \subseteq V_2$. Or there exists k such that $x \in X_{m_k} \cap G_{q_k}$; in this case, by regularity of initial topology on X, we can find two disjoint open sets W_1 and W_2 with $x \in W_1$ and $X \setminus X_{m_k} \subseteq W_2$. But then $W_1 \cap G_{q_k}$ and $W_2 \cup (X \setminus G_{q_k})$ are τ-open and disjoint, $x \in W_1 \cap G_{q_k}$ and $F \subseteq (X \setminus X_{m_k}) \cup (X \setminus G_{q_k}) \subseteq W_2 \cup (X \setminus G_{q_k})$.

(2) $\Rightarrow$ (3) Let τ be the finer topology. Then identity map from X, equipped with its initial topology, into X, equipped with τ, is Baire class one. Therefore, it is recoverable. So let (x_p) be a dense sequence of X such that, for each $x \in X$, $s_n[x, (x_p)]$ tends to x, in the sense of τ. Let $f \in A$. As f is continuous if X is equipped with τ, $f(s_n[x, (x_p)])$ tends to $f(x)$ for each $x \in X$. Therefore f is recoverable with respect to (x_p).

(2) $\Rightarrow$ (1) Let (X_n) be a basis for finer topology τ (therefore, we have $X_n \in \mathbf{\Sigma}^0_2(X)$). Let

$$C^p_n := \{f \in A \ / \ X_n \subseteq \phi^{-1}(Y_p)^f\}.$$

Then $\phi^{-1}(Y_p) = \bigcup_n X_n \times C^p_n \in (\mathbf{\Pi}^0_1(X) \times \mathcal{P}(A))_\sigma$. □

Remark. If X is a standard Borel space and A is a Polish space, conditions (1) and (2) of Proposition 22 are equivalent to "For each integer p, $\phi^{-1}(Y_p) \in (\mathbf{\Pi}^0_1(X) \times \mathbf{\Delta}^1_1(A))_\sigma$". Indeed, let P be a Polish space such that X is a Borel subset of P, and $f \in A$. As f is continuous if X is equipped with τ, $f^{-1}(Y_p) = \bigcup_k X_{n^{p,f}_k}$ for each integer p. Let $C^p_n := \{f \in A \ / \ X_n \subseteq \phi^{-1}(Y_p)^f\}$. Then C^p_n is $\mathbf{\Pi}^1_1(A)$, because ϕ is Baire class two:

$$f \in C^p_n \ \Leftrightarrow \ \forall x \in P \ \ x \notin X_n \ \text{ or } \ \phi(x, f) \in Y_p.$$

Moreover, $\phi^{-1}(Y_p) = \bigcup_n X_n \times C^p_n$. By $\mathbf{\Delta}^1_1$-selection (see 4B5 in [M]), there exists a Borel function $N_p : P \times A \to \omega$ such that $(x, f) \in X_{N_p(x,f)} \times C^p_{N_p(x,f)}$ if $f(x) \in Y_p$. Let

$$S^p_n := \{f \in A \ / \ \exists x \in X \ \ N_p(x, f) = n \ \text{ and } \ \phi(x, f) \in Y_p\}.$$

Then $S^p_n \in \mathbf{\Sigma}^1_1(A)$ and is a subset of C^p_n ; by the separation therem, there exists a Borel subset B^p_n of A such that $S^p_n \subseteq B^p_n \subseteq C^p_n$. Then we have $\phi^{-1}(Y_p) = \bigcup_n X_n \times B^p_n \in (\mathbf{\Pi}^0_1(X) \times \mathbf{\Delta}^1_1(A))_\sigma$.

Proposition 23 *If A has a countable basis, then there exists a finer metrizable separable topology on X making the functions of A continuous. Moreover, if X is Polish, we can have this topology Polish.*

Proof. Let (A_n) be a basis for the topology of A, and $X^p_n := \{x \in X / A_n \subseteq \phi^{-1}(Y_p)_x\}$. As

$$\phi^{-1}(Y_p)_x = \{f \in A \ / \ f(x) \in Y_p\} \in \mathbf{\Sigma}^0_1(A),$$

we have $\phi^{-1}(Y_p) = \bigcup_{n \in \omega} X^p_n \times A_n$ and $f^{-1}(Y_p) = \phi^{-1}(Y_p)^f = \bigcup_{n / f \in A_n} X^p_n$ for each $f \in A$. Thus it is enough to find a finer metrizable separable topology on X making X^p_n's open.

We use the same method as the one used to prove implication (1) $\Rightarrow$ (2) of Proposition 22. We notice that the algebra generated by the X^p_n's is countable (we let (G_q) be the elements of this algebra).

As ϕ is Baire class two, $\phi^{-1}(Y_p)$ is a $\mathbf{\Sigma}^0_3$ set with vertical sections in $\mathbf{\Sigma}^0_1(A)$. If X and A are Polish, we deduce from [L1] the existence of a finer Polish topology τ_p on X such that

$$\phi^{-1}(Y_p) \in (\mathbf{\Sigma}^0_1(X, \tau_p) \times \mathbf{\Sigma}^0_1(A))_\sigma.$$

Let $(B^p_n)_n$ be a basis for τ_p. Then there exists a finer Polish topology τ on X making the Borel sets B^p_n's open (see Exercises 15.4 and 13.5 in [Ke]). Then we are done, because τ is finer than the τ_p's.$\square$

Therefore, the problem is to find the finer topology in $\mathbf{\Sigma}^0_2(X)$. We have seen that it is not the case in general. If we look at Propositions 15 and 22, we can wonder whether conditions of Proposition 22 and the fact that ϕ is Baire class one are equivalent, especially in the case where Y is 0-dimensional. This question leads to the study of Borel subsets of $2^\omega \times 2^\omega$. The answer is no in general. First, because of Corollary 16. It shows that the fact that ϕ is Baire class one does not imply uniform recoverability (with A Polish, in fact homeomorphic to ω^ω). Secondly, let $A := \{f \in \mathcal{B}_1(2^\omega, 2) \ / \ f$ is recoverable with respect to $(x_p)\}$, where $(x_p) := P_f$ is dense in 2^ω. Then A is uniformly recoverable, but we cannot find a finer metrizable separable topology τ on 2^ω, made of $\mathbf{\Sigma}^0_2(2^\omega)$ and making the functions of A continuous. Otherwise, the characteristic functions of the compact sets $K_x := \{x\} \cup \{s_n[x, (x_p)] \ / \ n \in \omega\}$ would be continuous for τ, and this would contradict the Lindelöf property, with $\bigcup_{x \in P_\infty} K_x$. But A has no countable basis. Otherwise, the set of charateristic functions of the sets K_x (for $x \in P_\infty$) would also have one; this would contradict the Lindelöf property too (this last set is a subset of $\bigcup_{x \in P_\infty} \{f \in \mathcal{B}_1(2^\omega, 2) \ / \ f(x) = 1\}$). This leads us to assume that A is a K_σ and metrizable space, to hope for such an equivalence.

If ϕ is Baire class one, then $\phi^{-1}(Y_p)$ is a $\mathbf{\Sigma}^0_2$ subset of $X \times A$ with vertical sections in $\mathbf{\Sigma}^0_1(A)$. Thus it is natural to ask the

Question. Does every $\mathbf{\Sigma}^0_2$ subset of $X \times A$ with vertical sections in $\mathbf{\Sigma}^0_1(A)$ belong to the class $(\mathbf{\Pi}^0_1(X) \times \mathcal{P}(A))_\sigma$?

If the answer is yes, then the fact that ϕ is Baire class one implies condition (1) in Proposition 22, and the conditions of this proposition are equivalent to the fact that ϕ is Baire class one. The answer is negative, even if we assume that X and A are metrizable compact spaces:

Proposition 24 *There exists a $\check{D}_2(\mathbf{\Sigma}^0_1)$ subset of $2^\omega \times 2^\omega$ with vertical sections in $\mathbf{\Delta}^0_1(2^\omega)$ which is not $(\mathbf{\Pi}^0_1(2^\omega) \times \mathcal{P}(2^\omega))_\sigma$.*

Proof. Let $E := (P_\infty \times 2^\omega) \cup \bigcup_{s \in \mathcal{S}} \ \{s^\frown 0^\omega\} \times (\check{N}_s \cup N_{s^\frown 0})$ (we use again notations of the proof of Corollary 16). Clearly, vertical sections of E are $\mathbf{\Delta}^0_1(2^\omega)$. We set

$$G := \{\alpha \in 2^\omega \ / \ \forall n \ \exists m \geq n \ \ \alpha(m) = \alpha(m+1) = 1\}.$$

This is a dense G_δ subset of 2^ω, included in P_∞. If $\alpha \notin G$, then the horizontal section $\check{E}^\alpha$ is finite.

Otherwise, it is infinite and countable (it is a subset of P_f), and it is a sequence which tends to α. If $(s_n^\frown 0^\omega, s_n^\frown \gamma_n)_n \subseteq \check{E}$ tends to (β, α), then there are essentially two cases. Either the length of s_n is strictly increasing and $\alpha = \beta$. Or we may assume that (s_n) is constant and $(\beta, \alpha) \notin E$. As diagonal $\Delta(2^\omega) \subseteq E$, we can deduce from this that $\check{E} = \overline{\check{E}} \setminus \Delta(2^\omega) \in D_2(\mathbf{\Sigma}^0_1)(2^\omega \times 2^\omega)$. Assume that $E \in (\mathbf{\Pi}^0_1(2^\omega) \times \mathcal{P}(2^\omega))_\sigma$. We have $E = \bigcup_n F_n \times E_n$, where $F_n \in \mathbf{\Pi}^0_1(2^\omega)$ and $E_n \subseteq 2^\omega$. Let $C_n := \{\alpha \in 2^\omega \ / \ F_n \subseteq E^\alpha\}$. Then $C_n \in \mathbf{\Pi}^1_1(2^\omega)$ and $E = \bigcup_n F_n \times C_n$. As $\Delta(2^\omega) \subseteq E$, $2^\omega \subseteq \bigcup_n F_n \cap C_n$. So there exists an integer n such that $F_n \cap C_n$ is not meager, and a sequence $s \in 2^{<\omega}$ such that $N_s \cap F_n \cap C_n$ is a comeager subset of N_s. In particular, $N_s \subseteq F_n$. As G is comeager, there exists $\alpha \in G \cap N_s \cap C_n$. Let $(\beta_m) \subseteq \check{E}^\alpha$ converging to α. From some point m_0 on, we have $\beta_m \in N_s$. So $(\beta_m, \alpha) \in F_n \times C_n \subseteq E$ if $m \geq m_0$. But this is absurd because $(\beta_m, \alpha) \notin E$. $\square$

We can specify this result:

Proposition 25 *There exists a metrizable compact space $A \subseteq \mathcal{B}_1(2^\omega, 2)$ which is uniformly recoverable, but for which we cannot find any finer metrizable separable topology on 2^ω, made of $\mathbf{\Sigma}^0_2(2^\omega)$, making the functions of A continuous.*

Proof. We use again the notation of the proof of Proposition 24. Let $\psi : \omega \to \mathcal{S}$ be a bijective map such that for $s, t \in \mathcal{S}$, $s \prec_{\neq} t$ implies $\psi^{-1}(s) < \psi^{-1}(t)$. Such a bijection exists. Indeed, we take $\psi := (\theta \circ \phi_{\lceil \mathcal{S}})^{-1}$, where $\theta : \phi[\mathcal{S}] \to \omega$ is an increasing bijection, and where

$$\phi : \begin{cases} 2^{<\omega} \to \omega \\ s \quad \mapsto \begin{cases} 0 \text{ if } s = \emptyset, \\ q_0^{s(0)+1} \ldots q_{s(|s|-1)}^{s(|s|-1)+1} \text{ otherwise.} \end{cases} \end{cases}$$

(where (q_n) is sequence of prime numbers). We let $x_{2n} := \psi(n)^\frown 1^\omega$, $x_{2n+1} := \psi(n)^\frown 0^\omega$, and $x_s := x_{\min\{p \in \omega / s \prec x_p\}}$ if $s \in 2^{<\omega}$.

• Let us show that, if $s \in 2^{<\omega} \setminus \{\emptyset\}$, then $x_s \in P_\infty$ is equivalent to $s \in \mathcal{S}$. If $s \in \mathcal{S}$ and $x_s \in P_f$, then there exists u in $\mathcal{S}$ such that $x_s = s^\frown u^\frown 0^\omega$. Then $x_{2\psi^{-1}(s)}$ comes strictly before $x_{2\psi^{-1}(s)+1}$, which comes before $x_{2\psi^{-1}(s^\frown u)+1} = x_s$. But $s \prec s^\frown 1^\omega = x_{2\psi^{-1}(s)}$, which is absurd.

If $s \notin \mathcal{S}$ and $x_s \in P_\infty$, there exists u in $\mathcal{S}$ such that $x_s = s^\frown (1^{|u|} - u)^\frown 1^\omega$. Let $s' \in \mathcal{S}$ and m be an integer such that $s = s'^\frown 0^{m+1}$. Then $x_{2\psi^{-1}(s')+1}$ comes strictly before $x_{2\psi^{-1}(s^\frown (1^{|u|} - u)^\frown 1)}$, which comes before x_s. But $s \prec s^\frown 0^\omega = s'^\frown 0^\omega = x_{2\psi^{-1}(s')+1}$, which is absurd.

• We set

$$I : \begin{cases} 2^\omega \ \to \mathcal{B}_1(2^\omega, 2) \\ \alpha \quad \mapsto \begin{cases} 2^\omega \to 2 \\ \beta \ \mapsto \begin{cases} 0 \text{ if } \exists s \in \mathcal{S} \ \ \beta = s^\frown 0^\omega \text{ and } \alpha \in N_{s^\frown 1}, \\ 1 \text{ otherwise.} \end{cases} \end{cases} \end{cases}$$

Then I is defined because $\{\beta \in 2^\omega \ / \ \beta \notin I(\alpha)\}$ is $\overline{\{\beta \in 2^\omega \ / \ \beta \notin I(\alpha)\}} \setminus \{\alpha\} \in D_2(\mathbf{\Sigma}^0_1)(2^\omega)$ if $\alpha \in G$, and is finite otherwise. I is continuous because

$$I(\alpha)(\beta) = 1 \Leftrightarrow \begin{cases} \alpha \in 2^\omega \ \text{ if } \ \beta \in P_\infty, \\ \\ \check{N}_s \cup N_{s^\frown 0} \ \text{ if } \ \beta = s^\frown 0^\omega \ \text{ and } \ s \in \mathcal{S}. \end{cases}$$

Therefore, $A := I[2^\omega]$ is an analytic compact space and is metrizable.

As $E = (\mathrm{Id}_{2^\omega} \times I)^{-1}(\phi^{-1}(\{1\}))$, $\phi^{-1}(\{1\}) \notin (\mathbf{\Pi}^0_1(2^\omega) \times \mathcal{P}(A))_\sigma$, by Proposition 24. So there is no finer metrizable separable topology on 2^ω, made of $\mathbf{\Sigma}^0_2(2^\omega)$ and making the functions of A continuous, by Proposition 22. But A is uniformly recoverable with respect to (x_p).

Indeed, as $P_f \subseteq (x_p)$, it is enough to see that if $\alpha \in G$, then $I(\alpha)$ is recoverable with respect to $D := (x_p)$. The only thing to see is that from some integer n_0 on, $s_n[\alpha, D] \in E^\alpha$. We may assume that $\alpha \notin D$ because $G \subseteq P_\infty$.

We take $(W_m) := (N_s)_{s \in 2^{<\omega}}$ as a good basis for the topology of 2^ω. So that, if $\alpha \notin D$,

$$\begin{aligned} s_{n+1}[\alpha, D] \ &= x_{\min\{p \in \omega / \exists s \in 2^{<\omega} \ \alpha, x_p \in N_s \subseteq 2^\omega \setminus \{s_0[\alpha, D], \ldots, s_n[\alpha, D]\}\}} \\ &= x_{\min\{p \in \omega / \alpha \lceil (\max_{q \leq n} |\alpha \wedge s_q[\alpha, D]| + 1) \prec x_p\}}. \end{aligned}$$

But as the sequence $(|\alpha \wedge s_n[\alpha, D]|)_n$ is strictly increasing, $\max_{q \leq n} |\alpha \wedge s_q[\alpha, D]| = |\alpha \wedge s_n[\alpha, D]|$. Thus $s_{n+1}[\alpha, D] = x_{\alpha \lceil (|\alpha \wedge s_n[\alpha, D]| + 1)}$. By the previous facts, it is enough to get

$$x_{\alpha \lceil (|\alpha \wedge s_n[\alpha, D]| + 1)} \in E^\alpha.$$

Let $M_n := |\alpha \wedge s_n[\alpha, D]|$. If $\alpha(M_n) = 1$, then $s_{n+1}[\alpha, D]$ is in $P_\infty \subseteq E^\alpha$. Otherwise,

$$s_{n+1}[\alpha, D] = \alpha \lceil (M_n + 1)^\frown u^\frown 0^\omega,$$

where $u \in \mathcal{S}$. If $u \neq \emptyset$, then $s_{n+1}[\alpha, D]$ is minimal in $N_{\alpha \lceil (M_n+1)^\frown u} \subseteq N_{\alpha \lceil (M_n+1)}$, so $s_{n+1}[\alpha, D]$ is in P_∞, which is absurd. Thus $u = \emptyset$ and $s_{n+1}[\alpha, D] \in E^\alpha$. □

Now we will see some positive results for the very first classes of Borel sets. We know (see [L1]) that if X and A are Polish spaces, then every Borel subset of $X \times A$ with vertical sections in $\mathbf{\Sigma}^0_1(A)$ is $(\mathbf{\Delta}^1_1(X) \times \mathbf{\Sigma}^0_1(A))_\sigma$.

Proposition 26 *If A has a countable basis, then every $\mathbf{\Pi}^0_1(X \times A)$ with vertical sections in $\mathbf{\Sigma}^0_1(A)$ is $(\mathbf{\Pi}^0_1(X) \times \mathbf{\Sigma}^0_1(A))_\sigma$. If moreover A is 0-dimensional, then every $D_2(\mathbf{\Sigma}^0_1)(X \times A)$ with vertical sections in $\mathbf{\Sigma}^0_1(A)$ is $(\mathbf{\Pi}^0_1(X) \times \mathbf{\Delta}^0_1(A))_\sigma$.*

Proof. Let F be a closed subset of $X \times A$ with vertical sections in $\mathbf{\Sigma}^0_1(A)$. As in the proof of Proposition 23, $F = \bigcup_n X_n \times A_n$, where (A_n) is a basis for the topology of A. But as F is closed, we also have $F = \bigcup_n \overline{X_n} \times A_n \in (\mathbf{\Pi}^0_1(X) \times \mathbf{\Sigma}^0_1(A))_\sigma$.

If A is a 0-dimensional space, let U (respectively F) be an open (respectively closed) subset of $X \times A$ such that $U \cap F$ has vertical sections in $\mathbf{\Sigma}^0_1(A)$; then $U = \bigcup_n U_n$, where

$$U_n \in \mathbf{\Pi}^0_1(X) \times \mathbf{\Delta}^0_1(A).$$

For each $x \in X$, we have

$$(U \cap F)_x = U_x \cap F_x = \bigcup_n (U_n)_x \cap F_x = \bigcup_n (U_n \cap F)_x.$$

Moreover, $(U_n \cap F)_x = (U_n)_x \cap (U \cap F)_x$ is $\mathbf{\Sigma}^0_1(A)$, so $U_n \cap F$ is $\mathbf{\Pi}^0_1(X \times A)$ with vertical sections in $\mathbf{\Sigma}^0_1(A)$. By the previous facts, $U_n \cap F \in (\mathbf{\Pi}^0_1(X) \times \mathbf{\Delta}^0_1(A))_\sigma$ and $U \cap F = \bigcup_n U_n \cap F$ too. $\square$

Proposition 27 *There exists a $\check{D}_2(\mathbf{\Sigma}^0_1)$ subset of $2^\omega \times 2^\omega$ with sections in $\mathbf{\Delta}^0_1(2^\omega)$ which is not in $(\mathbf{\Pi}^0_1(2^\omega) \times \mathbf{\Sigma}^0_1(2^\omega))_\sigma$.*

Proof. This result is a consequence of Proposition 24. But we can find here a simpler counter-example. We will use it later. Let $\psi : \omega \to P_f$ be a bijective map, and

$$E := (2^\omega \times \{0^\omega\}) \cup \bigcup_p (2^\omega \setminus \{\psi(p)\} \times N_{0^p1}).$$

Then E is the union of a closed set and of an open set, so it is $\check{D}_2(\mathbf{\Sigma}^0_1)(2^\omega \times 2^\omega)$. If $\alpha \notin P_f$ (respectively $\alpha = \psi(p)$), then we have $E_\alpha = 2^\omega$ (respectively $2^\omega \setminus N_{0^p1}$); so E has vertical sections in $\mathbf{\Delta}^0_1(2^\omega)$. If $E = \bigcup_n F_n \times U_n$, then we have $E^{0^\omega} = 2^\omega = \bigcup_{n/0^\omega \in U_n} F_n$. By Baire's theorem, there exists $s \in 2^{<\omega}$ and an integer n_0 such that $0^\omega \in U_{n_0}$ and $N_s \subseteq F_{n_0}$. From some integer p_0 on, we have $N_{0^p1} \subseteq U_{n_0}$. As P_f is dense, there exists $p \geq p_0$ such that $\psi(p) \in N_s$. We have

$$(\psi(p), 0^p10^\omega) \in (N_s \times N_{0^p1}) \setminus E \subseteq (F_{n_0} \times U_{n_0}) \setminus E \subseteq E \setminus E.$$

This finishes the proof. $\square$

Now we will show that the example in Corollary 16 is in some way optimal. Recall that the Wadge hierarchy (the inclusion of classes obtained by continuous pre-images of a Borel subset of ω^ω; see [LSR]) is finer than that of Baire. The beginning of this hierarchy is the following:

$$\begin{array}{ccccccccc} \{\emptyset\} & & \mathbf{\Sigma}^0_1 & & D_2(\mathbf{\Sigma}^0_1) & & \mathbf{\Sigma}^0_2 \\ & \mathbf{\Delta}^0_1 & & \mathbf{\Sigma}^{0+}_1 & & \cdots & \\ \check{\{\emptyset\}} & & \mathbf{\Pi}^0_1 & & \check{D}_2(\mathbf{\Sigma}^0_1) & & \mathbf{\Pi}^0_2 \end{array}$$

The class $\mathbf{\Sigma}^{0+}_1$ is defined as follows: $\mathbf{\Sigma}^{0+}_1 := \{(U \cap O) \cup (F \setminus O) \ / \ U \in \mathbf{\Sigma}^0_1, F \in \mathbf{\Pi}^0_1,\ O \in \mathbf{\Delta}^0_1\}$.

Proposition 28 *Let A be a metrizable compact space, $B \subseteq X \times A$ with vertical (resp., horizontal) sections in $\mathbf{\Delta}^0_1(A)$ (resp., $\mathbf{\Sigma}^{0+}_1(X)$). Then $B \in (\mathbf{\Pi}^0_1(X) \times \mathcal{P}(A))_\sigma$. In particular, if $Y = 2$ and A is made of characteristic functions of $\mathbf{\Sigma}^{0+}_1(X)$, then conditions of Proposition 22 are satisfied and ϕ is Baire class one.*

Proof. For $f \in A$, we have $B^f = (U^f \cap O^f) \cup (F^f \setminus O^f)$. We set

$$B_1 := \{(x, f) \in X \times A \mathbin{/} x \in U^f \cap O^f\}, \qquad B_2 := \{(x, f) \in X \times A \mathbin{/} x \in F^f \setminus O^f\}.$$

Therefore we have $B = B_1 \cup B_2$. Let (X_n) be a basis for the topology of X. We have

$$B_1 = \bigcup_n X_n \times \{f \in A \mathbin{/} X_n \subseteq O^f \cap U^f\}.$$

Thus $B_1 \in (\mathbf{\Pi}^0_1(X) \times \mathcal{P}(A))_\sigma$. In the same way, $\{(x, f) \in X \times A \mathbin{/} x \notin O^f\} = \bigcup_n X_n \times E_n$, where $E_n := \{f \in A \mathbin{/} X_n \cap O^f = \emptyset\}$. Let us enumerate $\mathbf{\Delta}^0_1(A) := \{O_m \mathbin{/} m \in \xi\}$, where $\xi \in \omega + 1$. We have $B_2 = \bigcup_{n,m} \{x \in X_n \mathbin{/} O_m \cap E_n \subseteq B_x\} \times (O_m \cap E_n)$. It is enough to see that $\{x \in X_n / O_m \cap E_n \subseteq B_x\} \in \mathbf{\Pi}^0_1(X_n)$. Let $(f^n_p)_p$ be a dense sequence of E_n. If $x \in X_n$, then

$$\begin{aligned} O_m \cap E_n \subseteq B_x &\Leftrightarrow \forall p \in \omega \ \ f^n_p \notin O_m \cap E_n \ \text{ or } \ x \in B^{f^n_p} \\ &\Leftrightarrow \forall p \in \omega \ \ f^n_p \notin O_m \cap E_n \ \text{ or } \ x \in F^{f^n_p} \setminus O^{f^n_p}. \end{aligned}$$

Therefore, $B_2 \in (\mathbf{\Pi}^0_1(X) \times \mathcal{P}(A))_\sigma$ and B too. □

Proposition 29 *Assume that X and A are Polish spaces, that $Y = 2$, and that A is made of characteristic functions of $D_2(\mathbf{\Sigma}^0_1)(X)$. Then $\phi^{-1}(\{1\}) \in (\mathbf{\Pi}^0_1(X) \times \mathcal{P}(A))_\sigma$.*

Proof. As ϕ is Baire class two, $\phi^{-1}(\{1\})$ is $\mathbf{\Delta}^0_3(X \times A)$ with horizontal sections in $D_2(\mathbf{\Sigma}^0_1)(X)$. So there exists a finer Polish topology τ on A and some open subsets U_0 and U_1 of $X \times [A, \tau]$ such that $\phi^{-1}(\{1\}) = U_1 \setminus U_0$. The reader should see [L1] and [L2] to check this point (it is showed for Borel sets with sections in $\mathbf{\Sigma}^0_\xi$ in [L1]; we do the same thing here, using the fact, showed in [L2], that two disjoint Σ^1_1 which can be separated by a $D_2(\mathbf{\Sigma}^0_1)$ set can be separated by a $D_2(\mathbf{\Sigma}^0_1 \cap \Delta^1_1)$ set). Let (A_q) (resp., (X_n)) be a basis for the topology of A (resp., X). Let $E_n := \{f \in A \mathbin{/} X_n \subseteq U^f_1\}$. There exists $F^n_l \in \mathbf{\Pi}^0_1(X)$ such that $U_1 = \bigcup_n X_n \times E_n = \bigcup_{n,l} F^n_l \times E_n$. We set

$$\begin{aligned} F^{n,l} &:= [F^n_l \times E_n] \cap \phi^{-1}(\{1\}) = [F^n_l \times E_n] \setminus U_0 \\ &= \bigcup_q \{x \in F^n_l \mathbin{/} A_q \cap E_n \subseteq \phi^{-1}(\{1\})_x\} \times (A_q \cap E_n). \end{aligned}$$

This is a closed subset of $F^n_l \times [E_n, \tau\lceil E_n]$, and union of the $F^{n,l}$'s is $\phi^{-1}(\{1\})$. So we have $\phi^{-1}(\{1\}) = \bigcup_{n,l,q} \overline{\{x \in F^n_l \mathbin{/} A_q \cap E_n \subseteq \phi^{-1}(\{1\})_x\}} \times (A_q \cap E_n) \in (\mathbf{\Pi}^0_1(X) \times \mathcal{P}(A))_\sigma$. □

These last two propositions show that the example in Corollary 16 is optimal. In this example, one has $\phi^{-1}(\{0\}) \notin \mathbf{\Sigma}^0_2(X \times A) \cup (\mathbf{\Pi}^0_1(X) \times \mathcal{P}(A))_\sigma$.

(D) The case of Banach spaces.

The reader should see [DS] for basic facts about Banach spaces. Let E be a Banach space, $X := [B_{E^*}, w^*]$, $Y := \mathbb{R}$ and $A := \{G\lceil X \ / \ G \in B_{E^{**}}\}$. If E is separable, then X is a metrizable compact space. If moreover E contains no copy of l_1, Odell and Rosenthal's theorem gives, for every $G \in E^{**}$, a sequence (e_p) of E such that $f(e_p) \to G(f)$ for each $f \in E^*$ (see [OR]). Let $i : E \to E^{**}$ be the canonical map, and $G_p := i(e_p)$. Then (G_p) pointwise tends to G. By definition of the weak* topology, we have $i(e)\lceil X \in \mathcal{C}(X, Y)$ for each $e \in E$, thus $G\lceil X$ is the pointwise limit of a sequence of continuous functions. Therefore, $G\lceil X \in \mathcal{B}_1(X, Y)$ (see page 386 in [Ku]). We set

$$\Phi : \begin{cases} [B_{E^{**}}, w^*] \to [\mathcal{B}_1(X, Y), \text{p.c.}] \\ G \qquad\qquad \mapsto G\lceil X \end{cases}$$

By definition of weak* topology, Φ is continuous, and its range is A. So A is a compact space because Φ's domain is a compact space.

If E^* is separable, then E is separable and E contains no copy of l_1. Indeed, if ϕ was an embedding of l_1 into E, then the adjoint map $\phi^* : E^* \to l_1^*$ of ϕ would be onto, by the Hahn-Banach theorem. But $l_\infty \simeq l_1^*$ would be separable, which is absurd. The domain of Φ is a metrizable compact space, thus it is a Polish space. Therefore, A is an analytic compact space. So it is metrizable (see Corollary 2 page 77 of Chapter 9 in [Bo2]). In particular, every point of A is G_δ. Conversely, if E^* is not separable, then $\{0_{E^{**}}\}$ is not a G_δ subset of $B_{E^{**}}$. Indeed, if $(x_p) \subseteq E^*$, closed subspace spanned by $\{x_p \ / \ p \in \omega\}$ is not E^* (see page 5 in [B1]), and we use the Hahn-Banach theorem. Thus $\{0_{E^{**}}\lceil X\}$ is not a G_δ subset of A, because Φ is continuous. So the following are equivalent: E^* is separable, A is metrizable, and every point of A is G_δ.

Assume that E^{**} is separable. Then E^* is separable, and A is uniformly recoverable. Indeed, $A \subseteq \mathcal{C}([B_{E^*}, \|.\|], Y)$, and the following map is continuous:

$$\Phi' : \begin{cases} [E^{**}, \|.\|] \to [\mathcal{C}([B_{E^*}, \|.\|], Y), \|.\|_\infty] \\ G \qquad\qquad \mapsto G\lceil B_{E^*} \end{cases}$$

Therefore, $[\Phi'[E^{**}], \|.\|_\infty]$ is a separable metrizable space and contains A. Then we can apply Theorem 20. But we have a better result:

Theorem 30 *Let E be a Banach space, $X := [B_{E^*}, w^*]$, $A := \{G\lceil X/G \in B_{E^{**}}\}$, and $Y := \mathbb{R}$. The following statements are equivalent:*

(a) E^* is separable.

(b) A is metrizable.

(c) Every singleton of A is G_δ.

(d) A is uniformly recoverable.

Proof. We have seen that conditions (a), (b) and (c) are equivalent. So let us show that (a) $\Rightarrow$ (d). We have seen that X and A are metrizable compact spaces, and that $A \subseteq \mathcal{B}_1(X, Y)$. Thus we can apply Proposition 22, and it is enough to check that condition (2) is satisfied.

The finer topology is the norm topology. Let us check that it is made of $\mathbf{\Sigma}^0_2(X)$. We have $\|f-f_0\|<\varepsilon \Leftrightarrow \exists n\ \forall x\in B_E\ \ |f(x)-f_0(x)|\leq \varepsilon-2^{-n}$.

(d) $\Rightarrow$ (c) Let $G\in A$. Then $\{G\}=A\cap\bigcap_{p,q}\{g\in\mathbb{R}^X\ /\ |g(x_p)-G(x_p)|<2^{-q}\}$. Thus $\{G\}$ is $\mathbf{\Pi}^0_2(A)$. □

So we get a characterization of the separability of the dual space of an arbitrary Banach space. Notice that the equivalence between metrizability of the the compact space and the fact that each of its point is G_δ is not true for an arbitrary compact set of Baire class one functions (because of the "split interval").

This example of Banach spaces also shows that the converse of Theorem 20 is false. Indeed, we set $X:=[B_{l_1},\sigma(l_1,c_0)]$, $A:=\{G\lceil X/G\in B_{l_\infty}\}$, and $Y:=\mathbb{R}$. By Theorem 30, A is uniformly recoverable, since l_1 is separable. But since X is compact, compact open topology on A is the uniform convergence topology. If A was separable for compact open topology, l_∞ would be separable, which is absurd. Indeed, if $(G_n)\subseteq B_{l_\infty}$ is such that $\{G_n\lceil X\ /\ n\in\omega\}$ is a dense subset of A for the uniform convergence topology, we can easily check that $\{q.G_n\ /\ q\in\mathbb{Q}_+ \text{ and } n\in\omega\}$ is dense in l_∞. Notice that this gives an example of a metrizable compact space for the pointwise convergence topology which is not separable for the compact open topology.

Finally, notice that the map ϕ is Baire class one if E^* is separable. Indeed, it is the composition of the identity map from $X\times A$ into $[X,\|.\|]\times A$ (which is Baire class one), and of the map which associates $G(f)$ to $(f,G)\in[X,\|.\|]\times A$ (which is continuous).

(E) The notion of an equi-Baire class one set of functions.

We will give a characterization of Baire class one functions which lightly improves, in the sense (a) $\Rightarrow$ (b) of Corollary 33 below, the one we can find in [LTZ].

Definition 31 *Let X and Y be metric spaces, and $A\subseteq Y^X$. Then A is equi-Baire class one (EBC1) if, for each $\varepsilon>0$, there exists $\delta(\varepsilon)\in\mathcal{B}_1(X,\mathbb{R}^*_+)$ such that*

$$d_X(x,x')<\mathit{min}(\delta(\varepsilon)(x),\delta(\varepsilon)(x'))\Rightarrow\forall f\in A\ \ d_Y(f(x),f(x'))<\varepsilon.$$

Proposition 32 *Let X and Y be metric spaces. Assume that X is separable, that all the closed subsets of X are Baire spaces, and that $A\subseteq Y^X$. The following conditions are equivalent:*

(1) A is EBC1.

(2) For each $\varepsilon>0$, there exists a sequence $(G^\varepsilon_m)_m\subseteq\mathbf{\Pi}^0_1(X)$, whose union is X, such that for each $f\in A$ and for each integer m, diam$(f[G^\varepsilon_m])<\varepsilon$.

(3) There exists a finer metrizable separable topology on X, made of $\mathbf{\Sigma}^0_2(X)$, making A equicontinuous.

(4) Every nonempty closed subset F of X contains a point x such that $\{f_{|F}\ /\ f\in A\}$ is equicontinuous at x.

Proof. (1) $\Rightarrow$ (2) We set, for n integer, $H_n := \{x \in X \ / \ \delta(\varepsilon)(x) > 2^{-n}\}$. As $\delta(\varepsilon)$ is Baire class one, there exists $(F_q^n)_q \subseteq \mathbf{\Pi}_1^0(X)$ such that $H_n = \bigcup_q F_q^n$. We construct, for $\xi < \omega_1$, open subsets U_ξ of X, and integers n_ξ and q_ξ satisfying $\bigcup_{\eta<\xi} U_\eta \neq X \Rightarrow \emptyset \neq U_\xi \setminus (\bigcup_{\eta<\xi} U_\eta) \subseteq F_{q_\xi}^{n_\xi}$. It is clearly possible since $X = \bigcup_{n,q} F_q^n$ and $X \setminus (\bigcup_{\eta<\xi} U_\eta)$ is a Baire space. As X has a countable basis, there exists $\gamma < \omega_1$ such that $\bigcup_{\xi<\omega_1} U_\xi = \bigcup_{\xi\leq\gamma} U_\xi$. In particular we have $U_{\gamma+1} \subseteq \bigcup_{\xi\leq\gamma} U_\xi$, thus $X = \bigcup_{\xi\leq\gamma} U_\xi = \bigcup_{\xi\leq\gamma,\text{disj.}} U_\xi \setminus (\bigcup_{\eta<\xi} U_\eta)$. Let $(x_q^\xi)_q \subseteq X$ satisfying $U_\xi \subseteq \bigcup_q B(x_q^\xi, 2^{-n_\xi-1}[$. Let $G_{q,\xi}^\varepsilon := B(x_q^\xi, 2^{-n_\xi-1}[\cap U_\xi \setminus (\bigcup_{\eta<\xi} U_\eta)$. Then $G_{q,\xi}^\varepsilon \in \mathbf{\Sigma}_2^0(X)$ and X is the union of the sequence $(G_{q,\xi}^\varepsilon)_{q,\xi\leq\gamma}$. If $x, x' \in G_{q,\xi}^\varepsilon$, then we have $d_X(x,x') < 2^{-n_\xi} < \min(\delta(\varepsilon)(x), \delta(\varepsilon)(x'))$. Thus

$$d_Y(f(x), f(x')) < \varepsilon$$

for each function $f \in A$. It remains to write the $(G_{q,\xi}^\varepsilon)_{q,\xi\leq\gamma}$'s as countable unions of closed sets. So that we get the sequence $(G_m^\varepsilon)_m$.

(2) $\Rightarrow$ (3) Let us take a look at the proof of the implication (1) $\Rightarrow$ (2) in Proposition 22. There exists a finer metrizable separable topology on X, made of $\mathbf{\Sigma}_2^0(X)$, and making $G_m^{2^{-r}}$'s open. This is enough (notice that we do not use the fact that every closed subset of X is a Baire space to show this implication).

(3) $\Rightarrow$ (4) Let (X_n) be a basis for the finer topology. As $X_n \in \mathbf{\Sigma}_2^0(X)$, $F_n := (F \cap X_n) \setminus \text{Int}(F \cap X_n)$ is a meager $\mathbf{\Sigma}_2^0$ subset of F. Thus $F \setminus (\bigcup_n F_n)$ is a comeager G_δ subset of F. As F is a Baire space, this G_δ subset is nonempty. This gives the point x we were looking for. Indeed, let us fix $\varepsilon > 0$. Let n be an integer such that $x \in X_n$ and $\sup_{f\in A} \text{diam}(f[X_n]) < \varepsilon$. Then $x \in \text{Int}(F \cap X_n)$ and $\sup_{f\in A} \text{diam}(f_{|F}[\text{Int}(F \cap X_n)]) < \varepsilon$.

(4) $\Rightarrow$ (2) Let us fix $\varepsilon > 0$. We construct a sequence $(U_\xi)_{\xi<\omega_1}$ of open subsets of X such that $\sup_{f\in A} \text{diam}(f_{|X\setminus(\bigcup_{\eta<\xi} U_\eta)}[U_\xi \setminus (\bigcup_{\eta<\xi} U_\eta)]) < \varepsilon$ and $U_\xi \setminus (\bigcup_{\eta<\xi} U_\eta) \neq \emptyset$ if $\bigcup_{\eta<\xi} U_\eta \neq X$. As in the proof of the implication (1) $\Rightarrow$ (2), there exists $\gamma < \omega_1$ such that $X = \bigcup_{\xi\leq\gamma} U_\xi$. It remains to write the $(U_\xi \setminus (\bigcup_{\eta<\xi} U_\eta))_{\xi\leq\gamma}$'s as countable unions of closed sets to get the sequence $(G_m^\varepsilon)_m$.

(2) $\Rightarrow$ (1) For $x \in X$, we set $m^\varepsilon(x) := \min\{m \in \omega \ / \ x \in G_m^\varepsilon\}$, and

$$\delta(\varepsilon) : \begin{cases} X \to \mathbb{R}_+^* \\ x \mapsto d_X(x, \bigcup_{r<m^\varepsilon(x)} G_r^\varepsilon) \end{cases}$$

Then $\delta(\varepsilon)$ is Baire classe one since if $A, B > 0$, then we have

$$A < \delta(\varepsilon)(x) < B \ \Leftrightarrow \ \exists m \ [x \in G_m^\varepsilon \text{ and } \forall r < m \ \ x \notin G_r^\varepsilon \text{ and } A < d_X(x, \bigcup_{r<m} G_r^\varepsilon) < B].$$

If $d_X(x,x') < \min(\delta(\varepsilon)(x), \delta(\varepsilon)(x'))$, then we have $x' \notin \bigcup_{r<m^\varepsilon(x)} G_r^\varepsilon$, and conversely. Therefore, $m^\varepsilon(x) = m^\varepsilon(x')$ and $x, x' \in G_{m^\varepsilon(x)}^\varepsilon$. Thus $d_Y(f(x), f(x')) \leq \text{diam}(f[G_{m^\varepsilon(x)}^\varepsilon]) < \varepsilon$, for each function $f \in A$ (notice that we do not use the fact that every closed subset of X is a Baire space to show these last two implications). □

Corollary 33 *Let X and Y be metric spaces. Consider the following statements:*

(a) f is Baire class one.

(b) $\forall\varepsilon>0\ \exists\delta(\varepsilon)\in\mathcal{B}_1(X,\mathbb{R}^*_+)\ \ d_X(x,x')<min(\delta(\varepsilon)(x),\delta(\varepsilon)(x'))\ \Rightarrow\ d_Y(f(x),f(x'))<\varepsilon.$

(1) If Y is separable, then (a) implies (b).

(2) If X is separable and if every closed subset of X is a Baire space, then (b) implies (a).

Proof. To show condition (1), the only thing to notice is the following. Let $(y_n)\subseteq Y$ satisfying $Y=\bigcup_n B(y_n,\varepsilon/2[$. By condition (a), let $(F^n_q)_q\subseteq\mathbf{\Pi}^0_1(X)$ satisfying $f^{-1}(B(y_n,\varepsilon/2[)=\bigcup_q F^n_q$. We enumerate the sequence $(F^n_q)_{n,q}$, so that we get $(G^\varepsilon_m)_m$. We have $G^\varepsilon_m\in\mathbf{\Pi}^0_1(X)$, $X=\bigcup_m G^\varepsilon_m$, and $\text{diam}(f[G^\varepsilon_m])<\varepsilon$ for each integer m. Then we use the proof of implication (2) $\Rightarrow$ (1) in Proposition 32. □

Remark. Let X be a Polish space, $Y\subseteq\mathbb{R}$, and $A\subseteq Y^X$ be a Polish space. We assume that every nonempty closed subset F of X contains a point of equicontinuity of $\{f_{|F}\ /\ f\in A\}$. Then, by Proposition 32, $A\subseteq\mathcal{B}_1(X,Y)$ and by Proposition 17, J. Bourgain's ordinal rank is bounded on A. This result is true in a more general context :

Corollary 34 *Let X be a metrizable separable space, $Y\subseteq\mathbb{R}$, $A\subseteq Y^X$ and $a<b$ be reals. We assume that every nonempty closed subset F of X contains a point of equicontinuity of $\{f_{|F}\ /\ f\in A\}$. Then sup$\{L(f,a,b)\ /\ f\in A\}<\omega_1$. In particular, sup$\{L(f)\ /\ f\in A\}<\omega_1$.*

Proof. Using equicontinuity, we construct a sequence $(U_\xi)_{\xi<\omega_1}$ of open subsets of X satisfying $\sup_{f\in A}\text{diam}f_{|X\setminus(\bigcup_{\eta<\xi}U_\eta)}[U_\xi\setminus(\bigcup_{\eta<\xi}U_\eta)]<b-a$ and $U_\xi\setminus(\bigcup_{\eta<\xi}U_\eta)\neq\emptyset$ if $\bigcup_{\eta<\xi}U_\eta\neq X$. As X has a countable basis, there exists $\gamma<\omega_1$ such that $X=\bigcup_{\xi\leq\gamma}U_\xi$. Let $G_0:=\emptyset$, $G_{\alpha+1}:=\cup_{\xi\leq\alpha}U_\xi$ if $\alpha\leq\gamma$, $G_\lambda:=\cup_{\alpha<\lambda}G_\alpha$ if $0<\lambda\leq\gamma$ is a limit ordinal, and $G_{\gamma+2}:=X$. Let us check that, if $f\in A$, then $(G_\alpha)_{\alpha\leq\gamma+2}\in R(\{f\leq a\},\{f\geq b\})$ (this will be enough). By the proof of Proposition 32, f is Baire class one. So $\{f\leq a\}$ and $\{f\geq b\}$ are disjoint G_δ subsets of X. We have $G_\alpha\subseteq\cup_{\xi<\alpha}U_\xi$ if $\alpha\leq\gamma+1$, so the sequence $(G_\alpha)_{\alpha\leq\gamma+2}$ is increasing. If $\alpha\leq\gamma$ is the successor of ρ, then $G_{\alpha+1}\setminus G_\alpha=(\cup_{\xi\leq\alpha}U_\xi)\setminus(\cup_{\xi\leq\rho}U_\xi)=U_\alpha\setminus(\cup_{\xi<\alpha}U_\xi)$. So $G_{\alpha+1}\setminus G_\alpha$ is disjoint from $\{f\leq a\}$ or from $\{f\geq b\}$. If $\alpha\leq\gamma$ is a limit ordinal, then

$$G_{\alpha+1}\setminus G_\alpha=(\cup_{\xi\leq\alpha}U_\xi)\setminus(\cup_{\xi<\alpha}G_\xi)\subseteq(\cup_{\xi\leq\alpha}U_\xi)\setminus(\cup_{\xi<\alpha}U_\xi)$$

because $U_\xi\subseteq G_{\xi+1}$ if $\xi<\alpha$. Thus we have the same conclusion. Finally,

$$G_{\gamma+2}\setminus G_{\gamma+1}=X\setminus(\cup_{\xi\leq\gamma}U_\xi)=\emptyset,$$

and we are done. □

Now, we will study similar versions of Ascoli's theorems, for Baire class one functions. A similar version of the first of these three theorems is true:

Proposition 35 *If A is EBC1, then $\overline{A}^{p.c.}$ is EBC1.*

Proof. It is very similar to the classical one. We set $\delta_{\overline{A}^{\text{p.c.}}}(\varepsilon) := \delta_A(\varepsilon/3)$. Assume that

$$d_X(x,x') < \min(\delta_{\overline{A}^{\text{p.c.}}}(\varepsilon)(x), \delta_{\overline{A}^{\text{p.c.}}}(\varepsilon)(x')),$$

and let $g \in \overline{A}^{\text{p.c.}}$. The following set is an open neighborhood of g:

$$O := \{h \in Y^X \ / \ d_Y(h(x), g(x)) < \varepsilon/3 \ \text{ and } \ d_Y(h(x'), g(x')) < \varepsilon/3\}$$

(for the pointwise convergence topology). Set O meets A in f. Then we check that

$$d_Y(g(x), g(x')) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

This finishes the proof. □

A similar version of the third of Ascoli's theorem is true in one sense:

Proposition 36 *Assume that X and Y are separable metric spaces, and that X is locally compact. If $A \subseteq \mathcal{B}_1(X,Y)$, equipped with the compact open topology, is relatively compact in Y^X, then A is EBC1 and $A(x)$ is relatively compact for each $x \in X$.*

Proof. As X is metrizable, X is paracompact (see Theorem 4, page 51 of Chapter 9 in [Bo2]). By Corollary page 71 of Chapter 1 in [Bo1], there exists a locally finite open covering $(V_j)_{j\in\omega}$ of X made of relatively compact sets (we use the fact that X is separable). For $x \in X$, we set

$$J_x := \{j \in \omega \ / \ x \in V_j\}.$$

It is a finite subset of ω. Let $e(x) \in \omega$ be minimal such that $B(x, 2^{-e(x)}[\subseteq \bigcap_{j\in J_x} V_j$. Notice that $e \in \mathcal{B}_1(X,\omega)$. Indeed, let $(x^j_q)_q$ be a dense sequence of $X \setminus V_j$. We have

$$e(x)=p \ \Leftrightarrow \left\{\begin{array}{l} \exists k \ \{\forall j > k \ \ x \notin V_j\} \text{ and } x \in V_k \text{ and } \forall j \leq k \ \ \{\forall q \ \ x^j_q \notin B(x, 2^{-p}[\text{ or } x \notin V_j\} \\ \\ \text{and } \ \forall l < p \ \exists j \leq k \ \ \{\exists q \ \ x^j_q \in B(x, 2^{-l}[\text{ and } x \in V_j\}. \end{array}\right.$$

• Let us show that $\overline{A}^{\text{c.o.}} \subseteq \mathcal{B}_1(X,Y)$. For $x \in X$, we let U_x be a relatively compact open neighborhood of x. As X is a Lindelöf space, $X = \bigcup_n U_{x_n}$; let $K_n := \bigcup_{p\leq n} \overline{U_{x_p}}$. Then (K_n) is an increasing sequence of compact subsets of X and every compact subset of X is a subset of one of the K_n's. By Corollary page 20 of Chapter 10 in [Bo2], Y^X, equipped with the compact open topology, is metrizable.

So let $f \in \overline{A}^{\text{c.o.}}$. By the previous facts there exists a sequence $(f_n) \subseteq A$ which tends to f, uniformly on each compact subset of X. So we have

$$\forall m \in \omega \ \ \exists (p^m_n)_n \in \omega^\omega \ \ \forall x \in K_m \ \forall n \in \omega \ \ d_Y(f(x), f_{p^m_n}(x)) < 2^{-n}.$$

Therefore, if $F \in \mathbf{\Pi}^0_1(X)$, then

$$f^{-1}(F) = \bigcup_m K_m \setminus (\bigcup_{p<m} K_p) \cap \{x \in K_m \ / \ \forall n \in \omega \ \ d_Y(F, f_{p^m_n}(x)) \leq 2^{-n}\}.$$

We deduce from this that $f^{-1}(F)$ is G_δ, because it is union of countably many G_δ's, partitionned by some $\mathbf{\Delta}^0_2(X)$. So f is Baire class one and $\overline{A}^{\text{c.o.}} \subseteq \mathcal{B}_1(X,Y)$.

• Let $f \in \overline{A}^{\text{c.o.}}$, $\varepsilon > 0$ and K be a compact subset of X. We set

$$U(f,\varepsilon,K) := \{g \in \overline{A}^{\text{c.o.}} \ / \ \forall x \in K \ \ d_Y(f(x),g(x)) < \varepsilon/3\}.$$

Then $U(f,\varepsilon,K)$ is an open neighborhood of f for the compact open topology, so there exists an integer $p_{\varepsilon,K}$ and $(f_i^{\varepsilon,K})_{i \leq p_{\varepsilon,K}} \subseteq \overline{A}^{\text{c.o.}}$ such that $\overline{A}^{\text{c.o.}} = \bigcup_{i \leq p_{\varepsilon,K}} U(f_i^{\varepsilon,K},\varepsilon,K)$, because $\overline{A}^{\text{c.o.}}$ is compact.

• By Corollary 33, if $f \in \overline{A}^{\text{c.o.}}$, then there exists $\delta(f,\varepsilon) \in \mathcal{B}_1(X,\mathbb{R}_+^*)$ such that $d_Y(f(x),f(x')) < \varepsilon$ if $d_X(x,x') < \text{min}(\delta(f,\varepsilon)(x),\delta(f,\varepsilon)(x'))$. We set

$$\delta(\varepsilon) : \begin{cases} X \rightarrow \mathbb{R}_+^* \\ x \ \mapsto \text{min}(2^{-e(x)}, \text{min}_{j \in J_x, i \leq p_{\varepsilon/3,\overline{V_j}}}[\delta(f_i^{\varepsilon/3,\overline{V_j}},\varepsilon/3)(x)]) \end{cases}$$

If $d_X(x,x') < \text{min}(\delta(\varepsilon)(x),\delta(\varepsilon)(x'))$ and $f \in A$, then $d_X(x,x') < 2^{-e(x)}$ and $x' \in \bigcap_{j \in J_x} V_j$. Let $j \in J_x$ (so $j \in J_{x'}$) and $i \leq p_{\varepsilon/3,\overline{V_j}}$ be such that $f \in U(f_i^{\varepsilon/3,\overline{V_j}},\varepsilon/3,\overline{V_j})$. As $x,x' \in \overline{V_j}$ and $d_X(x,x') < \text{min}(\delta(f_i^{\varepsilon/3,\overline{V_j}},\varepsilon/3)(x),\delta(f_i^{\varepsilon/3,\overline{V_j}},\varepsilon/3)(x'))$, we have $d_Y(f(x),f(x')) < 3.\varepsilon/3 = \varepsilon$. Let us check that $\delta(\varepsilon)$ is Baire class one. If $A,B > 0$, $A < \delta(\varepsilon)(x) < B$ is equivalent to

$$\begin{cases} \exists \ k \ \{\forall \ j > k \ \ x \notin \ V_j\} \text{ and } x \in V_k \\ \text{and } \{e(x) > -\text{ln}(B)/\text{ln}(2) \text{ or } \exists \ j \leq k \ \ x \in V_j \text{ and } \exists \ i \leq p_{\varepsilon/3,\overline{V_j}} \ \ \delta(f_i^{\varepsilon/3,\overline{V_j}},\varepsilon/3)(x) < B\} \\ \text{and } \{e(x) < -\text{ln}(A)/\text{ln}(2) \text{ and } \forall \ j \leq k \ \ x \notin V_j \text{ or } \forall \ i \leq p_{\varepsilon/3,\overline{V_j}} \ \ \delta(f_i^{\varepsilon/3,\overline{V_j}},\varepsilon/3)(x) > A\}. \end{cases}$$

• The last point comes from the continuity of $\phi(x,.)$, for each $x \in X$; this implies that $\overline{A}^{\text{c.o.}}(x)$ is compact and contains $A(x)$. □

Counter-example. A similar version of the second of Ascoli's theorem is false, in the sense that there are some metric spaces X and Y, X being compact, and a metrizable compact space

$$A \subseteq [\mathcal{B}_1(X,Y), \text{p.c.}]$$

which is EBC1 and such that, on A, the compact open topology (i.e., the uniform convergence topology) and the pointwise convergence topology are different. Indeed, we set $X := [B_{l_1}, \sigma(l_1,c_0)]$, $A := \{G\lceil X/G \in B_{l_\infty}\}$, and $Y := \mathbb{R}$. We have seen that A is not separable for the uniform convergence topology. So this topology is different on A from that of pointwise convergence. Nevertheless, A is EBC1. Indeed, the norm topology makes A uniformly equicontinuous, and we just have to apply Proposition 32. Moreover, $A(x)$ is compact for each $x \in X$ and A is a closed subset of $[\mathbb{R}^X, \text{c.o.}]$ (we check it in a standard way). As A is metrizable and not separable in this space, it is not relatively compact. Therefore, the converse of Proposition 36 is false in general.

Corollary 37 *Assume that X and Y are separable metric spaces and that X is locally compact. If moreover $A \subseteq \mathcal{B}_1(X,Y)$, equipped with the compact open topology, is relatively compact in Y^X, then A is uniformly recoverable.*

Proof. By Proposition 13 page 66 of Chapter 1 in [Bo1] and Theorem 1 page 55 of Chapter 9 in [Bo2], we can apply Propositions 32 and 36, and use Proposition 22. □

Remarks. There is another proof of this corollary. Indeed, as in the proof of Proposition 36, Y^X, equipped with the compact open topology, is metrizable and $\overline{A}^{\text{c.o.}} \subseteq \mathcal{B}_1(X,Y)$. Thus $\overline{A}^{\text{c.o.}}$ is a metrizable compact space for the compact open topology. Thus it is separable for this topology. Then we apply Theorem 20.

Let X and Y be separable metric spaces. Assume that every closed subset of X is a Baire space, and that $A \subseteq Y^X$. If A is EBC1, then $A \subseteq \mathcal{B}_1(X,Y)$ and the conditions of Proposition 22 are satisfied, by Proposition 32. The converse of this is false. To see this, we use the example of Proposition 27 : $X := 2^\omega$, $Y := 2$ et $A := \bigcup_p \{\mathbb{1}_{2^\omega \setminus \{\psi(p)\}}\}$. By the proof of (1) ⇒ (2) in Proposition 22, there exists a finer metrizable separable topology τ on 2^ω, made of $\mathbf{\Sigma}^0_2(2^\omega)$, and making the $\{\psi(p)\}$'s open, for $p \in \omega$. Thus τ makes the functions of A continuous. But assume that τ' is a finer metrizable separable topology on 2^ω, made of $\mathbf{\Sigma}^0_2(2^\omega)$, and makes A equicontinuous. We would have $P_\infty \notin \mathbf{\Sigma}^0_1([2^\omega, \tau'])$. So we could find $\alpha \in P_\infty$ in the closure of P_f for τ'. If V is a neighborhood of α for τ', we could choose $\psi(p) \in V \cap P_f$. We would have $|\mathbb{1}_{2^\omega \setminus \{\psi(p)\}}(\alpha) - \mathbb{1}_{2^\omega \setminus \{\psi(p)\}}(\psi(p))| = 1$. But this contradicts the equicontinuity of A. Then we apply Proposition 32. This also shows the utility of the assumption of relative compactness in Proposition 36 (A is an infinite countable discrete closed set; so it is not compact, in $\mathcal{B}_1(2^\omega, 2)$ equipped with the compact open topology).

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