On minimal non-potentially closed subsets of the plane.

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Abstract. We study the Borel subsets of the plane that can be made closed by refining the Polish topology on the real line. These sets are called potentially closed. We first compare Borel subsets of the plane using products of continuous functions. We show the existence of a perfect antichain made of minimal sets among non-potentially closed sets. We apply this result to graphs, quasi-orders and partial orders. We also give a non-potentially closed set minimum for another notion of comparison. Finally, we show that we cannot have injectivity in the Kechris-Solecki-Todorčević dichotomy about analytic graphs.

1 Introduction.

The reader should see [K] for the descriptive set theoretic notation used in this paper. This work is the continuation of a study made in [L1]-[L4]. The usual way of comparing Borel equivalence relations $E \subseteq X \times X$ and $E' \subseteq X' \times X'$ on Polish spaces is the Borel reducibility quasi-order:

$$E \leq_B E' \;\Leftrightarrow\; \exists u\!:\!X \!\rightarrow\! X' \text{ Borel with } E\!=\!(u\!\times\! u)^{-1}(E')$$

(recall that a quasi-order is a reflexive and transitive relation). Note that this makes sense even if E, E' are not equivalence relations. It is known that if (B_n) is a sequence of Borel subsets of X, then we can find a finer Polish topology on X making the B_n's clopen (see Exercise 13.5 in [K]). So assume that $E \leq_B E'$ and let σ be a finer Polish topology on X making u continuous. If E' is in some Baire class Γ, then $E \in \Gamma([X,\sigma]^2)$. This motivates the following (see [Lo2]):

Definition 1 *(Louveau) Let X, Y be Polish spaces, A a Borel subset of $X \times Y$, and Γ a Baire class. We say that A is $potentially\ in$ Γ (denoted $A \in$ pot(Γ)) iff we can find a finer Polish topology σ (resp., τ) on X (resp., Y) such that $A \in \Gamma([X,\sigma] \times [Y,\tau])$.*

This notion is a natural invariant for $\leq_B$: if $E' \in$ pot(Γ) and $E \leq_B E'$, then $E \in$ pot(Γ). Using this notion, A. Louveau showed that the collection of $\mathbf{\Sigma}^0_\xi$ equivalence relations is not cofinal for $\leq_B$, and deduces from this the inexistence of a maximum Borel equivalence relation for $\leq_B$.

A. Louveau has also more recently noticed that one can associate a quasi-order $R_A \subseteq (X \times 2)^2$ to $A \subseteq X^2$ as follows:

$$(x,i)\ R_A\ (y,j) \;\;\Leftrightarrow\;\; (x,i) = (y,j) \;\text{ or }\; [(x,y) \in A \;\text{ and }\; (i,j) = (0,1)].$$

Note that R_A is also antisymmetric, so that it is actually a partial order.

A. Louveau noticed the following facts, using the following notion of comparison between Borel subsets $A\subseteq X\times Y$, $A'\subseteq X'\times Y'$ of products of two Polish spaces:

$$A \sqsubseteq^r_B A' \ \Leftrightarrow\ \exists u\!:\!X\rightarrow X'\ \exists v\!:\!Y\rightarrow Y' \text{ one-to-one Borel with } A=(u\times v)^{-1}(A').$$

Here the letter r means "rectangle" (u and v may be different).

- Assume that $A\subseteq X^2$ has full projections, and that $A'\subseteq (X')^2$. Then $A\sqsubseteq^r_B A'$ is equivalent to $R_A\leq_B R_{A'}$.

- If $A\subseteq X^2$ is $\sqsubseteq^r_B$-minimal among non-potentially closed sets, then R_A is $\leq_B$-minimal among non-potentially closed partial orders.

- Conversely, if R_A is $\leq_B$-minimal among non-potentially closed partial orders and if A has full projections, then A is $\sqsubseteq^r_B$-minimal among non-potentially closed sets.

These facts show that, from the point of view of Borel reducibility, the study of Borel partial orders is essentially the study of arbitrary Borel subsets of the plane. This strengthens the motivation for studying arbitrary Borel subsets of the plane, from the point of view of potential complexity.

• A standard way to see that a set is complicated is to notice that it is more complicated than a well-known example. For instance, we have the following result (see [SR]):

Theorem 2 *(Hurewicz) Let* $P_f:=\{\alpha\in 2^\omega/\exists n\in\omega\ \forall m\geq n\ \ \alpha(m)=0\}$, *$X$ be a Polish space and A a Borel subset of X. Then exactly one of the following holds:*
(a) The set A is $\mathbf{\Pi}^0_2(X)$.
(b) There is a continuous injection $u:2^\omega\rightarrow X$ such that $P_f=u^{-1}(A)$.

This theorem has been generalized to all Baire classes in [Lo-SR]. We try to adapt this result to the Borel subsets of the plane. In this direction, we have the following result for equivalence relations (see [H-K-Lo]):

Theorem 3 *(Harrington-Kechris-Louveau) Let X be a Polish space, E a Borel equivalence relation on X, and* $E_0:=\{(\alpha,\beta)\in 2^\omega\times 2^\omega/\exists n\in\omega\ \forall m\geq n\ \ \alpha(m)=\beta(m)\}$. *Then exactly one of the following holds:*
(a) The relation E is pot($\mathbf{\Pi}^0_1$).
(b) We have $E_0\leq_B E$ (with u continuous and one-to-one).

• We will study structures other than equivalence relations (for example quasi-orders), and even arbitrary Borel subsets of the plane. We need some other notions of comparison.

Recall that Wadge's quasi-order $\leq_W$ on Borel subsets of ω^ω is defined by

$$A\leq_W A'\ \ \Leftrightarrow\ \ \exists u\!:\!\omega^\omega\rightarrow\omega^\omega \text{ continuous with } A=u^{-1}(A').$$

It is known that this quasi-order is well-founded (in the sense that there is no sequence (B_n) with $B_{n+1}\leq_W B_n$ and $B_n\not\leq_W B_{n+1}$ for each n). Moreover, any $\leq_W$-antichain is of cardinality at most 2 (in fact of the form $\{A,\neg A\}$). It follows that any class $\mathbf{\Delta}^1_1\setminus\mathbf{\Pi}^0_\xi$ admits a unique (up to the equivalence associated to $\leq_W$) minimal element.

• There are several natural ways of comparing Borel subsets $A\subseteq X\times Y$, $A'\subseteq X'\times Y'$ of products of two Polish spaces. All of them will have the same behavior here. The one we will use is the following:

$$A\leq^r_c A' \ \Leftrightarrow\ \exists u\colon X\to X'\ \exists v\colon Y\to Y' \text{ continuous with } A=(u\times v)^{-1}(A').$$

Here the letter c is for "continuous". We have the following (see [L1]):

Theorem 4 *Let $\Delta(2^\omega):=\{(\alpha,\beta)\in 2^\omega\times 2^\omega/\alpha=\beta\}$, $L_0:=\{(\alpha,\beta)\in 2^\omega\times 2^\omega/\alpha<_{lex}\beta\}$, X and Y be Polish spaces, and A a $pot(\check{D}_2(\mathbf{\Sigma}^0_1))$ subset of $X\times Y$. Then exactly one of the following holds:*
(a) The set A is pot($\mathbf{\Pi}^0_1$).
(b) $\neg\Delta(2^\omega)\leq^r_c A$ or $L_0\leq^r_c A$ (with u and v one-to-one).

• Things become much more complicated at the level $D_2(\mathbf{\Sigma}^0_1)$ (differences of two open sets; $\check{D}_2(\mathbf{\Sigma}^0_1)$ is the dual Wadge class of unions of a closed set and an open set; notice that we can extend Definition 1 to the class $\check{D}_2(\mathbf{\Sigma}^0_1)$). We will show the following:

Theorem 5 *There is a perfect $\leq^r_c$-antichain $(A_\alpha)_{\alpha\in 2^\omega}\subseteq D_2(\mathbf{\Sigma}^0_1)(2^\omega\times 2^\omega)$ such that A_α is $\leq^r_c$-minimal among $\mathbf{\Delta}^1_1\setminus$ pot($\mathbf{\Pi}^0_1$) sets, for any $\alpha\in 2^\omega$.*

In particular, unlike for classical Baire classes and $\leq_W$, one cannot characterize non-pot($\mathbf{\Pi}^0_1$) sets by an obstruction condition involving only one (or even countably many) set(s). We will also show that $[D_2(\mathbf{\Sigma}^0_1)\setminus \text{pot}(\mathbf{\Pi}^0_1),\leq^r_c]$ is ill-founded.

Theorem 5 can be applied to structures. We will show the following:

Theorem 6 *There is a perfect $\leq_B$-antichain $(R_\alpha)_{\alpha\in 2^\omega}\subseteq D_2(\mathbf{\Sigma}^0_1)((2^\omega\times 2)^2)$ such that R_α is $\leq_B$-minimal among $\mathbf{\Delta}^1_1\setminus$pot($\mathbf{\Pi}^0_1$) sets, for any $\alpha\in 2^\omega$. Moreover, $(R_\alpha)_{\alpha\in 2^\omega}$ can be taken to be a subclass of any of the following classes:*
- Directed graphs (i.e., irreflexive relations).
- Graphs (i.e., irreflexive and symmetric relations).
- Oriented graphs (i.e., irreflexive and antisymmetric relations).
- Quasi-orders.
- Strict quasi-orders (i.e., irreflexive and transitive relations).
- Partial orders.
- Strict partial orders (i.e., irreflexive, antisymmetric and transitive relations).

Remarks. (a) Theorem 6 shows that Harrington, Kechris and Louveau's Theorem is very specific, and that the combination of symmetry and transitivity is very strong.

(b) We produce concrete examples of such antichains. These examples must be in any complete family of minimal sets, up to bi-reducibility.

• Theorem 5 shows that any complete family of minimal sets for $[\mathbf{\Delta}^1_1\setminus \text{pot}(\mathbf{\Pi}^0_1), \leq^r_c]$ has size continuum. So we must find another notion of comparison. In [K-S-T], the following notion is defined. Let X, X' be Polish spaces, and $A\subseteq X\times X$, $A'\subseteq X'\times X'$ be analytic sets. We set

$$(X,A)\preceq_c(X',A')\Leftrightarrow \exists u\!:\!X\rightarrow X' \text{ continuous with } A\subseteq (u\times u)^{-1}(A').$$

When u is Borel we write $\preceq_B$ instead of $\preceq_c$.

Let $\psi:\omega\rightarrow 2^{<\omega}$ be the natural bijection ($\psi(0)=\emptyset$, $\psi(1)=0$, $\psi(2)=1$, $\psi(3)=0^2$, $\psi(4)=01$, $\psi(5)=10$, $\psi(6)=1^2$, $\ldots$). Note that $|\psi(n)|\leq n$, so that we can define $s_n:=\psi(n)0^{n-|\psi(n)|}$. The crucial properties of (s_n) are that it is $dense$ (there is n such that $t\prec s_n$, for each $t\in 2^{<\omega}$), and that $|s_n|=n$. We set

$$A_1:=\{(s_n0\gamma,s_n1\gamma)/n\in\omega\ \text{ and }\ \gamma\in 2^\omega\}.$$

The symmetric set $s(A_1)$ generated by A_1 is considered in [K-S-T], where the following is essentially proved:

Theorem 7 *(Kechris, Solecki, Todorčević) Let X be a Polish space and A an analytic subset of $X\times X$. Then exactly one of the following holds:*
(a) $(X,A)\preceq_B(\omega,\neq)$.
(b) $(2^\omega,A_1)\preceq_c(X,A)$.

Actually, the original statement in [K-S-T] is when A is a graph, and with $s(A_1)$ instead of A_1. But we can get Theorem 7 without any change in the proof in [K-S-T].

• In [L3] the following is shown (see Theorem 2.9):

Theorem 8 *Let X, Y be Polish spaces, and A a pot($\mathbf{\Delta}^0_3$) subset of $X\times Y$. Then exactly one of the following holds:*
(a) The set A is pot($\mathbf{\Pi}^0_1$).
(b) There are $u:2^\omega\rightarrow X$, $v:2^\omega\rightarrow Y$ continuous with $A_1=(u\times v)^{-1}(A)\cap\overline{A_1}$.

(We can replace A_1 in [L3] by what we call A_1 here.) We generalize this result to arbitrary Borel subsets of $X\times Y$:

Theorem 9 *Let X, Y be Polish spaces, and A, B be disjoint analytic subsets of $X\times Y$. Then exactly one of the following holds:*
(a) The set A is separable from B by a pot($\mathbf{\Pi}^0_1$) set.
(b) There are $u:2^\omega\rightarrow X$ and $v:2^\omega\rightarrow Y$ continuous such that the inclusions $A_1\subseteq(u\times v)^{-1}(A)$ and $\overline{A_1}\setminus A_1\subseteq(u\times v)^{-1}(B)$ hold.
Moreover, we can neither replace $\overline{A_1}\setminus A_1$ with $(2^\omega\times 2^\omega)\setminus A_1$, nor ensure that u and v are one-to-one.

So we get a minimum non-potentially closed set if we do not ask for a reduction on the whole product.

• In [K-S-T], it is conjectured that we can have u one-to-one in Theorem 7.(b). This is not the case:

Theorem 10 *There is no graph (X_0, R_0) with X_0 Polish and $R_0 \in \mathbf{\Sigma}^1_1(X_0^2)$ such that for every graph (X, A) of the same type, exactly one of the following holds:*
(a) $(X, A) \preceq_B (\omega, \neq)$.
(b) $(X_0, R_0) \preceq_{c,1-1} (X, A)$.

The proof is based on the counterexample used in [L3] to show that we cannot have injectivity in Theorem 2.9.

• The paper is organized as follows.

- In Section 2, we prove Theorem 9.

- In Section 3, we prove Theorem 10.

- In Section 4, we give a sufficient condition for minimality among non-potentially closed sets. We use it to prove Theorems 5 and 6.

- In Section 5, we give conditions on A which allow us to replace $\overline{A_1} \setminus A_1$ with $(2^\omega \times 2^\omega) \setminus A_1$ in Theorem 9 (and therefore come back to $\leq^r_c$). We can write $A_1 = \bigcup_n \text{Gr}(f_n)$, where $f_n(s_n 0\gamma) := s_n 1\gamma$. Roughly speaking, we require that the f_n's do not induce cycles. This is really the key property making the A_α's appearing in the statement of Theorem 5 pairwise orthogonal. We will deduce from this the minimality of A_1 among non-potentially closed sets for $\leq^r_c$, using the sufficient condition for minimality in Section 4.

2 A minimum non-potentially closed set.

We will prove Theorem 9. The proof illustrates the link between the dichotomy results in [K-S-T] and the notion of potential Baire class. We will see another link in Section 3. The next lemma is essentially Lemma 3.5 in [L1], and the crucial point of its proof.

Lemma 11 *Let X be a nonempty Polish space, n be an integer, D_{f_n} and $f_n[D_{f_n}]$ be dense G_δ subsets of some open subsets of X, and $f_n : D_{f_n} \to f_n[D_{f_n}]$ a continuous and open map.*
(a) Let G be a dense G_δ subset of X. Then $Gr(f_n) \subseteq \overline{Gr(f_n) \cap G^2}$, for each n.
(b) Let $A := \bigcup_n Gr(f_n)$. If $\Delta(X) \subseteq \overline{A} \setminus A$, then A is not pot($\mathbf{\Pi}^0_1$).

Proof. (a) Let U (resp., V) be an open neighborhood of $x \in D_{f_n}$ (resp., $f_n(x)$). Then $f_n[D_{f_n}] \cap V \cap G$ is a dense G_δ subset of $f_n[D_{f_n}] \cap V$, thus $f_n^{-1}(V \cap G)$ is a dense G_δ subset of $f_n^{-1}(V)$. Therefore $G \cap f_n^{-1}(V)$ and $G \cap f_n^{-1}(V \cap G)$ are dense G_δ subsets of $f_n^{-1}(V)$. So we can find

$$y \in U \cap G \cap f_n^{-1}(V \cap G).$$

Now $(y, f_n(y))$ is in the intersection $(U \times V) \cap \text{Gr}(f_n) \cap G^2$, so this set is non-empty.

(b) We argue by contradiction: we can find a finer Polish topology on X such that A becomes closed. By 15.2, 11.5 and 8.38 in [K], the new topology and the old one agree on a dense G_δ subset of X, say G: $A \cap G^2 \in \mathbf{\Pi}^0_1(G^2)$. Let $x \in G$. We have $(x, x) \in G^2 \cap \overline{A} \setminus A$. By (a) we get $\overline{A} \subseteq \overline{A \cap G^2}$. Thus $(x, x) \in G^2 \cap \overline{A \cap G^2} \setminus (A \cap G^2)$, which is absurd. □

Corollary 12 *The set* $A_1 = \overline{A_1} \setminus \Delta(2^\omega)$ *is* $D_2(\mathbf{\Sigma}^0_1) \setminus$ *pot*$(\mathbf{\Pi}^0_1)$*, and* $\overline{A_1} = A_1 \cup \Delta(2^\omega)$.

Proof. As we saw in the introduction, we can write $A_1 = \bigcup_n \text{Gr}(f_n)$, where $f_n(s_n 0\gamma) := s_n 1\gamma$. Notice that f_n is a partial homeomorphism with clopen domain and range. Moreover, we have

$$\Delta(2^\omega) \subseteq \overline{A_1} \setminus A_1$$

(in fact, the equality holds). Indeed, if $t \in 2^{<\omega}$, we have $(s_{\psi^{-1}(t)}0^\infty, s_{\psi^{-1}(t)}10^\infty) \in N^2_t \cap A_1$. Thus $A_1 = \overline{A_1} \setminus \Delta(2^\omega)$ is $D_2(\mathbf{\Sigma}^0_1)$, and the corollary follows from Lemma 11. □

Proof of Theorem 9. We cannot have (a) and (b) simultaneously. For if D is potentially closed and separates A from B, then we get $A_1 = (u \times v)^{-1}(D) \cap \overline{A_1}$, thus $A_1 \in$ pot$(\mathbf{\Pi}^0_1)$, which contradicts Corollary 12.

• Let $f : \omega^\omega \to X \times Y$ be a continuous map with $f[\omega^\omega] = B$, and f_0 (resp., f_1) be the first (resp., second) coordinate of f, so that $(f_0 \times f_1)[\Delta(\omega^\omega)] = B$. We set $R := (f_0 \times f_1)^{-1}(A)$, which is an irreflexive analytic relation on ω^ω. By Theorem 7, either there exists a Borel map $c : \omega^\omega \to \omega$ such that $(\alpha, \beta) \in R$ implies $c(\alpha) \neq c(\beta)$, or there is a continuous map $u_0 : 2^\omega \to \omega^\omega$ such that $(\alpha, \beta) \in A_1$ implies $(u_0(\alpha), u_0(\beta)) \in R$.

• In the first case, we define $C_n := c^{-1}(\{n\})$. We get $\Delta(\omega^\omega) \subseteq \bigcup_n C^2_n \subseteq \neg R$, so that

$$B \subseteq \bigcup_n f_0[C_n] \times f_1[C_n] \subseteq \neg A.$$

By a standard reflection argument there is a sequence (X_n) (resp., (Y_n)) of Borel subsets of X (resp., Y) with

$$\bigcup_n f_0[C_n] \times f_1[C_n] \subseteq \bigcup_n X_n \times Y_n \subseteq \neg A.$$

But $\bigcup_n X_n \times Y_n$ is pot$(\mathbf{\Sigma}^0_1)$, so we are in the case (a).

• In the second case, let $u := f_0 \circ u_0$, $v := f_1 \circ u_0$. These maps satisfy the conclusion of condition (b) because $\overline{A_1} \setminus A_1 \subseteq \Delta(2^\omega)$, by Corollary 12.

• By the results in [L3], we can neither replace $\overline{A_1} \setminus A_1$ with $(2^\omega \times 2^\omega) \setminus A_1$, nor can we ensure that u and v are one-to-one. □

Remarks. (a) In Theorem 9, we cannot ensure that $u = v$ when $X = Y$: take $X := 2^\omega$,

$$A := \{(\alpha, \beta) \in N_0 \times N_1 / \alpha <_{\text{lex}} \beta\}$$

and $B := (N_0 \times N_1) \setminus A$.

(b) This proof cannot be generalized, in the sense that we used the fact that the range of a countable union of Borel rectangles (a pot$(\mathbf{\Sigma}^0_1)$ set) by a product function is still a countable union of rectangles, so more or less a pot$(\mathbf{\Sigma}^0_1)$ set. This fails completely for the dual level. Indeed, we saw that the range of the diagonal (which is closed) by a product function can be any analytic set. So in view of generalizations, it is better to have another proof of Theorem 9.

3 The non-injectivity in the Kechris-Solecki-Todorčević dichotomy.

Now we will prove Theorem 10. The proof we give is not the original one, which used effective descriptive set theory, and a reflection argument. The proof we give here is due to B. D. Miller, and is a simplification of the original proof.

Notation. If $A \subseteq X^2$, $A^{-1} := \{(y, x) \in X^2/(x, y) \in A\}$ and $s(A) := A \cup A^{-1}$ is the symmetric set generated by A.

• Fix sets $S_0 \supseteq S_1 \supseteq \ldots$ of natural numbers such that

(1) $S_n \setminus S_{n+1}$ is infinite for each integer n.

(2) $\bigcap_{n\in\omega} S_n = \emptyset$.

• For each $n \in \omega$, fix $f_n : S_n \to S_n \setminus S_{n+1}$ injective, and define $g_n : 2^\omega \to 2^\omega$ by

$$[g_n(\alpha)](k) := \begin{cases} \alpha[f_n(k)] \text{ if } k \in S_n, \\ \\ \alpha(k) \text{ otherwise.} \end{cases}$$

• It is clear that each of the closed sets $M_n := \{\alpha \in 2^\omega/g_n(\alpha) = \alpha\}$ is meager, and since each g_n is continuous and open, it follows that the F_σ set

$$M := \bigcup_{s\in\omega^{<\omega}, n\in\omega} (g_{s(0)} \circ \ldots \circ g_{s(|s|-1)})^{-1}(M_n)$$

is also meager, so that $X := 2^\omega \setminus M$ is a comeager, dense G_δ set which is invariant with respect to each g_n. Put $G_1 := \bigcup_{n\in\omega} s[\text{Gr}(g_n|_X)]$.

Proof of Theorem 10. We argue by contradiction: this gives (X_0, R_0).

Claim 1. *Let X be a Polish space, and $g_0, g_1, \ldots : X \to X$ fixed-point free Borel functions such that $g_m \circ g_n = g_m$ if $m < n$. Then every locally countable Borel directed subgraph of the Borel directed graph $G := \bigcup_{n\in\omega} Gr(g_n)$ has countable Borel chromatic number, i.e., satisfies Condition (a) in Theorem 7.*

Suppose that H is a locally countable Borel directed subgraph of G. By the Lusin-Novikov uniformization theorem, there are Borel partial injections h_n on X such that $H = \bigcup_{n\in\omega} \text{Gr}(h_n)$. By replacing each h_n with its restrictions to the sets $\{x \in D_{h_n}/h_n(x) = g_m(x)\}$, for $m \in \omega$, we can assume that for all $n \in \omega$, there is $k_n \in \omega$ such that $h_n = g_{k_n}|_{D_{h_n}}$. It is easily seen that the directed graph associated with a Borel function has countable Borel chromatic number (see also Proposition 4.5 of [K-S-T]), so by replacing h_n with its restriction to countably many Borel sets, we can assume also that for all $n \in \omega$, $D^2_{h_n} \cap \bigcup_{k\leq k_n} \text{Gr}(g_k) = \emptyset$. It only remains to note that $D^2_{h_n} \cap \bigcup_{k>k_n} \text{Gr}(g_k) = \emptyset$. To see this, simply observe that if $k > k_n$ and $x, g_k(x) \in D_{h_n}$, then $h_n(x) = g_{k_n}(x) = g_{k_n} \circ g_k(x) = h_n \circ g_k(x)$, which contradicts the fact that h_n is a partial injection. This proves the claim. $\diamond$

Claim 2. *The Borel graph G_1 has uncountable Borel chromatic number, but if $H \subseteq G_1$ is a locally countable Borel directed graph, then H has countable Borel chromatic number.*

Condition (1) implies that $g_m \circ g_n = g_m$ if $m < n$, so Claim 1 ensures that if $H \subseteq G_1$ is a locally countable Borel directed graph, then H has countable Borel chromatic number.

To see that G_1 has uncountable Borel chromatic number, it is enough to show that if $B \in \mathbf{\Delta}^1_1(2^\omega)$ is non-meager, then $B \cap G_1^2 \neq \emptyset$. Let $s \in 2^{<\omega}$ such that B is comeager in N_s. It follows from condition (2) that there is $n \in \omega$ such that $|s| < k$ for each $k \in S_n$. Then g_n is a continuous, open map which sends N_s into itself, thus $B \cap X \cap N_s \cap g_n^{-1}(B \cap X \cap N_s)$ is comeager in N_s. Letting x be any element of this set, it follows that x, $g_n(x)$ are G_1-related elements of B. $\diamond$

We are now ready to prove the theorem: as (X_0, R_0) satisfies (b), it does not satisfy (a). Therefore R_0 has uncountable Borel chromatic number. As $s(A_1)$ and G_1 have uncountable Borel chromatic number, we get $(X_0, R_0) \preceq_{c,1-1} [2^\omega, s(A_1)]$ and $(X_0, R_0) \preceq_{c,1-1} (2^\omega, G_1)$ (with witness π). As $s(A_1)$ is locally countable, R_0 is also locally countable. Therefore $(\pi \times \pi)[R_0]$ is a locally countable Borel subgraph of G_1 with uncountable Borel chromatic number, which contradicts Claim 2. $\square$

Remark. This proof also shows a similar theorem for irreflexive analytic relations, by considering $\bigcup_{n\in\omega}$ Gr$(g_n|_X)$ (resp., A_1) instead of G_1 (resp., $s(A_1)$).

4 Perfect antichains made of sets minimal among non-pot$(\mathbf{\Pi}^0_1)$ sets.

As mentioned in the introduction, a great variety of very different examples appear at level $D_2(\mathbf{\Sigma}^0_1)$, all of the same type. Let us make this more specific.

Definition 13 *We say that $(X, (f_n))$ is a $converging\ situation$ if*

(a) X is a nonempty 0-dimensional perfect Polish space.

(b) f_n is a partial homeomorphism with $\mathbf{\Delta}^0_1(X)$ domain and range.

(c) The diagonal $\Delta(X) = \overline{A^f} \setminus A^f$, where $A^f := \bigcup_n$ Gr(f_n).

This kind of situation plays an important role in the theory of potential complexity (see, for example, Definition 2.4 in [L3]).

Remarks. (a) Note that if $(X, (f_n))$ is a converging situation, then Lemma 11 ensures that A^f is $D_2(\mathbf{\Sigma}^0_1) \setminus$ pot$(\mathbf{\Pi}^0_1)$, since $A^f = \overline{A^f} \setminus \Delta(X)$.

(b) It is clear that an analytic graph (X, A) has countable Borel chromatic number if and only if A is separable from $\Delta(X)$ by a pot$(\mathbf{\Delta}^0_1)$ set. By Remark (a), this implies that $(2^\omega, s(A^f))$ does not have countable Borel chromatic number if $(X, (f_n))$ is a converging situation.

Notation. In the sequel, we set $f_n^B := f_n|_{B \cap f_n^{-1}(B)}$ if $B \subseteq X$ and $(X, (f_n))$ is a converging situation, so that Gr$(f_n^B) =$ Gr$(f_n) \cap B^2$.

The reader should see [Mo] for the basic notions of effective descriptive set theory. Let Z be a recursively presented Polish space.

• The topology Δ_Z is the topology on Z generated by $\Delta^1_1(Z)$. This topology is Polish (see the proof of Theorem 3.4 in [Lo2]).

• The Gandy-Harrington topology Σ_Z on Z is generated by $\Sigma^1_1(Z)$. Recall that

$$\Omega_Z := \{z \in Z / \omega_1^z = \omega_1^{\text{CK}}\}$$

is Borel and Σ^1_1, and $[\Omega_Z, \Sigma_Z]$ is a 0-dimensional Polish space (in fact, the intersection of Ω_Z with any nonempty Σ^1_1 set is a nonempty clopen subset of $[\Omega_Z, \Sigma_Z]$-see [L1]).

Lemma 14 *Let $(X, (f_n))$ be a converging situation, P a Borel subset of X such that $A^f \cap P^2$ is not pot($\mathbf{\Pi}^0_1$), and σ a finer Polish topology on P. Then we can find a Borel subset S of P and a topology τ on S finer than σ such that $([S, \tau], (f^S_n)_n)$ is a converging situation.*

Proof. We may assume that $[P, \sigma]$ is recursively presented and f^P_n, $A^f \cap P^2$ are Δ^1_1. We set $D := \{x \in P / x \in \Delta^1_1\}$, and $S := \{x \in P / (x, x) \in \overline{A^f \cap P^2}^{\Delta^2_P}\} \cap \Omega_P \setminus D$. As $S \in \Sigma^1_1$, $[S, \Sigma_P]$ is a 0-dimensional perfect Polish space. We set $E := A^f \cap (P \setminus D)^2$. Note that D is countable. By Remark 2.1 in [L1], E is not potentially closed since

$$A^f \cap P^2 = [A^f \cap ((P \cap D) \times P)] \cup [A^f \cap (P \times (P \cap D))] \cup E.$$

Therefore $\overline{E}^{\Delta^2_P} \setminus E$ is a nonempty subset of $(P \setminus D)^2 \cap \overline{A^f} \setminus A^f \subseteq \Delta(X)$. Thus $S \neq \emptyset$. Note also that $(x, x) \in \overline{A^f \cap P^2}^{\Delta^2_P} \cap S^2 = \overline{A^f \cap P^2}^{\Sigma^2_P} \cap S^2 = \overline{A^f \cap S^2}^{[S, \Sigma_P]^2}$ if $x \in S$. Conversely, we have $\overline{A^f \cap S^2}^{[S, \Sigma_P]^2} \setminus (A^f \cap S^2) \subseteq S^2 \cap \overline{A^f} \setminus A^f \subseteq \Delta(S)$. We have proved that S is a Borel subset of P such that $([S, \Sigma_P], (f^S_n)_n)$ is a converging situation. □

Theorem 15 *Let Y, Y' be Polish spaces, $A \in \mathbf{\Delta}^1_1(Y \times Y')$, $(X, (f_n))$ a converging situation. We assume that $A \leq^r_c A^f$. Then exactly one of the following holds:*
(a) The set A is pot($\mathbf{\Pi}^0_1$).
(b) We can find a Borel subset B of X and a finer topology τ on B such that $([B, \tau], (f^B_n)_n)$ is a converging situation and $A^f \cap B^2 \leq^r_c A$.

Proof. Let u and v be continuous functions such that $A = (u \times v)^{-1}(A^f)$. We assume that A is not potentially closed. By Theorem 9 we can find continuous maps $u' : 2^\omega \to Y$ and $v' : 2^\omega \to Y'$ such that $A_1 = (u' \times v')^{-1}(A) \cap \overline{A_1}$. We set $H := u[u'[2^\omega]]$, $K := v[v'[2^\omega]]$ and $P := H \cap K$. Then H, K and P are compact and $A^f \cap (H \times K)$ is not pot($\mathbf{\Pi}^0_1$) since

$$A_1 = [(u \circ u') \times (v \circ v')]^{-1}(A^f \cap (H \times K)) \cap \overline{A_1}$$

(we have $A_1 \notin$ pot($\mathbf{\Pi}^0_1$) by Corollary 12). Therefore $A^f \cap P^2$ is not pot($\mathbf{\Pi}^0_1$), since

$$\begin{aligned} A^f \cap (H \times K) &= [A^f \cap ((H \setminus K) \times K)] \cup [A^f \cap (H \times (K \setminus H))] \cup [A^f \cap P^2] \\ &= [\overline{A^f} \cap ((H \setminus K) \times K)] \cup [\overline{A^f} \cap (H \times (K \setminus H))] \cup [A^f \cap P^2]. \end{aligned}$$

By Lemma 14 we can find a Borel subset S of P and a finer topology σ on S such that $([S, \sigma], (f^S_n)_n)$ is a converging situation.

By the Jankov-von Neumann Theorem there is $f':S\rightarrow u^{-1}(S)$ (respectively, $g':S\rightarrow v^{-1}(S)$) Baire measurable such that $u(f'(x))=x$ (respectively, $v(g'(x))=x$), for each $x\in S$. Notice that f' and g' are one-to-one. Let G be a dense G_δ subset of S such that $f'|_G$ and $g'|_G$ are continuous. These functions are witnesses to the inequality $A^f\cap G^2\ \leq^r_c\ A$. By Lemma 11, we get $\text{Gr}(f^S_n)\subseteq\overline{\text{Gr}(f^S_n)\cap G^2}$. Therefore $\overline{A^f\cap S^2}=\overline{A^f\cap G^2}$, $\Delta(G)\ =\ G^2\cap\overline{A^f\cap G^2}\setminus(A^f\cap G^2)$, and $A^f\cap G^2$ is not pot($\mathbf{\Pi}^0_1$) by Lemma 11.

By Lemma 14 we can find a Borel subset B of G, equipped with some topology τ finer than σ, such that $([B,\tau],(f^B_n)_n)$ is a converging situation. $\square$

Corollary 16 *Let $(X,(f_n))$ be a converging situation. The following statements are equivalent:*
(a) A^f is $\leq^r_c$-minimal among $\mathbf{\Delta}^1_1\setminus$ pot($\mathbf{\Pi}^0_1$) sets.
(b) For any Borel subset B of X and any finer Polish topology τ on B, $A^f\ \leq^r_c\ A^f\cap B^2$ if $A^f\cap B^2$ is not pot($\mathbf{\Pi}^0_1$).
(c) For any Borel subset B of X and for each finer topology τ on B, $A^f\ \leq^r_c\ A^f\cap B^2$ if $([B,\tau],(f^B_n)_n)$ is a converging situation.

Proof. (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) are obvious. So let us show that (c) $\Rightarrow$ (a). Let Y, Y' be Polish spaces, $A\in\mathbf{\Delta}^1_1(Y\times Y')\setminus\text{pot}(\mathbf{\Pi}^0_1)$. We assume that $A\leq^r_c A^f$. By Theorem 15 we get a Borel subset B of X and a finer topology τ on B such that $([B,\tau],(f^B_n)_n)$ is a converging situation and $A^f\cap B^2\leq^r_c A$. By (c) we get $A^f\ \leq^r_c\ A^f\cap B^2$. Therefore $A^f\ \leq^r_c\ A$. $\square$

This is the sufficient condition for minimality that we mentioned in the introduction. The following definitions, notation and facts will be used here and in Section 5 to build the reduction functions in the minimality results that we want to show.

Definition 17 *Let R be a relation on a set E.*
- *An $R-path$ is a finite sequence $(e_i)_{i\leq n}\subseteq E$ such that $(e_i,e_{i+1})\in R$ for $i<n$.*
- *We say that E is $R-connected$ if there is an R-path $(e_i)_{i\leq n}$ with $e_0=e$ and $e_n=e'$ for each $e,\ e'\in E$.*
- *An $R-cycle$ is an R-path $(e_i)_{i\leq n}$ such that $n\geq 3$ and*

$$[0\leq i\neq j\leq n \text{ and } e_i=e_j]\ \Leftrightarrow\ \{i,j\}=\{0,n\}.$$

- *We say that R is $acyclic$ if there is no R-cycle.*

Recall that if R is symmetric and acyclic, $e,\ e'\in E$ and $(e_i)_{i\leq n}$ is an R-path with $e_0=e$ and $e_n=e'$, then we can find a unique R-path $p_{e,e'}:=(f_j)_{j\leq m}$ without repetition with $f_0=e$ and $f_m=e'$. We will write $|p_{e,e'}|=m+1$.

Notation. Let $\Theta:=(\theta_n)\subseteq 2^{<\omega}$ with $|\theta_n|=n$. We will use two examples of such Θ's: $\theta_n=0^n$ and $\theta_n=s_n$ (where s_n has been defined in the introduction to build A_1). We define a tree $\mathfrak{R}_\Theta$ on 2×2:

$$\mathfrak{R}_\Theta:=\{(e,e')\in(2\times 2)^{<\omega}/e=e' \text{ or } \exists n\in\omega\ \exists w\in 2^{<\omega}\ \ (e,e')=(\theta_n 0w,\theta_n 1w)\}.$$

Recall that $s(\mathfrak{R}_\Theta)$ is the symmetric set generated by $\mathfrak{R}_\Theta$.

Proposition 18 *(a)* $(2^n, s(\mathfrak{R}_\Theta))$ *is connected, for each* $n \in \omega$.
(b) The relation $s(\mathfrak{R}_\Theta)$ *is acyclic.*
(c) If $e,\ e' \in 2^n$ *and* $l < n$ *is maximal with* $e(l) \neq e'(l)$*, the coordinate* l *is changed only once in* $p_{e,e'}$*, and the other changed coordinates are at a level less than* l*.*

Proof. (a) We argue by induction on n. As $(\emptyset)$ is an $s(\mathfrak{R}_\Theta)$-path from $\emptyset$ to $\emptyset$, the statement is true for $n = 0$. Assume that it is true at the level n, and let $e,\ e' \in 2^{n+1}$. We can write $e = s\epsilon$ and $e' = s'\epsilon'$, where $s, t \in 2^n$ and $\epsilon, \epsilon' \in 2$. If $\epsilon = \epsilon'$, then let $(f_i)_{i \leq m}$ be an $s(\mathfrak{R}_\Theta)$-path with $f_0 = s$ and $f_m = s'$. Let $e_i := f_i\epsilon$. Then $(e_i)_{i \leq m}$ is an $s(\mathfrak{R}_\Theta)$-path with $e_0 = e$ and $e_m = e'$. If $\epsilon \neq \epsilon'$, then let $(f_i)_{i \leq m}$ be an $s(\mathfrak{R}_\Theta)$-path with $f_0 = s$ and $f_m = \theta_n$, and $(g_j)_{j \leq p}$ be an $s(\mathfrak{R}_\Theta)$-path with $g_0 = \theta_n$ and $g_p = s'$. We set $e_i := f_i\epsilon$ if $i \leq m$, $g_{i-m-1}\epsilon'$ if $m < i \leq m + p + 1$. Then $(e_i)_{i \leq m+p+1}$ is an $s(\mathfrak{R}_\Theta)$-path with $e_0 = e$ and $e_{m+p+1} = e'$.

(b) We argue by contradiction. Let $(e_i)_{i \leq n}$ be an $s(\mathfrak{R}_\Theta)$-cycle, $p > 0$ be the common length of the e_i's, and $l < p$ maximal such that the sequence $(e_i(l))_{i \leq n}$ is not constant. We can find i_1 minimal with $e_{i_1}(l) \neq e_{i_1+1}(l)$. We have $e_{i_1}(l) = e_0(l) = e_n(l)$. We can find $i_2 > i_1 + 1$ minimal with $e_{i_1+1}(l) \neq e_{i_2}(l)$. Then $e_{i_1}(l) = e_{i_2}(l)$ and $e_{i_1} = e_{i_2}$, because $|\theta_l| = l$. Thus $i_1 = 0$ and $i_2 = n$. But $e_{i_1+1} = e_{i_2-1}$, which is absurd. Note that this proof of (b) is essentially in [L3], Theorem 2.7.

(c) This follows from (b) and the proof of (a). □

Now we come to some examples of converging situations, with some cycle relations involved.

Notation. Let $S \subseteq \omega$, and

$$A^S := \{(s0\gamma, s1\gamma)/\ s \in 2^{<\omega} \text{and Card}(s) \in S \text{ and } \gamma \in 2^\omega\}.$$

(Card(s) is the number of ones in s.) We define partial homeomorphisms

$$f_n^S : \bigcup_{s \in 2^n,\ \text{Card}(s) \in S} N_{s0} \to \bigcup_{s \in 2^n,\ \text{Card}(s) \in S} N_{s1}$$

by $f_n^S(s0\gamma) := s1\gamma$. Notice that $A^S = A^{f^S}$ is Borel. One can show the existence of $\mathfrak{A} : 2^\omega \to 2^\omega$ continuous such that $\mathfrak{A}(S)$ is a Borel code for A^S, for each $S \subseteq \omega$. Notice that $(2^\omega, (f_n^S)_n)$ is a converging situation if and only if S is infinite. This is also equivalent to $A^S \notin$ pot$(\mathbf{\Pi}_1^0)$. Indeed, if S is finite, $\overline{A^S} \setminus A^S$ is a countable subset of $\Delta(2^\omega)$. **So in the sequel we will assume that** S **is infinite.**

Let $n_S := \min S$, and $S' := \{n - n_S / n \in S\}$. Then $0 \in S'$ and the maps u and v defined by $u(\alpha) = v(\alpha) := 1^{n_S}\alpha$ are witnesses to $A^{S'} \leq_c^r A^S$. **So in the sequel we will also assume that** $0 \in S$**.**

• If $S \subseteq \omega$ and $t \in \omega^{<\omega} \setminus \{\emptyset\}$, then we set $f_t^S := f_{t(0)}^S \ldots f_{t(|t|-1)}^S$, when it makes sense. We will also use the following tree $\mathfrak{R}$ on 2×2. If $s, t \in 2^{<\omega}$, then we set

$$s\ \mathfrak{R}\ t \ \Leftrightarrow\ |s| = |t| \text{ and } (N_s \times N_t) \cap A^S \neq \emptyset.$$

In particular, if $n_0 < n_1$ and $1 \in S$, then we get $f_{<n_0,n_1>}^S(0^\infty) = f_{<n_1,n_0>}^S(0^\infty)$. This is the kind of cycle relation we mentioned in the introduction. In this case $s(\mathfrak{R})$ is not acyclic since $< 0^{n_1+1}, 0^{n_0}10^{n_1-n_0}, 0^{n_0}10^{n_1-n_0-1}1, 0^{n_1}1, 0^{n_1+1} >$ is an $s(\mathfrak{R})$-cycle. We set $f_n^C := f_n^S|_{C \cap {f_n^S}^{-1}(C)}$ for each Borel subset C of 2^ω, when S is fixed.

• Let (H) be the following hypothesis on S:

$$(H)\left\{\begin{array}{l}\quad\text{Let } C\in\mathbf{\Delta}^1_1(2^\omega),\ \sigma \text{ be a finer topology on } C \text{ such that } ([C,\sigma],(f^C_n)_n)\\ \text{is a converging situation, } l,p\in\omega. \text{ Then we can find } n\geq l \text{ and } \gamma\in D_{f^C_n}\\ \text{with } \text{Card}(\gamma\lceil n)+(S\cap[0,p])=S\cap(\text{Card}(\gamma\lceil n)+[0,p]).\end{array}\right.$$

The next result will lead to a combinatorial condition on S implying the minimality of A^S among non-potentially closed sets.

Theorem 19 *Let S satisfy (H), $B\in\mathbf{\Delta}^1_1(2^\omega)$, and τ a finer topology on B such that $([B,\tau],(f^B_n)_n)$ is a converging situation. Then $A^S\leq^r_c A^S\cap B^2$.*

Proof. Let $X:=[B,\tau]$, $f_n:=f^B_n$. We are trying to build continuous maps $u,\ v:2^\omega\rightarrow X$ such that $A^S=(u\times v)^{-1}(A^f)$. We will actually have more: $u=v$ will be one-to-one. We set $s\wedge t:=s\lceil\max\{n\in\omega/s\lceil n=t\lceil n\}$, for $s,t\in 2^{<\omega}$.

• We construct a sequence $(U_s)_{s\in 2^{<\omega}}$ of nonempty clopen subsets of X, $\phi:\omega\rightarrow\omega$ strictly increasing, and $\theta:\omega\rightarrow\omega$ such that

$(i)\qquad U_{s^\frown i}\subseteq U_s.$

$(ii)\qquad \text{diam}(U_s)\leq 1/|s|$ if $s\neq\emptyset$.

$$(iii)\quad (s\,\mathfrak{R}\,t \text{ and } s\neq t)\Rightarrow\left\{\begin{array}{l}U_t=f_{\phi(|s\wedge t|)}[U_s],\\ \theta(|s\wedge t|)+(S\cap[0,|s\wedge t|])=S\cap(\theta(|s\wedge t|)+[0,|s\wedge t|]),\\ \forall z\in U_s\ \ \text{Card}(z\lceil\phi(|s\wedge t|))=\theta(|s\wedge t|)+\text{Card}(s\lceil|s\wedge t|).\end{array}\right.$$

$(iv)\qquad (\neg\, s\,\mathfrak{R}\,t \text{ and } |s|=|t|)\ \Rightarrow\ (U_s\times U_t)\cap[\bigcup_{q<|s|}\text{Gr}(f_q)\cup\Delta(X)]=\emptyset.$

• First we show that this construction is sufficient to get the theorem. We define a continuous map $u:2^\omega\rightarrow X$ by $\{u(\alpha)\}:=\bigcap_n U_{\alpha\lceil n}$. If $\alpha<_{\text{lex}}\beta$, then we have $\neg\beta\lceil r\ \mathfrak{R}\ \alpha\lceil r$ if r is big enough, thus by condition (iv), $(u(\beta),u(\alpha))$ is in $U_{\beta\lceil r}\times U_{\alpha\lceil r}\subseteq X^2\setminus\Delta(X)$. Therefore u is one-to-one. If $(\alpha,\beta)\in A^S$, fix n such that $\beta=f^S_n(\alpha)$. Then $\alpha\lceil r$ and $\beta\lceil r$ satisfy the hypothesis in condition (iii) for each $r>n$. Therefore $u(\beta)=f_{\phi(n)}(u(\alpha))$ and $(u(\alpha),u(\beta))\in A^f$. If $\alpha=\beta$, then $(\alpha,\beta)\notin A^S$ and $(u(\alpha),u(\beta))\in\Delta(X)\subseteq\neg A^f$. Otherwise, $(\alpha,\beta)\notin\overline{A^S}$ and there is r_0 such that $\alpha\lceil r$ and $\beta\lceil r$ satisfy the hypothesis in condition (iv) for $r\geq r_0$. This shows that $(u(\alpha),u(\beta))\notin A^f$. So it is enough to do the construction.

• We set $U_\emptyset:=X$. Suppose that $(U_s)_{s\in 2^{\leq p}}$, $(\phi(j))_{j<p}$ and $(\theta(j))_{j<p}$ satisfying conditions (i)-(iv) have been constructed, which is done for $p=0$.

• We will use the relation $\mathfrak{R}_\Theta$ defined before Proposition 18 with $\theta_n:=0^n$. Notice that $\mathfrak{R}_\Theta\subseteq\mathfrak{R}$. We set $t_0:=\theta_p 0$. We define a partition of 2^{p+1} as follows. Using Proposition 18.(b) we set, for $k\in\omega$,

$$H_k:=\{t\in 2^{p+1}/|p_{t,t_0}|=k+1\}.$$

If H_{k+1} is non-empty, then H_k is non-empty. Thus we can find an integer q such that H_0, ..., H_q are not empty and H_k is empty if $k>q$. We order 2^{p+1} as follows: t_0, then H_1 in any order with $\theta_p 1$ first, H_2 in any order, ..., H_q in any order. This gives $t_0,\ldots,t_{2^{p+1}-1}$. Notice that we can find $j<n$ such that $t_j\ s(\mathfrak{R}_\Theta)\ t_n$ if $0<n<2^{p+1}$. In particular, if $E^n:=\{t_j/j\leq n\}$, then $(E^n,s(\mathfrak{R}_\Theta))$ is connected for each $n<2^{p+1}$.

• We will construct integers $\phi(p)$, $\theta(p)$ and nonempty clopen subsets U_k^n of X, for $n < 2^{p+1}$ and $k \leq n$, satisfying

(1) $U_k^n \subseteq U_{t_k\lceil p}$.
(2) $\mathrm{diam}(U_k^n) \leq 1/p+1$.
(3) $(t_k \,\mathfrak{R}\, t_l$ and $t_k \neq t_l) \Rightarrow$
$$\begin{cases} U_l^n = f_{\phi(|t_k\wedge t_l|)}[U_k^n], \\ \theta(|t_k\wedge t_l|)+(S\cap[0,|t_k\wedge t_l|]) = S\cap(\theta(|t_k\wedge t_l|)+[0,|t_k\wedge t_l|]), \\ \forall z\in U_k^n \ \mathrm{Card}(z\lceil\phi(|t_k\wedge t_l|)) = \theta(|t_k\wedge t_l|)+\mathrm{Card}(t_k\lceil|t_k\wedge t_l|). \end{cases}$$
(4) $\neg\, t_k \,\mathfrak{R}\, t_l \ \Rightarrow\ (U_k^n \times U_l^n) \cap [\bigcup_{q\leq p} \mathrm{Gr}(f_q) \cup \Delta(X)] = \emptyset$.
(5) $U_k^{n+1} \subseteq U_k^n$.

We will then set $U_{t_k} := U_k^{2^{p+1}-1}$ for $k < 2^{p+1}$, so that conditions (i)-(iv) are fullfilled.

• Let $C \in \mathbf{\Delta}_1^0(U_{t_0\lceil p}) \setminus \{\emptyset\}$ such that $C^2 \cap \bigcup_{q\leq p} \mathrm{Gr}(f_q) = \emptyset$. Apply hypothesis (H) to C and $\sigma := \tau$. This gives $n_0 \geq \sup\{\phi(q)+1/q<p\}$ and $\gamma \in D_{f^C_{n_0}}$ such that

$$\mathrm{Card}(\gamma\lceil n_0) + (S\cap[0,p]) = S \cap (\mathrm{Card}(\gamma\lceil n_0) + [0,p]).$$

We set $\phi(p) := n_0$, $\theta(p) := \mathrm{Card}(\gamma\lceil n_0)$.

We then choose $U_0^0 \in \mathbf{\Delta}_1^0(C\cap f_{n_0}^{-1}(C)) \setminus \{\emptyset\}$ with suitable diameter such that $f_{n_0}[U_0^0] \cap U_0^0 = \emptyset$, and $z\lceil n_0 = \gamma\lceil n_0$ for each $z \in U_0^0$. Assume that $U_0^0,\ \ldots,\ U_0^{n-1},\ \ldots, U_{n-1}^{n-1}$ satisfying conditions (1)-(5) have been constructed (which has already been accomplished for $n=1$). As $n \geq 1$, we have $t_n \neq t_0$ and $|p_{t_n,t_0}| \geq 2$. So fix $r < n$ such that $p_{t_n,t_0}(1) = t_r$. Notice that U_r^{n-1} has been constructed.

Case 1. $t_n\lceil p = t_r\lceil p$.

- We have $t_n\lceil p = \theta_p$, thus $|p_{t_n,t_0}| = 2$, $r=0$, $t_n = \theta_p 1$ and $n=1$. Moreover, U_0^0 is a subset of $f_{\phi(p)}^{-1}(U_{t_1\lceil p})$, so we can choose a nonempty clopen subset U_1^1 of $f_{\phi(p)}[U_0^0]$ with suitable diameter. Then we set $U_0^1 := f_{\phi(p)}^{-1}(U_1^1) \subseteq U_0^0$. So $U_0^n, \ldots, U_n^n$ are constructed and fullfill (1)-(3) and (5). It remains to check condition (4).

- Fix $k,\ l \leq 1$ such that $\neg\, t_k \,\mathfrak{R}\, t_l$. Then $k = 1 = 1-l$. We have $U_1^1 = f_{\phi(p)}[U_0^1]$. Thus

$$U_1^1 \times U_0^1 = f_{\phi(p)}[U_0^1] \times U_0^1 = f_{n_0}[U_0^1] \times U_0^1 \subseteq f_{n_0}[U_0^0] \times U_0^0 \subseteq C^2,$$

so we are done by the choice of C and U_0^0.

Case 2. $t_n\lceil p \neq t_r\lceil p$.

2.1. $t_r \,\mathfrak{R}_\Theta\, t_n$.

- By the induction hypothesis we have $U_{t_n\lceil p} = f_{\phi(|t_r\wedge t_n|)}[U_{t_r\lceil p}]$ and $U_r^{n-1} \subseteq U_{t_r\lceil p}$. We choose a nonempty clopen subset U_n^n of $f_{\phi(|t_r\wedge t_n|)}[U_r^{n-1}]$ with suitable diameter, so that conditions (1)-(5) for $k = l = n$ are fullfilled.

- We then define the U_q^n's for $q < n$, by induction on $|p_{t_q,t_n}|$: fix $m \leq n$ with $p_{t_q,t_n}(1) = t_m$. Notice that $q = r$ if $m = n$.

2.1.1. $t_m \ \mathfrak{R}_\Theta \ t_q$.

We have $m < n$ since we cannot have $p_{t_q,t_n}(1) \ \mathfrak{R}_\Theta \ t_q$ and $t_q \ \mathfrak{R}_\Theta \ p_{t_q,t_n}(1)$ ($\tilde{s} \leq_{\text{lex}} \tilde{t}$ if $\tilde{s} \ \mathfrak{R}_\Theta \ \tilde{t}$). So $U_q^{n-1} = f_{\phi(|t_m \wedge t_q|)}[U_m^{n-1}]$. We put

$$U_q^n := f_{\phi(|t_m \wedge t_q|)}[U_m^n].$$

The set U_q^n is a nonempty clopen subset of U_q^{n-1} since $U_m^n \subseteq U_m^{n-1}$.

2.1.2. $t_q \ \mathfrak{R}_\Theta \ t_m$.

If $m < n$, then we have $U_m^{n-1} = f_{\phi(|t_q \wedge t_m|)}[U_q^{n-1}]$. We put

$$U_q^n := f^{-1}_{\phi(|t_q \wedge t_m|)}(U_m^n),$$

so that U_q^n is a nonempty clopen subset of U_q^{n-1}. If $m = n$, then $q = r$ and the same conclusion holds, by the choice of U_n^n.

- So condition (5) is fullfilled in both cases. Conditions (1) and (2) are fullfilled for $k = q$, too. Let us check that the first part of condition (3) restricted to $\mathfrak{R}_\Theta$ is fullfilled. Fix $k \neq l \leq n$ with $t_k \ \mathfrak{R}_\Theta \ t_l$. If $|p_{t_k,t_n}| = 1$ and $|p_{t_l,t_n}| = 2$, then the link between t_k and t_l has already been considered. The argument is similar if $|p_{t_k,t_n}| = 2$ and $|p_{t_l,t_n}| = 1$. If $|p_{t_k,t_n}|$ and $|p_{t_l,t_n}|$ are at least 2, then $p_{t_k,t_n}(1) = p_{t_l,t_n}(0)$ or $p_{t_k,t_n}(0) = p_{t_l,t_n}(1)$, by Proposition 18.(b). Here again, the link has already been considered. So condition (3) restricted to $\mathfrak{R}_\Theta$ is fullfilled. It remains to check conditions (3) and (4).

- Fix $k \neq l$ such that $t_k \ \mathfrak{R} \ t_l$. Then t_k, t_l differ at one coordinate only, and $t_k <_{\text{lex}} t_l$.

Claim. *Assume that t_k, t_l differ at one coordinate only, and that $t_k <_{lex} t_l$. Then*

$$\mathit{Card}(z\lceil\phi(|t_k \wedge t_l|)) = \theta(|t_k \wedge t_l|) + \mathit{Card}(t_k\lceil|t_k \wedge t_l|)$$

for each $z \in U_k^n$.

We can write

$$t_k := 0^{n_0}10^{n_1}1 \ldots 0^{n_{j-1}}10^{n_j}10^{n_{j+1}}1 \ldots 0^{n_{q-1}}10^{n_q},$$

$$t_l := 0^{n_0}10^{n_1}1 \ldots 0^{n_{j-1}}10^{n'_j}10^{m}10^{n_{j+1}}1 \ldots 0^{n_{q-1}}10^{n_q} \ \ (n'_j + 1 + m = n_j).$$

By construction we have

$$U_k^n = f_{\phi(n_0)} f_{\phi(\Sigma_{r\leq 1}\ (n_r+1)-1)} \cdots f_{\phi(\Sigma_{r\leq q-1}\ (n_r+1)-1)}[U_0^n],$$

$$U_l^n = f_{\phi(n_0)} \cdots f_{\phi(\Sigma_{r\leq j-1}\ (n_r+1)-1)} f_{\phi(\Sigma_{r\leq j-1}\ (n_r+1)+(n'_j+1)-1)} f_{\phi(\Sigma_{r\leq j}\ (n_r+1)-1)} \cdots f_{\phi(\Sigma_{r\leq q-1}\ (n_r+1)-1)}[U_0^n].$$

Notice that the length of $t_k \wedge t_l$ is equal to $\Sigma_{r \leq j-1}\ (n_r+1)+(n'_j+1)-1$. Set

$$f := f_{\phi(n_0)} \cdots f_{\phi(\Sigma_{r \leq j-1}\ (n_r+1)-1)}.$$

Then $U^n_l = f f_{\phi(|t_k \wedge t_l|)} f^{-1}(U^n_k)$. Fix $k' \neq l' \leq n$ such that

$$t_{k'} := 0^{\Sigma_{r \leq j-1}\ (n_r+1)+n_j} 1 0^{n_{j+1}} 1 \ldots 0^{n_{q-1}} 1 0^{n_q},$$

$$t_{l'} := 0^{\Sigma_{r \leq j-1}\ (n_r+1)+n'_j} 1 0^m 1 0^{n_{j+1}} 1 \ldots 0^{n_{q-1}} 1 0^{n_q}.$$

Note that Card$(y\lceil\phi(|t_k \wedge t_l|)) = \theta(|t_k \wedge t_l|) +$ Card$(t_{k'}\lceil|t_k \wedge t_l|)$, for each $y \in U^n_{k'}$, since $t_{k'}\ \mathfrak{R}_\Theta\ t_{l'}$. But Card$(z\lceil\phi(|t_k \wedge t_l|)) =$ Card$(y\lceil\phi(|t_k \wedge t_l|)) + j$, for each z in $U^n_k = f[U^n_{k'}]$. As

$$\text{Card}(t_{k'}\lceil|t_k \wedge t_l|) = \text{Card}(t_k\lceil|t_k \wedge t_l|) - j,$$

we get

$$(+) \qquad \text{Card}(z\lceil\phi(|t_k \wedge t_l|)) = \theta(|t_k \wedge t_l|) + \text{Card}(t_k\lceil|t_k \wedge t_l|).$$

This proves the claim. $\diamond$

- The second assertion in condition (3) is clearly fullfilled since $|t_k \wedge t_l| = |t_{k'} \wedge t_{l'}|$. As $t_k\ \mathfrak{R}\ t_l$ and $t_k \neq t_l$ we get Card$(t_k\lceil|t_k \wedge t_l|) \in S$. This implies that S contains $\theta(|t_k \wedge t_l|) +$ Card$(t_k\lceil|t_k \wedge t_l|)$. By the claim we get

$$U^n_l = f_{\phi(|t_k \wedge t_l|)} f f^{-1}(U^n_k) = f_{\phi(|t_k \wedge t_l|)}[U^n_k]$$

(the compositions $f f_{\phi(|t_k \wedge t_l|)} f^{-1}$ and $f_{\phi(|t_k \wedge t_l|)} f f^{-1}$ are defined on U^n_k, so they are equal on this set). Thus condition (3) is fullfilled.

- To get condition (4), fix $k, l \leq n$ with $\neg\ t_k\ \mathfrak{R}\ t_l$, $v(i) := |p_{t_k,t_l}(i) \wedge p_{t_k,t_l}(i+1)|$, and $\varepsilon(i) := 1$ (resp., -1) if $p_{t_k,t_l}(i)\ \mathfrak{R}_\Theta\ p_{t_k,t_l}(i+1)$ (resp., $p_{t_k,t_l}(i+1)\ \mathfrak{R}_\Theta\ p_{t_k,t_l}(i)$), for $i+1 < |p_{t_k,t_l}|$. We set $f^\varepsilon_v := f^{\varepsilon(|v|-1)}_{\phi(v(|v|-1))} \ldots f^{\varepsilon(0)}_{\phi(v(0))}$, so that $U^n_l = f^\varepsilon_v(U^n_k)$. Let m be maximal such that $t_k(m) \neq t_l(m)$. As ϕ is strictly increasing, we get $(U^n_k \times U^n_l) \cap \Delta(X) = \emptyset$, by Proposition 18.(c).

- If t_k, t_l differ in at least two coordinates $m \neq m'$, then the number of appearances of m and m' in v is odd. As ϕ is strictly increasing, this is also true for $\phi(m) \neq \phi(m')$ in $\{\phi(v(i))/i < |v|\}$. This implies that $(U^n_k \times U^n_l) \cap [\bigcup_{q \leq p}\ \text{Gr}(f_q)] = \emptyset$.

- If t_k, t_l differ at only one coordinate m and $t_k >_{\text{lex}} t_l$, then $\alpha(\phi(m)) > \beta(\phi(m))$ if (α, β) is in $U^n_k \times U^n_l$, and $(U^n_k \times U^n_l) \cap [\bigcup_{q \leq p}\ \text{Gr}(f_q)] = \emptyset$.

- So we may assume that t_k, t_l differ only at coordinate $\phi^{-1}(q)$, and that $t_k <_{\text{lex}} t_l$. By the Claim we have $(+)$ for each $z \in U^n_k$. But Card$(t_k\lceil|t_k \wedge t_l|) \notin S$, since $\neg\ t_k\ \mathfrak{R}\ t_l$. So Card$(z\lceil q) \notin S$ if $z \in U^n_k$, and f_q is not defined on U^n_k.

2.2. $t_n \,\Re_\Theta\, t_r$.

This cannot hold since $t_r \Re_\Theta t_n$. Indeed, if $t_n = 0^{n_0}10^{n_1}1\ldots 0^{n_{q-1}}10^{n_q}$, then

$$\begin{aligned} p_{t_n,t_0}(1) &= 0^{n_0+n_1+1}1\ldots 0^{n_{q-1}}10^{n_q},\\ &\;\;\vdots\\ p_{t_n,t_0}(|p_{t_n,t_0}|-2) &= 0^{n_0+n_1+\ldots+n_{q-1}+q-1}10^{n_q}.\end{aligned}$$

This finishes the proof. □

Lemma 20 *The set S satisfies hypothesis (H) if the following is fullfilled:*

$$(M) \qquad \forall p\in\omega\ \exists k\in\omega\ \forall q\in\omega\ \exists c\in\omega\cap[q,q+k]\ \ c+(S\cap[0,p])=S\cap(c+[0,p]).$$

In particular, condition (M) implies that A^S is minimal among non-potentially closed sets for $\leq^r_c$.

Proof. Note that $\emptyset\neq\Delta(C)\subseteq\overline{\bigcup_{q\geq l}\ \text{Gr}(f^C_q)}$, since $([C,\sigma],(f^C_n)_n)$ is a converging situation. So fix $q_0\geq l$ such that $D_{f^C_{q_0}}\neq\emptyset$, and $O_0:=D_{f^C_{q_0}}$. Assume that q_r and O_r have been constructed. We then choose $q_{r+1}>q_r$ such that $O_r\cap(f^C_{q_{r+1}})^{-1}(O_r)\neq\emptyset$, and we define $O_{r+1}:=O_r\cap(f^C_{q_{r+1}})^{-1}(O_r)$. This gives $(q_r)_{r<M}$ and $(O_r)_{r<M}$, where $M:=p+k$.

• For $t\in\omega^{<\omega}$, we let $f^C_t:=f^C_{t(0)}\ldots f^C_{t(|t|-1)}$, when it makes sense. We choose

$$n\geq\max(\sup\{q_r+1/r<M\},l)$$

with $f^C_{q_0,\ldots,q_{M-1}}[O_{M-1}]\cap f^{C^{-1}}_n(f^C_{q_0,\ldots,q_{M-1}}[O_{M-1}])\neq\emptyset$. Let $\beta:=f^C_{q_0,\ldots,q_{M-1}}(\alpha)$ be in the intersection. Notice that $q:=\text{Card}(\beta\lceil n)-M\in\omega$. This gives c in $\omega\cap[q,q+k]$. As $0\in S$, there is $j\leq k$ with $c=\text{Card}(\beta\lceil n)-p-j\in S$. Notice that $\beta=f^C_{q_0,\ldots,q_{p+j-1}}(\gamma)$, where $\gamma=f^C_{q_{p+j},\ldots,q_{M-1}}(\alpha)$. As $\text{Card}(\gamma\lceil n)=c$, $f^S_n(\gamma)$ is defined. But $f^C_n(\beta)$ is in $f^C_{q_0,\ldots,q_{M-1}}[O_{M-1}]$ and $f^S_n(\gamma)$ is in C. So $f^C_n(\gamma)$ is defined.

• The lemma now follows from Corollary 16 and Theorem 19. □

Example. We set $S_{m,F}:=\{n\in\omega/n\ (\text{mod}\ m)\in\{0\}\cup F\}$, where $m\in\omega\setminus\{0\}$ and $F\subseteq m\setminus\{0\}$. Then $S_{m,F}$ fullfills condition (M). In particular, A^ω is minimal. But this gives only countably many examples. To get more, we need some more notation:

Notation. For $\beta\in\omega^\omega$, we set $S_\beta:=\{\Sigma_{i<l}\ (1+\beta(i))/l\in\omega\}$. Notice that $0\in S_\beta$, S_β is infinite, and that any infinite S containing 0 is of this form. Moreover, the map $\beta\mapsto S_\beta$ is continuous since $n\in S_\beta\Leftrightarrow\exists l\leq n\ \ n=\Sigma_{i<l}\ (1+\beta(i))$. We will define a family $(\beta_\alpha)_{\alpha\in 2^\omega}$. Actually, we can find at least two examples:

• The original example is the following. For $\alpha\in 2^\omega$, we recursively define a sequence $(s_{\alpha,n})_n\subseteq 2^{<\omega}$ as follows: $s_{\alpha,0}:=0$, $s_{\alpha,1}:=1$, $s_{\alpha,n+2}:=s^{\alpha(n)+1}_{\alpha,n}s^{\alpha(n+1)+1}_{\alpha,n+1}$. Notice that $s_{\alpha,n}\prec_{\neq}s_{\alpha,n+2}$, so that $\beta_\alpha:=\lim_{n\to\infty}s_{\alpha,2n}\in 2^\omega$ is defined.

• A. Louveau found another example for which it is simpler to check property (M) (and ($\perp$) later), and in the sequel we will work with it. For $\alpha \in 2^\omega$, $n \in \omega$ and $\varepsilon \in 2$, we set

$$\gamma_\alpha(4n + 2\varepsilon) := \varepsilon,$$

$\gamma_\alpha(2n + 1) := \alpha(n)$ (so that γ_α has infinitely many zeros and ones, and the map $\alpha \mapsto \gamma_\alpha$ is continuous). For $i \in \omega$, we then set $(i)_0 := \max\{m \in \omega / 2^m \text{ divides } i+1\}$. Finally, we put $\beta_\alpha(i) := \gamma_\alpha((i)_0)$.

Notice that the map $\alpha \mapsto \beta_\alpha$ is continuous, so that the map $\alpha \mapsto A^{S_{\beta_\alpha}}$ is continuous in the codes.

Corollary 21 *Let $\alpha \in 2^\omega$. Then S_{β_α} satisfies condition (M). In particular, $A^{S_{\beta_\alpha}}$ is minimal among non-potentially closed sets for $\leq_c^r$.*

Proof. First notice that it is enough to show that the following is fullfilled:

$$(MM) \qquad \forall P \in \omega \ \exists K \in \omega \ \forall Q \in \omega \ \exists C \in \omega \cap [Q, Q+K] \ \ \beta_\alpha \lceil P \prec \beta_\alpha - \beta_\alpha \lceil C.$$

Indeed, this condition associates K to $P := p$. Set $k := 2K + 1$. For $q \in \omega$, let Q be minimal with $\Sigma_{i<Q}\ (1 + \beta_\alpha(i)) \geq q$, and fix $C \in \omega \cap [Q, Q + K]$ such that $\beta_\alpha \lceil P \prec \beta_\alpha - \beta_\alpha \lceil C$. We put

$$c := \Sigma_{i<C}\ (1 + \beta_\alpha(i)).$$

Notice that $c \leq q+k$ since $c \leq \Sigma_{i<Q-1}\ (1+\beta_\alpha(i)) + \Sigma_{Q-1 \leq i<C}\ (1+\beta_\alpha(i)) < q+2(C-Q+1) \leq q+2(K+1)$. Finally, note that $c + \Sigma_{i<j}\ (1 + \beta_\alpha(i)) = \Sigma_{i<C+j}\ (1 + \beta_\alpha(i))$, by induction on $j \leq p$.

Notice that for any integers $n,\ i$ and l with $i < 2^n - 1$, we have $(2^n \cdot l + i)_0 = (i)_0$. Indeed, we can find N with $i = 2^{(i)_0}(2N + 1) - 1$, and $(i)_0 < n$. Thus $2^n \cdot l + i = 2^{(i)_0}(2^{n-(i)_0} \cdot l + 2N + 1) - 1$ and $(2^n \cdot l + i)_0 = (i)_0$. Now, if $P \in \omega$, then let n_0 be minimal with $K := 2^{n_0} - 1 \geq P$. If $Q \in \omega$, then let $l \in \omega \cap [\frac{Q}{2^{n_0}}, \frac{Q}{2^{n_0}} + 1[$ and $C := 2^{n_0} \cdot l$. If $i < P$, then $i < 2^{n_0} - 1$, so $(2^{n_0} \cdot l + i)_0 = (i)_0 = (C + i)_0$. Thus $\beta_\alpha(i) = \beta_\alpha(C + i)$. $\square$

Now we come to the study of the cardinality of complete families of minimal sets.

Lemma 22 *Let $(X, (f_n)), (X', (f'_n))$ be converging situations, and $u, v : X \to X'$ continuous maps such that $A^f = (u \times v)^{-1}(A^{f'})$. Then $u = v$.*

Proof. For $x \in X$, fix $x_k \in X$ and $n_k \in \omega$ such that $(x_k, f_{n_k}(x_k))$ tends to (x, x). Note that $(u(x), v(x)) \notin A^{f'}$. Moreover, $(u[x_k], v[f_{n_k}(x_k)]) \in A^{f'}$. Thus $(u(x), v(x)) \in \overline{A^{f'}} \setminus A^{f'} = \Delta(X')$, therefore $u = v$. $\square$

Recall that $A^{-1} := \{(y, x) \in X^2 / (x, y) \in A\}$ if $A \subseteq X^2$.

Theorem 23 *Fix $S,\ S'$ satisfying condition (M). Then*
(a) $A^S \perp_c^r A^{S'}$, provided that the following condition is fullfilled:

$$(\perp) \qquad \exists p \in \omega \ \forall c \in \omega \ \ c+(S \cap [0, p]) \neq S' \cap (c+[0, p]).$$

(b) $A^S \perp_c^r (A^{S'})^{-1}$, provided that the following condition is fullfilled:

$$(\perp^{-1}) \qquad \exists p \in \omega \ \forall c \in \omega \ \ c-(S \cap [0, p]) \neq S' \cap (c-[0, p]).$$

Proof. (a) We argue by contradiction: by Lemma 20, we can find continuous maps $u,\ v:2^\omega\rightarrow 2^\omega$ such that $A^S=(u\times v)^{-1}(A^{S'})$. By Lemma 22, we have $u=v$.

Claim. *Let n, q be integers and $N\in\boldsymbol{\Sigma}^0_1(2^\omega)\setminus\{\emptyset\}$. Then we can find integers $n'>n$, $q'>q$ and a nonempty open subset N' of $N\cap {f^S_{n'}}^{-1}(N)$ with $f^{S'}_{q'}[u(\gamma)]=u[f^S_{n'}(\gamma)]$, and*

$$\mathit{Card}(\gamma\lceil n')+(S\cap[0,p])=S\cap(\mathit{Card}(\gamma\lceil n')+[0,p]),$$

for each $\gamma\in N'$.

Indeed, let $\delta\in u[N]$. As (δ,δ) is not in $\bigcup_{q'\leq q}\mathrm{Gr}(f^{S'}_{q'})$, we can find a clopen neighborhood W of δ such that $W^2\cap\bigcup_{q'\leq q}\mathrm{Gr}(f^{S'}_{q'})=\emptyset$. Let $\tilde N\in\boldsymbol{\Delta}^0_1(2^\omega)\setminus\{\emptyset\}$ with $\tilde N\subseteq N\cap u^{-1}(W)$. By Lemma 20, we can find $n'>n$ and $\gamma_0\in\tilde N\cap {f^S_{n'}}^{-1}(\tilde N)$ with

$$\mathrm{Card}(\gamma_0\lceil n')+(S\cap[0,p])=S\cap(\mathrm{Card}(\gamma_0\lceil n')+[0,p]).$$

Now there is $q'(\gamma)$ such that $f^{S'}_{q'(\gamma)}[u(\gamma)]=u[f^S_{n'}(\gamma)]$, for $\gamma\in\tilde N\cap {f^S_{n'}}^{-1}(\tilde N)\cap N_{\gamma_0\lceil n'}$. We have $q'(\gamma)>q$, by the choice of W. By Baire's Theorem we get q' and N'. $\diamond$

By the Claim we get n_1, q_1 and $N_1\subseteq D_{f^S_{n_1}}$ with $f^{S'}_{q_1}[u(\gamma)]=u[f^S_{n_1}(\gamma)]$ and

$$\mathrm{Card}(\gamma\lceil n_1)+(S\cap[0,p])=S\cap(\mathrm{Card}(\gamma\lceil n_1)+[0,p]),$$

for each $\gamma\in N_1$.

We then get $n_2>n_1$, $q_2>q_1$, and a nonempty open subset N_2 of $N_1\cap {f^S_{n_2}}^{-1}(N_1)$ with $f^{S'}_{q_2}[u(\gamma)]=u[f^S_{n_2}(\gamma)]$ and

$$\mathrm{Card}(\gamma\lceil n_2)+(S\cap[0,p])=S\cap(\mathrm{Card}(\gamma\lceil n_2)+[0,p]),$$

for each γ in N_2. We continue in this fashion, until we get n_{p+1}, q_{p+1} and N_{p+1}. Fix $\gamma\in N_{p+1}$ and set $c:=\mathrm{Card}(u(\gamma)\lceil q_{p+1})$.

• Fix $m\in S\cap[0,p]$. For $t\in\omega^{<\omega}$, we set $f^S_t:=f^S_{t(0)}\ldots f^S_{t(|t|-1)}$, when it makes sense. Notice that $f^S_{n_{p-m+1},\ldots,n_{p+1}}(\gamma)=f^S_{n_{p+1},n_{p-m+1},\ldots,n_p}(\gamma)$ is defined. Therefore, A^S contains

$$(f^S_{n_{p-m+1},\ldots,n_p}(\gamma),f^S_{n_{p-m+1},\ldots,n_{p+1}}(\gamma)),$$

which implies that $A^{S'}$ contains $(u[f^S_{n_{p-m+1},\ldots,n_p}(\gamma)],u[f^S_{n_{p-m+1},\ldots,n_{p+1}}(\gamma)])$. This shows that $A^{S'}$ contains $(f^{S'}_{q_{p-m+1},\ldots,q_p}[u(\gamma)],f^{S'}_{q_{p-m+1},\ldots,q_{p+1}}[u(\gamma)])$, thus

$$f^{S'}_{q_{p-m+1},\ldots,q_{p+1}}[u(\gamma)]=f^{S'}_{q_{p+1},q_{p-m+1},\ldots,q_p}[u(\gamma)],$$

so $c+(S\cap[0,p])\subseteq S'\cap(c+[0,p])$.

• Conversely, let $m:=c+m'\in S'\cap(c+[0,p])$. Again $f^S_{n_{p-m'+1},\dots,n_{p+1}}(\gamma)$ is defined. Notice that

$$u[f^S_{n_{p-m'+1},\dots,n_{p+1}}(\gamma)]=f^{S'}_{q_{p-m'+1},\dots,q_{p+1}}[u(\gamma)]=f^{S'}_{q_{p+1},q_{p-m'+1},\dots,q_p}[u(\gamma)].$$

Therefore $(u[f^S_{n_{p-m'+1},\dots,n_p}(\gamma)],u[f^S_{n_{p-m'+1},\dots,n_{p+1}}(\gamma)])\in A^{S'}$, A^S contains the pair

$$(f^S_{n_{p-m'+1},\dots,n_p}(\gamma),f^S_{n_{p-m'+1},\dots,n_{p+1}}(\gamma)),$$

and $f^S_{n_{p-m'+1},\dots,n_{p+1}}(\gamma)=f^S_{n_{p+1},n_{p-m'+1},\dots,n_p}(\gamma)$. Therefore $\text{Card}(\gamma\lceil n_{p+1})+m'\in S$ and $m'\in S$, so $S'\cap(c+[0,p])\subseteq c+(S\cap[0,p])$. But this contradicts condition $(\perp)$ since we actually have the equality.

(b) The proof is similar to that of (a). This time $A^S=(u\times v)^{-1}((A^{S'})^{-1})$. We construct sequences $(n_j)_{1\leq j\leq p+1}$, $(q_j)_{1\leq j\leq p+1}$ and $(N_j)_{1\leq j\leq p+1}$ satisfying the equality $f^{S'^{-1}}_{q_j}[u(\gamma)]=u[f^S_{n_j}(\gamma)]$ and

$$\text{Card}(\gamma\lceil n_j)+(S\cap[0,p])=S\cap(\text{Card}(\gamma\lceil n_j)+[0,p]),$$

for each $\gamma\in N_j$. This gives

$$(f^{S'}_{q_{p-m+1}})^{-1}\dots(f^{S'}_{q_{p+1}})^{-1}[u(\gamma)]=(f^{S'}_{q_{p+1}})^{-1}(f^{S'}_{q_{p-m+1}})^{-1}\dots(f^{S'}_{q_p})^{-1}[u(\gamma)],$$

thus $c-(S\cap[0,p])\subseteq S'\cap(c-[0,p])$, and we complete the proof as we did for (a). $\square$

Corollary 24 *Let $\alpha\neq\alpha'\in 2^\omega$. Then S_{β_α}, $S_{\beta_{\alpha'}}$ satisfy conditions (M), $(\perp)$ and $(\perp^{-1})$. In particular, $A^{S_{\beta_\alpha}}\perp^r_c A^{S_{\beta_{\alpha'}}}$ and $A^{S_{\beta_\alpha}}\perp^r_c (A^{S_{\beta_{\alpha'}}})^{-1}$.*

Theorem 5 is a corollary of this result. We saw that the map $\alpha\mapsto A^{S_{\beta_\alpha}}$ is continuous in the codes, and it is injective by Corollary 24. This implies that $(A^{S_{\beta_\alpha}})_{\alpha\in 2^\omega}$ is a perfect antichain for $\leq^r_c$ made of minimal sets (we use Corollaries 21 and 24).

Proof. If $s\in 2^{<\omega}$ and $t\in 2^{\leq\omega}$, we say that $s\subseteq t$ if we can find an integer $l\leq|t|$ such that $s\prec t-t\lceil l$. We define $s^{-1}\in 2^{|s|}$ by $s^{-1}(i):=s(|s|-1-i)$, for $i<|s|$. We say that s is $symmetric$ if $s=s^{-1}$.

• It is enough to prove the following condition:

$$(\perp\perp)\qquad \exists P\in\omega\ \ \beta_\alpha\lceil P\not\subseteq\beta_{\alpha'}\ \text{ and }\ (\beta_\alpha\lceil P)^{-1}\not\subseteq\beta_{\alpha'}.$$

Indeed, we will see that $(\perp\perp)$ implies $(\perp)$ and $(\perp^{-1})$ of Theorem 23. Condition $(\perp\perp)$ gives $P>0$. Let $p:=2P$ and $c\in\omega$. We argue by contradiction.

$(\perp)$ Assume that $c+(S_{\beta_\alpha}\cap[0,p])=S_{\beta_{\alpha'}}\cap(c+[0,p])$. As $0\in S_{\beta_\alpha}$, we can find l with

$$c=\Sigma_{i<l}\ (1+\beta_{\alpha'}(i)).$$

It is enough to prove that if $n<P$, then $\beta_\alpha(n)=\beta_{\alpha'}(l+n)$.

We argue by induction on n.

- Notice that $\beta_\alpha(0)=0$ is equivalent to $1\in S_{\beta_\alpha}$ and to $\beta_{\alpha'}(l)=0$. Therefore $\beta_\alpha(0)=\beta_{\alpha'}(l)$.

- Now suppose that $n+1<P$ and $\beta_\alpha(m)=\beta_{\alpha'}(l+m)$, for each $m\leq n$. As

$$2+\Sigma_{m\leq n}\ (1+\beta_\alpha(m))\leq p,$$

we get $\beta_\alpha(n+1)=\beta_{\alpha'}(l+n+1)$.

$(\perp^{-1})$ Assume that $c-(S_{\beta_\alpha}\cap[0,p])=S_{\beta_{\alpha'}}\cap(c-[0,p])$. Let $l':=l-P$ (as $2P-1$ or $2P$ is in $S_{\beta_\alpha}\cap[0,p]$, $c>2P-2$ and $l'\geq 0$). As $(\beta_\alpha\lceil P)^{-1}\not\subseteq\beta_{\alpha'}$ we can find $n<P$ such that $\beta_\alpha(n)\neq\beta_{\alpha'}(l-1-n)$, since $(\beta_\alpha\lceil P)^{-1}\not\prec\beta_{\alpha'}-\beta_{\alpha'}\lceil l'$. We conclude as in the case $(\perp)$.

• First notice that $\beta_\alpha\lceil(2^n-1)=[\beta_\alpha\lceil(2^n-1)]^{-1}$ for each integer n. Indeed, let $i<2^n-1$. It is enough to see that $(i)_0=(2^n-2-i)_0$. But we have

$$2^n-2-i=2^n-2-2^{(i)_0}(2N+1)+1=2^{(i)_0}(2^{n-(i)_0}-2N-1)-1,$$

so we are done, since $2^{n-(i)_0}-2N-1$ is odd and positive. So it is enough to find n such that $\beta_\alpha\lceil(2^n-1)\not\subseteq\beta_{\alpha'}$.

• Let n_0 minimal with $\gamma_\alpha(n_0)\neq\gamma_{\alpha'}(n_0)$, and $n_1>n_0+1$ with $\gamma_{\alpha'}(n_0+1)\neq\gamma_{\alpha'}(n_1)$. We put $n:=n_1+2$. We argue by contradiction: we get l with $\gamma_\alpha((i)_0)=\gamma_{\alpha'}((l+i)_0)$, for each $i<2^n-1$.

• Notice that for each $m<n-1$ we can find $i<2^{n-1}$ with $(l+i)_0=m$. Indeed, let

$$N\in\omega\cap[2^{-m-1}(l+1)-2^{-1},2^{-m-1}(2^{n-1}+l+1)-2^{-1}[.$$

It is clear that $i:=2^m(2N+1)-l-1$ is suitable.

• Let $M\geq n_0$ and $(\varepsilon_j)_{j\leq M}\subseteq 2$ with $l=\Sigma_{j\leq M}\ \varepsilon_j\cdot 2^j$. For $k\leq n_0$ we define

$$i_k:=\Sigma_{j<k}\ (1-\varepsilon_j)\cdot 2^j+\varepsilon_k\cdot 2^k.$$

Note that $i_k<2^{k+1}$ and $l+i_k\equiv 2^k-1 \pmod{2^{k+1}}$. We will show the following, by induction on k:

- The sequence $(\beta_\alpha(i))_{i<2^{n-1},i\equiv 2^k-1 \pmod{2^{k+1}}}$ is constant with value $\gamma_\alpha(k)$, and equal to

$$(\beta_{\alpha'}(l+i))_{i<2^{n-1},i\equiv 2^k-1 \pmod{2^{k+1}}}.$$

- The sequence $(\beta_{\alpha'}(l+i))_{i<2^{n-1},i\equiv i_k \pmod{2^{k+1}}}$ is constant with value $\gamma_{\alpha'}(k)$.

- The sequence $(\beta_{\alpha'}(l+i))_{i<2^{n-1},i\equiv i_k+2^k \pmod{2^{k+1}}}$ is not constant.

- $\varepsilon_k=0$ and $\gamma_\alpha(k)=\gamma_{\alpha'}(k)$.

This will give the desired contradiction with $k=n_0$.

So assume that these facts have been shown for $j < k \leq n_0$.

- The first point is clear.

- The second one comes from the fact that $l+i$ is of the form $2^k(2K+1)-1$ if $i \equiv i_k \pmod{2^{k+1}}$, since $l+i_k \equiv 2^k - 1 \pmod{2^{k+1}}$.

- To see the third one, choose $i < 2^{n-1}$ such that $(l+i)_0 = n_0+1$ (or n_1). We have to see that $i \equiv i_k + 2^k \pmod{2^{k+1}}$. We can find $(\eta_j)_{j<n-1}$ with $i = \Sigma_{j<n-1}\ \eta_j \cdot 2^j$, so that

$$l+i+1 \equiv 1 + \Sigma_{j<k}\ \eta_j \cdot 2^j + (\varepsilon_k + \eta_k) \cdot 2^k \pmod{2^{k+1}},$$

by the induction hypothesis. This inductively shows that $\eta_j = 1$ if $j < k$ and $\eta_k = 1 - \varepsilon_k$. Thus $i \equiv 2^k - 1 + (1-\varepsilon_k) \cdot 2^k \pmod{2^{k+1}}$. But $i_k + 2^k \equiv 2^k - 1 + \varepsilon_k \cdot 2^k + 2^k \pmod{2^{k+1}}$. Thus $i_k + 2^k \equiv -1 + \varepsilon_k \cdot 2^k \pmod{2^{k+1}}$. Finally, $2^k - 1 \equiv i_k$ (resp., $i_k + 2^k$) $\pmod{2^{k+1}}$ if $\varepsilon_k = 0$ (resp., $\varepsilon_k = 1$).

- So $\varepsilon_k = 0$ and $\gamma_\alpha(k) = \gamma_{\alpha'}(k)$.

This finishes the proof. □

Now we prove that $[D_2(\boldsymbol{\Sigma}^0_1) \setminus \text{pot}(\boldsymbol{\Pi}^0_1), \leq^r_c]$ is not well-founded.

Notation. Let $\mathcal{S} : \omega^\omega \rightarrow \omega^\omega$ be the shift map: $\mathcal{S}(\alpha)(k) := \alpha(k+1)$, β_0 be the sequence $(0,1,2,\ldots)$, and $\beta_n := \mathcal{S}^n(\beta_0)$. Notice that $\beta_n(i) = i+n$, by induction on n. We put $B_n := A^{S_{\beta_n}}$.

Proposition 25 *We have $B_{n+1} \leq^r_c B_n$ and $B_n \not\leq^r_c B_{n+1}$ for each integer n.*

Proof. We define injective continuous maps $u = v : 2^\omega \rightarrow 2^\omega$ by $u(\alpha) := 1^{1+n}\alpha$. They are clearly witnesses for $B_{n+1} \leq^r_c B_n$.

• Conversely, we argue by contradiction. This gives continuous maps u and v such that

$$B_n = (u \times v)^{-1}(B_{n+1}).$$

By Lemma 22, we have $u = v$. We set $f^n_m := f^{S_{\beta_n}}_m$, and $f^n_t := f^n_{t(0)} \cdots f^n_{t(|t|-1)}$ for $t \in \omega^{<\omega} \setminus \{\emptyset\}$, when it makes sense. Let $\alpha \in N_{0^{n+3}}$, so that $\alpha = 0^{n+3}\gamma$.

• If $f^n_t(\alpha)$ is defined, then fix $m_t \in \omega$ with $u[f^n_t(\alpha)] = f^{n+1}_{m_t}(u[f^n_{t-t\lceil 1}(\alpha)])$, and set $U := u(\alpha)$ (with the convention that $f^n_\emptyset := \text{Id}_{2^\omega}$). Then $u[f^n_t(\alpha)] = f^{n+1}_{m_t, m_{t-t\lceil 1}, \ldots, m_{t-t\lceil(|t|-1)}}(U)$. In particular,

$$f^{n+1}_{m_{(1,\ldots,n+2)}, m_{(2,\ldots,n+2)}, \ldots, m_{n+2}}(U) = f^{n+1}_{m_{(n+2,1,\ldots,n+1)}, m_{(1,\ldots,n+1)}, \ldots, m_{n+1}}(U).$$

Therefore

$$\{m_{(1,\ldots,n+2)}, m_{(2,\ldots,n+2)}, \ldots, m_{n+2}\} = \{m_{(n+2,1,\ldots,n+1)}, m_{(1,\ldots,n+1)}, \ldots, m_{n+1}\}.$$

If $m_{n+2}=m_{n+1}$, then we get $u(0^{n+2}1\gamma)=u(0^{n+1}10\gamma)$. As $f_0^n(0^{n+1}10\gamma)=10^n10\gamma$, we get $(u(0^{n+1}10\gamma),v(10^n10\gamma))\in B_{n+1}$ and $(0^{n+2}1\gamma,10^n10\gamma)\in B_n$, which is absurd. Now suppose that $M:=\max(m_{(1,\ldots,n+2)},m_{(2,\ldots,n+2)},\ldots,m_{n+2})$ is in $\{m_{n+1},m_{n+2}\}$. Then we can find $1\leq k\leq n+1$ such that

$$\text{Card}(U\lceil M),\ \text{Card}(U\lceil M)+k\in\{\Sigma_{i<l}\ (1+\beta_{n+1}(i))/l\in\omega\}.$$

But this is not possible, since $\Sigma_{i<l+1}\ (1+\beta_{n+1}(i))-\Sigma_{i<l}\ (1+\beta_{n+1}(i))=l+n+2$.

• We then get the contradiction by induction, since we can remove M from both

$$\{m_{(1,\ldots,n+2)},m_{(2,\ldots,n+2)},\ldots,m_{n+2}\},$$

$\{m_{(n+2,1,\ldots,n+1)},m_{(1,\ldots,n+1)},\ldots,m_{n+1}\}$. □

Remarks. (a) We showed that $(A^{S_{\beta_\alpha}})_{\alpha\in 2^\omega}$ is a perfect antichain made of sets minimal among non-pot($\mathbf{\Pi}^0_1$) sets for $\leq^r_c$. There are other natural notions of reduction. We defined $\leq^r_c$ in the introduction. If we moreover ask that u and v are one-to-one, this defines a new quasi-order that we denote $\sqsubseteq^r_c$. If u and v are only Borel, we have two other quasi-orders, denoted $\leq^r_B$ and $\sqsubseteq^r_B$. If $X=Y$, $X'=Y'$ and $u=v$, we get the usual notions $\leq_c$, $\sqsubseteq_c$, $\leq_B$ and $\sqsubseteq_B$. Let $\leq$ be any of these eight quasi-orders. Then $(A^{S_{\beta_\alpha}})_{\alpha\in 2^\omega}$ **is a perfect antichain made of sets minimal among non-pot($\mathbf{\Pi}^0_1$) sets for** $\leq$:

• Let us go back to Theorem 15 first. Assume this time that $A\leq A^f$. Then in the second case we can have $A^f\cap B^2\sqsubseteq^r_c A$ if $\leq$ is rectangular, and $A^f\cap B^2\sqsubseteq_c A$ otherwise. The changes to make in the proof are the following. Let ν (resp., ν') be a finer Polish topology on Y (resp., Y') making u (resp., v) continuous. We get continuous maps $u':2^\omega\rightarrow[Y,\nu]$ and $v':2^\omega\rightarrow[Y',\nu']$. The proof shows that $f|_G$ and $g|_G$ are actually witnesses for $A^f\cap G^2\sqsubseteq^r_c A$ if $\leq$ is rectangular, and $A^f\cap G^2\sqsubseteq_c A$ otherwise.

• In Corollary 16, we can replace $\leq^r_c$ with $\leq$.

• The proof of Theorem 19 shows that, in its statement, we can write $A^S\sqsubseteq_c A^S\cap B^2$.

• The proof of Lemma 20 shows that, in its statement, we can replace $\leq^r_c$ with $\leq$.

• It follows from Corollary 21 that $A^{S_{\beta_\alpha}}$ is, in fact, minimal among non-pot($\mathbf{\Pi}^0_1$) sets for $\leq$.

• To see that $(A^{S_{\beta_\alpha}})_{\alpha\in 2^\omega}$ is an antichain for $\leq^r_B$, it is enough to see that in the statement of Theorem 23, we can replace $\perp^r_c$ with $\perp^r_B$. We only have to change the beginning of the proof of Theorem 23. This time u and v are Borel. Let τ be a finer Polish topology on 2^ω making u and v continuous, and $X:=[2^\omega,\tau]$. By Lemma 20, A^S is $\leq^r_c$-minimal, so $(2^\omega,A^S)\leq^r_c(X,A^S)\leq^r_c A^{S'}$, and we may assume that u and v are continuous.

(b) We proved that $[D_2(\mathbf{\Sigma}^0_1)\setminus\text{pot}(\mathbf{\Pi}^0_1),\leq^r_c]$ is not well-founded. Let $\leq$ be any of the eight usual quasi-orders. Then $[D_2(\mathbf{\Sigma}^0_1)\setminus\text{pot}(\mathbf{\Pi}^0_1),\leq]$ **is not well-founded**:

• The proof of Proposition 25 shows that $B_{n+1}\sqsubseteq_c B_n$, thus $B_{n+1}\leq B_n$.

• We have to see that $B_n \not\leq_B^r B_{n+1}$. We argue by contradiction, so that we get u and v Borel.

• Let us show that we can find a dense G_δ subset G of 2^ω such that $u|_G = v|_G$ is continuous, and $f_m^n(\alpha) \in G$, for each $\alpha \in G \cap D_{f_m^n}$.

Claim. *The set* $H := \{\alpha \in 2^\omega / \forall p\ \exists m \geq p\ \ \alpha \in D_{f_m^n}\}$ *is a dense* G_δ *subset of* 2^ω.

We argue by contradiction. We can find a nonempty clopen set V disjoint from H. The set $B_n \cap V^2$ has finite sections, so is pot($\mathbf{\Pi}_1^0$) (see Theorem 3.6 in [Lo1]). But $(V, (f_m^n|_{V \cap f_m^{n\ -1}(V)}))$ is a converging situation, so that $B_n \cap V^2$ is not pot($\mathbf{\Pi}_1^0$). $\diamond$

So we can find a dense G_δ subset K of 2^ω such that $u|_K$, $v|_K$ are continuous and $K \subseteq H$. Now let $K_0 := K$, $K_{p+1} := K_p \setminus (\bigcup_m D_{f_m^n} \setminus f_m^{n\ -1}(K_p))$, and $G := \bigcap_p K_p$. If $\alpha \in K_1$, fix (m_k) infinite such that $\alpha \in \bigcap_k D_{f_{m_k}^n}$. We have $f_{m_k}^n(\alpha) \in K_0$, so $(u(\alpha), v[f_{m_k}^n(\alpha)])$ tends to $(u(\alpha), v(\alpha)) \in \overline{B_{n+1}} \setminus B_{n+1} = \Delta(2^\omega)$. So $u|_{K_1} = v|_{K_1}$. Now it is clear that G is suitable.

• We take $\alpha \in G \cap N_{0^{n+3}}$ and complete the proof as we did for Proposition 25.

Proof of Theorem 6. We will actually prove a stronger statement. We set

$$(P_0, P_1, P_2, P_3, P_4) := (\text{reflexive, irreflexive, symmetric, antisymmetric, transitive}).$$

Let $\sigma \in 2^5 \setminus \{\{2,4\}, \{0,2,4\}\}$ such that the class Γ of $\mathbf{\Delta}_1^1 \setminus$ pot($\mathbf{\Pi}_1^0$) relations satisfying $\wedge_{j \in \sigma} P_j$ is not empty. Then we can find a perfect $\leq_B$-antichain $(R_\alpha)_{\alpha \in 2^\omega}$ in $D_2(\mathbf{\Sigma}_1^0) \cap \Gamma$ such that R_α is $\leq_B$-minimal among $\mathbf{\Delta}_1^1 \setminus$ pot($\mathbf{\Pi}_1^0$) sets, for any $\alpha \in 2^\omega$.

• First, notice that if $\{0,1\} \subseteq \sigma$ or $\sigma = \{1,2,4\}$, then every relation satisfying $\wedge_{j \in \sigma} P_j$ is empty, thus pot($\mathbf{\Pi}_1^0$). If $\{2,3\} \subseteq \sigma$, then every Borel relation satisfying $\wedge_{j \in \sigma} P_j$ is a subset of the diagonal, and is therefore pot($\mathbf{\Pi}_1^0$). If $\sigma = \{0,2,4\}$, we are in the case of Borel equivalence relations, and by Harrington, Kechris and Louveau's Theorem, E_0 is minimum among non-pot($\mathbf{\Pi}_1^0$) equivalence relations. If $\sigma = \{2,4\}$, then any Borel relation $A \subseteq X^2$ satisfying $\wedge_{j \in \sigma} P_j$ is reflexive on its domain $\{x \in X/(x,x) \in A\}$, which is a Borel set. Thus we are reduced to the case of equivalence relations. In the sequel, we will avoid these cases and show the existence of a perfect antichain made of minimal sets for $[\Gamma, \leq_B]$.

• Let $\mathbb{A} := \{A^{S_{\beta_\alpha}} / \alpha \in 2^\omega\}$. In the introduction, we defined R_A for $A \subseteq 2^\omega \times 2^\omega$.

Claim 1. $\{R_A / A \in \mathbb{A}\}$ *is a* $\leq_B$*-antichain.*

Assume that $A \neq A' \in \mathbb{A}$ satisfy $R_A \leq_B R_{A'}$. Then there is $f : 2^\omega \times 2 \to 2^\omega \times 2$ with

$$R_A = (f \times f)^{-1}(R_{A'}).$$

We set $F_\varepsilon := \{x \in 2^\omega \times 2 / x_1 = \varepsilon\}$ and $b_\varepsilon := R_A \cap (F_\varepsilon \times F_\varepsilon)$, for $\varepsilon \in 2$. We then put $a := R_A \cap (F_0 \times F_1)$. We have $R_A = a \cup b_0 \cup b_1$, and $b_\varepsilon = \{(x,y) \in F_\varepsilon \times F_\varepsilon / x_0 = y_0\}$ is pot($\mathbf{\Pi}_1^0$). We set

$$F'_\varepsilon := \{x \in 2^\omega \times 2 / f_1(x) = \varepsilon\}$$

and $b'_\varepsilon := R_A \cap (F'_\varepsilon \times F'_\varepsilon)$, for $\varepsilon \in 2$.

We then put $a' := R_A \cap (F'_0 \times F'_1)$. We have $R_A = a' \cup b'_0 \cup b'_1$, and

$$b'_\varepsilon = (f|_{F'_\varepsilon} \times f|_{F'_\varepsilon})^{-1}(\Delta(2^\omega \times 2)) \in \text{pot}(\mathbf{\Pi}^0_1).$$

Notice that $A \leq^r_B R_A$, so that R_A is not pot$(\mathbf{\Pi}^0_1)$. So $R_A = (a \cap a') \cup b_0 \cup b_1 \cup b'_0 \cup b'_1$, and $a \cap a'$ is not pot$(\mathbf{\Pi}^0_1)$. It remains to define $C := a \cap a'$, viewed as a subset of $(F_0 \cap F'_0) \times (F_1 \cap F'_1)$. We equip $F_0 \cap F'_0$ (resp., $F_1 \cap F'_1$) with a finer Polish topology making $f|_{F_0 \cap F'_0}$ (resp., $f|_{F_1 \cap F'_1}$) continuous. Then $C \leq^r_c A$ and $C \leq^r_c A'$, which contradicts Corollaries 21 and 24. $\diamond$

Claim 2. *Let $A = A^{S_{\beta_\alpha}} \in \mathbb{A}$. Then R_A is minimal for $\leq_B$ among $\mathbf{\Delta}^1_1 \setminus$ pot$(\mathbf{\Pi}^0_1)$ relations.*

- Assume that $R \leq_B R_A$. This gives $f : X \to 2^\omega \times 2$ Borel with $R = (f \times f)^{-1}(R_A)$. Again we set $F_\varepsilon := \{x \in X / f_1(x) = \varepsilon\}$ for $\varepsilon \in 2$, and we see that $R \cap (F_0 \times F_1)$ is not pot$(\mathbf{\Pi}^0_1)$.

- Let τ be a finer Polish topology on X making f continuous. By Theorem 9 there are

$$u : 2^\omega \to [F_0, \tau],$$

$v : 2^\omega \to [F_1, \tau]$ continuous with $A_1 = (u \times v)^{-1}(R \cap (F_0 \times F_1)) \cap \overline{A_1}$. We define $H := f_0[u[2^\omega]]$, $K := f_0[v[2^\omega]]$ and $P := H \cap K$; this defines compact subsets of 2^ω. Then $A \cap (H \times K)$ is not pot$(\mathbf{\Pi}^0_1)$ since

$$A_1 = [(f_0 \circ u) \times (f_0 \circ v)]^{-1}(A \cap (H \times K)) \cap \overline{A_1}.$$

As in the proof of Theorem 15, this implies that $A \cap P^2$ is not pot$(\mathbf{\Pi}^0_1)$. By Lemma 14, we can find a Borel subset S of P and a finer topology σ on S such that $([S, \sigma], (f^S_n)_n)$ is a converging situation.

- By 18.3 in [K], we can find a Baire measurable map $g_\varepsilon : S \to f^{-1}(S \times \{\varepsilon\})$ such that

$$f_0(g_\varepsilon(\alpha)) = \alpha,$$

for α in S and $\varepsilon \in 2$. Let G be a dense G_δ subset of S such that each $g_\varepsilon|_G$ is continuous. Now we define $F : G \times 2 \to X$ by $F(\alpha, \varepsilon) := g_\varepsilon(\alpha)$. Then $R_A \cap (G \times 2)^2 = (F \times F)^{-1}(R)$, so $R_A \cap (G \times 2)^2 \leq_B R$. As in the proof of Theorem 15, we see that $A \cap G^2$ is not pot$(\mathbf{\Pi}^0_1)$. But $A \cap G^2 \sqsubseteq_c A$. By Remark (a) above, we get $A \sqsubseteq_c A \cap G^2$. Thus $R_A \sqsubseteq_c R_A \cap (G \times 2)^2$ and $R_A \leq_B R$. $\diamond$

Finally, one easily checks the existence of a continuous map $c : 2^\omega \to 2^\omega$ such that $c(\delta)$ is a Borel code for R_A if δ is a Borel code for A. So there is a continuous map $r : 2^\omega \to 2^\omega$ such that $r(\alpha)$ is a Borel code for $R_{A^{S_{\beta_\alpha}}}$. This shows, in particular, the existence of a perfect antichain made of minimal sets for $[\mathbf{\Delta}^1_1 \setminus$ pot$(\mathbf{\Pi}^0_1)$ quasi-orders, $\leq_B]$ and $[\mathbf{\Delta}^1_1 \setminus$ pot$(\mathbf{\Pi}^0_1)$ partial orders, $\leq_B]$. More generally, this works if $\sigma \subseteq \{0, 3, 4\}$.

• Similarly, we define, for $A \subseteq X^2$, a strict partial order relation R'_A on $X \times 2$ by

$$(x, i)\ R'_A\ (y, j) \ \Leftrightarrow\ [(x, y) \in A \text{ and } i = 0 \text{ and } j = 1].$$

The proof of the previous point shows that if $\sigma \subseteq \{1, 3, 4\}$, then $\{R'_A / A \in \mathbb{A}\}$ is a perfect antichain made of minimal sets for $[\Gamma, \leq_B]$. Notice that this applies when Γ is the class of $\mathbf{\Delta}^1_1 \setminus$ pot$(\mathbf{\Pi}^0_1)$ strict quasi-orders, strict partial orders, directed graphs or oriented graphs.

• Similarly again, we can define, for $A \subseteq X^2$, S_A reflexive symmetric on $X \times 2$ by

$$(x,i)\ S_A\ (y,j) \ \Leftrightarrow\ \begin{array}{l}(x,i) = (y,j) \text{ or } [(x,y) \in A \text{ and } i = 0 \text{ and } j = 1] \text{ or}\\ [(y,x) \in A \text{ and } i = 1 \text{ and } j = 0].\end{array}$$

Let $A_0 := A$ and $A_1 := A^{-1}$. The proof of Claim 1 shows that if $A \neq A' \in \mathbb{A}$ satisfy $S_A \leq_B S_{A'}$, then we can find $C \notin \text{pot}(\mathbf{\Pi}^0_1)$ and $\varepsilon,\ \varepsilon' \in 2$ such that $C \leq^r_c A_\varepsilon$ and $C \leq^r_c A'_{\varepsilon'}$. But this contradicts Corollaries 21 and 24. This shows that if $\sigma = \{0, 2\}$, then $\{S_A / A \in \mathbb{A}\}$ is a perfect antichain made of minimal sets for $[\Gamma, \leq_B]$.

• Similarly again, we can define, for $A \subseteq X^2$, a graph relation S'_A on $X \times 2$ by

$$(x,i)\ S'_A\ (y,j) \ \Leftrightarrow\ [(x,y) \in A \text{ and } i{=}0 \text{ and } j{=}1] \text{ or } [(y,x) \in A \text{ and } i{=}1 \text{ and } j{=}0].$$

The proof of the previous point shows that if $\sigma \subseteq \{1, 2\}$, then $\{S'_A / A \in \mathbb{A}\}$ is a perfect antichain made of minimal sets for $[\Gamma, \leq_B]$. Notice that this applies when Γ is the class of $\mathbf{\Delta}^1_1 \setminus \text{pot}(\mathbf{\Pi}^0_1)$ graphs. This finishes the proof. □

Remarks. (a) We showed that $(R_A)_{A\in\mathbb{A}}$ is a perfect antichain made of sets minimal among non-pot$(\mathbf{\Pi}^0_1)$ sets for $\leq_B$. Fix $\leq$ in $\{\leq_c, \sqsubseteq_c, \leq_B, \sqsubseteq_B\}$. Then $(R_A)_{A\in\mathbb{A}}$ **is a perfect antichain made of sets minimal among non-pot**$(\mathbf{\Pi}^0_1)$ **sets for** $\leq$**.** It is enough to check the minimality. The only thing to notice, in the proof of Claim 2 of the proof of Theorem 6, is that we have $R_A \cap (G \times 2)^2 \sqsubseteq_c R$ and $R_A \sqsubseteq_c R$. Similarly, R'_A, S_A and S'_A ($A \in \mathbb{A}$) are minimal for $\leq_c$, $\sqsubseteq_c$ and $\sqsubseteq_B$.

(b) We have $\neg\Delta(2^\omega) \perp^r_B L_0$. Indeed, assume that $\neg\Delta(2^\omega) = (u \times v)^{-1}(L_0)$. Then $u(\alpha) <_{\text{lex}} v(\beta)$ if $\alpha \neq \beta$, and $v(\alpha) \leq_{\text{lex}} u(\alpha)$. Thus

$$u(\beta) <_{\text{lex}} v(\alpha) \leq_{\text{lex}} u(\alpha) <_{\text{lex}} v(\beta) \leq_{\text{lex}} u(\beta),$$

which is absurd. Now assume that $L_0 = (u \times v)^{-1}(\neg\Delta(2^\omega))$. Then $\beta \leq_{\text{lex}} \alpha$ implies $u(\alpha) = v(\beta)$, thus $u = v$ has to be constant. Thus $\alpha <_{\text{lex}} \beta$ implies that $u(\alpha)$ and $v(\beta)$ are different and equal.

In the introduction, we saw that $\{\neg\Delta(2^\omega), L_0\}$ is a complete family of minimal sets for

$$[\text{pot}(\check{D}_2(\mathbf{\Sigma}^0_1)) \setminus \text{pot}(\mathbf{\Pi}^0_1), \sqsubseteq^r_c].$$

We just saw that $\{\neg\Delta(2^\omega), L_0\}$ is an antichain for $\leq^r_B$, and therefore for any of the eight usual quasi-orders. These facts imply that $\neg\Delta(2^\omega)$ and L_0 are minimal among non-pot$(\mathbf{\Pi}^0_1)$ sets for $\leq^r_c$, $\sqsubseteq^r_c$, $\leq^r_B$ and $\sqsubseteq^r_B$. But $\neg\Delta(2^\omega)$ and L_0 are also minimal for $\leq_c$, $\sqsubseteq_c$, $\leq_B$ and $\sqsubseteq_B$. Indeed, if O is any of these two open sets, we have $\overline{O} \setminus O = \Delta(2^\omega)$. This gives G such that $O \cap G^2 \sqsubseteq_c A$, as in the proof of Theorem 15 (and Remark (a) after Proposition 25). Then any increasing continuous injection $\phi : 2^\omega \to G$ is a witness to $O \sqsubseteq_c O \cap G^2$.

5 The minimality of A_1 for the classical notions of comparison.

As announced in the introduction, we will show a result implying that A_1 is minimal among non-potentially closed sets. The following definition specifies the meaning of the expression "the f_n's do not induce cycles" mentioned in the introduction. This kind of notion has already been used in the theory of potential complexity (see Definition 2.10 in [L3]).

Definition 26 *We say that* $(X,(f_n))$ *is an* $acyclic\ situation$ *if*

(a) $(X,(f_n))$ *is a converging situation, with only* $\Delta(X)\subseteq\overline{A^f}\setminus A^f$ *in condition (c).*

(b) For $v\in\omega^{<\omega}\setminus\{\emptyset\}$ *and* $\varepsilon\in\{-1,1\}^{|v|}$*, the following implication holds:*

$$(\forall i<|v|-1\ \ v(i)\neq v(i+1) \text{ or } \varepsilon(i)\neq-\varepsilon(i+1))\Rightarrow(\forall U\in\mathbf{\Delta}^0_1(X)\setminus\{\emptyset\}\ \exists V\in\mathbf{\Delta}^0_1(U)\setminus\{\emptyset\}$$
$$\forall x\in V\ [f^{\varepsilon(|v|-1)}_{v(|v|-1)}\ldots f^{\varepsilon(0)}_{v(0)}(x) \text{ is not defined or not in } V]).$$

Notation. We define $f^1_n:N_{s_n0}\rightarrow N_{s_n1}$ by $f^1_n(s_n0\gamma):=s_n1\gamma$ (where s_n is as defined in the introduction, to build $A_1=\bigcup_n\ \mathrm{Gr}(f^1_n)$).

Lemma 27 *Let* $\alpha\in 2^\omega$, $v\in\omega^{<\omega}\setminus\{\emptyset\}$ *and* $\varepsilon\in\{-1,1\}^{|v|}$. *Assume that* $v(i)\neq v(i+1)$ *or* $\varepsilon(i)\neq-\varepsilon(i+1)$ *if* $i<|v|-1$. *Then* ${f^1_{v(|v|-1)}}^{\varepsilon(|v|-1)}\ldots{f^1_{v(0)}}^{\varepsilon(0)}(\alpha)$ *is either undefined, or of value different than* α.

Proof. We argue by contradiction. Let v be a counter-example of minimal length. Note that $|v|\geq 3$. Set $l:=\mathrm{max}_{i<|v|}\ v(i)$, $e_0:=e_{|v|}:=\alpha\lceil(l+1)$, and, for $0<i<|v|$:

$$e_i:=[{f^1_{v(i-1)}}^{\varepsilon(i-1)}\ldots{f^1_{v(0)}}^{\varepsilon(0)}(\alpha)]\lceil(l+1).$$

Set $\Theta:=(\theta_n)$, where $\theta_n:=s_n$. Then $(e_i)_{i\leq|v|}$ is an $s(\mathfrak{R}_\Theta)$-cycle, which contradicts Proposition 18.(b). □

Example. $(2^\omega,(f^1_n))$ is an acyclic situation. Indeed, $(2^\omega,(f^1_n))$ is a converging situation, by Corollary 12. Let us show that condition (b) in the definition of an acyclic situation is true for $(2^\omega,(f^1_n))$. The domain D of ${f^1_{v(|v|-1)}}^{\varepsilon(|v|-1)}\ldots{f^1_{v(0)}}^{\varepsilon(0)}$ is clopen. If U is not included in D, then we can take $V:=U\setminus D$. Otherwise, let $\alpha\in U$. By Lemma 27, and by continuity, we can find a clopen neighborhood V of α included in U such that ${f^1_{v(|v|-1)}}^{\varepsilon(|v|-1)}\ldots{f^1_{v(0)}}^{\varepsilon(0)}[V]\cap V=\emptyset$.

Theorem 28 *Let* $(X,(f_n))$ *be an acyclic situation. Then* $A_1\leq^r_c A^f$.

Proof. It looks like those of Theorems 2.6 and 2.12 in [L3]. The main difference is that we want a reduction defined on the whole product. It is also similar to the proof of Theorem 19. Let us indicate the differences with the proof of Theorem 19. We replace $A^S=\bigcup_n\ \mathrm{Gr}(f^S_n)$ with $A_1=\bigcup_n\ \mathrm{Gr}(f^1_n)$.

• We only construct $(U_s)_{s\in 2^{<\omega}}$ and ϕ, so that (iii) becomes

$$(iii)\ (s\ \mathfrak{R}\ t \text{ and } s\neq t)\Rightarrow U_t=f_{\phi(|s\wedge t|)}[U_s].$$

• Here we choose $\Theta=(\theta_n)$ with $\theta_n:=s_n$. Notice that $\mathfrak{R}_\Theta=\mathfrak{R}$.

• Condition (3) becomes

$$(3)\ (t_k\ \mathfrak{R}\ t_l \text{ and } t_k\neq t_l)\Rightarrow U^n_l=f_{\phi(|t_k\wedge t_l|)}[U^n_k].$$

• We can find $C \in \mathbf{\Delta}^0_1(U_{t_0\lceil p}) \setminus \{\emptyset\}$ such that $C^2 \cap \bigcup_{q\leq p} \mathrm{Gr}(f_q) = \emptyset$, and also

$$n_0 \geq \sup \{\phi(q){+}1/q < p\}$$

with $C^2 \cap \mathrm{Gr}(f_{n_0}) \neq \emptyset$, since $\Delta(X) \subseteq \overline{A^f} \setminus A^f$. We set $\phi(p) := n_0$. We first construct clopen sets $\tilde{U}^n_k$ as in the proof of Theorem 19.

Case 2. $t_n\lceil p \neq t_r\lceil p$.

2.1. $t_r\ \mathfrak{R}_\Theta\ t_n$.

To get condition (4), fix $k, l \leq n$ with $\neg\ t_k\ \mathfrak{R}\ t_l$. Set $f^\varepsilon_v := f^{\varepsilon(|v|-1)}_{\phi(v(|v|-1))} \cdots f^{\varepsilon(0)}_{\phi(v(0))}$, so that $\tilde{U}^n_l = f^\varepsilon_v[\tilde{U}^n_k]$, and we have $\phi(v(i)) \neq \phi(v(i+1))$, since ϕ is strictly increasing. As $(X, (f_n))$ is without cycles, we can find $x \in \tilde{U}^n_k$ with $f^\varepsilon_v(x) \neq x$. We can therefore find a clopen neighborhood $\underline{U}^n_k$ of x, included in $\tilde{U}^n_k$, such that $\underline{U}^n_k \cap f^\varepsilon_v[\underline{U}^n_k] = \emptyset$. We construct clopen sets $\underline{U}^n_r$, for $k \neq r \leq n$, as before, ensuring condition (3). Notice that $\underline{U}^n_r \subseteq \tilde{U}^n_r$, so that the hereditary conditions (1), (2) and (5) remain fullfilled. In finitely many steps we get $(\underline{U}^n_k \times \underline{U}^n_l) \cap \Delta(X) = \emptyset$, for each pair (k, l). The argument is similar for $\mathrm{Gr}(f_q)$ instead of $\Delta(X)$.

2.2. $t_n\ \mathfrak{R}_\Theta\ t_r$.

This case is similar to case 2.1. □

Remark. We actually showed that $A_1 \sqsubseteq_c A^f$.

Corollary 29 *A_1 is minimal among non-potentially closed sets for the eight usual quasi-orders.*

Proof. Let $B \in \mathbf{\Delta}^1_1(2^\omega)$, τ a finer topology on B, $Z := [B, \tau]$ and $f_n := f^1_n|_{B\cap {f^1_n}^{-1}(B)}$. We assume that $(Z, (f_n))$ is a converging situation. By Corollary 16 and Remark (a) after Proposition 25, it is enough to show that $A_1 \sqsubseteq_c A_1 \cap Z^2 = A^f$. By Theorem 28 and the remark above, it is enough to check that $(Z, (f_n))$ is an acyclic situation, i.e., condition (b). Fix $\alpha \in U$ and $f^\varepsilon_v := f^{\varepsilon(|v|-1)}_{v(|v|-1)} \cdots f^{\varepsilon(0)}_{v(0)}$. If U is not included in $D_{f^\varepsilon_v}$, then we can take $V := U \setminus D_{f^\varepsilon_v}$, because the domain is a clopen subset of Z. As f^ε_v is continuous, it is enough to see that $f^\varepsilon_v(\alpha) \neq \alpha$, if U is included in $D_{f^\varepsilon_v}$. But this is clear, since ${f^1_{v(|v|-1)}}^{\varepsilon(|v|-1)} \cdots {f^1_{v(0)}}^{\varepsilon(0)}(\alpha)$ is different from α, by Lemma 27. □

Remarks. (a) Theorem 28 is also a consequence of the following result:

Theorem 30 *(Miller) Let X be a Polish space, and A a locally countable $\mathbf{\Sigma}^1_1$ oriented graph on X whose symmetrization is acyclic (in the sense of Definition 17). Then exactly one of the following holds:*

(a) A has countable Borel chromatic number.

(b) $A_1 \sqsubseteq_c A$.

Theorem 30 is actually a corollary of a more general result, motivated by the results of this paper, which gives a basis for locally countable Borel directed graphs of uncountable Borel chromatic number, with respect to $\sqsubseteq_c$. The proof of both Theorem 30 and the basis result appear in [M1].

(b) We saw that $A_1 \sqsubseteq_c A^f$ if $(X, (f_n))$ is an acyclic situation. There is another example of a

$$D_2(\mathbf{\Sigma}^0_1) \setminus \text{pot}(\mathbf{\Pi}^0_1)$$

set, which seems more "natural" than A_1. It is

$$C_1 := \{(\alpha, \beta) \in 2^\omega \times 2^\omega / \exists s \in 2^{<\omega}\ \exists \gamma \in 2^\omega\ \ (\alpha, \beta) = (s0\gamma, s1\gamma)\}.$$

Its symmetric version plays an important role in the theory of potential complexity (see for example Theorem 3.7 and Corollary 4.14 in [L1]). We wonder what $\{C_1\}$ is a basis for. Roughly speaking, $\{C_1\}$ will be a basis for situations where commuting relations between the f_n's are involved. More specifically,

Definition 31 *We say that $(X, (f_n))$ is a $commuting\ situation$ if*
(a) X is a nonempty perfect closed subset of ω^ω.
(b) f_n is a partial homeomorphism with disjoint $\mathbf{\Delta}^0_1(X)$ domain and range. Moreover $\alpha <_{lex} f_n(\alpha)$ if $\alpha \in D_{f_n}$.
(c) $\Delta(X) \subseteq \overline{A^f} \setminus A^f$, and $A^f \in \mathbf{\Pi}^0_2(X^2)$.
(d) For each $\alpha \in f^{-1}_m(D_{f_n})$ we have $\alpha \in f^{-1}_n(D_{f_m})$ and $f_m(f_n(\alpha)) = f_n(f_m(\alpha))$. Moreover the graphs of the f_n's are pairwise disjoint.

A 0-dimensional Polish space is homeomorphic to a closed subset of ω^ω. So condition (a) is essentially the same as condition (a) of a converging situation. We use this formulation for the last part of condition (b). The disjunction of the domain and the range of f_n, and the inequality $\alpha <_{\text{lex}} f_n(\alpha)$ come from symmetry problems. We will come back later to this. We will also come back to the $\mathbf{\Pi}^0_2$ condition. It is linked with transitivity properties. The first part of condition (d) expresses the commutativity of the functions. One can show the following result, whose proof contains a part quite similar to the proof of Theorems 19 and 28.

Theorem 32 *Let $(X, (f_n))$ be a commuting situation. Then $C_1 \sqsubseteq_c A^f$.*

The proof of this uses the fact that $C_1 = A^f$, where $(2^\omega, (f_n))$ is a commuting situation. Let

$$g_n : 2^\omega \to 2^\omega$$

be defined by $g_n(\alpha)(k) := \alpha(k)$ if $k \neq n$, $1 - \alpha(n)$ otherwise. Then $s(C_1) = \bigcup_n\ \text{Gr}(g_n)$, so $(2^\omega, (g_n))$ is not a commuting situation, since otherwise we would have $C_1 \sqsubseteq_C s(C_1)$, which is absurd since $s(C_1)$ is symmetric and C_1 is not. But the two reasons for that are that $\alpha \not<_{\text{lex}} g_n(\alpha)$, and that the domain and the range of the bijections g_n are not disjoint.

Similarly, let $\phi : \omega \to P_f \setminus \{0^\infty\}$ be a bijective map. We let $g'_n(\alpha)(p) := \alpha(p)$ if $\phi(n)(p) = 0$, 1 otherwise. This defines $g'_n : \{\alpha \in 2^\omega / \forall p\ \phi(n)(p) = 0\ or\ \alpha(p) = 0\} \to 2^\omega$. Note that

$$E_0 \cap L'_0 = \bigcup_q \text{Gr}(g'_n),$$

where $L'_0 := \{(\alpha, \beta) \in 2^\omega \times 2^\omega / \forall i \in \omega\ \alpha(i) \leq \beta(i) \text{ and } \alpha \neq \beta\}$.

Then $(2^\omega, (g'_n))$ is not a commuting situation, since otherwise $C_1 \sqsubseteq_C E_0 \cap L'_0$, which is absurd since $E_0 \cap L'_0$ is transitive and C_1 is not. But the reason for that is that $E_0 \cap L'_0 \notin \mathbf{\Pi}^0_2$.

B. D. Miller has also a version of Theorem 32 for directed graphs of uncountable Borel chromatic number (in [M2]). Its proof uses some methods analogous to those in the proof of Theorem 30. All of this shows the existence of numerous analogies between non potentially closed directed graphs and directed graphs of uncountable Borel chromatic number.

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