There Exist Some ω-Powers of Any Borel Rank

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Abstract. The operation $V \to V^\omega$ is a fundamental operation over finitary languages leading to ω-languages. Since the set Σ^ω of infinite words over a finite alphabet Σ can be equipped with the usual Cantor topology, the question of the topological complexity of ω-powers of finitary languages naturally arises and has been posed by Niwinski [Niw90], Simonnet [Sim92] and Staiger [Sta97a]. It has been recently proved that for each integer $n \geq 1$, there exist some ω-powers of context free languages which are $\mathbf{\Pi}^0_n$-complete Borel sets, [Fin01], that there exists a context free language L such that L^ω is analytic but not Borel, [Fin03], and that there exists a finitary language V such that V^ω is a Borel set of infinite rank, [Fin04]. But it was still unknown which could be the possible infinite Borel ranks of ω-powers.

We fill this gap here, proving the following very surprising result which shows that ω-powers exhibit a great topological complexity: for each non-null countable ordinal ξ, there exist some $\mathbf{\Sigma}^0_\xi$-complete ω-powers, and some $\mathbf{\Pi}^0_\xi$-complete ω-powers.

Keywords: Infinite words; ω-languages; ω-powers; Cantor topology; topological complexity; Borel sets; Borel ranks; complete sets.

1 Introduction

The operation $V \to V^\omega$ is a fundamental operation over finitary languages leading to ω-languages. It produces ω-powers, i.e. ω-languages in the form V^ω, where V is a finitary language. This operation appears in the characterization of the class REG_ω of ω-regular languages (respectively, of the class CF_ω of context free ω-languages) as the ω-Kleene closure of the family REG of regular finitary languages (respectively, of the family CF of context free finitary languages) [Sta97a].

Since the set Σ^ω of infinite words over a finite alphabet Σ can be equipped with the usual Cantor topology, the question of the topological complexity of ω-powers of finitary languages naturally arises and has been posed by Niwinski [Niw90], Simonnet

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[Sim92], and Staiger [Sta97a]. A first task is to study the position of ω-powers with regard to the Borel hierarchy (and beyond to the projective hierarchy) [Sta97a, PP04].

It is easy to see that the ω-power of a finitary language is always an analytic set because it is either the continuous image of a compact set $\{0, 1, \ldots, n\}^\omega$ for $n \geq 0$ or of the Baire space ω^ω.

It has been recently proved, that for each integer $n \geq 1$, there exist some ω-powers of context free languages which are $\mathbf{\Pi}^0_n$-complete Borel sets, [Fin01], and that there exists a context free language L such that L^ω is analytic but not Borel, [Fin03]. Notice that amazingly the language L is very simple to describe and it is accepted by a simple 1-counter automaton.

The first author proved in [Fin04] that there exists a finitary language V such that V^ω is a Borel set of infinite rank. However the only known fact on their complexity is that there is a context free language W such that W^ω is Borel above $\mathbf{\Delta}^{\mathbf{0}}_\omega$, [DF06].

We fill this gap here, proving the following very surprising result which shows that ω-powers exhibit a great topological complexity: for each non-null countable ordinal ξ, there exist some $\mathbf{\Sigma}^0_\xi$-complete ω-powers, and some $\mathbf{\Pi}^0_\xi$-complete ω-powers. For that purpose we use a theorem of Kuratowski which is a level by level version of a theorem of Lusin and Souslin stating that every Borel set $B \subseteq 2^\omega$ is the image of a closed subset of the Baire space ω^ω by a continuous bijection. This theorem of Lusin and Souslin had already been used by Arnold in [Arn83] to prove that every Borel subset of Σ^ω, for a finite alphabet Σ, is accepted by a non-ambiguous finitely branching transition system with Büchi acceptance condition and our first idea was to code the behaviour of such a transition system. This way, in the general case, we can manage to construct an ω-power of the same complexity as B.

The paper is organized as follows. In Section 2 we recall basic notions of topology and in particular definitions and properties of Borel sets. We proved our main result in Section 3.

2 Topology

We first give some notations for finite or infinite words we shall use in the sequel, assuming the reader to be familiar with the theory of formal languages and of ω-languages, see [Tho90, Sta97a, PP04]. Let Σ be a finite or countable alphabet whose elements are called letters. A non-empty finite word over Σ is a finite sequence of letters: $x = a_0.a_1.a_2 \ldots a_n$ where $\forall i \in [0; n]\ a_i \in \Sigma$. We shall denote $x(i) = a_i$ the $(i+1)^{th}$ letter of x and $x\lceil(i+1) = x(0) \ldots x(i)$ for $i \leq n$, is the beginning of length $i+1$ of x. The length of x is $|x| = n+1$. The empty word will be denoted by $\emptyset$ and has 0 letters. Its length is 0. The set of finite words over Σ is denoted $\Sigma^{<\omega}$. A (finitary) language L over Σ is a subset of $\Sigma^{<\omega}$. The usual concatenation product of u and v will be denoted by $u^\frown v$ or just uv. If $l \in \omega$ and $(a_i)_{i<l} \in (\Sigma^{<\omega})^l$, then $^\frown{}_{i<l}\ a_i$ is the concatenation $a_0 \ldots a_{l-1}$.

The first infinite ordinal is ω. An ω-word over Σ is an ω -sequence $a_0 a_1 \ldots a_n \ldots$, where for all integers $i \geq 0 \quad a_i \in \Sigma$. When σ is an ω-word over Σ, we write $\sigma = \sigma(0)\sigma(1)\ldots\sigma(n)\ldots$ and $\sigma\lceil(n+1) = \sigma(0)\sigma(1)\ldots\sigma(n)$ the finite word of length $n+1$, prefix of σ. The set of ω-words over the alphabet Σ is denoted by Σ^ω. An

ω-language over an alphabet Σ is a subset of Σ^ω. If $\forall i\in\omega \quad a_i\in\Sigma^{<\omega}$, then $\frown_{i\in\omega} a_i$ is the concatenation $a_0a_1\ldots$. The concatenation product is also extended to the product of a finite word u and an ω-word v: the infinite word $u.v$ or $u^\frown v$ is then the ω-word such that: $(uv)(k) = u(k)$ if $k < |u|$, and $(u.v)(k) = v(k - |u|)$ if $k \geq |u|$.

The prefix relation is denoted $\prec$: the finite word u is a prefix of the finite word v (respectively, the infinite word v), denoted $u \prec v$, if and only if there exists a finite word w (respectively, an infinite word w), such that $v = u^\frown w$.

If $s\prec\alpha=\alpha(0)\alpha(1)...$, then $\alpha-s$ is the sequence $\alpha(|s|)\alpha(|s|+1)...$

For a finitary language $V \subseteq \Sigma^{<\omega}$, the ω-power of V is the ω-language

$$V^\omega = \{u_1\ldots u_n\ldots \in \Sigma^\omega \mid \forall i \geq 1 \;\; u_i \in V\}$$

We recall now some notions of topology, assuming the reader to be familiar with basic notions which may be found in [Kur66, Mos80, Kec95, LT94, Sta97a, PP04].

There is a natural metric on the set Σ^ω of infinite words over a countable alphabet Σ which is called the prefix metric and defined as follows. For $u, v \in \Sigma^\omega$ and $u \neq v$ let $d(u, v) = 2^{-l_{pref(u,v)}}$ where $l_{pref(u,v)}$ is the first integer n such that the $(n + 1)^{th}$ letter of u is different from the $(n + 1)^{th}$ letter of v. The topology induced on Σ^ω by this metric is just the product topology of the discrete topology on Σ. For $s \in \Sigma^{<\omega}$, the set $N_s:=\{\alpha\in\Sigma^\omega \mid s\prec\alpha\}$ is a basic clopen (i.e., closed and open) set of Σ^ω. More generally open sets of Σ^ω are in the form $W^\frown\Sigma^\omega$, where $W \subseteq \Sigma^{<\omega}$.

The topological spaces in which we will work in this paper will be subspaces of Σ^ω where Σ is either finite having at least two elements or countably infinite.

When Σ is a finite alphabet, the prefix metric induces on Σ^ω the usual Cantor topology and Σ^ω is compact.

The Baire space ω^ω is equipped with the product topology of the discrete topology on ω. It is homeomorphic to $P_\infty:=\{\alpha\in 2^\omega \mid \forall i\in\omega\; \exists j\geq i \;\; \alpha(j)=1\}\subseteq 2^\omega$, via the map defined on ω^ω by $H(\beta):=0^{\beta(0)}10^{\beta(1)}1\ldots$

We define now the **Borel Hierarchy** on a topological space X:

Definition 1. *The classes $\mathbf{\Sigma}^0_n(X)$ and $\mathbf{\Pi}^0_n(X)$ of the Borel Hierarchy on the topological space X are defined as follows:*

$\mathbf{\Sigma}^0_1(X)$ is the class of open subsets of X.
$\mathbf{\Pi}^0_1(X)$ is the class of closed subsets of X.
And for any integer $n \geq 1$:
$\mathbf{\Sigma}^0_{n+1}(X)$ is the class of countable unions of $\mathbf{\Pi}^0_n$-subsets of X.
$\mathbf{\Pi}^0_{n+1}(X)$ is the class of countable intersections of $\mathbf{\Sigma}^0_n$-subsets of X.
The Borel Hierarchy is also defined for transfinite levels. The classes $\mathbf{\Sigma}^0_\xi(X)$ and $\mathbf{\Pi}^0_\xi(X)$, for a non-null countable ordinal ξ, are defined in the following way:
$\mathbf{\Sigma}^0_\xi(X)$ is the class of countable unions of subsets of X in $\cup_{\gamma<\xi}\mathbf{\Pi}^0_\gamma$.
$\mathbf{\Pi}^0_\xi(X)$ is the class of countable intersections of subsets of X in $\cup_{\gamma<\xi}\mathbf{\Sigma}^0_\gamma$.

Suppose now that $X\subseteq Y$; then $\mathbf{\Sigma}^0_\xi(X)=\{A\cap X \mid A\in\mathbf{\Sigma}^0_\xi(Y)\}$, and similarly for $\mathbf{\Pi}^0_\xi$, see [Kec95, Section 22.A]. Notice that we have defined the Borel classes $\mathbf{\Sigma}^0_\xi(X)$ and $\mathbf{\Pi}^0_\xi(X)$ mentioning the space X. However when the context is clear we will sometimes omit X and denote $\mathbf{\Sigma}^0_\xi(X)$ by $\mathbf{\Sigma}^0_\xi$ and similarly for the dual class.

The Borel classes are closed under finite intersections and unions, and continuous preimages. Moreover, $\boldsymbol{\Sigma}^0_\xi$ is closed under countable unions, and $\boldsymbol{\Pi}^0_\xi$ under countable intersections. As usual the ambiguous class $\boldsymbol{\Delta}^0_\xi$ is the class $\boldsymbol{\Sigma}^0_\xi \cap \boldsymbol{\Pi}^0_\xi$.

The class of **Borel sets** is $\boldsymbol{\Delta}^1_1 := \bigcup_{\xi<\omega_1} \boldsymbol{\Sigma}^0_\xi = \bigcup_{\xi<\omega_1} \boldsymbol{\Pi}^0_\xi$, where ω_1 is the first uncountable ordinal.

The **Borel hierarchy** is as follows:

$$\begin{array}{ccccccccc} & \boldsymbol{\Sigma}^0_1 = \text{open} & & \boldsymbol{\Sigma}^0_2 & \dots & \boldsymbol{\Sigma}^0_\omega & \dots & \\ \boldsymbol{\Delta}^0_1 = \text{clopen} & & \boldsymbol{\Delta}^0_2 & & & \boldsymbol{\Delta}^0_\omega & & \boldsymbol{\Delta}^1_1 \\ & \boldsymbol{\Pi}^0_1 = \text{closed} & & \boldsymbol{\Pi}^0_2 & \dots & \boldsymbol{\Pi}^0_\omega & \dots & \end{array}$$

This picture means that any class is contained in every class to the right of it, and the inclusion is strict in any of the spaces Σ^ω.

For a countable ordinal α, a subset of Σ^ω is a Borel set of *rank* α iff it is in $\boldsymbol{\Sigma}^0_\alpha \cup \boldsymbol{\Pi}^0_\alpha$ but not in $\bigcup_{\gamma<\alpha}(\boldsymbol{\Sigma}^0_\gamma \cup \boldsymbol{\Pi}^0_\gamma)$.

We now define completeness with regard to reduction by continuous functions. For a countable ordinal $\alpha \geq 1$, a set $F \subseteq \Sigma^\omega$ is said to be a $\boldsymbol{\Sigma}^0_\alpha$ (respectively, $\boldsymbol{\Pi}^0_\alpha$)-*complete set* iff for any set $E \subseteq Y^\omega$ (with Y a finite alphabet): $E \in \boldsymbol{\Sigma}^0_\alpha$ (respectively, $E \in \boldsymbol{\Pi}^0_\alpha$) iff there exists a continuous function $f : Y^\omega \to \Sigma^\omega$ such that $E = f^{-1}(F)$. $\boldsymbol{\Sigma}^0_n$ (respectively, $\boldsymbol{\Pi}^0_n$)-complete sets, with n an integer ≥ 1, are thoroughly characterized in [Sta86].

Recall that a set $X \subseteq \Sigma^\omega$ is a $\boldsymbol{\Sigma}^0_\alpha$ (respectively $\boldsymbol{\Pi}^0_\alpha$)-complete subset of Σ^ω iff it is in $\boldsymbol{\Sigma}^0_\alpha$ but not in $\boldsymbol{\Pi}^{\mathbf{0}}_\alpha$ (respectively in $\boldsymbol{\Pi}^0_\alpha$ but not in $\boldsymbol{\Sigma}^0_\alpha$), [Kec95].

For example, the singletons of 2^ω are $\boldsymbol{\Pi}^0_1$-complete subsets of 2^ω. The set P_∞ is a well known example of a $\boldsymbol{\Pi}^0_2$-complete subset of 2^ω.

If $\boldsymbol{\Gamma}$ is a class of sets, then $\check{\boldsymbol{\Gamma}} := \{\neg A \mid A \in \boldsymbol{\Gamma}\}$ is the class of complements of sets in $\boldsymbol{\Gamma}$. In particular, for every non-null countable ordinal α, $\check{\boldsymbol{\Sigma}}^0_\alpha = \boldsymbol{\Pi}^0_\alpha$ and $\check{\boldsymbol{\Pi}}^0_\alpha = \boldsymbol{\Sigma}^0_\alpha$.

There are some subsets of the topological space Σ^ω which are not Borel sets. In particular, there exists another hierarchy beyond the Borel hierarchy, called the projective hierarchy. The first class of the projective hierarchy is the class $\boldsymbol{\Sigma}^1_1$ of **analytic** sets. A set $A \subseteq \Sigma^\omega$ is analytic iff there exists a Borel set $B \subseteq (\Sigma \times Y)^\omega$, with Y a finite alphabet, such that $x \in A \leftrightarrow \exists y \in Y^\omega$ such that $(x, y) \in B$, where $(x, y) \in (\Sigma \times Y)^\omega$ is defined by: $(x, y)(i) = (x(i), y(i))$ for all integers $i \geq 0$.

A subset of Σ^ω is analytic if it is empty, or the image of the Baire space by a continuous map. The class of analytic sets contains the class of Borel sets in any of the spaces Σ^ω. Notice that $\boldsymbol{\Delta}^1_1 = \boldsymbol{\Sigma}^1_1 \cap \boldsymbol{\Pi}^1_1$, where $\boldsymbol{\Pi}^1_1$ is the class of co-analytic sets, i.e. of complements of analytic sets.

The ω-power of a finitary language V is always an analytic set because if V is finite and has n elements then V^ω is the continuous image of a compact set $\{0, 1, \dots, n-1\}^\omega$ and if V is infinite then there is a bijection between V and ω and V^ω is the continuous image of the Baire space ω^ω, [Sim92].

3 Main Result

We now state our main result, showing that ω-powers exhibit a very surprising topological complexity.

Theorem 2. *Let ξ be a non-null countable ordinal.*
(a) There is $A\subseteq 2^{<\omega}$ such that A^ω is $\mathbf{\Sigma}^0_\xi$-complete.
(b) There is $A\subseteq 2^{<\omega}$ such that A^ω is $\mathbf{\Pi}^0_\xi$-complete.

To prove Theorem 2, we shall use a level by level version of a theorem of Lusin and Souslin stating that every Borel set $B \subseteq 2^\omega$ is the image of a closed subset of the Baire space ω^ω by a continuous bijection, see [Kec95, p.83]. It is the following theorem, proved by Kuratowski in [Kur66, Corollary 33.II.1]:

Theorem 3. *Let ξ be a non-null countable ordinal, and $B\in\mathbf{\Pi}^0_{\xi+1}(2^\omega)$. Then there is $C\in\mathbf{\Pi}^0_1(\omega^\omega)$ and a continuous bijection $f : C \rightarrow B$ such that f^{-1} is $\mathbf{\Sigma}^0_\xi$-measurable (i.e., $f[U]$ is $\mathbf{\Sigma}^0_\xi(B)$ for each open subset U of C).*

The existence of the continuous bijection $f : C \rightarrow B$ given by this theorem (without the fact that f^{-1} is $\mathbf{\Sigma}^0_\xi$-measurable) has been used by Arnold in [Arn83] to prove that every Borel subset of Σ^ω, for a finite alphabet Σ, is accepted by a non-ambiguous finitely branching transition system with Büchi acceptance condition. Notice that the sets of states of these transition systems are countable.

Our first idea was to code the behaviour of such a transition system. In fact this can be done on a part of ω-words of a special compact set $K_{0,0}$. However we shall have also to consider more general sets $K_{N,j}$ and then we shall need the hypothesis of the $\mathbf{\Sigma}^0_\xi$-measurability of the function f.

We now come to the proof of Theorem 2.

Let $\mathbf{\Gamma}$ be the class $\mathbf{\Sigma}^0_\xi$, or $\mathbf{\Pi}^0_\xi$. We assume first that $\xi\geq 3$.

Let $B \subseteq 2^\omega$ be a $\mathbf{\Gamma}$-complete set. Then B is in $\mathbf{\Gamma}(2^\omega)$ but not in $\check{\mathbf{\Gamma}}(2^\omega)$. As $B\in\mathbf{\Pi}^0_{\xi+1}$, Theorem 3 gives $C\in\mathbf{\Pi}^0_1(P_\infty)$ and f. By Proposition 11 in [Lec05], it is enough to find $A\subseteq 4^{<\omega}$. The language A will be made of two pieces: we will have $A=\mu\cup\pi$. The set π will code f, and π^ω will look like B on some nice compact sets $K_{N,j}$. Outside this countable family of compact sets we will hide f, so that A^ω will be the simple set μ^ω.

• We set $Q := \{(s, t) \in 2^{<\omega} \times 2^{<\omega} \mid |s| = |t|\}$. We enumerate Q as follows. We start with $q_0 := (\emptyset, \emptyset)$. Then we put the sequences of length 1 of elements of 2×2, in the lexicographical ordering: $q_1 := (0, 0)$, $q_2 := (0, 1)$, $q_3 := (1, 0)$, $q_4 := (1, 1)$. Then we put the 16 sequences of length 2: $q_5 := (0^2, 0^2)$, $q_6 := (0^2, 01)$, ... And so on. We will sometimes use the coordinates of $q_N := (q^0_N, q^1_N)$. We put $M_j := \Sigma_{i<j}\ 4^{i+1}$. Note that the sequence $(M_j)_{j\in\omega}$ is strictly increasing, and that q_{M_j} is the last sequence of length j of elements of 2×2.

• Now we define the "nice compact sets". We will sometimes view 2 as an alphabet, and sometimes view it as a letter. To make this distinction clear, we will use the boldface notation $\mathbf{2}$ for the letter, and the lightface notation 2 otherwise. We will have the same distinction with 3 instead of 2, so we have $2 = \{0, 1\}, 3 = \{0, 1, \mathbf{2}\}, 4 = \{0, 1, \mathbf{2}, \mathbf{3}\}$. Let N, j be non-negative integers with $N\leq M_j$. We set

$$K_{N,j} := \{\ \gamma = \mathbf{2}^N \frown [\ {}^\frown{}_{i\in\omega}\ \ m_i\ \mathbf{2}^{M_{j+i+1}}\ \mathbf{3}\ \mathbf{2}^{M_{j+i+1}}\ ]\in 4^\omega \mid \forall i\in\omega\ \ m_i\in 2=\{0,1\}\}.$$

As the map $\varphi_{N,j} : K_{N,j} \to 2^\omega$ defined by $\varphi_{N,j}(\gamma) := {}^\frown_{i\in\omega} m_i$ is a homeomorphism, $K_{N,j}$ is compact.

• Now we will define the sets that "look like B".

- Let $l \in \omega$. We define a function $c_l : B \to Q$ by $c_l(\alpha) := [f^{-1}(\alpha), \alpha]\lceil l$. Note that Q is countable, so that we equip it with the discrete topology. In these conditions, we prove that c_l is $\mathbf{\Sigma}^0_\xi$-measurable.

If $l \neq |q^0| = |q^1|$ then $c_l^{-1}(q)$ is the empty set. And for any $q \in Q$, and $l = |q^0| = |q^1|$, it holds that $c_l^{-1}(q) = \{\alpha \in B \mid [f^{-1}(\alpha), \alpha]\lceil l = q\} = \{\alpha \in B \mid \alpha\lceil l = q^1 \text{ and } f^{-1}(\alpha)\lceil l = q^0\}$. But $\alpha\lceil l = q^1$ means that α belongs to the basic open set N_{q^1} and $f^{-1}(\alpha)\lceil l = q^0$ means that $f^{-1}(\alpha)$ belongs to the basic open set N_{q^0} or equivalently that $\alpha = f(f^{-1}(\alpha))$ belongs to $f(N_{q^0})$ which is a $\mathbf{\Sigma}^0_\xi$-subset of B. So $c_l^{-1}(q) = N_{q^1} \cap f(N_{q^0})$ is a $\mathbf{\Sigma}^0_\xi$-subset of B and c_l is $\mathbf{\Sigma}^0_\xi$-measurable.

- Let N be an integer. We put

$$E_N := \{\ \alpha\in 2^\omega \mid q^1_N\alpha \in B \ \text{ and } \ c_{|q^1_N|}(q^1_N\alpha) = q_N\ \}.$$

Notice that $E_0 = \{\ \alpha\in 2^\omega \mid \alpha\in B \text{ and } c_0(\alpha) = \emptyset\} = B$.

As $c_{|q^1_N|}$ is $\mathbf{\Sigma}^0_\xi$-measurable and $\{q_N\} \in \mathbf{\Delta}^0_1(Q)$, we get $c^{-1}_{|q^1_N|}(\{q_N\}) \in \mathbf{\Delta}^0_\xi(B) \subseteq \mathbf{\Gamma}(B)$. Therefore there is $G \in \mathbf{\Gamma}(2^\omega)$ with $c^{-1}_{|q^1_N|}(\{q_N\}) = G \cap B$. Thus $c^{-1}_{|q^1_N|}(\{q_N\}) \in \mathbf{\Gamma}(2^\omega)$ since $\mathbf{\Gamma}$ is closed under finite intersections. Note that the map S associating $q^1_N\alpha$ with α is continuous, so that $E_N = S^{-1}[c^{-1}_{|q^1_N|}(\{q_N\})]$ is in $\mathbf{\Gamma}(2^\omega)$.

• Now we define the transition system obtained from f.

- If $m\in 2$ and $n, p\in\omega$, then we write $n \xrightarrow{m} p$ if $q^0_n \prec q^0_p$ and $q^1_p = q^1_n m$.

- As f is continuous on C, the graph Gr(f) of f is a closed subset of $C \times 2^\omega$. As C is $\mathbf{\Pi}^0_1(P_\infty)$, Gr$(f)$ is also a closed subset of $P_\infty \times 2^\omega$. So there is a closed subset F of $2^\omega \times 2^\omega$ such that Gr$(f) = F \cap (P_\infty \times 2^\omega)$. We identify $2^\omega \times 2^\omega$ with $(2\times 2)^\omega$, i.e., we view (β, α) as $[\beta(0), \alpha(0)], [\beta(1), \alpha(1)], \ldots$ By [Kec95, Proposition 2.4], there is $R \subseteq (2\times 2)^{<\omega}$, closed under initial segments, such that $F = \{(\beta, \alpha) \in 2^\omega \times 2^\omega \mid \forall k \in \omega \ \ (\beta, \alpha)\lceil k \in R\}$; notice that R is a tree whose infinite branches form the set F. In particular, we get

$$(\beta, \alpha) \in \text{Gr}(f) \ \Leftrightarrow\ \beta\in P_\infty \ \text{ and } \ \forall k\in\omega \ \ (\beta, \alpha)\lceil k \in R.$$

- Set $Q_f := \{(t, s) \in R \mid t \neq \emptyset \ \text{ and } \ t(|t|-1) = 1\}$. Notice that Q_f is simply the set of pairs $(t, s) \in R$ such that the last letter of t is a 1.

We have in fact already defined the transition system $\mathcal{T}$ obtained from f. This transition system has a countably infinite set Q of states and a set Q_f of accepting states. The initial state is $q_0 := (\emptyset, \emptyset)$. The input alphabet is $2 = \{0, 1\}$ and the transition relation

$\delta \subseteq Q \times 2 \times Q$ is given by: if $m \in 2$ and $n, p \in \omega$ then $(q_n, m, q_p) \in \delta$ iff $n \xrightarrow{m} p$. Recall that a run of $\mathcal{T}$ is said to be Büchi accepting if final states occur infinitely often during this run. Then the set of ω-words over the alphabet 2 which are accepted by the transition system $\mathcal{T}$ from the initial state q_0 with Büchi acceptance condition is exactly the Borel set B.

• Now we define the finitary language π.

- We set

$$\pi := \left\{ \begin{array}{ll} s \in 4^{<\omega} \mid \exists j, l \in \omega \;\; \exists (m_i)_{i \leq l} \in 2^{l+1} \;\; \exists (n_i)_{i \leq l}, (p_i)_{i \leq l}, (r_i)_{i \leq l} \in \omega^{l+1} & \\ & n_0 \leq M_j \\ & \quad \text{and} \\ & \forall i \leq l \;\; n_i \xrightarrow{m_i} p_i \;\text{ and }\; p_i + r_i = M_{j+i+1} \\ & \quad \text{and} \\ & \forall i < l \;\; p_i = n_{i+1} \\ & \quad \text{and} \\ & q_{p_l} \in Q_f \\ & \quad \text{and} \\ & s = {}^\frown_{i \leq l} \;\; \mathbf{2}^{n_i} \; m_i \; \mathbf{2}^{p_i} \; \mathbf{2}^{r_i} \; \mathbf{3} \; \mathbf{2}^{r_i} \end{array} \right\}.$$

• Let us show that $\varphi_{N,j}[\pi^\omega \cap K_{N,j}] = E_N$ if $N \leq M_j$.

Let $\gamma \in \pi^\omega \cap K_{N,j}$, and $\alpha := \varphi_{N,j}(\gamma)$. We can write

$$\gamma = {}^\frown_{k \in \omega} \left[{}^\frown_{i \leq l_k} \;\; \mathbf{2}^{n_i^k} \; m_i^k \; \mathbf{2}^{p_i^k} \; \mathbf{2}^{r_i^k} \; \mathbf{3} \; \mathbf{2}^{r_i^k} \; \right].$$

As this decomposition of γ is in π, we have $n_i^k \xrightarrow{m_i^k} p_i^k$ if $i \leq l_k$, $p_i^k = n_{i+1}^k$ if $i < l_k$, and $q_{p_{l_k}^k} \in Q_f$, for each $k \in \omega$. Moreover, $p_{l_k}^k = n_0^{k+1}$, for each $k \in \omega$, since $\gamma \in K_{N,j}$ implies that $p_{l_k}^k + r_{l_k}^k = r_{l_k}^k + n_0^{k+1} = M_{j+1+m}$ for some integer m. So we get

$$N \xrightarrow{\alpha(0)} p_0^0 \xrightarrow{\alpha(1)} \ldots \xrightarrow{\alpha(l_0)} p_{l_0}^0 \xrightarrow{\alpha(l_0+1)} p_0^1 \xrightarrow{\alpha(l_0+2)} \ldots \xrightarrow{\alpha(l_0+l_1+1)} p_{l_1}^1 \ldots$$

In particular we have

$$q_N^0 \prec q_{p_0^0}^0 \prec \ldots \prec q_{p_{l_0}^0}^0 \prec q_{p_0^1}^0 \prec \ldots \prec q_{p_{l_1}^1}^0 \ldots$$

because $n \xrightarrow{m} p$ implies that $q_n^0 \prec q_p^0$. Note that $|q_{p_{l_k}^k}^1| = |q_N^1| + \Sigma_{j \leq k} \; (l_j + 1)$ because $n \xrightarrow{m} p$ implies that $|q_p^1| = |q_n^1| + 1$, so that the sequence $(|q_{p_{l_k}^k}^0|)_{k \in \omega}$ is strictly increasing since $|q_n^0| = |q_n^1|$ for each integer n. This implies the existence of $\beta \in P_\infty$ such that $q_{p_{l_k}^k}^0 \prec \beta$ for each $k \in \omega$. Note that $\beta \in P_\infty$ because, for each integer k, $q_{p_{l_k}^k} \in Q_f$. Note also that $(\beta, q_N^1 \alpha) \lceil k \in R$ for infinitely many k's. As R is closed under initial segments, $(\beta, q_N^1 \alpha) \lceil k \in R$ for every $k \in \omega$, so that $q_N^1 \alpha = f(\beta) \in B$. Moreover,

$$c_{|q_N^1|}(q_N^1\alpha) = (\beta\lceil|q_N^1|, q_N^1) = (q_N^0, q_N^1) = q_N,$$

and $\alpha \in E_N$.

Conversely, let $\alpha \in E_N$. We have to see that $\gamma := \varphi_{N,j}^{-1}(\alpha) \in \pi^\omega$. As $\gamma \in K_{N,j}$, we are allowed to write $\gamma = \mathbf{2}^N \frown [\frown_{i\in\omega} \ \alpha(i)\ \mathbf{2}^{M_{j+i+1}}\ \mathbf{3}\ ^{M_{j+i+1}}]$. Set $\beta := f^{-1}(q_N^1\alpha)$. There is a sequence of integers $(k_l)_{l\in\omega}$ such that $q_{k_l} = (\beta, q_N^1\alpha)\lceil l$. Note that $N \overset{\alpha(0)}{\rightarrow} k_{|q_N^1|+1} \overset{\alpha(1)}{\rightarrow} k_{|q_N^1|+2} \ldots$ As $N \leq M_j$ we get $k_{|q_N^1|+i+1} \leq M_{j+i+1}$. So we can define $n_0 := N$, $p_0 := k_{|q_N^1|+1}$, $r_0 := M_{j+1} - p_0$, $n_1 := p_0$. Similarly, we can define $p_1 := k_{|q_N^1|+2}$, $r_1 := M_{j+2} - p_1$. We go on like this until we find some q_{p_i} in Q_f. This clearly defines a word in π. And we can go on like this, so that $\gamma \in \pi^\omega$.

Thus $\pi^\omega \cap K_{N,j}$ is in $\mathbf{\Gamma}(K_{N,j}) \subseteq \mathbf{\Gamma}(4^\omega)$. Notice that we proved, among other things, the equality $\varphi_{0,0}[\pi^\omega \cap K_{0,0}] = B$. In particular, $\pi^\omega \cap K_{0,0}$ is not in $\check{\mathbf{\Gamma}}(4^\omega)$.

Notice that π^ω codes on $K_{0,0}$ the behaviour of the transition system accepting B. In a similar way π^ω codes on $K_{N,j}$ the behaviour of the same transition system but starting this time from the state q_N instead of the initial state q_0. But some ω-words in π^ω are not in $K_{0,0}$ and even not in any $K_{N,j}$ and we do not know what is exactly the complexity of this set of ω-words. However we remark that all words in π have the same form $\mathbf{2}^N \frown [\frown_{i\leq l}\ m_i\ \mathbf{2}^{P_i}\ \mathbf{3}\ \mathbf{2}^{R_i}]$.

• We are ready to define μ. The idea is that an infinite sequence containing a word in μ cannot be in the union of the $K_{N,j}$'s. We set

$$\mu^0 = \left\{ \begin{array}{ll} s \in 4^{<\omega} \mid \exists l \in \omega\ \exists (m_i)_{i\leq l+1} \in 2^{l+2}\ \exists N \in \omega\ \exists (P_i)_{i\leq l+1}, (R_i)_{i\leq l+1} \in \omega^{l+2} & \\ & \forall i \leq l+1\ \exists j \in \omega\ \ P_i = M_j \\ & \quad \text{and} \\ & P_l \neq R_l \\ & \quad \text{and} \\ & s = \mathbf{2}^N \frown [\frown_{i\leq l+1}\ m_i\ \mathbf{2}^{P_i}\ \mathbf{3}\ \mathbf{2}^{R_i}] \end{array} \right\},$$

$$\mu^1 = \left\{ \begin{array}{ll} s \in 4^{<\omega} \mid \exists l \in \omega\ \exists (m_i)_{i\leq l+1} \in 2^{l+2}\ \exists N \in \omega\ \exists (P_i)_{i\leq l+1}, (R_i)_{i\leq l+1} \in \omega^{l+2} & \\ & \forall i \leq l+1\ \exists j \in \omega\ \ P_i = M_j \\ & \quad \text{and} \\ & \exists j \in \omega\ \ (P_l = M_j \ \text{and}\ P_{l+1} \neq M_{j+1}) \\ & \quad \text{and} \\ & s = \mathbf{2}^N \frown [\frown_{i\leq l+1}\ m_i\ \mathbf{2}^{P_i}\ \mathbf{3}\ \mathbf{2}^{R_i}] \end{array} \right\},$$

$$\mu := \mu^0 \cup \mu^1.$$

All the words in A will have the same form $\mathbf{2}^N \frown [\frown_{i\leq l}\ m_i\ \mathbf{2}^{P_i}\ \mathbf{3}\ \mathbf{2}^{R_i}]$. Note that any finite concatenation of words of this form still has this form. Moreover, such a concatenation is in μ^i if its last word is in μ^i.

• Now we show that μ^ω is "simple". The previous remarks show that

$$\mu^\omega = \{ \gamma \in 4^\omega \mid \exists i \in 2 \;\; \forall j \in \omega \;\; \exists k, n \in \omega \;\; \exists t_0, t_1, \ldots, t_n \in \mu^i \;\; n \geq j \text{ and } \gamma\lceil k = ^\frown{}_{l \leq n} \, t_l \}.$$

This shows that $\mu^\omega \in \mathbf{\Pi}^0_2(4^\omega)$.

Notice again that all words in A have the same form $\mathbf{2}^N \frown [\frown_{i \leq l} \;\; m_i \; \mathbf{2}^{P_i} \; \mathbf{3} \; \mathbf{2}^{R_i} \,]$. We set

$$P := \{ \mathbf{2}^N \frown [\frown_{i \in \omega} \;\; m_i \; \mathbf{2}^{P_i} \; \mathbf{3} \; \mathbf{2}^{R_i} \,] \in 4^\omega \mid N \in \omega \text{ and } \forall i \in \omega \;\; m_i \in 2, \;\; P_i, R_i \in \omega$$
$$\text{and } \forall i \in \omega \; \exists j \in \omega \; P_i = M_j \}.$$

We define a map $F : P \setminus \mu^\omega \to (\{\emptyset\} \cup \mu) \times \omega^2$ as follows.
Let $\gamma := \mathbf{2}^N \frown [\frown_{i \in \omega} \;\; m_i \; \mathbf{2}^{P_i} \; \mathbf{3} \; \mathbf{2}^{R_i} \,] \in P \setminus \mu^\omega$, and $j_0 \in \omega$ with $P_0 = M_{j_0}$. If $\gamma \in K_{N, j_0 - 1}$, then we put $F(\gamma) := (\emptyset, N, j_0)$. If $\gamma \notin K_{N, j_0 - 1}$, then there is an integer l maximal for which $P_l \neq R_l$ or there is $j \in \omega$ with $P_l = M_j$ and $P_{l+1} \neq M_{j+1}$. Let $j_1 \in \omega$ with $P_{l+2} = M_{j_1}$. We put

$$F(\gamma) := (\mathbf{2}^N \frown [\frown_{i \leq l} \;\; m_i \; \mathbf{2}^{P_i} \; \mathbf{3} \; \mathbf{2}^{R_i} \,] \frown m_{l+1} \; \mathbf{2}^{P_{l+1}} \; \mathbf{3}, R_{l+1}, j_1).$$

• Fix $\gamma \in A^\omega$. If $\gamma \notin \mu^\omega$, then $\gamma \in P \setminus \mu^\omega$, $F(\gamma) := (t, S, j)$ is defined. Note that $t \; \mathbf{2}^S \prec \gamma$, and that $j > 0$. Moreover, $\gamma - t \; \mathbf{2}^S \in K_{0, j-1}$. Note also that $S \leq M_{j-1}$ if $t = \emptyset$, and that $t \; \mathbf{2}^S \; \gamma(|t| + S) \; \mathbf{2}^{M_j} \; \mathbf{3} \notin \mu$. Moreover, there is an integer $N \leq \min(M_{j-1}, S)$ ($N = S$ if $t = \emptyset$) such that $\gamma - t \; \mathbf{2}^{S-N} \in \pi^\omega \cap K_{N, j-1}$, since the last word in μ in the decomposition of γ (if it exists) ends before $t \; \mathbf{2}^S$.

• In the sequel we will say that $(t, S, j) \in (\{\emptyset\} \cup \mu) \times \omega^2$ is <u>*suitable*</u> if $S \leq M_j$ if $t = \emptyset$, $t(|t| - 1) = \mathbf{3}$ if $t \in \mu$, and $t \; \mathbf{2}^S \; m \; \mathbf{2}^{M_{j+1}} \; \mathbf{3} \notin \mu$ if $m \in 2$. We set, for (t, S, j) suitable,

$$P_{t,S,j} := \left\{ \gamma \in 4^\omega \mid t \; \mathbf{2}^S \prec \gamma \;\text{ and }\; \gamma - t \; \mathbf{2}^S \in K_{0,j} \right\}.$$

Note that $P_{t,S,j}$ is a compact subset of $P \setminus \mu^\omega$, and that $F(\gamma) = (t, S, j+1)$ if $\gamma \in P_{t,S,j}$. This shows that the $P_{t,S,j}$'s, for (t, S, j) suitable, are pairwise disjoint. Note also that μ^ω is disjoint from $\bigcup_{(t,S,j) \text{ suitable}} \; P_{t,S,j}$.

• We set, for (t, S, j) suitable and $N \leq \min(M_j, S)$ ($N = S$ if $t = \emptyset$),

$$A_{t,S,j,N} := \left\{ \gamma \in P_{t,S,j} \mid \gamma - t \; \mathbf{2}^{S-N} \in \pi^\omega \cap K_{N,j} \right\}.$$

Note that $A_{t,S,j,N} \in \mathbf{\Gamma}(4^\omega)$ since $N \leq M_j$.

• The previous discussion shows that

$$A^\omega = \mu^\omega \cup \bigcup_{(t,S,j) \text{ suitable}} \;\; \bigcup_{\substack{N \leq \min(M_j, S) \\ N = S \text{ if } t = \emptyset}} A_{t,S,j,N}.$$

As $\mathbf{\Gamma}$ is closed under finite unions, the set

$$A_{t,S,j} := \bigcup_{\substack{N \leq \min(M_j, S) \\ N = S \text{ if } t = \emptyset}} A_{t,S,j,N}$$

is in $\mathbf{\Gamma}(4^\omega)$. On the other hand we have proved that $\mu^\omega \in \mathbf{\Pi}^0_2(4^\omega) \subseteq \mathbf{\Gamma}(4^\omega)$, thus we get $A^\omega \in \mathbf{\Gamma}(4^\omega)$ if $\mathbf{\Gamma} = \mathbf{\Sigma}^0_\xi$.

Consider now the case $\mathbf{\Gamma} = \mathbf{\Pi}^0_\xi$. We can write

$$A^\omega = \mu^\omega \setminus \left(\bigcup_{(t,S,j) \text{ suitable}} P_{t,S,j} \right) \cup \bigcup_{(t,S,j) \text{ suitable}} A_{t,S,j} \cap P_{t,S,j}.$$

Thus

$$\neg A^\omega = \neg \left[\mu^\omega \cup \left(\bigcup_{(t,S,j) \text{ suitable}} P_{t,S,j} \right) \right] \cup \bigcup_{(t,S,j) \text{ suitable}} P_{t,S,j} \setminus A_{t,S,j}.$$

Here $\neg \left[\mu^\omega \cup \left(\bigcup_{(t,S,j) \text{ suitable}} P_{t,S,j} \right) \right] \in \mathbf{\Delta}^0_3(4^\omega) \subseteq \check{\mathbf{\Gamma}}(4^\omega)$ because μ^ω is a $\mathbf{\Pi}^0_2$-subset of 4^ω and $(\bigcup_{(t,S,j) \text{ suitable}} P_{t,S,j})$ is a $\mathbf{\Sigma}^0_2$-subset of 4^ω as it is a countable union of compact hence closed sets. On the other hand $P_{t,S,j} \setminus A_{t,S,j} \in \check{\mathbf{\Gamma}}(4^\omega)$, thus $\neg A^\omega$ is in $\check{\mathbf{\Gamma}}(4^\omega)$ and $A^\omega \in \mathbf{\Gamma}(4^\omega)$. Moreover, the set $A^\omega \cap P_{\emptyset,0,0} = \pi^\omega \cap P_{\emptyset,0,0} = \pi^\omega \cap K_{0,0}$ is not in $\check{\mathbf{\Gamma}}$. This shows that A^ω is not in $\check{\mathbf{\Gamma}}$. Thus A^ω is in $\mathbf{\Gamma}(4^\omega) \setminus \check{\mathbf{\Gamma}}$.

We can now end the proof of Theorem 2.

(a) If $\xi = 1$, then we can take $A := \{s \in 2^{<\omega} \mid 0 \prec s \;\text{ or }\; \exists k \in \omega \;\; 10^k 1 \prec s\}$ and $A^\omega = 2^\omega \setminus \{10^\omega\}$ is $\mathbf{\Sigma}^0_1 \setminus \mathbf{\Pi}^0_1$.

• If $\xi = 2$, then we will see in Theorem 4 the existence of $A \subseteq 2^{<\omega}$ such that A^ω is $\mathbf{\Sigma}^0_2 \setminus \mathbf{\Pi}^0_2$.

• So we may assume that $\xi \geq 3$, and we are done.

(b) If $\xi = 1$, then we can take $A := \{0\}$ and $A^\omega = \{0^\omega\}$ is $\mathbf{\Pi}^0_1 \setminus \mathbf{\Sigma}^0_1$.

• If $\xi = 2$, then we can take $A := \{0^k 1 \mid k \in \omega\}$ and $A^\omega = P_\infty$ is $\mathbf{\Pi}^0_2 \setminus \mathbf{\Sigma}^0_2$.

• So we may assume that $\xi \geq 3$, and we are done. □

As we have said above it remains a Borel class for which we have not yet got a complete ω-power: the class $\mathbf{\Sigma}^0_2$. Notice that it is easy to see that the classical example of $\mathbf{\Sigma}^0_2$-complete set, the set $2^\omega \setminus P_\infty$, is not an ω-power. However we are going to prove the following result.

Theorem 4. *There is a context-free language $A \subseteq 2^{<\omega}$ such that $A^\omega \in \mathbf{\Sigma}^0_2 \setminus \mathbf{\Pi}^0_2$.*

Proof. By Proposition 11 in [Lec05], it is enough to find $A \subseteq 3^{<\omega}$. We set, for $j<3$ and $s \in 3^{<\omega}$,

$$n_j(s) := \text{ Card}\{i<|s| \mid s(i)=j\},$$

$$T := \{\alpha \in 3^{\leq\omega} \mid \forall l<1+|\alpha| \;\; n_2(\alpha\lceil l) \leq n_1(\alpha\lceil l)\}.$$

• We inductively define, for $s \in T \cap 3^{<\omega}$, $s^{\hookleftarrow} \in 2^{<\omega}$ as follows:

$$s^{\hookleftarrow} := \begin{cases} \emptyset \text{ if } s=\emptyset, \\ t^{\hookleftarrow}\varepsilon \text{ if } s=t\varepsilon \text{ and } \varepsilon<2, \\ t^{\hookleftarrow}\text{, except that its last 1 is replaced with 0, if } s=t\mathbf{2}. \end{cases}$$

• We will extend this definition to infinite sequences. To do this, we introduce a notion of limit. Fix $(s_n)_{n\in\omega}$ a sequence of elements in $2^{<\omega}$. We define $\lim\limits_{n\to\infty} s_n \in 2^{\leq\omega}$ as follows. For each $t \in 2^{<\omega}$,

$$t \prec \lim_{n\to\infty} s_n \;\Leftrightarrow\; \exists n_0 \in \omega \;\; \forall n \geq n_0 \;\; t \prec s_n.$$

• If $\alpha \in T \cap 3^\omega$, then we set $\alpha^{\hookleftarrow} := \lim\limits_{n\to\infty} (\alpha\lceil n)^{\hookleftarrow}$. We define $e : T \cap 3^\omega \to 2^\omega$ by $e(\alpha) := \alpha^{\hookleftarrow}$. Note that $T \cap 3^\omega \in \mathbf{\Pi}^0_1(3^\omega)$, and e is a $\mathbf{\Sigma}^0_2$-measurable partial function on $T \cap 3^\omega$, since for $t \in 2^{<\omega}$ we have

$$t \prec e(\alpha) \;\Leftrightarrow\; \exists n_0 \in \omega \;\; \forall n \geq n_0 \;\; t \prec (\alpha\lceil n)^{\hookleftarrow}.$$

• We set $E := \{s \in T \cap 3^{<\omega} \mid n_2(s)=n_1(s) \text{ and } s \neq \emptyset \text{ and } 1 \prec [s\lceil(|s|-1)]^{\hookleftarrow}\}$. Note that $\emptyset \neq s^{\hookleftarrow} \prec 0^\omega$, and that $s(|s|-1)=\mathbf{2}$ changes $s(0)=[s\lceil(|s|-1)]^{\hookleftarrow}(0)=1$ into 0 if $s \in E$.

• If $S \subseteq 3^{<\omega}$, then $S^* := \{{}^\frown{}_{i<l}\, s_i \in 3^{<\omega} \mid l \in \omega \text{ and } \forall i<l \;\; s_i \in S\}$. We put

$$A := \{0\} \cup E \cup \{{}^\frown{}_{j\leq k}\,(c_j 1) \in 3^{<\omega} \mid [\forall j \leq k \;\; c_j \in (\{0\} \cup E)^*] \text{ and } [k>0 \text{ or } (k=0 \text{ and } c_0 \neq \emptyset)]\}.$$

• In the proof of Theorem 2.(b) we met the set $\{s \in 2^{<\omega} \mid 0 \prec s \text{ or } \exists k \in \omega \;\; 10^k 1 \prec s\}$. We shall denoted it by B in the sequel. We have seen that $B^\omega = 2^\omega \setminus \{10^\omega\}$ is $\mathbf{\Sigma}^0_1 \setminus \mathbf{\Pi}^0_1$. Let us show that $A^\omega = e^{-1}(B^\omega)$.

- By induction on $|t|$, we get $(st)^{\hookleftarrow} = s^{\hookleftarrow} t^{\hookleftarrow}$ if $s, t \in T \cap 3^{<\omega}$. Let us show that $(s\beta)^{\hookleftarrow} = s^{\hookleftarrow}\beta^{\hookleftarrow}$ if moreover $\beta \in T \cap 3^\omega$.

Assume that $t \prec (s\beta)^{\hookleftarrow}$. Then there is $m_0 \geq |s|$ such that, for $m \geq m_0$,

$$t \prec [(s\beta)\lceil m]^{\hookleftarrow} = [s\beta\lceil(m-|s|)]^{\hookleftarrow} = s^{\hookleftarrow}[\beta\lceil(m-|s|)]^{\hookleftarrow}.$$

This implies that $t \prec s^{\hookleftarrow}\beta^{\hookleftarrow}$ if $|t|<|s^{\hookleftarrow}|$. If $|t|\geq|s^{\hookleftarrow}|$, then there is $m_1\in\omega$ such that, for $m\geq m_1$, $\beta^{\hookleftarrow}\lceil(|t|-|s^{\hookleftarrow}|)\prec[\beta\lceil(m-|s|)]^{\hookleftarrow}$. Here again, we get $t\prec s^{\hookleftarrow}\beta^{\hookleftarrow}$. Thus $(s\beta)^{\hookleftarrow}=s^{\hookleftarrow}\beta^{\hookleftarrow}$.

Let $(s_i)_{i\in\omega}$ be a sequence such that for each integer $i\in\omega$, $s_i\in T\cap 3^{<\omega}$. Then $\frown_{i\in\omega}\, s_i\in T$, and $(\frown_{i\in\omega}\, s_i)^{\hookleftarrow}=\frown_{i\in\omega}\, s_i^{\hookleftarrow}$, by the previous facts.

- Let $(a_i)_{i\in\omega}$ be a sequence such that for each integer $i\in\omega$, $a_i\in A\setminus\{\emptyset\}$ and $\alpha:=\frown_{i\in\omega}\, a_i$. As $A\subseteq T$, $e(\alpha)=(\frown_{i\in\omega}\, a_i)^{\hookleftarrow}=\frown_{i\in\omega}\, a_i^{\hookleftarrow}$.
If $a_0\in\{0\}\cup E$, then $\emptyset\neq a_0^{\hookleftarrow}\prec 0^\omega$, thus $e(\alpha)\in N_0\subseteq 2^\omega\setminus\{10^\omega\}=B^\omega$.
If $a_0\notin\{0\}\cup E$, then $a_0=\frown_{j\leq k}\,(c_j1)$, thus $a_0^{\hookleftarrow}=\frown_{j\leq k}\,(c_j^{\hookleftarrow}1)$.
If $c_0\neq\emptyset$, then $e(\alpha)\in B^\omega$ as before.
If $c_0=\emptyset$, then $k>0$, so that $e(\alpha)\neq 10^\omega$ since $e(\alpha)$ has at least two coordinates equal to 1.
We proved that $A^\omega\subseteq e^{-1}(B^\omega)$.

- Assume now that $e(\alpha)\in B^\omega$. We have to find $(a_i)_{i\in\omega}\subseteq A\setminus\{\emptyset\}$ with $\alpha=\frown_{i\in\omega}\, a_i$. We split into cases:

1. $e(\alpha)=0^\omega$.
1.1. $\alpha(0)=0$.
In this case $\alpha-0\in T$ and $e(\alpha-0)=0^\omega$. Moreover, $0\in A$. We put $a_0:=0$.

1.2. $\alpha(0)=1$.
In this case there is a coordinate j_0 of α equal to **2** ensuring that $\alpha(0)$ is replaced with a 0 in $e(\alpha)$. We put $a_0:=\alpha\lceil(j_0+1)$, so that $a_0\in E\subseteq A$, $\alpha-a_0\in T$ and $e(\alpha-a_0)=0^\omega$.

Now the iteration of the cases 1.1 and 1.2 shows that $\alpha\in A^\omega$.

2. $e(\alpha)=0^{k+1}10^\omega$ for some $k\in\omega$.

As in case 1, there is $c_0\in(\{0\}\cup E)^*$ such that $c_0\prec\alpha$, $c_0^{\hookleftarrow}=0^{k+1}$, $\alpha-c_0\in T$ and $e(\alpha-c_0)=10^\omega$. Note that $\alpha(|c_0|)=1$, $\alpha-(c_01)\in T$ and $e[\alpha-(c_01)]=0^\omega$. We put $a_0:=c_01$, and argue as in case 1.

3. $e(\alpha)=(\frown_{j\leq l+1}\, 0^{k_j}1)0^\omega$ for some $l\in\omega$.

The previous cases show the existence of $(c_j)_{j\leq l+1}$, where for each $j\leq l+1$ $c_j\in(\{0\}\cup E)^*$ such that :
$a_0:=\frown_{j\leq l+1}\, c_j1\prec\alpha$, $\alpha-a_0\in T$ and $e(\alpha-a_0)=0^\omega$. We are done since $a_0\in A$.

4. $e(\alpha)=\frown_{j\in\omega}\, 0^{k_j}1$.

An iteration of the discussion of case 3 shows that we can take a_i of the form $\frown_{j\leq l+1}\, c_j1$.

• The previous discussion shows that $A^\omega = e^{-1}(B^\omega)$. As B^ω is an open subset of 2^ω and e is $\mathbf{\Sigma}^0_2$-measurable, the ω-power $A^\omega = e^{-1}(B^\omega)$ is in $\mathbf{\Sigma}^0_2(3^\omega)$.

It remains to see that $A^\omega = e^{-1}(B^\omega) \notin \mathbf{\Pi}^0_2$. We argue by contradiction.

Assume on the contrary that $e^{-1}(B^\omega) \in \mathbf{\Pi}^0_2(3^\omega)$. We know that $B^\omega = 2^\omega \setminus \{10^\omega\}$ so $e^{-1}(\{10^\omega\}) = (T \cap 3^\omega) \setminus e^{-1}(B^\omega)$ would be a $\mathbf{\Sigma}^0_2$-subset of 3^ω since $T \cap 3^\omega$ is closed in 3^ω. Thus $e^{-1}(\{10^\omega\})$ would be a countable union of compact subsets of 3^ω.

Consider now the **cartesian product** $(\{0\} \cup E)^{\mathbb{N}}$ of countably many copies of $(\{0\} \cup E)$. The set $(\{0\} \cup E)$ is countable and it can be equipped with the discrete topology. Then the product $(\{0\} \cup E)^{\mathbb{N}}$ is equipped with the product topology of the discrete topology on $(\{0\} \cup E)$. The topological space $(\{0\} \cup E)^{\mathbb{N}}$ is homeomorphic to the Baire space ω^ω.

Consider now the map $h : (\{0\} \cup E)^{\mathbb{N}} \to e^{-1}(\{10^\omega\})$ defined by $h(\gamma) := 1[\frown_{i\in\omega} \gamma_i]$ for each $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_i, \ldots) \in (\{0\} \cup E)^{\mathbb{N}}$. The map h is a homeomorphism by the previous discussion. As $(\{0\} \cup E)^{\mathbb{N}}$ is homeomorphic to the Baire space ω^ω, the Baire space ω^ω is also homeomorphic to the space $e^{-1}(\{10^\omega\})$, so it would be also a countable union of compact sets. But this is absurd by [Kec95, Theorem 7.10].

It remains to see that A is context-free. It is easy to see that the language E is in fact accepted by a 1-counter automaton: it is the set of words $s \in 3^{<\omega}$ such that :

$$\forall l \in [1; |s|[\;\; n_2(s\lceil l) < n_1(s\lceil l) \text{ and } n_2(s) = n_1(s) \text{ and } s(0) = 1 \text{ and } s(|s|-1) = \mathbf{2}.$$

This implies that A is also accepted by a 1-counter automaton because the class of 1-counter languages is closed under concatenation and star operation. In particular A is a context-free language because the class of languages accepted by 1-counter automata form a strict subclass of the class of context-free languages, [ABB96]. □

Remark 5. *The operation $\alpha \to \alpha^{\hookleftarrow}$ we have defined is very close to the erasing operation defined by Duparc in his study of the Wadge hierarchy, [Dup01]. However we have modified this operation in such a way that $\alpha^{\hookleftarrow}$ is always infinite when α is infinite, and that it has the good property with regard to ω-powers and topological complexity.*

4 Concluding Remarks and Further Work

It is natural to wonder whether the ω-powers obtained in this paper are effective. For instance could they be obtained as ω-powers of recursive languages ?

In the long version of this paper we prove effective versions of the results presented here. Using tools of effective descriptive set theory, we first prove an effective version of Kuratowski's Theorem 3. Then we use it to prove the following effective version of Theorem 2, where Σ^0_ξ and Π^0_ξ denote classes of the hyperarithmetical hierarchy and ω_1^{CK} is the first non-recursive ordinal, usually called the Church-kleene ordinal.

Theorem 6. *Let ξ be a non-null ordinal smaller than ω_1^{CK}.*
(a) There is a recursive language $A \subseteq 2^{<\omega}$ such that $A^\omega \in \Sigma_\xi^0 \setminus \mathbf{\Pi}_\xi^0$.
(b) There is a recursive language $A \subseteq 2^{<\omega}$ such that $A^\omega \in \Pi_\xi^0 \setminus \mathbf{\Sigma}_\xi^0$.

The question, left open in [Fin04], also naturally arises to know what are all the possible infinite Borel ranks of ω-powers of finitary languages belonging to some natural class like the class of context free languages (respectively, languages accepted by stack automata, recursive languages, recursively enumerable languages, ...).
We know from [Fin06] that there are ω-languages accepted by Büchi 1-counter automata of every Borel rank (and even of every Wadge degree) of an effective analytic set. Every ω-language accepted by a Büchi 1-counter automaton can be written as a finite union $L = \bigcup_{1 \leq i \leq n} U_i^\frown V_i^\omega$, where for each integer i, U_i and V_i are finitary languages accepted by 1-counter automata. And the supremum of the set of Borel ranks of effective analytic sets is the ordinal γ_2^1. This ordinal is defined by A.S. Kechris, D. Marker, and R.L. Sami in [KMS89] and it is proved to be strictly greater than the ordinal δ_2^1 which is the first non Δ_2^1 ordinal. Thus the ordinal γ_2^1 is also strictly greater than the first non-recursive ordinal ω_1^{CK}. From these results it seems plausible that there exist some ω-powers of languages accepted by 1-counter automata which have Borel ranks up to the ordinal γ_2^1, although these languages are located at the very low level in the complexity hierarchy of finitary languages.

Another question concerns the Wadge hierarchy which is a great refinement of the Borel hierarchy. It would be interesting to determine the Wadge hierarchy of ω-powers. In the full version of this paper we give many Wadge degrees of ω-powers and this confirms the great complexity of these ω-languages.

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