Borel chromatic number of closed graphs

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Abstract. We construct, for each countable ordinal ξ, a closed graph with Borel chromatic number two and Baire class ξ chromatic number $\aleph_0$.

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1 Introduction

The study of the Borel chromatic number of analytic graphs on Polish spaces was initiated in [K-S-T]. In particular, the authors prove in this paper that the Borel chromatic number of the graph generated by a partial Borel function has to be in $\{1, 2, 3, \aleph_0\}$. They also provide a minimum graph $\mathcal{G}_0$ of uncountable Borel chromatic number. This last result had a lot of developments. For example, B. Miller gave in [Mi] some other versions of it, which helped him to generalize a number of known dichotomy theorems in descriptive set theory. The first author generalized in [L2] the $\mathcal{G}_0$-dichotomy to any dimension making sense in classical descriptive set theory, and also used versions of $\mathcal{G}_0$ to study the non-potentially closed subsets of a product of two Polish spaces (see [L1]).

A study of the $\mathbf{\Delta}^0_\xi$ chromatic number of analytic graphs on Polish spaces was initiated in [L-Z1] and was motivated by the $\mathcal{G}_0$-dichotomy. More precisely, let B be a Borel binary relation, on a Polish space X, having a Borel countable coloring (i.e., a Borel map $c : X \to \omega$ such that $c(x) \neq c(y)$ if $(x, y) \in B$). Is there a relation between the Borel class of B and that of the coloring? In other words, is there a map $k : \omega_1 \setminus \{0\} \to \omega_1 \setminus \{0\}$ such that any $\mathbf{\Pi}^0_\xi$ binary relation having a Borel countable coloring has in fact a $\mathbf{\Delta}^0_{k(\xi)}$-measurable countable coloring, for each $\xi \in \omega_1 \setminus \{0\}$?

In [L-Z2], the authors give a negative answer: for each countable ordinal $\xi \geq 1$, there is a partial injection with disjoint domain and range $i : \omega^\omega \to \omega^\omega$, whose graph

- is $D_2(\mathbf{\Pi}^0_1)$ (i.e., the difference of two closed sets),
- has Borel chomatic number two,
- has no $\mathbf{\Delta}^0_\xi$-measurable countable coloring.

On the other hand, they note that an open binary relation having a finite coloring c has also a $\mathbf{\Delta}^0_2$-measurable finite coloring (consider the differences of the $\overline{c^{-1}(\{n\})}$'s, for n in the range of the coloring). Note that an irreflexive closed binary relation on a zero-dimensional space has a continuous countable coloring (this coloring is $\mathbf{\Delta}^0_2$-measurable in non zero-dimensional spaces). So they wonder whether we can build, for each countable ordinal $\xi \geq 1$, a closed binary relation with a Borel finite coloring but no $\mathbf{\Delta}^0_\xi$-measurable finite coloring. This is indeed the case:

Theorem *Let $\xi \geq 1$ be a countable ordinal. Then there exists a partial injection with disjoint domain and range $f : \omega^\omega \to \omega^\omega$ whose graph is closed (and thus has Borel chromatic number two), and has no $\mathbf{\Delta}^0_\xi$-measurable finite coloring (and thus has $\mathbf{\Delta}^0_\xi$ chromatic number $\aleph_0$).*

The previous discussion shows that this result is optimal. Its proof uses, among other things, the method used in [L-Z2] improving Theorem 4 in [M]. This method relates topological complexity and Baire category.

2 Mátrai sets

Before proving our main result, we recall some material from [L-Z2].

Notation. The symbol τ denotes the usual product topology on the Baire space ω^ω.

Definition 2.1 *We say that a partial map* $f:\omega^\omega\rightarrow\omega^\omega$ *is* **nice** if its graph $\text{Gr}(f)$ is a $(\tau\times\tau)$-closed subset of $\omega^\omega\times\omega^\omega$.

The construction of P_ξ and τ_ξ, and the verification of the properties (1)-(3) from the next lemma (a corollary of Lemma 2.6 in [L-Z2]), can be found in [M], up to minor modifications.

Lemma 2.2 *Let* $1\leq\xi<\omega_1$*. Then there are* $P_\xi\subseteq\omega^\omega$*, and a topology* τ_ξ *on* ω^ω *such that*

(1) τ_ξ *is zero-dimensional perfect Polish and* $\tau\subseteq\tau_\xi\subseteq\mathbf{\Sigma}^0_\xi(\tau)$*,*

(2) P_ξ *is a nonempty* τ_ξ*-closed nowhere dense set,*

(3) if $S\in\mathbf{\Sigma}^0_\xi(\omega^\omega,\tau)$ *is* τ_ξ*-nonmeager in* P_ξ*, then* S *is* τ_ξ*-nonmeager in* ω^ω*,*

(4) if V,W *are nonempty* τ_ξ*-open subsets of* ω^ω*, then we can find a* τ_ξ*-dense* G_δ *subset* H *of* $V\setminus P_\xi$*, a* τ_ξ*-dense* G_δ *subset* L *of* $W\setminus P_\xi$*, and a nice* (τ_ξ,τ_ξ)*-homeomorphism from* H *onto* L*.*

The following lemma (a corollary of Lemma 2.7 in [L-Z2]) is a consequence of the previous one. It provides, among other things, a topology T_ξ that we will use in the sequel.

Lemma 2.3 *Let* $1\leq\xi<\omega_1$*. Then there are a disjoint countable family* $\mathcal{G}_\xi$ *of subsets of* ω^ω *and a topology* T_ξ *on* ω^ω *such that*

(a) T_ξ *is zero-dimensional perfect Polish and* $\tau\subseteq T_\xi\subseteq\mathbf{\Sigma}^0_\xi(\tau)$*,*

(b) for any nonempty T_ξ*-open sets* V,V'*, there are distinct* $G,G'\in\mathcal{G}_\xi$ *with* $G\subseteq V$*,* $G'\subseteq V'$*, and there is a nice* (T_ξ,T_ξ)*-homeomorphism from* G *onto* G'*,*

and, for every $G\in\mathcal{G}_\xi$*,*

(c) G *is nonempty,* T_ξ*-nowhere dense, and in* $\mathbf{\Pi}^0_2(T_\xi)$*,*

(d) if $S\in\mathbf{\Sigma}^0_\xi(\omega^\omega,\tau)$ *is* T_ξ*-nonmeager in* G*, then* S *is* T_ξ*-nonmeager in* ω^ω*.*

The construction of $\mathcal{G}_\xi$ and T_ξ ensures that T_ξ is $(\tau_\xi)^\omega$, where τ_ξ is as in Lemma 2.2. This topology is on $(\omega^\omega)^\omega$, identified with ω^ω. We will need the following consequence of the construction of $\mathcal{G}_\xi$ and T_ξ.

Lemma 2.4 *Let* $1\leq\xi<\omega_1$*, and* V *be a nonempty* T_ξ*-open set. Then* $\overline{V}^\tau$ *is not* τ*-compact.*

Proof. The fact that T_ξ is $(\tau_\xi)^\omega$ gives a finite sequence $U_0,...,U_n$ of nonempty open subsets of (ω^ω,τ_ξ) with $U_0\times...\times U_n\times(\omega^\omega)^\omega\subseteq V$. Thus $\overline{V}^\tau$ contains the τ-closed set $\overline{U_0}^\tau\times...\times\overline{U_n}^\tau\times(\omega^\omega)^\omega$, and it is enough to see that this last set is not τ-compact. This comes from the fact that the Baire space (ω^ω,τ) is not compact. □

3 Proof of the main result

Before proving our main result, we give an example giving the flavour of the sequel. In [Za], the author gives a Hurewicz-like test to see when two disjoint subsets A,B of a product $Y\times Z$ of Polish spaces can be separated by an open rectangle. We set $\mathbb{A}:=\{(n^\infty,n^\infty)\mid n\in\omega\}$,

$$\mathbb{B}_0:=\big\{\big(0^{m+1}(n+1)^\infty,(m+1)^{n+1}0^\infty\big)\mid m,n\in\omega\big\}$$

and $\mathbb{B}_1:=\big\{\big((m+1)^{n+1}0^\infty,0^{m+1}(n+1)^\infty\big)\mid m,n\in\omega\big\}$. Then A is not separable from B by an open rectangle exactly when there are $\varepsilon\in 2$ and continuous maps $g:\omega^\omega\rightarrow Y$, $h:\omega^\omega\rightarrow Z$ such that $\mathbb{A}\subseteq(g\times h)^{-1}(A)$ and $\mathbb{B}_\varepsilon\subseteq(g\times h)^{-1}(B)$.

Example. Here we are looking for closed graphs with Borel chromatic number two and of arbitrarily high finite $\mathbf{\Delta}^0_\xi$ chromatic number n. There is an example with $\xi=1$ and $n=3$ where $\mathbb{B}_0$ is involved. We set $C:=\big\{\big((2m)^\infty,(2m{+}1)^\infty\big)\mid m\in\omega\big\}\cup\mathbb{B}_0$,

$$D:=\{(2m)^\infty\mid m\in\omega\}\cup\{0^{m+1}(n{+}1)^\infty\mid m,n\in\omega\},$$

$$R:=\{(2m{+}1)^\infty\mid m\in\omega\}\cup\{(m{+}1)^{n+1}0^\infty\mid m,n\in\omega\},$$

$$f\big((2m)^\infty\big):=(2m{+}1)^\infty \text{ and } f\big(0^{m+1}(n{+}1)^\infty\big):=(m{+}1)^{n+1}0^\infty.$$

This defines $f:D\rightarrow R$ whose graph is C. The first part of C is discrete, and thus closed. Assume that $(\alpha_k,\beta_k):=\big(0^{m_k+1}(n_k+1)^\infty,(m_k+1)^{n_k+1}0^\infty\big)\in\mathbb{B}_0$ and converges to $(\alpha,\beta)\in\omega^\omega\times\omega^\omega$ as k goes to infinity. We may assume that (m_k) is constant, and (n_k) too, so that $(\alpha,\beta)\in\mathbb{B}_0$, which is therefore closed. This shows that C is closed. Note that D,R are disjoint and Borel, so that C has Borel chromatic number two. Let Δ be a clopen subset of ω^ω. Let us prove that $C\cap\Delta^2$ or $C\cap(\neg\Delta)^2$ is not empty. We argue by contradiction. Then Δ or $\neg\Delta$ has to contain 0^∞. Assume that it is Δ, the other case being similar. Then $0^{m+1}(n{+}1)^\infty\in\Delta$ if m is big enough. Thus $(m{+}1)^{n+1}0^\infty\notin\Delta$ if m is big enough. Therefore $(m{+}1)^\infty\notin\Delta$ if m is big enough. Thus $\big((2m)^\infty,(2m{+}1)^\infty\big)\in C\cap(\neg\Delta)^2$ if m is big enough, which is absurd.

We now turn to the general case. Our main lemma is as follows. We equip ω^m with the discrete topology τ_d, for each $m>0$.

Lemma *Let $\xi\geq 1$ be a countable ordinal, $n\geq 1$ be a natural number, and $X:=\omega\times\omega^\omega$. Then we can find a partial injection $f:X\rightarrow X$ and a disjoint countable family $\mathcal{F}$ of subsets of X such that*

(a) f has disjoint domain and range,

(b) Gr(f) is $\big((\tau_d\times\tau)\times(\tau_d\times\tau)\big)$-closed,

(c) there is no sequence $(\Delta_i)_{i<n}$ of $\mathbf{\Delta}^0_\xi$ subsets of $(X,\tau_d\times\tau)$ such that

(i) $\forall i<n\ \ Gr(f)\cap\Delta_i^2=\emptyset$,

(ii) $\bigcup_{i<n}\ \Delta_i$ is $(\tau_d\times T_\xi)$-comeager in X,

(d) $\mathcal{F}$ has the properties (b)-(d) in Lemma 2.3, where $\mathcal{G}_\xi$, ω^ω, T_ξ and τ are respectively replaced with $\mathcal{F}$, X, $\tau_d\times T_\xi$ and $\tau_d\times\tau$,

(e) $(\bigcup\mathcal{F})\cap\big(Domain(f)\cup Range(f)\big)=\emptyset$.

Proof. We argue by induction on n.

The basic case $n=1$

Let $\mathcal{G}_\xi$ be the family given by Lemma 2.3. We split $\mathcal{G}_\xi$ into two disjoint subfamilies $\mathcal{G}^0_\xi$ and $\mathcal{G}^1_\xi$ having the property (b) in Lemma 2.3. This is possible since the elements of $\mathcal{G}_\xi$ are T_ξ-nowhere dense. Let $G_0,G_1\in\mathcal{G}^0_\xi$ be distinct, and φ be a nice (T_ξ,T_ξ)-homeomorphism from G_0 onto G_1. We then set $f(0,\alpha):=\big(0,\varphi(\alpha)\big)$ if $\alpha\in G_0$, and $\mathcal{F}:=\big\{\{n\}\times G\mid n\in\omega\wedge G\in\mathcal{G}^1_\xi\big\}$. It remains to check that the property (c) is satisfied. We argue by contradiction, which gives $\Delta_0\in\mathbf{\Delta}^0_\xi$. By property (d) in Lemma 2.3, $\Delta_0\cap(\{0\}\times G_\varepsilon)$ is $(\tau_d\times T_\xi)$-comeager in $\{0\}\times G_\varepsilon$ for each $\varepsilon\in 2$. As f is a $(\tau_d\times T_\xi,\tau_d\times T_\xi)$-homeomorphism, $\Delta_0\cap(\{0\}\times G_0)\cap f^{-1}\big(\Delta_0\cap(\{0\}\times G_1)\big)$ is $(\tau_d\times T_\xi)$-comeager in $\{0\}\times G_0$, which contradicts the fact that $\text{Gr}(f)\cap\Delta_0^2=\emptyset$.

The induction step from n to $n+1$

The induction assumption gives f and $\mathcal{F}$. Here again, we split $\mathcal{F}$ into two disjoint subfamilies $\mathcal{F}^0$ and $\mathcal{F}^1$ having the property (b) in Lemma 2.3, where $\mathcal{G}_\xi$, ω^ω, T_ξ and τ are respectively replaced with $\mathcal{F}^\varepsilon$, X, $\tau_d\times T_\xi$ and $\tau_d\times\tau$. Let (V_p) be a basis for the topology $\tau_d\times T_\xi$ made of nonempty sets. Fix $p\in\omega$. By Lemma 2.4, there is a countable family $(W^p_q)_{q\in\omega}$, with $(\tau_d\times\tau)$-closed union, and made of pairwise disjoint $(\tau_d\times\tau)$-clopen subsets of X intersecting V_p.

• Let $b:\omega\rightarrow\omega^2$ be a bijection. We construct, for $\vec{v}=(p,q)\in\omega^2$ and $\varepsilon\in 2$, and by induction on $b^{-1}(\vec{v})$,

- $G^{\vec{v}}_\varepsilon\in\mathcal{F}^0$,
- a nice $(\tau_d\times T_\xi,\tau_d\times T_\xi)$-homeomorphism $\varphi^{\vec{v}}:G^{\vec{v}}_0\rightarrow G^{\vec{v}}_1$.

We want these objects to satisfy the following:

- $G^{\vec{v}}_0\subseteq(V_p\cap W^p_q)\setminus(\bigcup_{m<b^{-1}(\vec{v})}\ \overline{G^{b(m)}_0\cup G^{b(m)}_1}^{\tau_d\times T_\xi})$,

- $G^{\vec{v}}_1\subseteq V_q\setminus(G^{\vec{v}}_0\cup\bigcup_{m<b^{-1}(\vec{v})}\ \overline{G^{b(m)}_0\cup G^{b(m)}_1}^{\tau_d\times T_\xi})$.

• We now define the desired partial map $\tilde{f}:\omega\times\omega\times\omega^\omega\rightarrow\omega\times\omega\times\omega^\omega$, as well as $\tilde{\mathcal{F}}\subseteq 2^{\omega\times\omega\times\omega^\omega}$, as follows:

$$\tilde{f}(l,x):=\begin{cases}\big(p+1,\varphi^{p,q}(x)\big)\text{ if } l=0\ \wedge\ x\in G^{p,q}_0,\\ \\ \big(l,f(x)\big)\text{ if } l>0\ \wedge\ x\in\text{Domain}(f).\end{cases}$$

and $\tilde{\mathcal{F}}:=\big\{\{l\}\times G\mid l\in\omega\ \wedge\ G\in\mathcal{F}^1\big\}$. Note that $\tilde{f}$ is well-defined and injective, by disjointness of the $(G^{\vec{v}}_0\cup G^{\vec{v}}_1)$'s. Identifying X with $\omega\times\omega\times\omega^\omega$, we can consider $\tilde{f}$ as a partial map from X into itself and $\tilde{\mathcal{F}}$ as a family of subsets of X (this identification is based on the identification of ω with $\omega\times\omega$).

(a), (d) and (e) are clearly satisfied.

(b) Assume that $\big((l_k,x_k),(m_k,y_k)\big)\in\text{Gr}(\tilde{f})$ tends to $\big((l,x),(m,y)\big)\in(\omega\times X)^2$ as k goes to infinity. We may assume that (l_k) and (m_k) are constant.

If $l=0$, then there is p such that $p+1=m$ and $(x_k,y_k)\in G^{p,q_k}_0\times G^{p,q_k}_1$. As $G^{p,q_k}_0\subseteq W^p_{q_k}$, we may also assume that (q_k) is also constant and equals q. As $\varphi^{p,q}$ is nice, $\big((l,x),(m,y)\big)\in\text{Gr}(\tilde{f})$.

If $l>0$, then $(x_k,y_k)\in\text{Gr}(f)$. As $\text{Gr}(f)$ is $\big((\tau_d\times\tau)\times(\tau_d\times\tau)\big)$-closed, $\big((l,x),(m,y)\big)\in\text{Gr}(\tilde{f})$.

(c) We argue by contradiction, which gives $(\Delta_i)_{i\leq n}$. We may assume, without loss of generality, that $(\{0\}\times\omega\times\omega^\omega)\cap\Delta_n$ is not meager in $(\{0\}\times\omega\times\omega^\omega,\tau_d\times T_\xi)$. This gives $p\in\omega$ such that $(\{0\}\times V_p)\cap\Delta_n$ is $(\tau_d\times T_\xi)$-comeager in $V'_p:=\{0\}\times V_p$. As $V'_p\setminus\Delta_n\in\mathbf{\Sigma}^0_\xi(\tau_d\times\tau)$, $(\{0\}\times G^{p,q}_0)\cap\Delta_n$ is $(\tau_d\times T_\xi)$-comeager in $\{0\}\times G^{p,q}_0$ for each $q\in\omega$.

As $\mathrm{Gr}(\tilde{f})\cap\Delta_n^2=\emptyset$ and the $\varphi^{\vec{v}}$'s are $(\tau_d\times T_\xi,\tau_d\times T_\xi)$-homeomorphisms, $(\{p+1\}\times G_1^{p,q})\cap\Delta_n$ is $(\tau_d\times T_\xi)$-meager in $\{p+1\}\times G_1^{p,q}$, for each q.

As $(\omega\times\omega\times\omega^\omega)\setminus(\bigcup_{i\leq n}\ \Delta_i)$ is $(\tau_d\times T_\xi)$-meager in $\omega\times\omega\times\omega^\omega$ and $\mathbf{\Delta}^0_\xi(\tau_d\times\tau)$,

$$(\{p+1\}\times G_1^{p,q})\setminus(\bigcup_{i\leq n}\ \Delta_i)$$

is $(\tau_d\times T_\xi)$-meager in $\{p+1\}\times G_1^{p,q}$, for each q. Thus $(\{p+1\}\times G_1^{p,q})\cap(\bigcup_{i<n}\ \Delta_i)$ is $(\tau_d\times T_\xi)$-comeager in $\{p+1\}\times G_1^{p,q}$, for each q.

Claim *The set* $(\{p+1\}\times\omega\times\omega^\omega)\cap(\bigcup_{i<n}\ \Delta_i)$ *is* $(\tau_d\times T_\xi)$*-comeager in* $\{p+1\}\times\omega\times\omega^\omega$.

Indeed, we argue by contradiction. This gives $W\in(\tau_d\times T_\xi)\setminus\{\emptyset\}$ such that

$$(\{p+1\}\times W)\cap(\bigcup_{i<n}\ \Delta_i)$$

is $(\tau_d\times T_\xi)$-meager in $W':=\{p+1\}\times W$. Let $q\in\omega$ be such that $V_q\subseteq W$. Then $G_1^{p,q}\subseteq W$ and $\{p+1\}\times G_1^{p,q}\subseteq W'$. As $W'\cap(\bigcup_{i<n}\ \Delta_i)\in\mathbf{\Sigma}^0_\xi(\tau_d\times\tau)$ and $(\{p+1\}\times G_1^{p,q})\cap W'\cap(\bigcup_{i<n}\ \Delta_i)$ is $(\tau_d\times T_\xi)$-comeager in $\{p+1\}\times G_1^{p,q}$, $W'\cap(\bigcup_{i<n}\ \Delta_i)$ is not $(\tau_d\times T_\xi)$-meager in W', which is absurd. $\diamond$

Now we set $\Delta'_i:=(\{p+1\}\times\omega\times\omega^\omega)\cap\Delta_i$ if $i<n$. Note that $\Delta'_i\in\mathbf{\Delta}^0_\xi(\{p+1\}\times\omega\times\omega^\omega,\tau_d\times\tau)$, $\mathrm{Gr}(\tilde{f})\cap(\Delta'_i)^2=\emptyset$, and $\bigcup_{i<n}\ \Delta'_i$ is $(\tau_d\times T_\xi)$-comeager in $\{p+1\}\times\omega\times\omega^\omega$, which contradicts the induction assumption. $\square$

In order to get our main result, it is enough to apply the main lemma to each $n\geq 1$. This gives $f_n:\omega\times\omega^\omega\to\omega\times\omega^\omega$. It remains to define $f:\bigcup_{n\geq 1}\ (\{n\}\times\omega\times\omega^\omega)\to\bigcup_{n\geq 1}\ (\{n\}\times\omega\times\omega^\omega)$ by $f(n,x):=f_n(x)$ (we identify $(\omega\setminus\{0\})\times\omega\times\omega^\omega$ with ω^ω).

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