Continuous colorings on compact spaces

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Abstract. We study several natural classes of graphs on a zero-dimensional metrizable compact space having no continuous coloring. We compare these graphs with the quasi-order $\preceq^i_c$ associated with injective continuous homomorphisms. We prove the existence of an antichain basis for these classes. We determine the size of such an antichain basis. We provide a concrete antichain basis when there is a countable one. We also provide some related quasi-orders and equivalence relations which are analytic complete as sets.

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1 Introduction

The present work is the continuation of the study of continuous 2-colorings initiated in [L]. All our relations will be binary. A **coloring** of a relation R on a set X is a map c from X into a set κ with the property that $c(x)\neq c(y)$ if $(x,y)\in R$. We will call this a κ-coloring. In practice, κ will be a countable cardinal, equipped with the discrete topology. We say that R (or (X,R)) is a **graph** if R is symmetric and does not meet the **diagonal** $\Delta(X):=\{(x,x)\mid x\in X\}$ of X. We set $R^{-1}:=\{(x,y)\in X^2\mid (y,x)\in R\}$, and $s(R):=R\cup R^{-1}$ is the **symmetrization** of R. We compare our relations with the following quasi-order:

$$(X,R)\preceq^i (Y,S)\Leftrightarrow \exists h:X\rightarrow Y \text{ injective with } R\subseteq (h\times h)^{-1}(S).$$

If this holds, then we say that h is an injective **homomorphism** from (X,R) into (Y,S). In the present article, we work with the quasi-order $\preceq^i_c$ associated with injective continuous homomorphisms. All our topological spaces will be zero-dimensional, except where indicated, to ensure the existence of enough continuous functions between them. We write $(X,R)\prec^i_c (Y,S)$ when $(X,R)\preceq^i_c (Y,S)$ and $(Y,S)\not\preceq^i_c (X,R)$. The material in [L] shows that the structure of $\preceq^i_c$ is complex on a number of classes of graphs. Recall that a **basis** for a quasi-order $(\mathcal{Q},\leq)$ is a subclass $\mathcal{B}$ of $\mathcal{Q}$ such that any element of $\mathcal{Q}$ is $\leq$-above an element of $\mathcal{B}$. We are interested in basis as small as possible for the inclusion, which means that their elements are pairwise $\leq$-incomparable (if this last property is satisfied, then we say that we have a $\leq$-**antichain**). Note that an antichain basis is always made of minimal elements of the considered class. Conversely, let $\equiv^i_c\ :=\ \preceq^i_c\cap(\preceq^i_c)^{-1}$ be the equivalence relation associated with $\preceq^i_c$. Note that we can derive an antichain basis from a basis made of minimal elements by choosing an element in each $\equiv^i_c$-equivalence class, using the axiom of choice if necessary.

• Theorem 1.10 in [L] shows that there is no antichain basis for the class of graphs on a zero-dimensional metrizable compact space (**0DMC** for short; we will also use similar abbreviations like MC or 0DM) having no continuous 2-coloring. This theorem in fact gives the same result for graphs (X,R) with R countable. The situation is completely different for **closed graphs**, which leads to the first class we study. A compactness argument shows that any closed graph on a 0DMC space has a continuous $\aleph_0$-coloring.

Theorem 1.1 *Let $\kappa<\aleph_0$ be a cardinal.*
(a) There is a $\preceq^i_c$-basis made of minimal elements for the class of closed graphs on a 0DMC space having no continuous κ-coloring.
(b) Such a basis can be $\{(1,\emptyset)\}$ if $\kappa=0$, $\{(2,\{(0,1),(1,0)\})\}$ if $\kappa=1$, and has size $2^{\aleph_0}$ if $\kappa\geq 2$.

• The case of graphs induced by a function has been considered since the very beginning of the study of definable colorings in [K-S-T], and also in [Co-M], [L], [P] and [T-V] for instance. If

$$f:\text{Domain}(f)\subseteq X\rightarrow \text{Range}(f)\subseteq X$$

is a partial function, then the graph **induced** by f is $G_f:=s\big(\text{Graph}(f)\big)\setminus\Delta(X)$. The end of Section 9 in [L] shows that there is no antichain basis for the class of graphs induced by a partial homeomorphism on a 0DMC space with countable domain having no continuous 2-coloring. So we will focus on **graphs induced by a total homeomorphism**.

The following example was essentially introduced in [L-Z]. We consider a converging sequence with its limit in the Cantor space 2^ω, for instance $\mathbb{X}_1:=\{0^n1^\infty\mid n\in\omega\}\cup\{0^\infty\}$, which is a countable MC space. We define a homeomorphism f_1 of $\mathbb{X}_1$ by $f_1(0^\infty):=0^\infty$ and $f_1(0^{2n+\varepsilon}1^\infty):=0^{2n+1-\varepsilon}1^\infty$ if $\varepsilon\in 2$. We will see that $(\mathbb{X}_1,G_{f_1})$ has no continuous $\aleph_0$-coloring.

Theorem 1.2 *Let $\kappa \leq \aleph_0$ be a cardinal.*

(a) There is a $\preceq_c^i$-basis made of minimal elements for the class of graphs, induced by a homeomorphism of a 0DMC space, having no continuous κ-coloring.

(b) ([L], Theorems 1.17 (b) and 1.13 (d)) Such a basis can be $\{(1, G_{0\mapsto 0})\}$ if $\kappa = 0$, $\{(2, G_{\varepsilon\mapsto 1-\varepsilon})\}$ if $\kappa = 1$, has size $2^{\aleph_0}$ if $\kappa = 2$, and can be $\{(\mathbb{X}_1, G_{f_1})\}$ if $\kappa \geq 3$.

This result can be refined when $\kappa = 2$ if we consider the Cantor-Bendixson rank of the considered spaces. Recall from 6.C in [K1] that if X is a topological space, then the **Cantor-Bendixson derivative** of X is $X' := \{x \in X \mid x \text{ is a limit point of } X\}$. The **iterated Cantor-Bendixson derivatives** are defined by $X^0 := X$, $X^{\alpha+1} := (X^\alpha)'$, and, when λ is a limit ordinal, $X^\lambda := \bigcap_{\alpha<\lambda} X^\alpha$. Note that if f is a homeomorphism of X, then all the derivatives are f-**invariant**, i.e., $f[X^\alpha] = X^\alpha$ if α is an ordinal. If X is a countable MC space, then the **Cantor-Bendixson rank** of X is the least countable ordinal α_0 such that $X^{\alpha_0} = \emptyset$. If moreover X is nonempty, then this rank is a successor ordinal, by compactness. More generally, the **Cantor-Bendixson rank** of a Polish space is the least countable ordinal α_0 such that $X^\alpha = X^{\alpha_0}$ for each $\alpha \geq \alpha_0$, so that the Cantor space 2^ω has Cantor-Bendixson rank zero.

The following examples are of particular interest here.

- The odd cycles $(2q+3, C_{2q+3})$, for $q \in \omega$. In this case, the formula $f(i) := (i+1) \text{ mod } (2q+3)$ defines a homeomorphism of the discrete MC space $2q+3$, whose Cantor-Bendixson rank is one. We set $C_{2q+3} := G_f$, and the fact that $(2q+3, C_{2q+3})$ has no (continuous) 2-coloring is classical.

- $\mathbb{X}_1$ has Cantor-Bendixson rank two.

- We also consider subshifts, which are particular dynamical systems widely studied in symbolic dynamics. We refer to the book [Ku] for basic notions and definitions.

Definition 1.3 *Let A be a finite set of cardinality at least two.*
(a) The **shift map** $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ *is defined by the formula* $\sigma(\alpha)(k) := \alpha(k+1)$.
(b) A **two-sided subshift** *is a closed σ-invariant subset Σ of $A^{\mathbb{Z}}$.*

The restriction of the homeomorphism σ to a two-sided subshift Σ induces a graph $(\Sigma, G_{\sigma_{|\Sigma}})$ that we will denote by (Σ, G_σ). If f is a bijection of the set X and $x \in X$, then the f-**orbit** of x is

$$\text{Orb}_f(x) := \{f^k(x) \mid k \in \mathbb{Z}\}$$

(also denoted by $\text{Orb}(x)$ when the context is clear). If $x \in A^{-\omega}$ and $y \in A^\omega$, then $z := x \cdot y \in A^{\mathbb{Z}}$ is defined by $z(i) := y(i)$ and $z(-i-1) := x(-i)$ when $i \in \omega$. If $w \in A^{<\omega} \setminus \{\emptyset\}$, then $w^{-\infty} := \cdots ww$ is in $A^{-\omega}$, $w^\infty := ww\cdots$ is in A^ω and $w^{\mathbb{Z}} := w^{-\infty} \cdot w^\infty$. Note that $(2q+3, C_{2q+3})$ can be seen as a two-sided subshift by putting ${}_{2q+3}\Sigma := \text{Orb}_\sigma\Big(\big(0 \cdots (2q+2)\big)^{\mathbb{Z}}\Big)$. Recall that if X, Y are metrizable compact spaces and f, g are homeomorphisms of X, Y respectively, then $(X, f), (Y, g)$ (or f, g) are **conjugate** (resp., **flip-conjugate**) if there is a homeomorphism $\varphi : X \to Y$ such that $\varphi \circ f = g \circ \varphi$ (resp., $\varphi \circ f = g \circ \varphi$ or $\varphi \circ f = g^{-1} \circ \varphi$). We will see that $(\mathbb{X}_1, G_{f_1})$ is not conjugate to the shift of a two-sided subshift. We set

$$\mathcal{P} = \big\{\mathbf{p} := (l, \lambda_0, \cdots, \lambda_l, m, \varepsilon_0, \cdots, \varepsilon_{l-1}) \in \omega^{l+3} \times 2^l \mid \forall i \leq l \ \lambda_i > 0 \text{ is even and } m < \lambda_0 \text{ is odd}\big\}.$$

We associate to each $\mathbf{p} \in \mathcal{P}$ a two-sided subshift as follows. We fix disjoint injective families of symbols $(a_j^i)_{i,j\in\omega}$ and $(b_i)_{i\in\omega}$, and set $A_{\mathbf{p}} := \{a_j^i \mid i \leq l \wedge j < \lambda_i\} \cup \{b_i \mid i < m\}$, which is finite of cardinality at least two (in fact at least three).

We then set, for $i\leq l$, $w_i:=a^i_0\cdots a^i_{\lambda_i-1}\in A_{\mathbf{p}}^{<\omega}\setminus\{\emptyset\}$, and define

$$\Sigma_{\mathbf{p}}:=\bigcup_{i\leq l}\ \text{Orb}_\sigma(w_i^{\mathbb{Z}})\cup\bigcup_{i<l}\ \text{Orb}_\sigma(w_{i+\varepsilon_i}^{-\infty}\cdot w_{i+1-\varepsilon_i}^{\infty})\cup\text{Orb}_\sigma\big(w_l^{-\infty}\cdot b_0\cdots b_{m-1}(w_0^{\infty})\big),$$

a countable MC space with Cantor-Bendixson rank two. We will see that $(\Sigma_{\mathbf{p}},G_\sigma)$ has no continuous 2-coloring. We set, for each $\mathbf{p}\in\mathcal{P}$, $\mathcal{F}_{\mathbf{p}}:=\{\mathbf{p}'\in\mathcal{P}\mid(\Sigma_{\mathbf{p}'},G_\sigma)\equiv^i_c(\Sigma_{\mathbf{p}},G_\sigma)\}$.

For $\kappa=2$, we prove the following.

Theorem 1.4 *(a) The family* $\{({}_{2q+3}\Sigma,G_\sigma)\mid q\in\omega\}\cup\{(\mathbb{X}_1,G_{f_1})\}\cup\{(\Sigma_{\mathbf{p}},G_\sigma)\mid\mathbf{p}\in\mathcal{P}\}$ *is a concrete* $\preceq^i_c$*-basis of size* $\aleph_0$ *for the class of graphs, induced by a homeomorphism of a countable MC space with Cantor-Bendixson rank at most two, having no continuous 2-coloring. Moreover, for each* $\mathbf{p}\in\mathcal{P}$*, the set* $\mathcal{F}_{\mathbf{p}}$ *is finite, and choosing* $\text{min}_{lex}\,\mathcal{F}_{\mathbf{p}}$ *provides a* $\preceq^i_c$*-antichain basis of size* $\aleph_0$.
(b) (see [L], Theorem 1.17 (b)) If $\xi\geq 3$ *is a countable ordinal, then there is a* $\preceq^i_c$*-basis made of minimal elements for the class of graphs, induced by a homeomorphism of a countable MC space with Cantor-Bendixson rank at most* ξ*, having no continuous 2-coloring, and any such basis must have size* $2^{\aleph_0}$.
(c) ([L], Theorem 1.15) If ξ *is a countable ordinal, then any* $\preceq^i_c$*-basis made of minimal elements for the class of graphs, induced by a homeomorphism of a 0DMC space with Cantor-Bendixson rank at most* ξ*, having no continuous 2-coloring, must have size* $2^{\aleph_0}$.

Note that the proof of Theorem 1.1 (b) will show that it has no such refinement when $\kappa\geq 2$.

• The class of **graphs induced by the shift of a two-sided subshift** is a natural subclass of the previous one.

Theorem 1.5 *Let* $\kappa\leq\aleph_0$ *be a cardinal.*
(a) There is a $\preceq^i_c$*-basis made of minimal elements for the class of graphs, induced by the shift of a two-sided subshift, having no continuous* κ*-coloring.*
(b) Such a basis can be $\{(Orb_\sigma(0^{\mathbb{Z}}),G_\sigma)\}$ *if* $\kappa=0$, $\big\{\big(Orb_\sigma\big((01)^{\mathbb{Z}}\big),G_\sigma\big)\big\}$ *if* $\kappa=1$*, and has size* $2^{\aleph_0}$ *if* $\kappa\geq 2$.

Here again, this result can be refined when $\kappa=2$ if we consider the Cantor-Bendixson rank of the considered spaces. As $(\mathbb{X}_1,G_{f_1})$ is not conjugate to the shift of a two-sided subshift, we have to introduce some other examples.

We set ${}_0\Sigma:=\text{Orb}_\sigma(0^{\mathbb{Z}})\cup\text{Orb}_\sigma(0^{-\infty}\cdot 10^{\infty})$, ${}_1\Sigma:=\text{Orb}_\sigma(0^{\mathbb{Z}})\cup\text{Orb}_\sigma(1^{\mathbb{Z}})\cup\text{Orb}_\sigma(0^{-\infty}\cdot 1^{\infty})$ and, for $q\in\omega$, ${}_{2q+2}\Sigma:=\text{Orb}_\sigma(0^{\mathbb{Z}})\cup\text{Orb}_\sigma\big((1,\cdots,2q+2)^{\mathbb{Z}}\big)\cup\text{Orb}_\sigma(0^{-\infty}\cdot(1,\cdots,2q+2)^{\infty})$.

Theorem 1.6 *(a) The family* $\{({}_n\Sigma,G_\sigma)\mid n\in\omega\}\cup\{(\Sigma_{\mathbf{p}},G_\sigma)\mid\mathbf{p}\in\mathcal{P}\}$ *is a concrete* $\preceq^i_c$*-basis of size* $\aleph_0$ *for the class of graphs, induced by the shift of a countable two-sided subshift with Cantor-Bendixson rank at most two, having no continuous 2-coloring. Moreover, choosing* $\text{min}_{lex}\,\mathcal{F}_{\mathbf{p}}$ *for each* $\mathbf{p}\in\mathcal{P}$ *provides a* $\preceq^i_c$*-antichain basis of size* $\aleph_0$.
(b) ([L], Theorem 1.17 (b) and Corollary 10.12) If $\xi\geq 3$ *is a countable ordinal, then there is a* $\preceq^i_c$*-basis made of minimal elements for the class of graphs, induced by the shift of a countable two-sided subshift with Cantor-Bendixson rank at most* ξ*, having no continuous 2-coloring, and any such basis must have size* $2^{\aleph_0}$.
(c) ([L], Corollary 10.12) If ξ *is a countable ordinal, then any* $\preceq^i_c$*-basis made of minimal elements for the class of graphs, induced by the shift of a two-sided subshift with Cantor-Bendixson rank at most* ξ*, having no continuous 2-coloring, must have size* $2^{\aleph_0}$.

A **dynamical system** (X, f) is given by a homeomorphism f of a metrizable compact space X. If X is homeomorphic to 2^ω, then we say that (X, f) is a **Cantor dynamical system**. A dynamical system (or f) is **minimal** if $\text{Orb}_f(x)$ is dense in X for each $x \in X$. The set of homeomorphisms of 2^ω is denoted by $\mathcal{H}(2^\omega)$. It is a Polish group when equipped with the topology whose basic neighbourhoods of the identity are of the form $\{h \in \mathcal{H}(2^\omega) \mid \forall i < n \;\; h[O_i] = O_i\}$, where $(O_i)_{i<n}$ ranges over all finite families of clopen subsets of 2^ω. By Lemma 4.1 in [Me], the space $\mathbb{M}$ of minimal homeomorphisms of 2^ω is a Polish space. The equivalence relation of flip-conjugacy on $\mathbb{M}$ is denoted by FCO. The standard way to compare analytic equivalence relations on standard Borel spaces is the Borel reducibility quasi-order $\leq_B$ (see, for instance, [G]). Recall that if X, Y are standard Borel spaces and E, F are analytic equivalence relations on X, Y respectively, then

$$(X, E) \leq_B (Y, F) \Leftrightarrow \exists \varphi\colon X \to Y \text{ Borel with } E = (\varphi \times \varphi)^{-1}(F).$$

Theorem 13.2 in [L] essentially shows that FCO is Borel reducible to the (analytic) restriction of $\equiv^i_c$ to the set of irreflexive relations G on a fixed countable dense subset of 2^ω such that $(2^\omega, G)$ has no continuous 2-coloring. A very recent result, in [De-GR-Ka-Kun-Kw], asserts that FCO is analytic complete as a set. As a consequence, this restriction is also analytic complete. We make such statement more systematic, and partly in relation with the classes $\mathcal{C}_\kappa$ (resp., $\mathcal{H}_\kappa$) of closed graphs (resp., of graphs induced by a homeomorphism) introduced in Section 3.

Let $\kappa < \aleph_0$, $\mathfrak{C}_\kappa$ be the set of closed graphs on 2^ω having no continuous κ-coloring, and

$$E^{\mathfrak{C}}_\kappa := \{(G, H) \in \mathfrak{C}^2_\kappa \mid (2^\omega, G) \equiv^i_c (2^\omega, H)\}.$$

Now let $\kappa \leq 3$, $\mathfrak{H}_\kappa$ be the set of homeomorphisms of 2^ω whose induced graph has no continuous κ-coloring, and $E^{\mathfrak{H}}_\kappa := \{(f, g) \in \mathfrak{H}^2_\kappa \mid (2^\omega, G_f) \equiv^i_c (2^\omega, G_g)\}$. Now let $\kappa \leq \aleph_0$, D be a countable dense subset of 2^ω, $\mathfrak{D}_\kappa$ be the set of graphs G on D such that $(2^\omega, G)$ has no continuous κ-coloring, and $E^{\mathfrak{D}}_\kappa := \{(G, H) \in \mathfrak{D}^2_\kappa \mid (2^\omega, G) \equiv^i_c (2^\omega, H)\}$.

Theorem 1.7 *The spaces $\mathfrak{C}_\kappa$, $\mathfrak{H}_\kappa$ and $\mathfrak{D}_\kappa$ are Polish, and FCO is Borel reducible to the analytic equivalence relations $E^{\mathfrak{C}}_\kappa$, $E^{\mathfrak{H}}_\kappa$ and $E^{\mathfrak{D}}_\kappa$. In particular, these relations are analytic complete as sets.*

2 Fixed points

The set $F_f := \{x \in \text{Domain}(f) \mid f(x) = x\}$ of fixed points of f is very much related to the continuous colorings of G_f. The next two results are essentially Proposition 7.2 and Corollary 7.3 in [L]. We recall them for the convenience of the reader.

Proposition 2.1 *Let X be a first countable space, and $f\colon X \to X$ be a partial continuous function. If F_f is not an open subset of Domain(f), then there is no continuous $\aleph_0$-coloring of G_f.*

Proof. We argue by contradiction, which gives $c : X \to \aleph_0$. Let $(C_i)_{i\in\aleph_0}$ be the partition of X into clopen sets given by $C_i := c^{-1}(\{i\})$. As F_f is not open in Domain(f), we can find $x \in F_f$ and $(x_n)_{n\in\omega} \in (\text{Domain}(f)\backslash F_f)^\omega$ converging to x. Note that $f(x_n)$ is different from x_n, and $\big(f(x_n)\big)_{n\in\omega}$ converges to $f(x) = x$. Let i with $x \in C_i$. Then we may assume that $x_n, f(x_n) \in C_i$. This implies that $\big(x_n, f(x_n)\big) \in G_f \cap C^2_i$, which is the desired contradiction. $\square$

Corollary 2.2 *Let X be a 0DM space, and $f : X \to X$ be a partial continuous function with closed domain.*

(a) Exactly one of the following holds:

(1) F_f is an open subset of Domain(f),

(2) there is no continuous $\aleph_0$-coloring of G_f.

(b) If Domain(f) is clopen in X, F_f is an open subset of Domain(f), f is injective and $1 \leq \kappa \leq \aleph_0$, then (X, G_f) has a continuous κ-coloring if and only if $(X \setminus F_f, G_f \cap (X \setminus F_f)^2)$ has a continuous κ-coloring.

Proof. (a) Assume that (1) holds. Note that $s\big(\text{Graph}(f_{|\text{Domain}(f)\setminus F_f})\big)$ and $\Delta(X)$ are disjoint and closed in X^2. 22.16 in [K1] gives $C \subseteq X^2$ clopen with $\Delta(X) \subseteq C \subseteq X^2 \setminus s\big(\text{Graph}(f_{|\text{Domain}(f)\setminus F_f})\big)$. The relation C gives a continuous $\aleph_0$-coloring of G_f since X is zero-dimensional and second countable. So (2) does not hold.

If F_f is not an open subset of Domain(f), then we apply Proposition 2.1.

(b) Let $c : X \setminus F_f \to \kappa$ be a continuous κ-coloring of $(X \setminus F_f, G_f \cap (X \setminus F_f)^2)$. We extend c to X by setting $c(x) := 0$ if $x \in F_f$. As F_f is a clopen subset of the clopen set Domain(f), this extension is continuous. If $\big(x, f(x)\big) \in G_f$, then $x \notin F_f$. As f is injective and F_f is f-invariant, $f[\text{Domain}(f) \setminus F_f] \cap F_f = \emptyset$, so that $f(x) \notin F_f$. Thus $c(x) \neq c\big(f(x)\big)$, showing that c is a κ-coloring of (X, G_f). Conversely, any continuous κ-coloring of (X, G_f) defines a continuous κ-coloring of $(X \setminus F_f, G_f \cap (X \setminus F_f)^2)$ by restriction. □

In particular, $({}_n\Sigma, G_\sigma)$ has no continuous $\aleph_0$-coloring if n is 1 or even.

Corollary 2.3 *Let X be a 0DM space, and f be a homeomorphism of X. Then exactly one of the following holds:*

(1) F_f is an open subset of X,

(2) $(\mathbb{X}_1, G_{f_1}) \preceq_c^i (X, G_f)$.

Proof. If (1) holds, then Corollary 2.2 provides a continuous $\aleph_0$-coloring of G_f. If (2) also holds, then G_{f_1} also has such a coloring c. As $F_{f_1} = \{0^\infty\}$ is not an open subset of $\mathbb{X}_1$, this contradicts Proposition 2.1. So assume that (1) does not hold. Then we can find an injective sequence $(x_n)_{n\in\omega}$ of points of $X \setminus F_f$ converging to a point x of F_f. Moreover, we may assume that $\{x_m, f(x_m)\} \cap \{x_n, f(x_n)\} = \emptyset$ if $m \neq n$. We then set $\varphi(0^\infty) := x$, $\varphi(0^{2n}1^\infty) := x_n$ and $\varphi(0^{2n+1}1^\infty) := f(x_n)$, so that φ is a witness for the fact that $(\mathbb{X}_1, G_{f_1}) \preceq_c^i (X, G_f)$. □

Remark. In the introduction, we mentioned the fact that $(\mathbb{X}_1, G_{f_1})$ is not conjugate to the shift of a two-sided subshift. Here's the argument. We argue by contradiction. By Proposition 3.68 in [Ku],

$$\exists n \in \omega \ \ \forall x \neq y \in \mathbb{X}_1 \ \ \exists k \in \mathbb{Z} \ \ f_1^k(x)|n \neq f_1^k(y)|n.$$

Fix $n \in \omega$, and choose $(x, y) := (0^{2n}1^\infty, 0^{2n+1}1^\infty)$. Then $f_1^k(x)|n = f_1^k(y)|n = 0^n$ for each $k \in \mathbb{Z}$, which is the desired contradiction.

We will use the following fact, which is part of Proposition 7.6 in [L].

Proposition 2.4 $(\mathbb{X}_1, G_{f_1})$ *is $\preceq_c^i$-minimal in the class of graphs on a 0DMC space having no continuous 2-coloring.*

3 Basis made of minimal elements

The part (a) of Theorems 1.1, 1.2 and 1.5 is based on compactness. The first key fact is that it is possible to keep a big chromatic number when we take infinite decreasing sequences of graphs or spaces.

Lemma 3.1 *Assume that $2 \leq \kappa < \aleph_0$ is a cardinal, X is a 0DMC space, $(K_p)_{p\in\omega}$ is a decreasing sequence of closed subsets of X, $(G_p)_{p\in\omega}$ is a decreasing sequence of closed graphs on X such that, for each $p \in \omega$, $G_p \subseteq K_p^2$ and (K_p, G_p) has no continuous κ-coloring, $K := \bigcap_{p\in\omega} K_p$ and $G := \bigcap_{p\in\omega} G_p$. Then (K, G) has no continuous κ-coloring.*

Proof. We argue by contradiction, which gives a continuous coloring $c : K \to \kappa$ of (K, G). By 7.8 and 2.8 in [K1], there is a continuous extension $\overline{c} : X \to \kappa$ of c. We set, for $p \in \omega$ and $\varepsilon \in \kappa$, $K^p_\varepsilon := K_p \cap \overline{c}^{-1}(\{\varepsilon\})$, so that $(K^p_\varepsilon)_{\varepsilon\in\kappa}$ is a partition of K_p into clopen subsets. By assumption, $\overline{c}_{|K_p}$ is not a κ-coloring of (K_p, G_p), which gives $\varepsilon_p \in \kappa$ and $(x_p, y_p) \in G_p \cap (K^p_{\varepsilon_p})^2$. We may assume that $\varepsilon := \varepsilon_p$ does not depend on p. By compactness of X^2, we may assume that $\big((x_p, y_p)\big)_{p\in\omega}$ converges to some $(x, y) \in K^2$. It remains to note that $(x, y) \in G \cap \big(K \cap c^{-1}(\varepsilon)\big)^2$, which is the desired contradiction. □

We are now ready to prove Theorem 1.1 (a). Let $\mathcal{C}_\kappa$ be the class of closed graphs on a 0DMC space having no continuous κ-coloring. When we write $\sup_{p\in\omega} \lambda_p$, we always assume that $(\lambda_p)_{p\in\omega}$ is a strictly increasing sequence of ordinals.

Proof of Theorem 1.1 (a). Note that $(X, G) \in \mathcal{C}_0$ exactly when $X \neq \emptyset$, so that the singleton in (b) is convenient. Note then that $(X, G) \in \mathcal{C}_1$ exactly when $G \neq \emptyset$, so that the singleton in (b) is convenient.

So we may assume that $\kappa \geq 2$. We argue by contradiction, which gives $(X, G) \in \mathcal{C}_\kappa$ such that, for each $(X', G') \in \mathcal{C}_\kappa$ with $(X', G') \preceq^i_c (X, G)$, there is $(X'', G'') \in \mathcal{C}_\kappa$ with the property that $(X'', G'') \prec^i_c (X', G')$.

Claim. *For each $(X', G') \in \mathcal{C}_\kappa$ with $(X', G') \preceq^i_c (X, G)$, there is $(X'', G'') \in \mathcal{C}_\kappa$ such that $X'' \subseteq X'$ and $G'' \subsetneqq G'$.*

Indeed, let $\tilde{X}$ be the projection proj$[G']$ of G', and $\tilde{G} := G'$. Note that $\tilde{X} = \text{proj}[\tilde{G}]$, $(\tilde{X}, \tilde{G}) \in \mathcal{C}_\kappa$ and $(\tilde{X}, \tilde{G}) \preceq^i_c (X', G') \preceq^i_c (X, G)$. This gives $(\tilde{\tilde{X}}, \tilde{\tilde{G}}) \in \mathcal{C}_\kappa$ with the property that $(\tilde{\tilde{X}}, \tilde{\tilde{G}}) \prec^i_c (\tilde{X}, \tilde{G})$. Let h be a witness for the fact that $(\tilde{\tilde{X}}, \tilde{\tilde{G}}) \preceq^i_c (\tilde{X}, \tilde{G})$. We put $X'' := h[\tilde{\tilde{X}}]$ and $G'' := (h\times h)[\tilde{\tilde{G}}]$. Note that $(\tilde{\tilde{X}}, \tilde{\tilde{G}}) \preceq^i_c (X'', G'') \preceq^i_c (\tilde{\tilde{X}}, \tilde{\tilde{G}})$ with witnesses h, h^{-1} respectively. In particular, (X'', G'') is in $\mathcal{C}_\kappa$, $X'' \subseteq \tilde{X} = \text{proj}[G'] \subseteq X'$, and $G'' \subseteq \tilde{G} = G'$. If $G'' = G'$, then $X'' = \tilde{X}$ and

$$(\tilde{X}, \tilde{G}) = (X'', G'') \preceq^i_c (\tilde{\tilde{X}}, \tilde{\tilde{G}}) \prec^i_c (\tilde{X}, \tilde{G}),$$

which cannot be. ⋄

We inductively construct a $\subseteq$-decreasing sequence $(X_\xi)_{\xi<\aleph_1}$ and a strictly $\subseteq$-decreasing sequence $(G_\xi)_{\xi<\aleph_1}$ such that $(X_0, G_0) = (X, G)$ and $(X_\xi, G_\xi) \in \mathcal{C}_\kappa$, which will contradict the fact that G is a 0DMC space. If (X_ξ, G_ξ) is constructed, then the claim gives $(X_{\xi+1}, G_{\xi+1}) \in \mathcal{C}_\kappa$ such that $X_{\xi+1} \subseteq X_\xi$ and $G_{\xi+1} \subsetneqq G_\xi$. If $\lambda = \sup_{p\in\omega} \lambda_p$ is a limit ordinal, then Lemma 3.1 applied to X, $(X_{\lambda_p})_{p\in\omega}$ and $(G_{\lambda_p})_{p\in\omega}$ implies, setting $X_\lambda := \bigcap_{p\in\omega} X_{\lambda_p}$ and $G_\lambda := \bigcap_{p\in\omega} G_{\lambda_p}$, that $(X_\lambda, G_\lambda) \in \mathcal{C}_\kappa$. As $G_\lambda \subsetneqq G_{\lambda_p}$ for each $p \in \omega$, we are done. □

We now study the graphs induced by a homeomorphism. Things become more complex since

- fixed poins can exist; when they cannot be avoided, the induced graph is not closed,
- the intersection of such a graph G_f with a closed square C^2 is not necessarily of the form G_g; it is of this form if C is f-invariant.

The next lemma is a first step towards invariance. It is about the preservation of the size of orbits with at least three points under $\preceq^i_c$.

Lemma 3.2 *Let X be a topological space, f be a homeomorphism of X, Y, g having the corresponding properties and satisfying $(X,G_f)\preceq^i_c(Y,G_g)$ with h as a witness, and $x\in X$ with $|Orb_f(x)|\geq 3$. Then $h[Orb_f(x)]=Orb_g\big(h(x)\big)$, and either $h\circ f=g\circ h$ on $Orb_f(x)$, or $h\circ f=g^{-1}\circ h$ on $Orb_f(x)$.*

Proof. Let $O:=\text{Orb}_f(x)$. As $f_{|O}$ is fixed point free, $\big(x,f(x)\big)\in G_f$. Thus $\Big(h(x),h\big(f(x)\big)\Big)\in G_g$, showing that $h\big(f(x)\big)=g^{\pm 1}\big(h(x)\big)$. In particular, $h[O]\subseteq\text{Orb}_g\big(h(x)\big)$. We set

$$P:=\{z\in O\mid h\big(f(z)\big)=g\big(h(z)\big)\}$$

and $M:=\{z\in O\mid h\big(f(z)\big)=g^{-1}\big(h(z)\big)\}$. As $|O|\geq 3$, $|\text{Orb}_g\big(h(x)\big)|\geq 3$ by injectivity of h, and P and M are disjoint closed subsets of $O=P\cup M$. If $z\in P$, then $f(z)\in P$ since otherwise $f(z)\in M$, $h\big(f^2(z)\big)=g^{-1}\Big(h\big(f(z)\big)\Big)=h(z)$, $f^2(z)=z$ by injectivity of h, which contradicts the fact that $|O|\geq 3$. Thus $O=P$ or $O=M$. In particular, either $h\big(f^i(x)\big)=g^i\big(h(x)\big)$ for each $i\in\mathbb{Z}$, or $h\big(f^i(x)\big)=g^{-i}\big(h(x)\big)$ for each $i\in\mathbb{Z}$. In both cases, we get $h[O]=\text{Orb}_g\big(h(x)\big)$. $\square$

The next lemma is about the preservation of the size of orbits of size two under $\preceq^i_c$ (an orbit of size two could be sent into a bigger orbit since we consider symmetrizations). Let $\mathcal{H}_\kappa$ be the class of graphs, induced by a homeomorphism of a 0DMC space, having no continuous κ-coloring. We set, for $(X,G_f)\in\mathcal{H}_\kappa$, $F^{X,f}_2:=\{x\in X\mid f^2(x)=x\}$.

Lemma 3.3 *Let $(X,G_f),(Y,G_g)\in\mathcal{H}_\kappa$ such that $(X,G_f)\preceq^i_c(Y,G_g)$ with h as a witness and $F^{X,f}_2$ is nowhere dense in X, and $x\in X$ with $|Orb_f(x)|=2$. Then $h[Orb_f(x)]$ is a g-orbit of size two.*

Proof. As $F^{X,f}_2$ is nowhere dense in X, we can find a sequence $(x_n)_{n\in\omega}$ of points of $X\setminus F^{X,f}_2$ converging to x. Note that $|\text{Orb}_f(x_n)|\geq 3$ for each $n\in\omega$. We set $y_n:=h(x_n)$, so that $h[\text{Orb}_f(x_n)]$ is $\text{Orb}_g(y_n)$ by Lemma 3.2. We set $z:=f(x)$, so that $(x,z)\in G_f$ since $|\text{Orb}_f(x)|=2$. Thus $\big(h(x),h(z)\big)\in G_g$, which gives $\theta\in\{-1,1\}$ with $h(z)=g^\theta\big(h(x)\big)$. Similarly, there is, for each $n\in\omega$, $\theta_n\in\{-1,1\}$ with $h\big(f(x_n)\big)=g^{\theta_n}(y_n)$, and we may assume that $\theta_n=\theta_0$ for each $n\in\omega$. Thus $h(z)=g^{\theta_0}\big(h(x)\big)$. So we are done if $\theta\neq\theta_0$ since $h(z)\in\text{Orb}_g\big(h(x)\big)\setminus\{h(x)\}$. So we may assume that $\theta=\theta_0$. As $|\text{Orb}_f(x_n)|\geq 3$, $f^{-1}(x_n)\neq f(x_n)$ and $h\big(f^{-1}(x_n)\big)\neq h\big(f(x_n)\big)$. This implies that $h\big(f^{-1}(x_n)\big)=g^{-\theta_0}(y_n)$. Thus $h\big(f^{-1}(x)\big)=g^{-\theta_0}\big(h(x)\big)$. As $|\text{Orb}_f(x)|=2$, $f^{-1}(x)=f(x)=z$, so that $g^{\theta_0}\big(h(x)\big)=g^\theta\big(h(x)\big)=h(z)=g^{-\theta_0}\big(h(x)\big)$. $\square$

In the next proof, we also have to deal with orbits of size one. We are now ready to prove Theorem 1.2 (a).

Proof of Theorem 1.2 (a). As in the proof of Theorem 1.1 (a), we may assume that $\kappa\geq 2$. We argue by contradiction, which gives $(X,G_f)\in\mathcal{H}_\kappa$ such that, for each $(X',G_{f'})\in\mathcal{H}_\kappa$ with the property that $(X',G_{f'})\preceq^i_c(X,G_f)$, there is $(X'',G_{f''})\in\mathcal{H}_\kappa$ with $(X'',G_{f''})\prec^i_c(X',G_{f'})$.

If F_f is not an open subset of X, then $(\mathbb{X}_1, G_{f_1}) \preceq^i_c (X, G_f)$ by Corollary 2.3. By Corollaries 2.3 and 2.2, $(\mathbb{X}_1, G_{f_1}) \in \mathcal{H}_\kappa$. Our assumption gives $(X'', G_{f''}) \in \mathcal{H}_\kappa$ with the property that $(X'', G_{f''})$ is strictly $\preceq^i_c$-below $(\mathbb{X}_1, G_{f_1})$, which contradicts Proposition 2.4. This shows that F_f is an open subset of X. Corollary 2.2 then shows that we may assume that f is fixed point free. In particular, there is a $\aleph_0$-coloring of (X, G_f), by Corollary 2.2, so that $\kappa < \aleph_0$.

Claim. *For each $(X', G_{f'}) \in \mathcal{H}_\kappa$ with $(X', G_{f'}) \preceq^i_c (X, G_f)$, there is $(X'', G_{f''}) \in \mathcal{H}_\kappa$ such that f'' is fixed point free, $(X'', G_{f''}) \preceq^i_c (X', G_{f'})$ and $F_2^{X'', f''}$ is nowhere dense in X''.*

Indeed, we argue by contradiction, which gives $(X', G_{f'}) \in \mathcal{H}_\kappa$. As $(X', G_{f'}) \preceq^i_c (X, G_f)$, there is also a $\aleph_0$-coloring of $(X', G_{f'})$. Corollary 2.2 then shows that we may assume that f' is fixed point free. We inductively construct a strictly $\subseteq$-decreasing sequence $(X_\xi)_{\xi<\aleph_1}$ such that $X_0 = X'$, X_ξ is f'-invariant and $(X_\xi, G_{f'} \cap X_\xi^2) \in \mathcal{H}_\kappa$, which will contradict the fact that X' is a 0DMC space. Assume that X_ξ is constructed. Note that $F_2^{X_\xi, f'_{|X_\xi}}$ is closed and not nowhere dense in X_ξ. This gives a nonempty clopen subset C of X_ξ with the property that $C \subseteq F_2^{X_\xi, f'_{|X_\xi}}$. Note that the set $U := C \cup f'[C]$ is a nonempty clopen f'-invariant subset of X_ξ contained in $F_2^{X_\xi, f'_{|X_\xi}}$. In particular, U is a ODM separable space and $f'_{|U}$ is a fixed point free continuous involution. Proposition 7.5 in [L] provides a continuous 2-coloring of $(U, G_{f'_{|U}})$. All this implies that $X_{\xi+1} := X_\xi \setminus U \subsetneqq X_\xi$, $X_{\xi+1}$ is f'-invariant and $(X_{\xi+1}, G_{f'} \cap X_{\xi+1}^2) \in \mathcal{H}_\kappa$. If $\lambda = \sup_{p\in\omega} \lambda_p$ is a limit ordinal, then Lemma 3.1 applied to X', $G_{\lambda_p} := G_{f'} \cap X_{\lambda_p}^2$ and $(X_{\lambda_p})_{p\in\omega}$ implies, setting $X_\lambda := \bigcap_{p\in\omega} X_{\lambda_p}$, that X_λ is f'-invariant and $(X_\lambda, G_{f'} \cap X_\lambda^2) \in \mathcal{H}_\kappa$. As $X_\lambda \subsetneqq X_{\lambda_p}$ for each $p \in \omega$, we are done. $\diamond$

We inductively construct a strictly $\subseteq$-decreasing sequence $(X_\xi)_{\xi<\aleph_1}$ such that $X_0 = X$, X_ξ is f-invariant and $(X_\xi, G_f \cap X_\xi^2) \in \mathcal{H}_\kappa$, which will contradict the fact that X is a 0DMC space. Assume that X_ξ is constructed. Our assumption gives $(X', G_{f'}) \in \mathcal{H}_\kappa$ with the property that $(X', G_{f'})$ is stricly $\preceq^i_c$-below $(X_\xi, G_f \cap X_\xi^2)$. The claim gives $(X'', G_{f''}) \in \mathcal{H}_\kappa$ such that f'' is fixed point free, $(X'', G_{f''}) \preceq^i_c (X', G_{f'})$ and $F_2^{X'', f''}$ is nowhere dense in X''. In particular, $(X'', G_{f''})$ is strictly $\preceq^i_c$-below $(X_\xi, G_f \cap X_\xi^2)$. Let h be a witness for the fact that $(X'', G_{f''}) \preceq^i_c (X_\xi, G_f \cap X_\xi^2)$. The fact that f'' is fixed point free and Lemmas 3.2, 3.3 imply that $X_{\xi+1} := h[X''] \subseteq X_\xi$ is f-invariant. Moreover, $(h \times h)[G_{f''}] \subseteq G_f \cap X_{\xi+1}^2$ and $(X'', G_{f''}) \preceq^i_c (X_{\xi+1}, (h \times h)[G_{f''}])$ with h as a witness, so that $(X_{\xi+1}, G_f \cap X_{\xi+1}^2) \in \mathcal{H}_\kappa$. If $(y_0, y_1) \in G_f \cap X_{\xi+1}^2$, then let $x_0, x_1 \in X''$ with $y_\varepsilon = h(x_\varepsilon)$. Note that $\big(x_0, f''^\theta(x_0)\big) \in G_{f''}$ for each $\theta \in \{-1, 1\}$ since f'' is fixed point free, which implies that $\Big(h(x_0), h\big(f''^\theta(x_0)\big)\Big) \in G_f$. If $|\mathrm{Orb}_{f''}(x_0)| \geq 3$, then $h\big(f''(x_0)\big) \neq h\big(f''^{-1}(x_0)\big)$ is of the form $f^\eta\big(h(x_0)\big)$ for some $\eta \in \{-1, 1\}$. This gives $\eta_0, \theta_0 \in \{-1, 1\}$ with $y_1 = f^{\eta_0}\big(h(x_0)\big) = h\big(f''^{\theta_0}(x_0)\big)$. Thus $x_1 = f''^{\theta_0}(x_0)$, $(x_0, x_1) \in G_{f''}$, and $(y_0, y_1) \in (h \times h)[G_{f''}]$. If $|\mathrm{Orb}_{f''}(x_0)| < 3$, then $|\mathrm{Orb}_{f''}(x_0)| = 2$ since f'' is fixed point free, and $h[\mathrm{Orb}_{f''}(x_0)]$ is an f-orbit of size two by Lemma 3.3. The conclusion is the same, with $\eta_0 = \theta_0 = 1$. So we proved that $(h \times h)[G_{f''}] = G_f \cap X_{\xi+1}^2$ in any case. Now note that $(X_{\xi+1}, G_f \cap X_{\xi+1}^2) = (X_{\xi+1}, (h \times h)[G_{f''}]) \preceq^i_c (X'', G_{f''})$ with h^{-1} as a witness, so that $(X_{\xi+1}, G_f \cap X_{\xi+1}^2) \prec^i_c (X_\xi, G_f \cap X_\xi^2)$, proving that $X_{\xi+1} \subsetneqq X_\xi$. If $\lambda = \sup_{p\in\omega} \lambda_p$ is a limit ordinal, then Lemma 3.1 applied to X, $G_{\lambda_p} := G_f \cap X_{\lambda_p}^2$ and $(X_{\lambda_p})_{p\in\omega}$ implies, setting $X_\lambda := \bigcap_{p\in\omega} X_{\lambda_p}$, that X_λ is f-invariant and $(X_\lambda, G_f \cap X_\lambda^2) \in \mathcal{H}_\kappa$. As $X_\lambda \subsetneqq X_{\lambda_p}$ for each $p \in \omega$, we are done. $\square$

We now study subshifts. We have to find another solution when fixed points cannot be avoided since $(\mathbb{X}_1, G_{f_1})$ is not conjugate to the shift of a two-sided subshift. If $x\in A^{\mathbb{Z}}$ and $j\leq k$ are integers, then we define $x_{[j,k]}\in A^{k-j+1}$ by $x_{[j,k]}:=\big(x(j),\cdots,x(k)\big)$.

Lemma 3.4 *Let $\Sigma\subseteq A^{\mathbb{Z}}$ be a two-sided subshift, $l\in\omega$, $a_0,\cdots,a_l\in A$, and $(x_n)_{n\in\omega}$ be an injective sequence of points of Σ converging to $(a_0\cdots a_l)^{\mathbb{Z}}$. Then we can find $s\in A^{l+1}\setminus\{(a_0\cdots a_l)\}$ and $\gamma\in A^\omega$ with $(a_0\cdots a_l)^{-\infty}\cdot s\gamma\in\Sigma$ or $\gamma^{-1}s\cdot(a_0\cdots a_l)^{\infty}\in\Sigma$.*

Proof. We may assume, for example, that $x_{n[-k_n(l+1),k_n(l+1)-1]}=(a_0\cdots a_l)^{2k_n}$,

$$x_{n[k_n(l+1),(k_n+1)(l+1)-1]}$$

is a constant $s\neq(a_0\cdots a_l)$, and $k_n\to\infty$. By compactness, we may assume that the sequence $(x_{n[k_n(l+1),\infty)})_{n\in\omega}$ converges to some $s\gamma$ in A^ω. Note that $(a_0\cdots a_l)^{-\infty}\cdot s\gamma\in\Sigma$. □

We also need a version of Lemma 3.1 for subshifts. Let $\mathcal{S}_\kappa$ be the class of graphs, induced by the shift of a two-sided subshift, having no continuous κ-coloring.

Lemma 3.5 *Let $(\Sigma_p)_{p\in\omega}$ be a decreasing sequence of two-sided subshifts such that, for each $p\in\omega$, $(\Sigma_p, G_\sigma)\in\mathcal{S}_{\aleph_0}$, and $\Sigma:=\bigcap_{p\in\omega}\Sigma_p$. Then $(\Sigma, G_\sigma)\in\mathcal{S}_{\aleph_0}$.*

Proof. Assume that $\Sigma_0\subseteq A^{\mathbb{Z}}$. Note that $\sigma_{|\Sigma_0}$ has finitely many fixed points since these fixed points are of the form $a^{\mathbb{Z}}$ for $a\in A$ and A is finite. As $(\Sigma_p, G_\sigma)\in\mathcal{S}_{\aleph_0}$, we can find $a\in A$ such that, for any $p\in\omega$, $a^{\mathbb{Z}}\in\Sigma_p$ is not isolated in Σ_p. Lemma 3.4 provides $b\in A\setminus\{a\}$ and, for example and for each $p\in\omega$, $\gamma_p\in A^\omega$ such that $a^{-\infty}\cdot b\gamma_p$ is in Σ_p. Extracting a subequence if necessary, we may assume that $(a^{-\infty}\cdot b\gamma_p)_{p\in\omega}$ converges to $a^{-\infty}\cdot b\gamma\in\Sigma_0$, by compactness. Note that $a^{-\infty}\cdot b\gamma\in\Sigma$, so that $(\Sigma, G_\sigma)\in\mathcal{S}_{\aleph_0}$. □

We are now ready to prove Theorem 1.5 (a).

Proof of Theorem 1.5 (a). As in the proof of Theorem 1.1 (a), we may assume that $\kappa\geq 2$. We argue by contradiction, which gives $(\Sigma, G_\sigma)\in\mathcal{S}_\kappa$ such that, for each $(\Sigma', G_\sigma)\in\mathcal{S}_\kappa$ with the property that $(\Sigma', G_\sigma)\preceq^i_c(\Sigma, G_\sigma)$, there is $(\Sigma'', G_\sigma)\in\mathcal{S}_\kappa$ with $(\Sigma'', G_\sigma)\prec^i_c(\Sigma', G_\sigma)$.

Case 1. There is $(\Sigma', G_\sigma)\in\mathcal{S}_\kappa$ with $(\Sigma', G_\sigma)\preceq^i_c(\Sigma, G_\sigma)$ such that $\sigma_{|\Sigma'}$ is fixed point free.

We can copy the proof of Theorem 1.2 (a) to conclude.

Case 2. For each $(\Sigma', G_\sigma)\in\mathcal{S}_\kappa$ with $(\Sigma', G_\sigma)\preceq^i_c(\Sigma, G_\sigma)$, $\sigma_{|\Sigma'}$ is not fixed point free.

Claim 1. *For each $(\Sigma', G_\sigma)\in\mathcal{S}_\kappa$ with $(\Sigma', G_\sigma)\preceq^i_c(\Sigma, G_\sigma)$, there is $(\Sigma'', G_\sigma)\in\mathcal{S}_\kappa$ with $\Sigma''\subseteq\Sigma'$ and Σ'' has a dense infinite orbit, and $(\Sigma', G_\sigma)\in\mathcal{S}_{\aleph_0}$.*

Indeed, assume that $\Sigma'\subseteq A^{\mathbb{Z}}$. Note that $\sigma_{|\Sigma'}$ has finitely many fixed points since these fixed points are of the form $a^{\mathbb{Z}}$ for $a\in A$ and A is finite. Let $U:=\{a^{\mathbb{Z}}\in\Sigma'\mid a^{\mathbb{Z}}\text{ is isolated in }\Sigma'\}$. Note that U is a clopen $\sigma_{|\Sigma'}$-invariant subset of Σ', and $G_{\sigma_{|\Sigma'\setminus U}}=G_{\sigma_{|\Sigma'}}$, so that $(\Sigma'\setminus U, G_\sigma)\in\mathcal{S}_\kappa$. As we are in Case 2, $\sigma_{|\Sigma'\setminus U}$ is not fixed point free, which gives $a^{\mathbb{Z}}\in\Sigma'$ and a sequence $(x_n)_{n\in\omega}$ of points of $\Sigma'\setminus F_\sigma$ converging to $a^{\mathbb{Z}}$. Lemma 3.4 applied to Σ', $l:=0$ and $a_0:=a$ provides $b\in A\setminus\{a\}$ and $\gamma\in A^\omega$ such that, for example, $x:=a^{-\infty}\cdot b\gamma\in\Sigma'$. In particular, $a^{\mathbb{Z}}\in\overline{\text{Orb}_\sigma(x)}$. So we proved the existence of $x\in\Sigma'\setminus F_\sigma$ such that $a^{\mathbb{Z}}\in\overline{\text{Orb}_\sigma(x)}$. So $\Sigma'':=\overline{\text{Orb}_\sigma(x)}$ is as desired since $a^{\mathbb{Z}}$ is a witness for the fact that $(\Sigma'', G_\sigma)\in\mathcal{S}_{\aleph_0}\subseteq\mathcal{S}_\kappa$. ⋄

Claim 2. *There is $\Sigma'\subseteq\Sigma$ such that $(\Sigma',G_\sigma)\in\mathcal{S}_\kappa$, Σ' contains a dense infinite orbit O and, for each $(\Sigma'',G_\sigma)\in\mathcal{S}_\kappa$ with $\Sigma''\subseteq\Sigma'$, $\Sigma''\cap O$ is infinite.*

Indeed, we argue by contradiction. We inductively construct a strictly $\subseteq$-decreasing sequence $(\Sigma_\xi)_{\xi<\aleph_1}$ such that $\Sigma_0=\Sigma$ and $(\Sigma_\xi,G_\sigma)\in\mathcal{S}_\kappa$, which will contradict the fact that Σ is a 0DMC space. Assume that Σ_ξ is constructed, which is the case for $\xi=0$. Claim 1 gives $(\Sigma',G_\sigma)\in\mathcal{S}_\kappa$ with $\Sigma'\subseteq\Sigma_\xi$ and Σ' has a dense infinite orbit O. Our assumption gives $(\Sigma_{\xi+1},G_\sigma)\in\mathcal{S}_\kappa$ with $\Sigma_{\xi+1}\subseteq\Sigma'$ and $\Sigma_{\xi+1}\cap O$ is finite. In particular, $\Sigma_{\xi+1}\subsetneqq\Sigma_\xi$. If $(\lambda_p)_{p\in\omega}$ is strictly increasing and $\lambda=\sup_{p\in\omega}\lambda_p$ is a limit ordinal, then we set $\Sigma_\lambda:=\bigcap_{p\in\omega}\ \Sigma_{\lambda_p}$. By Lemma 3.5, $(\Sigma_\lambda,G_\sigma)\in\mathcal{S}_\kappa$. As $\Sigma_\lambda\subsetneqq\Sigma_{\lambda_p}$ for each $p\in\omega$, we are done. $\diamond$

Let $(\Sigma_0,G_\sigma)\in\mathcal{S}_\kappa$ such that $(\Sigma_0,G_\sigma)\prec^i_c(\Sigma',G_\sigma)$. Claim 1 provides $(\Sigma'_0,G_\sigma)\in\mathcal{S}_\kappa$ with $\Sigma'_0\subseteq\Sigma_0$ and Σ'_0 has a dense infinite orbit O_0. Note that $(\Sigma'_0,G_\sigma)\preceq^i_c(\Sigma',G_\sigma)$, with h as a witness, and $h[O_0]$ is an infinite orbit, by Lemma 3.2. We set $\Sigma'':=\overline{h[O_0]}$. Then $\Sigma''\subseteq\Sigma'$, h is a witness for the fact that $(\Sigma'_0,G_\sigma)\preceq^i_c(\Sigma'',G_\sigma)$, and thus $(\Sigma'',G_\sigma)\in\mathcal{S}_\kappa$. Let O be the orbit given by Claim 2. By Claim 2, $\Sigma''\cap O$ is infinite. As Σ'' is contained in the closed set $h[\Sigma'_0]$, $h[\Sigma'_0]\cap O$ is infinite. As Σ'_0 has a dense infinite orbit, $F_2^{\Sigma'_0,\sigma}$ is nowhere dense in Σ'_0. As Σ'_0 has finitely many fixed points and a finite orbit of size at least two is sent onto an orbit of the same size by h by Lemmas 3.2, 3.3, there is $z_0\in\Sigma'_0$ with an infinite orbit sent into O. This implies that $h[\text{Orb}(z_0)]=O$ and there is $\eta\in\{-1,1\}$ such that $h\circ\sigma=\sigma^\eta\circ h$ on $\text{Orb}(z_0)$, by Lemma 3.2. In particular, the set O is contained in the compact set $h[\Sigma'_0]$, showing that h is onto, and thus a homeomorphism, by compactness. In particular, $\text{Orb}(z_0)$ is dense in Σ'_0. This implies that $h\circ\sigma=\sigma^\eta\circ h$ on Σ'_0. If $y\neq\sigma(y)\in\Sigma'$ and, for example, $\eta=-1$, then we set $z:=\sigma(y)$. Let $x\in\Sigma'_0$ with $z=h(x)$. Note that $\big(y,\sigma(y)\big)=\big(\sigma^{-1}(z),z\big)=\big(\sigma^{-1}\big(h(x)\big),h(x)\big)=\big(h\big(\sigma(x)\big),h(x)\big)\in(h\times h)[G_\sigma]$, showing that $(h\times h)[G_\sigma]=G_\sigma$. Thus $(\Sigma',G_\sigma)\preceq^i_c(\Sigma'_0,G_\sigma)\preceq^i_c(\Sigma_0,G_\sigma)$, which is the desired contradiction concluding the proof. $\square$

4 Concrete countable basis

We already checked the part (b) of Theorems 1.1, 1.2 and 1.5 when $\kappa\leq 1$.

Proof of Theorem 1.2 (b) when $\kappa\geq 3$. Let X be a 0DMC space, and f be a homeomorphism of X such that (X,G_f) has no continuous κ-coloring. If F_f is an open subset of X, then there is a continuous $\aleph_0$-coloring of G_f, by Corollary 2.2. By Theorem 1.12 in [L], there is a continuous 3-coloring of G_f, which contradicts the fact that $\kappa\geq 3$. Thus F_f is not an open subset of X and we can apply Corollary 2.3. $\square$

In this section, it remains to study the part (a) of Theorems 1.4 and 1.6.

4.1 Some general facts about symmetric relations

Let X be a set, R be a relation on X, κ be a countable cardinal, and $\overline{x}:=(x_i)_{i<\kappa}$ be a sequence of elements of X. Recall that $\overline{x}$ is a R-**walk** if $(x_i,x_{i+1})\in R$ whenever $i+1<\kappa$. A R-**path** is an injective R-walk. We say that $\overline{x}$ is a R-**cycle** if $3\leq\kappa<\aleph_0$, $\overline{x}$ is a R-path and $(x_{\kappa-1},x_0)\in R$. A **connected component** of (X,R) is a subset C of X such that, for each $x\in C$,

$$C=\{y\in X\mid\exists\overline{x}\ R\text{-path with }1\leq\kappa<\aleph_0,\ x_0=x\text{ and }x_{\kappa-1}=y\}.$$

We say that (X,R) is **connected** if X is a connected component of (X,R).

The following fact is very classical.

Lemma 4.1.1 *Let X be a set, and G be a symmetric relation on X. Exactly one of the following holds:*

(1) there is a 2-coloring of (X,G),

(2) we can find $m\in\omega$ and $(x_i)_{i\leq 2m}\in X^{2m+1}$ such that $(x_i,x_{i+1})\in G$ for each $i\leq 2m$ (with the convention $x_{2m+1}:=x_0$).

In particular, $(2q+3,C_{2q+3})$ has no continuous 2-coloring.

4.2 Isolated finite 2-colorable connected components

The next two results will allow us to remove the isolated finite 2-colorable connected components.

Lemma 4.2.1 *Let X be a 0DMC space, G be a closed graph on X, and*

$$O:=\bigcup\{C\subseteq X\setminus X' \mid C \text{ finite } (X,G)\text{-connected component and } (C,G\cap C^2) \text{ has a 2-coloring}\}.$$

If $(X\setminus O,G\cap(X\setminus O)^2)$ has a continuous 2-coloring, then so does (X,G).

Proof. Let $c:X\setminus O\rightarrow 2$ be a continuous 2-coloring of $(X\setminus O,G\cap(X\setminus O)^2)$. As O is an open subset of the 0DM space X, $X\setminus O$ is closed and there is a clopen partition $(P_\varepsilon)_{\varepsilon\in 2}$ of X with $P_\varepsilon\setminus O=c^{-1}(\{\varepsilon\})$ (see 22.16 in [K1]).

Let us prove that if $x\notin O$, then there is an open neighbourhood N_x of x such that $N_x\subseteq P_{c(x)}$, and $y\in P_{1-c(x)}$ if there is $x'\in N_x$ with $(x',y)\in G$. We argue by contradiction. Let $(N_i)_{i\in\omega}$ be a decreasing basis of open neighbourhoods of x contained in $P_{c(x)}$. Then for each i there is $(x_i,y_i)\in G\cap(N_i\times P_{c(x)})$. Note that $(x_i)_{i\in\omega}$ converges to x, and we may assume that $(y_i)_{i\in\omega}$ converges to some $y\in P_{c(x)}$ by compactness of X. As G is closed, $(x,y)\in G$. As $x\notin O$ and O is a union of connected components, $y\notin O$, which contradicts the fact that c is a coloring.

We now set $N:=\bigcup\{N_x \mid x\notin O\}$, and define $\overline{c}:N\rightarrow 2$ by $\overline{c}(y):=c(x)$ if $x\notin O$ and $y\in N_x$. This definition is correct since $x,x'\notin O$ and $y\in N_x\cap N_{x'}$ imply that $c(x)=c(x')$. If $x,z\notin O$ and $(x',y)\in G\cap(N_x\times N_z)$, then $\overline{c}(x')=c(x)\neq c(z)=\overline{c}(y)$ since $y\in P_{1-c(x)}\cap P_{c(z)}$. Thus $\overline{c}$ is a 2-coloring of $(N,G\cap N^2)$. By definition, $\overline{c}$ is continuous.

Now note that $X\setminus N$ is finite, since otherwise there is an injective sequence $(w_i)_{i\in\omega}$ of elements of $X\setminus N$, and we may assume that it converges to some $w\in X$ by compactness of X. As $X\setminus O\subseteq N$, $X\setminus N\subseteq O\subseteq X\setminus X'$, so that $w\in X'\subseteq N$ and $w_i\in N$ if i is big enough, which is the desired contradiction.

As $X\setminus N$ is finite, the set $I:=\bigcup\{C \mid C \text{ appears in the definition of } O \text{ and } C\setminus N\neq\emptyset\}$ is finite and G-invariant. We restrict $\overline{c}$ to $N\setminus I$ and extend this restriction using any 2-coloring on each of the components of I to conclude. □

Corollary 4.2.2 *Let X be a 0DMC space, f be a fixed point free homeomorphism of X, and*

$$O:=\bigcup\{Orb(x) \text{ finite of even cardinality} \mid x\in X\setminus X'\}.$$

If $(X\setminus O,G_f\cap(X\setminus O)^2)$ has a continuous 2-coloring, then so does (X,G_f).

Note that in this section there is no upper bound on the Cantor-Bendixson rank of X.

4.3 Finiteness results

Convention. In this section, X is a countable MC space, and f is a homeomorphism of X.

The following classical fact will be used a lot.

Lemma 4.3.1 *Assume that X has Cantor-Bendixson rank $\beta{+}1$. Then X^β is finite.*

Proof. X^β is compact and discrete, and thus finite. $\square$

In practice, in our spaces of Cantor-Bendixson rank at most two, we will consider a partition of the Cantor-Bendixson derivative into finitely many (closed) invariant sets.

Lemma 4.3.2 *Let C be a clopen f-invariant subset of X', O be an open subset of X containing C, and $(x_n)_{n\in\omega}$ be a sequence of points of $X\setminus X'$ converging to a point of C. Then*

$$\{\mathit{Orb}(x_n) \mid n\in\omega \text{ and } \mathit{Orb}(x_n)\not\subseteq O\}$$

is finite.

Proof. Note that $C, X'\setminus C$ are disjoint and clopen in X', and thus closed in the compact space X. If $y\in C$, then $y, f(y), f^{-1}(y)$ are in the open set $V:=X\setminus(X'\setminus C)$, which gives an open neighbourhood N_y of y with $\overline{N_y}\subseteq O\cap V\cap f^{-1}(V)\cap f[V]$. The compactness of C provides $F\subseteq C$ finite such that $C\subseteq N:=\bigcup_{y\in F} N_y\subseteq\overline{N}=\bigcup_{y\in F}\overline{N_y}\subseteq V\cap f^{-1}(V)\cap f[V]$. In particular, $X'\setminus C=X\setminus V$ is contained in the open set $U:=X\setminus\big(\overline{N}\cup f[\overline{N}]\cup f^{-1}(\overline{N})\big)$. Note that we can find $n_0\in\omega$ such that $x_n\in N$ if $n\geq n_0$. We set $M:=\{n\geq n_0\mid \exists m\in\omega\ \ f^m(x_n)\in N\wedge f^{m+1}(x_n)\notin N\}$. If $n\in M$, then there is $m_n\in\omega$ with $f^{m_n+1}(x_n)\notin N\cup U$. As $X'\subseteq N\cup U$, $X\setminus(N\cup U)$ is finite. This shows that $\{\mathrm{Orb}_f(x_n)\mid n\in M\}$ is finite. Moreover, $\{f^m(x_n)\mid m\in\omega\}\subseteq N\subseteq O$ if $n_0\leq n\notin M$. We can argue similarly with f^{-1} instead of f, so that $\{\mathrm{Orb}_f(x_n)\mid n\in\omega\wedge\mathrm{Orb}_f(x_n)\not\subseteq O\}$ is finite. $\square$

Notation. If b is a bijection of a set S, $x\in S$ and $\mathbf{d}\in\{-,+\}$, then we set

$$\mathrm{Orb}_b^{\mathbf{d}}(x):=\{b^{\mathbf{d}i}(x)\mid i\in\omega\}.$$

Convention. In the rest of this section, we assume the existence of $\kappa\in\omega$ and a (finite) partition $(C_j)_{j\leq\kappa}$ of X' into closed f-invariant sets.

The next lemma controls the closures of the orbits and is a basic tool.

Lemma 4.3.3 *Assume that $x\in X\setminus X'$ has an infinite orbit. Then we can find $j^-, j^+\leq\kappa$ such that, for each $\mathbf{d}\in\{-,+\}$, $\overline{\mathit{Orb}^{\mathbf{d}}(x)}\subseteq\mathit{Orb}^{\mathbf{d}}(x)\cup C_{j^{\mathbf{d}}}$.*

If moreover the C_j's are orbits, then X' is finite and we can find $y^-, y^+\in X'$ such that, for each $\mathbf{d}\in\{-,+\}$, $\overline{\mathit{Orb}^{\mathbf{d}}(x)}=\mathit{Orb}^{\mathbf{d}}(x)\cup\mathit{Orb}(y^{\mathbf{d}})$ and $\big(f^{\mathbf{d}q|\mathit{Orb}(y^{\mathbf{d}})|}(x)\big)_{q\in\omega}$ converges to $y^{\mathbf{d}}$.

Proof. We first prove the following.

Claim. *Let S be a closed subset of X', and $1\leq\kappa'\in\omega$ such that the limit points of $\{f^{q\kappa'}(x)\mid q\in\omega\}$ are in S. Then it is not possible to find disjoint $f^{\kappa'}$-invariant subsets S_0, S_1 clopen in S for which we can find, for each $\varepsilon\in 2$, $y_\varepsilon\in S_\varepsilon$ and $(q_j^\varepsilon)_{j\in\omega}$ such that $\big(f^{q_j^\varepsilon\kappa'}(x)\big)_{j\in\omega}$ converges to y_ε.*

Indeed, we argue by contradiction. Note that $S_0, S\setminus S_0$ are disjoint and closed in S and X', and thus closed in the compact space X. If $y\in S_0$, then $y, f^{\kappa'}(y)$ are in the open set $O:=X\setminus(S\setminus S_0)$, which gives an open neighbourhood N_y of y with $\overline{N_y}\subseteq O\cap f^{-\kappa'}(O)$. The compactness of S_0 provides $F\subseteq S_0$ finite such that $S_0\subseteq N:=\bigcup_{y\in F}\ N_y\subseteq\overline{N}=\bigcup_{y\in F}\ \overline{N_y}\subseteq O\cap f^{-\kappa'}(O)$. In particular, $S\setminus S_0=X\setminus O$ is contained in the open set $N':=X\setminus(\overline{N}\cup f^{\kappa'}[\overline{N}])$. Note that the set $\{q\in\omega\mid f^{q\kappa'}(x)\in N\wedge f^{(q+1)\kappa'}(x)\notin N\}$, and thus $\{q\in\omega\mid f^{(q+1)\kappa'}(x)\notin N\cup N'\}$, are infinite. By compactness, a subsequence of these $f^{(q+1)\kappa'}(x)$'s has to converge to a point of $S\subseteq N\cup N'$, which is the desired contradiction. $\diamond$

As Orb(x) is infinite, the sequence $\big(f^n(x)\big)_{n\in\omega}$ is injective and contained in the discrete space $X\setminus X'$. The compactness of X provides a strictly increasing sequence $(n_q)_{q\in\omega}$ and $y^+\in X'$ such that $\big(f^{n_q}(x)\big)_{q\in\omega}$ converges to y^+. Fix $j^+\leq\kappa$ with $y^+\in C_{j^+}$. The claim applied to $S:=X'=\bigcup_{j\leq\kappa}\ C_j$ and $\kappa':=1$ shows that $\overline{\text{Orb}^+(x)}\subseteq\text{Orb}^+(x)\cup C_{j^+}$.

If the C_j's are orbits, then $C_{j^+}=\text{Orb}(y^+)$, so that $\overline{\text{Orb}^+(x)}\subseteq\text{Orb}^+(x)\cup\text{Orb}(y^+)$. As y^+ is in $\overline{\text{Orb}^+(x)}$, we actually have equality since f is a homeomorphism. As X' is a nonempty countable MC space, there is a countable ordinal β such that X' has Cantor-Bendixson rank β+1. Thus $(X')^\beta$ is nonempty finite by Lemma 4.3.1. If $\beta=0$, then X' is finite. If $\beta\geq 1$ and $z\in X'\setminus(X')^\beta$ has an infinite orbit, then this orbit is not closed, which contradicts our assumptions on the C_j's. This shows that X' has finite orbits and $\beta=0$. In particular, we may assume that y^+ is a limit point of $\{f^{q\kappa'}(x)\mid q\in\omega\}$, where $\kappa':=|\text{Orb}(y^+)|$. This gives $(q_l)_{l\in\omega}$ such that $\big(f^{q_l\kappa'}(x)\big)_{l\in\omega}$ converges to y^+. The claim applied to $S:=\text{Orb}(y^+)$ and κ' implies that $\big(f^{q\kappa'}(x)\big)_{q\in\omega}$ converges to y^+.

We argue similarly with Orb$^-(x)$ instead of Orb$^+(x)$. $\square$

There is no upper bound on the Cantor-Bendixson rank of X in Lemma 4.3.2, the first part of Lemma 4.3.3, and Lemmas 4.3.4, 4.3.5 to come. The next two results complete Lemma 4.3.3.

Lemma 4.3.4 *Let $j^-\neq j^+\leq\kappa$, O^-,O^+ be disjoint open subsets of X such that $C_{j^-}\subseteq O^-$ and $\bigcup_{j^-\neq j\leq\kappa}\ C_j\subseteq O^+$, and $(x_n)_{n\in\omega}$ be a sequence of points of $X\setminus X'$ such that Orb(x_n) is infinite and Orb$^+(x_n)$ meets O^- and O^+ for each n. Then the set $\{$Orb$(x_n)\mid n\in\omega\}$ is finite. In particular, the set $\{$Orb(x) infinite $\mid x\in X\setminus X'$ and $\forall\mathbf{d}\in\{-,+\}\ \overline{\text{Orb}^{\mathbf{d}}(x)}\subseteq\text{Orb}^{\mathbf{d}}(x)\cup C_{j^{\mathbf{d}}}\}$ is finite.*

Proof. Note that the sets C_{j^-}, $\bigcup_{j^-\neq j\leq\kappa}\ C_j$ are disjoint and clopen in X', and thus closed in the compact space X. We argue by contradiction, so that we may assume that $\big(\text{Orb}(x_n)\big)_{n\in\omega}$ is injective. If $n\in\omega$, then we can find, for each $\mathbf{d}\in\{-,+\}$, $(m_q^{n,\mathbf{d}})_{q\in\omega}$ strictly increasing and $y_n^{\mathbf{d}}\in O^{\mathbf{d}}$ such that $\big(f^{m_q^{n,\mathbf{d}}}(x_n)\big)_{q\in\omega}$ converges to $y_n^{\mathbf{d}}$. In particular, we may assume that $f^{m_q^{n,\mathbf{d}}}(x_n)\in O^{\mathbf{d}}$. This gives $(p_n)_{n\in\omega}$ such that $f^{p_n}(x_n)\in X\setminus O^+$ and $f^{p_n+1}(x_n)\in O^+\subseteq X\setminus O^-$. The compactness of X provides $y\in X'$ such that $\big(f^{p_n}(x_n)\big)_{n\in\omega}$ converges to $y\in X'\setminus O^+=C_{j^-}$. As C_{j^-} is f-invariant, $f(y)\in C_{j^-}\subseteq O^-$. On the other hand, $f(y)=\lim_{n\to\infty}\ f^{p_n+1}(x_n)\notin O^-$, which is the desired contradiction.

For the last assertion, assume that $\big(\text{Orb}(x_n)\big)_{n\in\omega}$ is a sequence of elements of our set. The compactness provides, for each n and each $\mathbf{d}\in\{-,+\}$, $(l_q^{n,\mathbf{d}})_{q\in\omega}$ strictly increasing and $y_n^{\mathbf{d}}$ in $C_{j^{\mathbf{d}}}$ such that $\big(f^{\mathbf{d}l_q^{n,\mathbf{d}}}(x_n)\big)_{q\in\omega}$ converges to $y_n^{\mathbf{d}}$. In particular, we may assume that $f^{\mathbf{d}l_q^{n,\mathbf{d}}}(x_n)\in O^{\mathbf{d}}$, so that $\text{Orb}\big(f^{-l_q^{n,-}}(x_n)\big)=\text{Orb}(x_n)$ and $\text{Orb}^+\big(f^{-l_q^{n,-}}(x_n)\big)$ meets O^- and O^+. $\square$

Convention. In the rest of this section, we assume the existence of a continuous 2-coloring $\overline{c}$ of $(X', G_f \cap (X')^2)$.

Notation. In order to simplify the notation, we will sometimes identify κ with $\mathbb{Z}/\kappa\mathbb{Z}$ when $\kappa \geq 2$ is finite. For example, the parity $\text{par}(n)$ of $n \in \mathbb{Z}$ will often be viewed as an element of $\mathbb{Z}/2\mathbb{Z}$. We set, for $j \leq \kappa$ and $\varepsilon \in 2$, $C_j^\varepsilon := C_j \cap \overline{c}^{-1}(\{\varepsilon\})$.

Lemma 4.3.5 *Let $\mathbf{d} \in \{-,+\}$, $j \leq \kappa$, $\varepsilon \in 2$, and $x \in X \setminus X'$ for which there is a sequence $(m_q)_{q \in \omega}$ of natural numbers of constant parity such that $\big(f^{\mathbf{d}m_q}(x)\big)_{q \in \omega}$ converges to a point of C_j^ε. Then*

$$\begin{cases} \overline{\mathit{Orb}^{\mathbf{d}}_{f^2}\big(f^{par(m_0)}(x)\big)} \subseteq \mathit{Orb}^{\mathbf{d}}_{f^2}\big(f^{par(m_0)}(x)\big) \cup C_j^\varepsilon, \\ \\ \overline{\mathit{Orb}^{\mathbf{d}}_{f^2}\big(f^{1-par(m_0)}(x)\big)} \subseteq \mathit{Orb}^{\mathbf{d}}_{f^2}\big(f^{1-par(m_0)}(x)\big) \cup C_j^{1-\varepsilon}. \end{cases}$$

Proof. Note that $\text{Orb}(x)$ is infinite since $x \in X \setminus X'$ and $C_j \subseteq X'$. It remains to apply Lemma 4.3.3 to f^2 and $(C_j^\varepsilon)_{j \leq \kappa, \varepsilon \in 2}$. □

4.4 Some general facts about homeomorphisms

The next lemma provides a sufficient condition for minimality.

Notation. Let $\mathfrak{G}$ be the class of graphs induced by a homeomorphism of a MC space having no continuous 2-coloring.

Lemma 4.4.1 *Let Y be a 0DMC space, h be a fixed point free homeomorphism of Y such that (Y, G_h) has no continuous 2-coloring, and S be a dense subset of Y with the property that for any $V \subseteq Y$, for any graph H on V contained in G_h such that (V, H) has no continuous 2-coloring, and for any $y \in S$, $\big(y, h(y)\big) \in H$ holds. Then (Y, G_h) is $\preceq_c^i$-minimal in $\mathfrak{G}$ and in the class of closed graphs on a MC space having no continuous 2-coloring.*

Proof. Assume that $(K, G) \in \mathfrak{G}$ and $(K, G) \preceq_c^i (Y, G_h)$ with φ as a witness. Corollary 2.3 in [Kr-St] shows that (Y, G_h) has a continuous 3-coloring, which implies that (K, G) too. By compactness, K is homeomorphic to a subspace of Y, so that K is 0D. As $(K, G) \in \mathfrak{G}$, there is a homeomorphism $g : K \to K$ with $G = G_g$. In particular, the set F_g of fixed points of g is a clopen subset of K, and $(K \setminus F_g, G_g \cap (K \setminus F_g)^2)$ has no continuous 2-coloring, by Corollary 2.2.(b). This implies that we may assume that g is fixed point free, so that G is compact. We set $V := \varphi[K]$ and $H := (\varphi \times \varphi)[G]$, so that $V \subseteq Y$, $H \subseteq G_h$ is a compact graph on V, $(K, G) \preceq_c^i (V, H)$ with φ as a witness, and $(V, H) \preceq_c^i (K, G)$ with φ^{-1} as a witness by compactness. Note that $\big(y, h(y)\big) \in H$ if $y \in S$, by our assumption. The density of S in Y and the compactness of H then imply that $\text{Graph}(h) \subseteq H$. As H is a graph, we get $H = G_h$ and therefore $V = Y$. Thus $(Y, G_h) \preceq_c^i (K, G)$ and (Y, G_h) is $\preceq_c^i$-minimal in $\mathfrak{G}$, and also in the class of closed graphs on a MC space having no continuous 2-coloring. □

Corollary 4.4.2 *Let q be a natural number. Then $(2q+3, C_{2q+3})$ is $\preceq_c^i$-minimal in $\mathfrak{G}$ and in the class of closed graphs on a MC space having no continuous 2-coloring.*

Proof. Note that $Y := 2q+3$, equipped with the discrete topology, is a 0DMC space. The formula $h(i) := (i+1) \bmod (2q+3)$ defines a fixed point free homeomorphism h of Y, and $C_{2q+3} = G_h$. Lemma 4.1.1 implies that (Y, G_h) has no continuous 2-coloring. Any dense subset of Y is equal to Y. If $V \subseteq Y$, H is a graph on V contained in G_h such that (V, H) has no continuous 2-coloring, and $y \in Y$, then $\big(y, h(y)\big) \in H$. Indeed, we argue by contradiction, and we may assume that $y = 2q+2$, the other cases being similar. Then the formula $c(x) := \text{par}(x)$ defines a continuous 2-coloring of (V, H), which cannot be. It remains to apply Lemma 4.4.1. $\square$

Remark. $(2q+3, C_{2q+3})$ is in fact $\preceq^i_c$-minimal in the class of graphs on a Hausdorff topological space having no continuous 2-coloring.

The following result is also without upper bound on the rank.

Convention. In the rest of this section, we assume that X is a countable 0DMC space, f is a homeomorphism of X such that $(X', G_f \cap (X')^2)$ has a continuous 2-coloring $\bar{c}$, $\kappa \in \omega$, and $(C_j)_{j \leq \kappa}$ is a partition of X' into closed f-invariant sets.

Notation. Let $F := \{C^\varepsilon_j \mid \varepsilon \in 2 \text{ and } j \leq \kappa\}$. We define relations D', E' on F by

$$(C^\varepsilon_j, C^{\varepsilon'}_{j'}) \in D' \Leftrightarrow (\varepsilon \neq \varepsilon' \text{ and } j = j') \text{ or}$$
$$\exists x \in X \setminus X' \;\; \exists (m_q)_{q\in\omega}, (n_q)_{q\in\omega} \in \omega^\omega \text{ with constant parity such that}$$
$$m_0 + n_0 \text{ is odd and } \lim_{q\to\infty} f^{-m_q}(x) \in C^\varepsilon_j \text{ and } \lim_{q\to\infty} f^{n_q}(x) \in C^{\varepsilon'}_{j'},$$

$$(C^\varepsilon_j, C^{\varepsilon'}_{j'}) \in E' \Leftrightarrow \exists x \in X \setminus X' \;\; \exists (m_q)_{q\in\omega}, (n_q)_{q\in\omega} \in \omega^\omega \text{ with constant parity such that}$$
$$m_0 + n_0 \text{ is even and } \lim_{q\to\infty} f^{-m_q}(x) \in C^\varepsilon_j \text{ and } \lim_{q\to\infty} f^{n_q}(x) \in C^{\varepsilon'}_{j'},$$

and set $D := s(D')$ and $E := s(E')$.

Lemma 4.4.3 *Assume that f is fixed point free and $X \setminus X'$ contains only infinite orbits. Then, with the notation just above, if there is $g : F \to 2$ satisfying*

$$\begin{cases} \forall (C^\varepsilon_j, C^{\varepsilon'}_{j'}) \in D \;\; g(C^\varepsilon_j) \neq g(C^{\varepsilon'}_{j'}), \\ \forall (C^\varepsilon_j, C^{\varepsilon'}_{j'}) \in E \;\; g(C^\varepsilon_j) = g(C^{\varepsilon'}_{j'}), \end{cases}$$

then (X, G_f) has a continuous 2-coloring.

Proof. We define $c : X \to 2$ as follows. If $y \in X'$, then there is a unique $C^\varepsilon_j \in F$ with $y \in C^\varepsilon_j$. We put $c(y) := g(C^\varepsilon_j)$. If $x \in X \setminus X'$, then Orb(x) is infinite, and Lemma 4.3.3 provides a unique $j^- \leq \kappa$ such that $\overline{\text{Orb}^-(x)} \subseteq \text{Orb}^-(x) \cup C_{j^-}$. As $X \setminus X'$ is discrete and Orb(x) is infinite, there is $(m_q)_{q\in\omega} \in \omega^\omega$ with constant parity strictly increasing such that $\big(f^{-m_q}(x)\big)_{q\in\omega}$ converges to a point of C_{j^-}. Replacing m_q with m_q+1 if necessary, we may assume that this limit is in $C^0_{j^-}$. Lemma 4.3.5 applied to $\mathbf{d} := -$, $j := j^-$ and $\varepsilon := 0$ implies that $\overline{\text{Orb}^-_{f^2}\big(f^{\text{par}(m_0)}(x)\big)} \subseteq \text{Orb}^-_{f^2}\big(f^{\text{par}(m_0)}(x)\big) \cup C^0_{j^-}$ and $\overline{\text{Orb}^-_{f^2}\big(f^{1-\text{par}(m_0)}(x)\big)} \subseteq \text{Orb}^-_{f^2}\big(f^{1-\text{par}(m_0)}(x)\big) \cup C^1_{j^-}$. Thus, if $\big(f^{-m'_q}(x)\big)_{q\in\omega}$ converges to a point of $C^0_{j^-}$ and the parity of m'_q is constant, then $\text{par}(m_0) = \text{par}(m'_0)$. This allows us to put $c(x) := g(C^0_{j^-}) + \text{par}(m_0)$.

If $y \in C_j^\varepsilon$, then $f(y) \in C_j^{1-\varepsilon}$ since f is fixed point free, so that $c(y) = g(C_j^\varepsilon) \neq g(C_j^{1-\varepsilon}) = c\big(f(y)\big)$ since $(C_j^\varepsilon, C_j^{1-\varepsilon}) \in D$. If $x \in X \setminus X'$, then $c(x) = g(C_{j^-}^0) + \text{par}(m_0) \neq g(C_{j^-}^0) + \text{par}(m_0+1) = c\big(f(x)\big)$. Thus c is a 2-coloring of (X, G_f).

Assume that $(x_n)_{n\in\omega} \in (X \setminus X')^\omega$ converges to $y \in C_j^\varepsilon$, so that $c(y) = g(C_j^\varepsilon)$. Let us prove that $c(x_n) = c(y)$ if n is big enough. As $\text{Orb}(x_n)$ is infinite, Lemma 4.3.3 provides a unique $j_n^- \leq \kappa$ such that $\overline{\text{Orb}^-(x_n)} \subseteq \text{Orb}^-(x_n) \cup C_{j_n^-}$. Splitting the sequence $(x_n)_{n\in\omega}$ into finitely many subsequences if necessary, we may assume that the sequence $(j_n^-)_{n\in\omega}$ is a constant j^-. Replacing x_0 with $f(x_0)$ if necessary, we may assume that $c(x_0) = g(C_{j^-}^0)$. As above, there is $(m_q^n)_{q\in\omega} \in \omega^\omega$ with constant parity strictly increasing such that $\big(f^{-m_q^n}(x_n)\big)_{q\in\omega}$ converges to a point y_n of C_{j^-}. If $n > 0$, then, replacing m_q^n with $m_q^n + 1$ if necessary, we may assume that $y_n \in C_{j^-}^0$, so that $c(x_n) = g(C_{j^-}^0) + \text{par}(m_0^n)$. If $n = 0$, then we choose m_0^0 even, and $y_0 \in C_{j^-}^0$ since $c(x_0) = g(C_{j^-}^0)$.

Let us prove that we can find disjoint clopen subsets O^0, O^1 of X satisfying, for each $\eta \in 2$,

- $C_j^\eta \subseteq O^\eta \subseteq X \setminus (X' \setminus C_j)$,
- $O^\eta \cap f^{-1}(O^\eta) = \emptyset$.

Note that C_j^0, C_j^1 are disjoint and clopen in X', and thus closed in the 0DMC space X. By 22.16 in [K1], there is a clopen subset C of X with $C_j^0 \subseteq C \subseteq X \setminus C_j^1$. Similarly, we can find clopen subsets C^0, C^1 of X with the properties that $C_j^0 \subseteq C^0 \subseteq C \setminus (X' \setminus C_j)$ and $C_j^1 \subseteq C^1 \subseteq X \setminus \big(C \cup (X' \setminus C_j)\big)$. Note that $C_j^\eta \subseteq f^{-1}(C^{1-\eta})$ since f is fixed point free, so that we can set $O^\eta := C^\eta \cap f^{-1}(C^{1-\eta})$.

We then put $O := O^0 \cup O^1$, so that, by Lemma 4.3.2 applied to C_j and O, the set

$$\{\text{Orb}(x_n) \mid n \in \omega \text{ and } \text{Orb}(x_n) \not\subseteq O\}$$

is finite. We set $I := \{n \in \omega \mid \text{Orb}(x_n) \not\subseteq O\}$. Note that we can find $n_0 \in \omega$ such that $x_n \in O^\varepsilon$ if $n \geq n_0$.

We first prove that $c(x_n) = c(y)$ if $n \notin I$ is big enough. If $n \notin I$, then $\text{Orb}(x_n) \subseteq O$, $f^{-m_q^n}(x_n)$ is in the clopen set O and thus $y_n \in C_{j^-}^0 \cap O \subseteq X' \setminus (X' \setminus C_j) = C_j$. This implies that $j^- = j$ and $c(x_n) = g(C_j^0) + \text{par}(m_0^n)$. If $q \geq q_0$ is big enough, then $f^{-m_q^n}(x_n)$ is in O^0. As $\text{Orb}(x_n)$ is contained in O and $O^0 \cap f^{-1}(O^0) = \emptyset$, $f^{1-m_q^n}(x_n) \in O \setminus O^0 = O^1$. As $\text{Orb}(x_n) \subseteq O$ and $O^1 \cap f^{-1}(O^1) = \emptyset$, $f^{2-m_q^n}(x_n) \in O \setminus O^1 = O^0$. Inductively, as $\big(\text{par}(m_q^n)\big)_{q\in\omega}$ is constant, $f^{-m_0^n}(x_n) \in O^0$. Similarly, $x_n \in O^{\text{par}(m_0^n)}$. This implies that $\text{par}(m_0^n) = \varepsilon$ if $n \geq n_0$. As $g(C_j^\varepsilon)$ is different from $g(C_j^{1-\varepsilon})$, $c(x_n) = g(C_j^0) + \text{par}(m_0^n) = g(C_j^0) + \varepsilon = g(C_j^\varepsilon) = c(y)$ if $n \geq n_0$, as desired.

As $\{\text{Orb}(x_n) \mid n \in \omega \text{ and } \text{Orb}(x_n) \not\subseteq O\}$ is finite, it remains to see that if $x_n \in \text{Orb}(x_0)$ and n is big enough, then $c(x_n) = c(y)$. As $x_n \in \text{Orb}(x_0)$, we may assume that either there is $(p_n)_{n\in\omega} \in \omega^\omega$ with constant parity such that $x_n = f^{p_n}(x_0)$ for each n, or there is $(r_n)_{n\in\omega} \in \omega^\omega$ with constant parity such that $x_n = f^{-r_n}(x_0)$ for each n.

Case 1. $x_n = f^{p_n}(x_0)$ for each n.

Subcase 1.1. $g(C_{j^-}^0) = g(C_j^\varepsilon)$.

Note that $\big(f^{-m_q^0}(x_0)\big)_{q\in\omega} = \big(f^{-(m_q^0+p_n)}(x_n)\big)_{q\in\omega}$ converges to $y_0 \in C_{j^-}^0$. Thus, by definition of c, $c(x_n) = g(C_{j^-}^0) + \text{par}(p_n)$. Note that $\big(f^{p_q}(x_0)\big)_{q\in\omega}$ converges to $y \in C_j^\varepsilon$. As $g(C_{j^-}^0) = g(C_j^\varepsilon)$, $(C_{j^-}^0, C_j^\varepsilon) \notin D$, and $m_q^0 + p_q$ is even, like p_n. Thus $c(x_n) = g(C_{j^-}^0) = g(C_j^\varepsilon) = c(y)$, as desired.

Subcase 1.2. $g(C^0_{j^-})\neq g(C^\varepsilon_j)$.

Arguing as in Subcase 1.1, as $g(C^0_{j^-})\neq g(C^\varepsilon_j)$, $(C^0_{j^-},C^\varepsilon_j)\notin E$, and thus $m^0_q+p_q$ is odd, just like p_n. This implies that $c(x_n)=g(C^0_{j^-})+1=g(C^\varepsilon_j)=c(y)$, as desired.

Case 2. $x_n=f^{-r_n}(x_0)$ for each n.

Note that $r_0=0$ since Orb(x_0) is infinite. Assume that $\varepsilon=1$, the other case being similar. Note that $\overline{\text{Orb}^-(x_0)}\subseteq \text{Orb}^-(x_0)\cup C_{j^-}$, so that $y=\lim_{n\to\infty} x_n\in C_j\cap C_{j^-}$ and $j=j^-$. In particular, $c(x_n)=g(C^0_j)+1-\text{par}(r_n)=g(C^0_j)+1=g(C^1_j)=c(y)$, as desired.

This proves the continuity of c. □

A key consequence of Lemma 4.4.3 is the following.

Lemma 4.4.4 *Assume that f is fixed point free, $X\setminus X'$ contains only infinite orbits, and (X,G_f) has no continuous 2-coloring. Then we can find $l\in\omega$, $C^{\varepsilon_0}_{j_0},\cdots,C^{\varepsilon_l}_{j_l}$ with $(j_i)_{i\leq l}$ injective, a sequence $(z'_i)_{i\leq l}$ of elements of $X\setminus X'$, and, for $i\leq l$, a sequence $(m^i_q)_{q\in\omega}$ of even natural numbers and a sequence $(n^i_q)_{q\in\omega}$ of natural numbers with constant parity satisfying the following:*

(a) if $i<l$, then n^i_0 is even and one of the following holds:

$(\alpha)_i$ $y^-_i:=\lim_{q\to\infty} f^{-m^i_q}(z'_i)\in C^{\varepsilon_i}_{j_i}$ *and* $y^+_i:=\lim_{q\to\infty} f^{n^i_q}(z'_i)\in C^{\varepsilon_{i+1}}_{j_{i+1}}$,

$(\beta)_i$ $y^-_i:=\lim_{q\to\infty} f^{-m^i_q}(z'_i)\in C^{\varepsilon_{i+1}}_{j_{i+1}}$ *and* $y^+_i:=\lim_{q\to\infty} f^{n^i_q}(z'_i)\in C^{\varepsilon_i}_{j_i}$,

(b) n^l_0 is odd and one of the following holds:

$(\alpha)_l$ $y^-_l:=\lim_{q\to\infty} f^{-m^l_q}(z'_l)\in C^{\varepsilon_l}_{j_l}$ *and* $y^+_l:=\lim_{q\to\infty} f^{n^l_q}(z'_l)\in C^{\varepsilon_0}_{j_0}$,

$(\beta)_l$ $y^-_l:=\lim_{q\to\infty} f^{-m^l_q}(z'_l)\in C^{\varepsilon_0}_{j_0}$ *and* $y^+_l:=\lim_{q\to\infty} f^{n^l_q}(z'_l)\in C^{\varepsilon_i}_{j_l}$.

Proof. By Lemma 4.4.3 and the notation just above its statement, it is not possible to find $g:F\to 2$ satisfying

$$\begin{cases}\forall (C^\varepsilon_j,C^{\varepsilon'}_{j'})\in D\ \ g(C^\varepsilon_j)\neq g(C^{\varepsilon'}_{j'}),\\ \forall (C^\varepsilon_j,C^{\varepsilon'}_{j'})\in E\ \ g(C^\varepsilon_j)=g(C^{\varepsilon'}_{j'}).\end{cases}$$

Note that if $(C^\varepsilon_j,C^{\varepsilon'}_{j'})\in E$, then $(C^\varepsilon_j,C^{1-\varepsilon}_j),(C^{1-\varepsilon}_j,C^{\varepsilon'}_{j'})\in D$. In particular, the first condition (on D) just above implies the second one (on E). So there is no g satisfying the first condition (on D). Lemma 4.1.1 provides $m\in\omega$ and $(\varepsilon_i,j_i)_{i\leq 2m}\in(2\times\kappa)^{2m+1}$ such that $(C^{\varepsilon_i}_{j_i},C^{\varepsilon_{i+1}}_{j_{i+1}})\in D$ for each $i\leq 2m$. We may assume that the sequence $\big((\varepsilon_i,j_i)\big)_{i<n}$ is injective. We set $l:=2m$. If $i\leq l$, then $(C^{\varepsilon_i}_{j_i},C^{\varepsilon_{i+1}}_{j_{i+1}})\in D\cup E$, so that ($\varepsilon_i\neq\varepsilon_{i+1}$ and $j_i=j_{i+1}$), or we can find $z'_i\in X\setminus X'$ and $(m^i_q)_{q\in\omega}$, $(n^i_q)_{q\in\omega}\in\omega^\omega$ with constant parity such that one of the following holds:

$$(\alpha)_i\ y^-_i:=\lim_{q\to\infty} f^{-m^i_q}(z'_i)\in C^{\varepsilon_i}_{j_i} \text{ and } y^+_i:=\lim_{q\to\infty} f^{n^i_q}(z'_i)\in C^{\varepsilon_{i+1}}_{j_{i+1}},$$

$$(\beta)_i\ y^-_i:=\lim_{q\to\infty} f^{-m^i_q}(z'_i)\in C^{\varepsilon_{i+1}}_{j_{i+1}} \text{ and } y^+_i:=\lim_{q\to\infty} f^{n^i_q}(z'_i)\in C^{\varepsilon_i}_{j_i}.$$

Note that, changing ε_{p+1} enough times if necessary, we may assume that $(C^{\varepsilon_i}_{j_i},C^{\varepsilon_{i+1}}_{j_{i+1}})\in E$ if $i<l$, so that $(C^{\varepsilon_l}_{j_l},C^{\varepsilon_0}_{j_0})\in D$. Note then that, canceling $C^{\varepsilon_l}_{j_l}$ if necessary, we may assume that the case when ($\varepsilon_i\neq\varepsilon_{i+1}$ and $j_i=j_{i+1}$) never holds. Also, replacing (z'_i,m^i_q,n^i_q) with $\big(f(z'_i),m^i_q+1,n^i_q-1\big)$ if necessary, we may assume that m^i_0 is even if $i\leq l$. □

4.5 General homeomorphisms

We first study the $\Sigma_{\mathbf{p}}$'s. Recall that the space $\Sigma_{\mathbf{p}}$ is MC with Cantor-Bendixson rank two, and $\sigma_{|\Sigma_{\mathbf{p}}}$ is a homeomorphism of the σ-invariant space $\Sigma_{\mathbf{p}}$. It will be convenient to set, for

$$\mathbf{p}=(l,\lambda_0,\cdots,\lambda_l,m,\varepsilon_0,\cdots,\varepsilon_{l-1})\in\omega^{l+3}\times 2^l,$$

$y_i:=w_i^{\mathbb{Z}}$ if $i\leq l$, $z_i:=w_{i+\varepsilon_i}^{-\infty}\cdot w_{i+1-\varepsilon_i}^{\infty}$ if $i<l$, $z_l:=w_l^{-\infty}\cdot b_0\cdots b_{m-1}(w_0^{\infty})$. It is important to note that $\lim_{q\rightarrow\infty}\ \sigma^{-q\lambda_{i+\varepsilon_i}}(z_i)=y_{i+\varepsilon_i}$ and

$$\lim_{q\rightarrow\infty}\ \sigma^{q\lambda_{i+1-\varepsilon_i}}(z_i)=y_{i+1-\varepsilon_i}$$

if $i<l$. Similarly, $\lim_{q\rightarrow\infty}\ \sigma^{-q\lambda_l}(z_l)=y_l$ and $\lim_{q\rightarrow\infty}\ \sigma^{q\lambda_0+m}(z_l)=y_0$.

Theorem 4.5.1 *Let X be a countable MC space with Cantor-Bendixson rank at most two, and f be a fixed point free homeomorphism of X such that (X,G_f) contains no odd cycle and has no continuous 2-coloring. Then there is $\mathbf{p}\in\mathcal{P}$ such that $(\Sigma_{\mathbf{p}},G_\sigma)\preceq_c^i(X,G_f)$.*

Proof. Note that X is countable, so that X is 0D by 7.12 in [K1]. By Corollary 4.2.2, we may assume that $X\setminus X'$ contains only infinite orbits. Note that X' is finite by Lemma 4.3.1, which gives $\kappa\in\omega$ and a partition $(C_j)_{j\leq\kappa}$ of X' into orbits, which are closed and f-invariant sets. Note that the C_j's have even cardinality, which gives a (continuous) 2-coloring $\overline{c}$ of $(X',G_f\cap(X')^2)$. Lemma 4.4.4 provides l, $C_{j_0}^{\varepsilon_0},\cdots,C_{j_l}^{\varepsilon_l}$, $(z_i')_{i\leq l}$, and, for $i\leq l$, $(m_q^i)_{q\in\omega}$ and $(n_q^i)_{q\in\omega}$. We set, for $i\leq l$, $\lambda_i:=|C_{j_i}|$, so that $\lambda_i>0$ is even. Note that f^{-1} is a homeomorphism and $G_{f^{-1}}=G_f$. So, replacing f and z_l' with f^{-1} and $f(z_l')$ respectively if necessary, we may assume that $(\alpha)_l$ holds. We set, for $i<l$,

$$\varepsilon_i:=\begin{cases}0\text{ if }(\alpha)_i\text{ holds,}\\1\text{ if }(\beta)_i\text{ holds.}\end{cases}$$

We also define, for $i\leq l$, $\overline{i+1}:=i+1\text{ mod }(l+1)$ and $\mathbf{d}_i\in\{-,+\}$ by

$$\mathbf{d}_i:=\begin{cases}-\text{ if }(\alpha)_i\text{ holds,}\\+\text{ if }(\beta)_i\text{ holds,}\end{cases}$$

and we will use the conventions $--=+$ and $-+=-$. We will now show that we may assume that m_q^i or n_q^i is equal to $q\lambda_i$ if the limit coming from Lemma 4.4.4 is in C_{j_i}, except n_q^l that will be $q\lambda_0+m$ with $m<\lambda_0$ odd. We will also ensure that $y_i^{-\mathbf{d}_i}=y_{\overline{i+1}}^{\mathbf{d}_{\overline{i+1}}}$. If x is in $C_{j_i}^0$, then $\text{Orb}_{f^2}(x)\subseteq C_{j_i}^0$ and $\text{Orb}_{f^2}\big(f(x)\big)\subseteq C_{j_i}^1$, showing that $C_{j_i}^{\varepsilon_i}$ is a f^2-orbit. The key fact is as follows.

Claim. *Let $x\in X\setminus X'$, $\mathbf{d}\in\{-,+\}$, $j\leq\kappa$ and $\varepsilon\in 2$ for which there is a sequence $(m_q)_{q\in\omega}$ of natural numbers of constant parity such that $\big(f^{\mathbf{d}m_q}(x)\big)_{q\in\omega}$ converges to a point of C_j^ε. Then there is $y\in C_j^{\varepsilon+par(m_0)}$ such that $\big(f^{\mathbf{d}q|C_j|}(x)\big)_{q\in\omega}$ converges to y.*

Indeed, Lemma 4.3.3 provides $y\in C_j$ such that $\big(f^{\mathbf{d}q|C_j|}(x)\big)_{q\in\omega}$ converges to y. Lemma 4.3.5 implies that $y\in C_j^{\varepsilon+\text{par}(m_0)}$. $\diamond$

Assume first that $l=0$. The claim provides $y^-\in C^{\varepsilon_0}_{j_0}, y^+\in C^{\varepsilon_0+1}_{j_0}$ such that $\big(f^{-q\lambda_0}(z'_0)\big)_{q\in\omega}$ converges to y^- and $\big(f^{q\lambda_0}(z'_0)\big)_{q\in\omega}$ converges to y^+. As $C^{\varepsilon_0}_{j_0}$ is an f^2-orbit, there is $p<|C^{\varepsilon_0}_{j_0}|$ such that $y^-=f^{2p+1}(y^+)$. As $\big(f^{q\lambda_0+2p+1}(z'_0)\big)_{q\in\omega}$ converges to y^-, we are done.

Assume now that $l\geq 1$ and $i=0$. The claim provides

- $y'_0\in C^{\varepsilon_0}_{j_0}$ such that $\big(f^{\mathbf{d}_0 q\lambda_0}(z'_0)\big)_{q\in\omega}$ converges to y'_0,
- $y'_1\in C^{\varepsilon_1}_{j_1}$ such that $\big(f^{-\mathbf{d}_0 q\lambda_1}(z'_0)\big)_{q\in\omega}$ converges to y'_1,
- $y''_1\in C^{\varepsilon_1}_{j_1}$ such that $\big(f^{\mathbf{d}_1 q\lambda_1}(z'_1)\big)_{q\in\omega}$ converges to y''_1.

As $C^{\varepsilon_1}_{j_1}$ is a f^2-orbit, there is $p'<|C^{\varepsilon_1}_{j_1}|$ such that $y'_1=f^{2p'}(y''_1)$. As $\Big(f^{\mathbf{d}_1 q\lambda_1}\big(f^{2p'}(z'_1)\big)\Big)_{q\in\omega}$ converges to y'_1, we are done if we replace z'_1 with $f^{2p'}(z'_1)$, which does not affect our convergence and parity properties. Iterating this process if necessary and arguing as in the case $l=0$, we complete our construction. In other words, possibly changing the z'_i's, we ensured the existence of $(y'_i)_{i\leq l}$ and $m<\lambda_0$ odd satisfying the following.

(a) if $i<l$, then one of the following holds:

$(\alpha)_i$ $y'_i:=\lim_{q\to\infty} f^{-q\lambda_i}(z'_i)\in C^{\varepsilon_i}_{j_i}$ and $y'_{i+1}:=\lim_{q\to\infty} f^{q\lambda_{i+1}}(z'_i)\in C^{\varepsilon_{i+1}}_{j_{i+1}}$,

$(\beta)_i$ $y'_{i+1}:=\lim_{q\to\infty} f^{-q\lambda_{i+1}}(z'_i)\in C^{\varepsilon_{i+1}}_{j_{i+1}}$ and $y'_i:=\lim_{q\to\infty} f^{q\lambda_i}(z'_i)\in C^{\varepsilon_i}_{j_i}$,

(b) $y'_l:=\lim_{q\to\infty} f^{-q\lambda_l}(z'_l)\in C^{\varepsilon_l}_{j_l}$ and $y'_0:=\lim_{q\to\infty} f^{q\lambda_0+m}(z'_l)\in C^{\varepsilon_0}_{j_0}$.

We now completely defined $\mathbf{p}:=(l,\lambda_0,\cdots,\lambda_l,m,\varepsilon_0,\cdots,\varepsilon_{l-1})\in\mathcal{P}$, and are ready to define $h:\Sigma_{\mathbf{p}}\to X$. We set $h\big(\sigma^j(y_i)\big):=f^j(y'_i)$ if $i\leq l$ and $j<\lambda_i$, and $h\big(\sigma^k(z_q)\big):=f^k(z'_q)$ if $q\leq l$ and $k\in\mathbb{Z}$ so that h is an injective homomorphism from $(\Sigma_{\mathbf{p}},G_\sigma)$ into (X,G_f). Assume that $i,q\leq l$ and $(\zeta_n)_{n\in\omega}$ is a sequence of points of Orb(z_q) converging to a point y of Orb(y_i). Let $k_n\in\mathbb{Z}$ with $\zeta_n=\sigma^{k_n}(z_q)$, and $j<\lambda_i$ with $y=\sigma^j(y_i)$. Note that $(k_n)_{n\in\omega}$ tends to ∞ or $-\infty$. Let $i_n\in\mathbb{Z}$ and $0\leq r_n<\lambda_i$ with $k_n=i_n\lambda_i+r_n$. We may assume that r_n is a constant r. Then $h(\zeta_n)=h\big(\sigma^{k_n}(z_q)\big)=f^{k_n}(z'_q)$, and $h(y)=h\big(\sigma^j(y_i)\big)=f^j(y'_i)$. If $q<l$ or k_n tends to $-\infty$, then $\sigma^{k_n}(z_q)$ tends to $\sigma^r(y_i)$, so that $j=r$ and $h(\zeta_n)$ tends to $f^r(y'_i)=h(y)$ as desired. If $q=l$ and k_n tends to ∞, then $\sigma^{k_n}(z_q)$ tends to $\sigma^{r-m}(y_i)$, so that $j=r-m$ and $h(\zeta_n)$ tends to $f^{r-m}(y'_i)=h(y)$ as desired. Thus h is continuous. □

Lemma 4.5.2 *Let $\mathbf{p}\in\mathcal{P}$. Then $(\Sigma_{\mathbf{p}},G_\sigma)$ is $\preceq^i_c$-minimal in $\mathfrak{G}$ and in the class of closed graphs on a MC space having no continuous 2-coloring.*

Proof. Note first that $\sigma_{|\Sigma_{\mathbf{p}}}$ is fixed point free, so that G_σ is closed.

Claim 1. *$(\Sigma_{\mathbf{p}},G_\sigma)$ has no continuous 2-coloring.*

Indeed, we argue by contradiction, which gives $c:\Sigma_{\mathbf{p}}\to 2$. Assume, for example, that $c(z_l)=0$. As λ_l is even, $c\big(\sigma^{-q\lambda_l}(z_l)\big)=0$, so that $c(y_l)=0$ by continuity. As λ_0 is even and m is odd, $c\big(\sigma^{q\lambda_0+m}(z_l)\big)=1$, so that $c(y_0)=1$ by continuity. On the other hand, if $i<l$ and $\varepsilon_i=0$, as $\lambda_{i+\varepsilon_i}$ is even, $c\big(\sigma^{-q\lambda_{i+\varepsilon_i}}(z_i)\big)=c(z_i)$, so that $c(y_i)=c(z_i)$ by continuity. Similarly, as $\lambda_{i+1-\varepsilon_i}$ is even, $c\big(\sigma^{q\lambda_{i+1-\varepsilon_i}}(z_i)\big)=c(z_i)$, so that $c(y_{i+1})=c(z_i)$ by continuity. This shows that $c(y_i)=c(y_{i+1})$ if $i<l$ (even if $\varepsilon_i=1$). Thus $c(y_0)=c(y_l)$, which is the desired contradiction. ◇

This proves that $(\Sigma_{\mathbf{p}}, G_\sigma)$ is in our classes. We now set $S := \bigcup_{i\leq l} \ \text{Orb}_\sigma(z_i)$, so that S is a dense open subset of $\Sigma_{\mathbf{p}}$, and $F_2^{\Sigma_{\mathbf{p}},\sigma}$ is nowhere dense in $\Sigma_{\mathbf{p}}$.

Claim 2. *Let $V \subseteq \Sigma_{\mathbf{p}}$, H be a graph on V contained in G_σ such that (V, H) has no continuous 2-coloring, and $x \in S$. Then $\big(x, \sigma(x)\big) \in H$.*

Indeed, we argue by contradiction. Recall that the sets of the form

$$[w]_q := \{y \in A_{\mathbf{p}}^{\mathbb{Z}} \mid \forall j < |w| \ \ w(j) = y(q+j)\},$$

where $w \in A_{\mathbf{p}}^{<\omega}$ and $q \in \mathbb{Z}$, form a basis made of clopen subsets of the space $A_{\mathbf{p}}^{\mathbb{Z}}$. Assume first that $x = z_l$. We set $C := (\bigcup_{i\leq l, j<\lambda_i \text{ even}} [a_j^i]_0 \cup [b_0]_0 \cup \bigcup_{j<m \text{ odd}} [b_j]_0) \cap V$, so that C is a clopen subset of V and $H \cap \big(C^2 \cup (V \setminus C)^2\big) = \emptyset$. Thus C defines a continuous 2-coloring of (V, H), which is the desired contradiction. If there is $k \in \mathbb{Z}$ with $x = \sigma^k(z_l)$, then we just replace the basic clopen sets of the form $[w]_0$ in the definition of C with $[w]_{-k}$, the rest of the argument is the same. Similarly, if $i < l$ and $x \in \text{Orb}(z_i)$, then we may assume that $x = z_i$. If $\varepsilon_i = 0$, then we set

$$C := \Big(\bigcup_{n\leq i, j<\lambda_n \text{ even}} [a_j^n]_0 \cup [a_{\lambda_i-1}^i a_0^{i+1}]_{-1} \cup \bigcup_{i<n\leq l, j<\lambda_n \text{ odd}} [a_j^n]_0 \cup \bigcup_{j<m \text{ odd}} [b_j]_0 \Big) \cap V$$

and conclude similarly. If $\varepsilon_i = 1$, then we set

$$C := \Big(\bigcup_{n\leq i, j<\lambda_n \text{ odd}} [a_j^n]_0 \cup [a_{\lambda_{i+1}-1}^{i+1} a_0^i]_{-1} \cup \bigcup_{i<n\leq l, j<\lambda_n \text{ even}} [a_j^n]_0 \cup \bigcup_{j<m \text{ even}} [b_j]_0 \Big) \cap V$$

and conclude similarly. $\diamond$

It remains to apply our claims and Lemma 4.4.1. $\square$

Proof of Theorem 1.4 (a). Let X be a countable MC space with Cantor-Bendixson rank at most two, and f be a homeomorphism of X such that (X, G_f) has no continuous 2-coloring. As X is countable, X is 0D by 7.12 in [K1]. By Corollary 4.4.2, we may assume that (X, G_f) contains no odd cycle. If F_f is not open in X, then $(\mathbb{X}_1, G_{f_1}) \preceq_c^i (X, G_f)$ by Corollary 2.3. So we may assume that F_f is open in X. This implies that $(X \setminus F_f, G_f \cap (X \setminus F_f)^2)$ has no continuous 2-coloring, by Corollary 2.2. Thus we may assume that f is fixed point free. It remains to apply Theorem 4.5.1 to get the basis result.

Assume that $\mathbf{p}, \mathbf{p}' \in \mathcal{P}$ and $(\Sigma_{\mathbf{p}}, G_\sigma) \preceq_c^i (\Sigma_{\mathbf{p}'}, G_\sigma)$ with h as a witness. As $\sigma_{|\Sigma_{\mathbf{p}}}$ is fixed point free and $F_2^{\Sigma_{\mathbf{p}},\sigma}$ is nowhere dense in $\Sigma_{\mathbf{p}}$, Lemmas 3.2 and 3.3 show that h sends any orbit onto an orbit of the same size. This shows that the number of orbits of $\Sigma_{\mathbf{p}}$ is at most the finite number of orbits of $\Sigma_{\mathbf{p}'}$, by injectivity of h. By Lemma 4.5.2, $(\Sigma_{\mathbf{p}'}, G_\sigma) \preceq_c^i (\Sigma_{\mathbf{p}}, G_\sigma)$, so that $\Sigma_{\mathbf{p}}$ and $\Sigma_{\mathbf{p}'}$ have the same number of orbits, in bijection by h. In particular, $\Sigma_{\mathbf{p}}$ and $\Sigma_{\mathbf{p}'}$ have the same number of finite orbits, i.e., $l = l'$. This also shows that if $\Lambda := \max_{i\leq l} \lambda_i$, then $\Lambda = \max_{i\leq l} \lambda'_i$. Note also that $m < \lambda_0 \leq \Lambda$ and, similarly, $m' < \Lambda$. This shows that $\mathbf{p}' \in \{l\} \times (\Lambda+1)^{l+1} \times \Lambda \times 2^l$, so that $\mathcal{F}_{\mathbf{p}}$ is finite, and $\mathbf{m}_{\mathbf{p}} := \min_{\text{lex}} \mathcal{F}_{\mathbf{p}}$ is defined.

Claim. *There is $\mathcal{P}_a \subseteq \mathcal{P}$, obtained by choosing $\min_{lex} \mathcal{F}_{\mathbf{p}}$ for each $\mathbf{p} \in \mathcal{P}$, with the properties that $\{(\Sigma_{\mathbf{m}}, G_\sigma) \mid \mathbf{m} \in \mathcal{P}_a\}$ is a basis for $\{(\Sigma_{\mathbf{p}}, G_\sigma) \mid \mathbf{p} \in \mathcal{P}\}$ and*

$$\{(2q+3, C_{2q+3}) \mid q \in \omega\} \cup \{(\mathbb{X}_1, G_{f_1})\} \cup \{(\Sigma_{\mathbf{m}}, G_\sigma) \mid \mathbf{m} \in \mathcal{P}_a\}$$

is an antichain.

Indeed, let $\mathcal{P}_a := \{\mathbf{m}_\mathbf{p} \mid \mathbf{p} \in \mathcal{P}\}$, so that $\{(\Sigma_\mathbf{m}, G_\sigma) \mid \mathbf{m} \in \mathcal{P}_a\}$ is an antichain basis for $\{(\Sigma_\mathbf{p}, G_\sigma) \mid \mathbf{p} \in \mathcal{P}\}$. As $\mathbb{X}_1$ and the $\Sigma_\mathbf{p}$'s are infinite, the odd cycles are not above the other graphs. As these graphs contain no odd cycle, they are not below them. As a homomorphism sends an odd cycle of cardinality l into an odd cycle of cardinality at most l and by injectivity, the odd cycles form antichain. We saw that $\sigma_{|\Sigma_\mathbf{p}}$ is fixed point free, so that $(\Sigma_\mathbf{p}, G_\sigma)$ has a continuous $\aleph_0$-coloring, by Corollary 2.2. Corollaries 2.2 and 2.3 imply that $(\mathbb{X}_1, G_{f_1})$ has no continuous $\aleph_0$-coloring. Thus the $(\Sigma_\mathbf{p}, G_\sigma)$'s are not above $(\mathbb{X}_1, G_{f_1})$. As $(\Sigma_\mathbf{p}, G_\sigma)$ has an infinite orbit, the orbits of $(\mathbb{X}_1, G_{f_1})$ have size at most two and an orbit has to be sent into an orbit, $(\mathbb{X}_1, G_{f_1})$ is not above the $(\Sigma_\mathbf{p}, G_\sigma)$'s, by injectivity. $\diamond$

By the claim, $\{(2q+3, C_{2q+3}) \mid q \in \omega\} \cup \{(\mathbb{X}_1, G_{f_1})\} \cup \{(\Sigma_\mathbf{m}, G_\sigma) \mid \mathbf{m} \in \mathcal{P}_a\}$ is our antichain basis. $\square$

4.6 Subshifts

We first prove a lemma useful to prove Theorem 1.6 (a). Note that the fixed points of the shift are the constant sequences, of the form $a^\mathbb{Z} \in \Sigma$ with $a \in A$.

Lemma 4.6.1 *Let $\Sigma \subseteq A^\mathbb{Z}$ be a countable two-sided subshift with Cantor-Bendixson rank at most two, $l \in \omega$, $a_0, \cdots, a_l \in A$, and $(x_n)_{n\in\omega}$ be an injective sequence of points of Σ converging to $(a_0 \cdots a_l)^\mathbb{Z}$. Then the sequence $\big(Orb(x_n)\big)_{n\in\omega}$ is eventually constant, and we can find $s \in A^{l+1} \setminus \{(a_0 \cdots a_l)\}$ and $\gamma \in A^\omega$ with $(a_0 \cdots a_l)^{-\infty} \cdot s\gamma \in \Sigma$ or $\gamma^{-1} s \cdot (a_0 \cdots a_l)^\infty \in \Sigma$.*

Proof. For the last assertion, we can apply Lemma 3.4. We argue by contradiction, so that we may assume that the sequence $\big(\text{Orb}(x_n)\big)_{n\in\omega}$ is injective. We may also assume, for example, that

$$x_{n[-k_n(l+1), k_n(l+1)-1]} = (a_0 \cdots a_l)^{2k_n},$$

$x_{n[k_n(l+1), (k_n+1)(l+1)-1]}$ is a constant $s \neq (a_0 \cdots a_l)$, and $k_n \to \infty$. By compactness, we may assume that the sequence $(x_{n[k_n(l+1), \infty)})_{n\in\omega}$ converges to some $s\gamma \in A^\omega$. We put $x := (a_0 \cdots a_l)^{-\infty} \cdot s\gamma$, so that $x \in \Sigma$. Note that x is the limit of $\big(\sigma^{k_n(l+1)}(x_n)\big)_{n\in\omega}$. As the sequence $\big(\text{Orb}(x_n)\big)_{n\in\omega}$ is injective, this sequence $\big(\sigma^{k_n(l+1)}(x_n)\big)_{n\in\omega}$ is also injective, so that $x \in \Sigma'$. Thus $\text{Orb}(x)$ is finite of cardinality κ, and $x = \sigma^{-\kappa(l+1)}(x)$, contradicting $s \neq (a_0 \cdots a_l)$. $\square$

We now provide an antichain basis when fixed points exist.

Lemma 4.6.2 *Let Σ be a countable two-sided subshift with Cantor-Bendixson rank at most two such that $F_{\sigma_{|\Sigma}}$ is not open in Σ. Then there is $n \in \omega$ with the property that $({}_n\Sigma, G_\sigma) \preceq^i_c (\Sigma, G_\sigma)$. Moreover, $\{({}_n\Sigma, G_\sigma) \mid n \in \omega\}$ is a $\preceq^i_c$-antichain.*

Proof. Assume that $\Sigma \subseteq A^\mathbb{Z}$ and $(x_n)_{n\in\omega}$ is an injective sequence of points of $\Sigma \setminus \{a^\mathbb{Z} \mid a \in A\}$ converging to $a^\mathbb{Z} \in \Sigma$. By Lemma 4.6.1, we may assume that $\text{Orb}(x_n) = \text{Orb}(x_0)$ for each n, and that x_0 is of the form $a^{-\infty} \cdot b\gamma$ or $\gamma^{-1} b \cdot a^\infty$ with $b \in A \setminus \{a\}$ and $\gamma \in A^\omega$, so that $x_0 \in \Sigma \setminus \Sigma'$. Lemma 4.3.3 provides $y^-, y^+ \in \Sigma'$, and at least one of them is $a^\mathbb{Z}$. The other one is of the form $(a_0 \cdots a_l)^\mathbb{Z}$, where $l \in \omega$ and $a_0, \cdots, a_l \in A$. If $l \geq 2$ is even, then the map $(0 \cdots l)^\mathbb{Z} \mapsto (a_0 \cdots a_l)^\mathbb{Z}$ is a witness for the fact that $({}_{l+1}\Sigma, G_\sigma) \preceq^i_c (\Sigma, G_\sigma)$.

If $l=0$ and $a_0=a$, then the map defined by $0^{\mathbb{Z}}\mapsto a^{\mathbb{Z}}$ and $\sigma^k(0^{-\infty}\cdot 10^{\infty})\mapsto\sigma^k(x_0)$ when $k\in\mathbb{Z}$ is a witness for the fact that $({}_0\Sigma, G_\sigma)\preceq_c^i(\Sigma, G_\sigma)$.

If $l=0$ and $a_0\neq a$, then the map defined by

- $0^{\mathbb{Z}}\mapsto a^{\mathbb{Z}}$, $1^{\mathbb{Z}}\mapsto a_0^{\mathbb{Z}}$ and $\sigma^k(0^{-\infty}\cdot 1^{\infty})\mapsto\sigma^k(x_0)$ when $k\in\mathbb{Z}$ and $x_0=a^{-\infty}\cdot b\gamma$,
- $0^{\mathbb{Z}}\mapsto a_0^{\mathbb{Z}}$, $1^{\mathbb{Z}}\mapsto a^{\mathbb{Z}}$ and $\sigma^k(0^{-\infty}\cdot 1^{\infty})\mapsto\sigma^k(x_0)$ when $k\in\mathbb{Z}$ and $x_0=\gamma^{-1}b\cdot a^{\infty}$,

is a witness for the fact that $({}_1\Sigma, G_\sigma)\preceq_c^i(\Sigma, G_\sigma)$.

If l is odd, then the map defined by

- $0^{\mathbb{Z}}\mapsto a^{\mathbb{Z}}$, $\sigma^i\big((1,\cdots,l+1)^{\mathbb{Z}}\big)\mapsto\sigma^i\big((a_0\cdots a_l)^{\mathbb{Z}}\big)$ when $i\leq l$ and $\sigma^k(0^{-\infty}\cdot(1,\cdots,l+1)^{\infty})\mapsto\sigma^k(x_0)$ when $k\in\mathbb{Z}$ and $x_0=a^{-\infty}\cdot b\gamma$,
- $0^{\mathbb{Z}}\mapsto a^{\mathbb{Z}}$, $\sigma^i\big((1,\cdots,l+1)^{\mathbb{Z}}\big)\mapsto\sigma^{-i}\big((a_0\cdots a_l)^{\mathbb{Z}}\big)$ when $i\leq l$ and $\sigma^k(0^{-\infty}\cdot(1,\cdots,l+1)^{\infty})\mapsto\sigma^{-k}(x_0)$ when $k\in\mathbb{Z}$ and $x_0=\gamma^{-1}b\cdot a^{\infty}$,

is a witness for the fact that $({}_{l+1}\Sigma, G_\sigma)\preceq_c^i(\Sigma, G_\sigma)$.

By Theorem 1.4, the odd cycles $({}_{2q+3}\Sigma, G_\sigma)$ form antichain. The other $({}_n\Sigma, G_\sigma)$'s are infinite and contain no odd cycle, so they are incomparable with the odd cycles. Assume, towards a contradiction, that $m\neq n$ and $({}_m\Sigma, G_\sigma)\preceq_c^i({}_n\Sigma, G_\sigma)$ with h as a witness. The previous discussion shows that we may assume that m, n are not of the form $2q+3$. This implies that the two subshifts have a unique infinite orbit, which is dense. Note that an orbit has to be sent into an orbit. In particular, the infinite orbit $\text{Orb}(x_m)$ of $({}_m\Sigma, G_\sigma)$ has to be sent into that $\text{Orb}(x_n)$ of $({}_n\Sigma, G_\sigma)$, by injectivity. Note that $h[\text{Orb}(x_m)]=\text{Orb}(x_n)$, and $h\circ\sigma=\sigma\circ h$ on $\text{Orb}(x_m)$ or $h\circ\sigma=\sigma^{-1}\circ h$ on $\text{Orb}(x_m)$, by Lemma 3.2. In particular, $h[{}_m\Sigma\backslash\text{Orb}(x_m)]\subseteq{}_n\Sigma\backslash\text{Orb}(x_n)$ by injectivity, and $h\circ\sigma=\sigma\circ h$ or $h\circ\sigma=\sigma^{-1}\circ h$, by density. Thus $m<n$, by injectivity. If $m=0$, then $\big(\sigma^{\mathbf{d}i}(x_m)\big)_{i\in\omega}$ converges to $0^{\mathbb{Z}}$ for each $\mathbf{d}\in\{-,+\}$, so that $\Big(\sigma^{-i}\big(h(x_m)\big)\Big)_{i\in\omega}$, $\Big(\sigma^{i}\big(h(x_m)\big)\Big)_{i\in\omega}$ have the same limit, which cannot be. If $m=1$, then $\big(\sigma^{\mathbf{d}i}(x_m)\big)_{i\in\omega}$ converges for each $\mathbf{d}\in\{-,+\}$, so that $\Big(\sigma^{\mathbf{d}i}\big(h(x_m)\big)\Big)_{i\in\omega}$ also converges for each $\mathbf{d}\in\{-,+\}$, which cannot be. If $m=2q+2$, then $h\big[\text{Orb}\big((1,\cdots,m)^{\mathbb{Z}}\big)\big]=\text{Orb}\big((1,\cdots,n)^{\mathbb{Z}}\big)$, which cannot be. □

Proof of Theorem 1.6 (a). Let Σ be a countable two-sided subshift with Cantor-Bendixson rank at most two such that (Σ, G_σ) has no continuous 2-coloring. If $F_{\sigma_{|\Sigma}}$ is not open in Σ, then Lemma 4.6.2 provides $n\in\omega$ such that $({}_n\Sigma, G_\sigma)\preceq_c^i(\Sigma, G_\sigma)$. If $F_{\sigma_{|\Sigma}}$ is open in Σ, then Corollary 2.2 implies that $(\Sigma\setminus F_{\sigma_{|\Sigma}}, G_\sigma)$ has no continuous 2-coloring. Thus we may assume that $\sigma_{|\Sigma}$ is fixed point free. We may also assume that (Σ, G_σ) contains no odd cycle. It remains to apply Theorem 4.5.1 to get the basis result.

The set $\mathcal{P}_a\subseteq\mathcal{P}$ provided by the claim in the proof of Theorem 1.4 has the properties that $\{(\Sigma_{\mathbf{m}}, G_\sigma)\mid\mathbf{m}\in\mathcal{P}_a\}$ is a basis for $\{(\Sigma_{\mathbf{p}}, G_\sigma)\mid\mathbf{p}\in\mathcal{P}\}$ and

$$\{({}_{2q+3}\Sigma, G_\sigma)\mid q\in\omega\}\cup\{(\Sigma_{\mathbf{m}}, G_\sigma)\mid\mathbf{m}\in\mathcal{P}_a\}$$

is an antichain. By Lemma 4.6.2, $\{({}_n\Sigma, G_\sigma)\mid n\in\omega\}$ is also an antichain. We saw that $\sigma_{|\Sigma_{\mathbf{p}}}$ is fixed point free, so that $(\Sigma_{\mathbf{p}}, G_\sigma)$ has a continuous $\aleph_0$-coloring, by Corollary 2.2. We also saw after Corollary 2.2 that $({}_n\Sigma, G_\sigma)$ has no continuous $\aleph_0$-coloring if n is 1 or even. Thus the $(\Sigma_{\mathbf{p}}, G_\sigma)$'s are not above $({}_n\Sigma, G_\sigma)$ if n is 1 or even.

Let us check that $({}_n\Sigma, G_\sigma)$ is not above the $(\Sigma_{\mathbf{p}}, G_\sigma)$'s if n is 1 or even. We argue by contradiction, which provides $h : \Sigma_{\mathbf{p}} \to {}_n\Sigma$. Note that $h[\Sigma_{\mathbf{p}}]$ is compact and does not contain a fixed point of $\sigma_{|{}_n\Sigma}$. This provides a continuous 2-coloring of $(h[\Sigma_{\mathbf{p}}], (h\times h)[G_\sigma])$, and thus one of $(\Sigma_{\mathbf{p}}, G_\sigma)$, which is not possible by Claim 1 in the proof of Lemma 4.5.2. This shows that $\{({}_n\Sigma, G_\sigma) \mid n\in\omega\} \cup \{(\Sigma_{\mathbf{m}}, G_\sigma) \mid \mathbf{m}\in\mathcal{P}_a\}$ is our antichain basis. □

5 Basis of size continuum

Recall the definition of $\mathcal{C}_\kappa$, $\mathcal{H}_\kappa$, and $\mathcal{S}_\kappa$, before the proof of Theorem 1.1 (a), Lemma 3.3, and Lemma 3.5, respectively. Theorem 1.17 (b) in [L] provides a $\preceq^i_c$-antichain $\big((\Sigma_\nu, G_\sigma)\big)_{\nu\in 2^\omega}$ made of countable two-sided subshifts with Cantor-Bendixson rank three such that $\sigma_{|\Sigma_\nu}$ is fixed point free (and thus $G_{\sigma_{|\Sigma_\nu}}$ is closed), (Σ_ν, G_σ) has a continuous 3-coloring and is $\preceq^i_c$-minimal in $\mathcal{C}_2$ and in $\mathcal{H}_2$. This proves Theorem 1.1 (b) for $\kappa=2$, finishes the proof of Theorem 1.2 (b), proves Theorem 1.5 (b) for $\kappa=2$, and proves the second part of Theorems 1.4 (b) (the first part comes from Theorem 1.2), and 1.6 (b) (the first part comes from Theorem 1.5). The proof of Theorem 1.5 (b) for $\kappa\geq 3$ is partly similar, so we recall the construction of Theorem 1.17 (b) in [L].

Notation. Let $\alpha_0 := (01)^{-\infty}\cdot(01)^\infty$, $\alpha_1 := (01)^{-\infty}\cdot 1^2(01)^\infty$, $Q := (q_j)_{j\in\omega}\in\omega^\omega$ converging to infinity, and $\beta_Q := (01)^{-\infty}\cdot 1{}^\frown{}_{j\in\omega}\,\big((01)^{q_j}1^2\big)$. This defines $\Sigma_Q = \bigcup_{m\leq 1}\ \text{Orb}_\sigma(\alpha_m)\cup\text{Orb}_\sigma(\beta_Q)$.

Note that Σ_Q is a countable two-sided subshift. By Claim 8 in the proof of Theorem 1.17 (b) in [L], Σ_Q has Cantor-Bendixson rank three, and the remark after this claim shows that (Σ_Q, G_σ) has a continuous 3-coloring. The proof of Theorem 1.17 (b) in [L] also shows the minimality of (Σ_Q, G_σ). In order to get the antichain, we consider the sequence $(p_n)_{n\in\omega}$ of prime numbers. We set, for $\nu\in 2^\omega$ and $n\in\omega$, $q^\nu_0 := 0$ and $q^\nu_{n+1} := p_0^{\nu(0)+2}\cdots p_n^{\nu(n)+2}-1$, which defines $Q^\nu\in\omega^\omega$ converging to infinity. Then $\Sigma_\nu := \Sigma_{Q_\nu}$.

Proof of Theorem 1.5 (b) for $\kappa\geq 3$. Let $L := (l_j)_{j\in\omega}\in\omega^\omega$ converging to infinity, $\gamma_0 := 0^{\mathbb{Z}}$, $\gamma_1 := (01)^{\mathbb{Z}}$, $\gamma_2 := (01)^{-\infty}\cdot 1^2(01)^\infty$, and $\delta_L := 0^{-\infty}\cdot{}^\frown{}_{j\in\omega}\,\big((01)^{l_j}1^2\big)$. This defines as above $\Sigma_L = \bigcup_{m\leq 2}\ \text{Orb}_\sigma(\gamma_m)\cup\text{Orb}_\sigma(\delta_L)$. Note that Σ_L is a countable two-sided subshift. As $\gamma_0\in F_{\sigma_{|\Sigma_L}}$ is the limit of $\big(\sigma^{-n}(\delta_L)\big)_{n\in\omega}\in(\Sigma_L\setminus F_{\sigma_{|\Sigma_L}})^\omega$, $F_{\sigma_{|\Sigma_L}}$ is not an open subset of Σ_L. By Corollary 2.2, there is no continuous $\aleph_0$-coloring of $G_{\sigma_{|\Sigma_L}}$, so that $(\Sigma_L, G_\sigma)\in\mathcal{S}_\kappa$.

Claim. (Σ_L, G_σ) *is* $\preceq^i_c$*-minimal in* $\mathcal{S}_\kappa$*.*

Indeed, let $(\Sigma, G_\sigma)\in\mathcal{S}_\kappa$ such that $(\Sigma, G_\sigma)\preceq^i_c(\Sigma_L, G_\sigma)$ with h as a witness. We first prove that there is $(\Sigma', G_\sigma)\in\mathcal{S}_\kappa$ with $(\Sigma', G_\sigma)\preceq^i_c(\Sigma, G_\sigma)$ and $F_2^{\Sigma',\sigma_{|\Sigma'}}$ is nowhere dense in Σ'. We argue as in the proof of the claim in the proof of Theorem 1.2 (a), by contradiction. We inductively construct a strictly $\subseteq$-decreasing sequence $(\Sigma_\xi)_{\xi<\aleph_1}$ such that $\Sigma_0=\Sigma$, Σ_ξ is σ-invariant and $(\Sigma_\xi, G_\sigma\cap\Sigma_\xi^2)\in\mathcal{S}_\kappa$, which will contradict the fact that Σ is a 0DMC space. Assume that Σ_ξ is constructed. Note that $F_{\sigma_{|\Sigma_\xi}}$ is finite. Let $I := \{x\in F_{\sigma_{|\Sigma_\xi}} \mid x \text{ is isolated in } \Sigma_\xi\}$. Note that I is a finite σ-invariant clopen subset of Σ_ξ, $G_{\sigma_{|\Sigma_\xi}} = G_{\sigma_{|\Sigma_\xi\setminus I}}$, $(\Sigma_\xi\setminus I, G_\sigma)\in\mathcal{S}_\kappa$, and $(\Sigma_\xi\setminus I, G_\sigma)\preceq^i_c(\Sigma_\xi, G_\sigma)$. So, restricting Σ_ξ to $\Sigma_\xi\setminus I$ if necessary, we may assume that $F_{\sigma_{|\Sigma_\xi}}$ is nowhere dense in Σ_ξ. Note that $F_2^{\Sigma_\xi,\sigma_{|\Sigma_\xi}}$ is closed and not nowhere dense in Σ_ξ.

This gives a nonempty clopen subset C of Σ_ξ with the property that $C\subseteq F_2^{\Sigma_\xi,\sigma_{|\Sigma_\xi}}\setminus F_{\sigma_{|\Sigma_\xi}}$. Note that $U:=C\cup\sigma[C]$ is a nonempty clopen σ-invariant subset of Σ_ξ contained in $F_2^{\Sigma_\xi,\sigma_{|\Sigma_\xi}}\setminus F_{\sigma_{|\Sigma_\xi}}$. In particular, U is a ODM separable space and $\sigma_{|U}$ is a fixed point free continuous involution. Proposition 7.5 in [L] provides a continuous 2-coloring of $(U,G_{\sigma_{|U}})$. All this implies that $\Sigma_{\xi+1}:=\Sigma_\xi\setminus U\subsetneqq\Sigma_\xi$, $\Sigma_{\xi+1}$ is σ-invariant and $(\Sigma_{\xi+1},G_\sigma\cap\Sigma_{\xi+1}^2)\in\mathcal{S}_\kappa$. If $(\lambda_p)_{p\in\omega}$ is strictly increasing and $\lambda=\sup_{p\in\omega}\lambda_p$ is a limit ordinal, then we set $\Sigma_\lambda:=\bigcap_{p\in\omega}\ \Sigma_{\lambda_p}$. As $\kappa\geq 3$, the $(\Sigma_{\lambda_p},G_\sigma)$'s are in $\mathcal{S}_{\aleph_0}$, by Theorem 1.12 in [L]. By Lemma 3.5, $(\Sigma_\lambda,G_\sigma)\in\mathcal{S}_\kappa$. As $\Sigma_\lambda\subsetneqq\Sigma_{\lambda_p}$ for each $p\in\omega$, we are done. In other words, we may assume that $F_2^{\Sigma,\sigma_{|\Sigma}}$ is nowhere dense in Σ. By Lemmas 3.2, 3.3, h sends an orbit of size at least two onto an orbit of the same size. We set $P:=h^{-1}\big(\text{Orb}_\sigma(\delta_L)\big)$. If P is contained in $F_{\sigma_{|\Sigma}}$, then P is finite since a two-sided subshift has only finitely many fixed points. Moreover, these points are isolated in Σ since so are the elements of $\text{Orb}_\sigma(\delta_L)$ in Σ_L. This shows that P is a finite σ-invariant clopen subset of Σ. In particular, $\Sigma\setminus P$ is a σ-invariant clopen subset of Σ and $(\Sigma\setminus P,G_\sigma)\in\mathcal{S}_\kappa$. On the other hand, $(\Sigma\setminus P,G_\sigma)\preceq_c^i(\bigcup_{m\leq 2}\ \text{Orb}_\sigma(\gamma_m),G_\sigma)$, which has a continuous 2-coloring, which cannot be. This shows that P contains an element of $\Sigma\setminus F_{\sigma_{|\Sigma}}$. Thus the dense set $\text{Orb}_\sigma(\delta_L)$ is contained in the compact set $h[\Sigma]$ by the previous size argument, proving that h is onto, and thus a homeomorphism by compactness. In particular, P is a dense orbit $\text{Orb}_\sigma(x)$. By Lemma 3.2, there is $\theta\in\{-1,1\}$ such that $h\circ\sigma_{|\Sigma}=\sigma^\theta_{|\Sigma_L}\circ h$ on $\text{Orb}_\sigma(x)$, and thus on Σ. In particular, $(h\times h)[G_{\sigma_{|\Sigma}}]=G_{\sigma_{|\Sigma_L}}$ and $(\Sigma_L,G_\sigma)\preceq_c^i(\Sigma,G_\sigma)$ with h^{-1} as a witness. $\diamond$

We now define L^ν as we defined Q^ν just before this proof. It remains to check that the family $\big((\Sigma_{L^\nu},G_{\sigma_{|\Sigma_{L^\nu}}})\big)_{\nu\in 2^\omega}$ is a $\preceq_c^i$-antichain. Assume, towards a contradiction, that $\nu\neq\nu'$ and

$$(\Sigma_{L^\nu},G_{\sigma_{|\Sigma_{L^\nu}}})\preceq_c^i(\Sigma_{L^{\nu'}},G_{\sigma_{|\Sigma_{L^{\nu'}}}})$$

with h as a witness. Let m_0 be minimal with the property that $\nu(m_0)\neq\nu'(m_0)$. By minimality of $(\Sigma_{L^{\nu'}},G_{\sigma_{|\Sigma_{L^{\nu'}}}})$, we may assume that $\nu(m_0)$ is smaller than $\nu'(m_0)$. By Lemma 3.2, $h[\text{Orb}_\sigma(\gamma_2)]$, $h[\text{Orb}_\sigma(\delta_{L^\nu})]$ are disjoint infinite orbits in $\Sigma_{L^{\nu'}}$, so they are $\text{Orb}_\sigma(\gamma_2)$, $\text{Orb}_\sigma(\delta_{L^{\nu'}})$. As $\text{Orb}_\sigma(\delta_{L^{\nu'}})$ is dense in $\Sigma_{L^{\nu'}}$, the compact set $h[\Sigma_{L^\nu}]$ is $\Sigma_{L^{\nu'}}$, so that h is a homeomorphism from Σ_{L^ν} onto $\Sigma_{L^{\nu'}}$. Moreover, h is a witness for the fact that $\sigma_{|\Sigma_{L^\nu}}$ and $\sigma_{|\Sigma_{L^{\nu'}}}$ are flip-conjugate, by density of $\text{Orb}_\sigma(\delta_{L^\nu})$ in Σ_{L^ν} and Lemma 3.2. In particular, $h[\Sigma'_{L^\nu}]=\Sigma'_{L^{\nu'}}$ and $h[\Sigma''_{L^\nu}]=\Sigma''_{L^{\nu'}}$, so that

$$h[\text{Orb}_\sigma(\gamma_1)]=\text{Orb}_\sigma(\gamma_1),$$

$h[\text{Orb}_\sigma(\gamma_2)]=\text{Orb}_\sigma(\gamma_2)$, $h[\text{Orb}_\sigma(\delta_{L^\nu})]$ is $\text{Orb}_\sigma(\delta_{L^{\nu'}})$ and $h(\gamma_0)=\gamma_0$. This gives $n_0,n_1\in\mathbb{Z}$ with $h(\gamma_2)=\sigma^{n_1}(\gamma_2)$ and $h(\delta_{L^\nu})=\sigma^{n_0}(\delta_{L^{\nu'}})$. We then set, for $r\in\omega$, $K_r^\nu:=\big(\Sigma_{j<r}\ (2l_j+2)\big)+2l_r$. Note that the sequence $\big(\sigma^{K_r^\nu}(\delta_{L^\nu})\big)_{r\in\omega}$ converges to γ_2, so that $\Big(h\big(\sigma^{K_r^\nu}(\delta_{L^\nu})\big)\Big)_{r\in\omega}$ converges to $h(\gamma_2)=\sigma^{n_1}(\gamma_2)$. As $h\big(\sigma^{K_r^\nu}(\delta_{L^\nu})\big)=\sigma^{n_0\pm K_r^\nu}(\delta_{L^{\nu'}})$, this implies that $\big(\sigma^{n_0-n_1\pm K_r^\nu}(\delta_{L^{\nu'}})\big)_{r\in\omega}$ converges to γ_2. As $(K_r^\nu)_{r\in\omega}$ is strictly increasing, this implies that $\sigma_{|\Sigma_{L^\nu}}$ and $\sigma_{|\Sigma_{L^{\nu'}}}$ are conjugate and $\big(\sigma^{n_0-n_1+K_r^\nu}(\delta_{L^{\nu'}})\big)_{r\in\omega}$ converges to γ_2. In particular,

$$\sigma^{n_0-n_1+K_r^\nu}(\delta_{L^{\nu'}})_{[-2,2]}=\big(\delta_{L^{\nu'}}(n_0-n_1+K_r^\nu-2),\cdots,\delta_{L^{\nu'}}(n_0-n_1+K_r^\nu+2)\big)=\gamma_{2[-2,2]}=01^30$$

if r is large enough. Using similar notation, this implies that $n_0-n_1+K_r^\nu\in\{K_m^{\nu'}\mid m\in\omega\}$ if r is large enough.

In particular, this gives, for r large enough, $m<M\in\omega$ with $n_0-n_1+K^\nu_r=K^{\nu'}_m$ and

$$n_0-n_1+K^\nu_{r+1}=K^{\nu'}_M.$$

Thus $K^\nu_{r+1}-K^\nu_r=2l^\nu_{r+1}+2=\Sigma_{m<j\leq M}\ (2l^{\nu'}_j+2)$ and

$$p_0^{\nu(0)+2}\cdots p_r^{\nu(r)+2}=l^\nu_{r+1}+1=\Sigma_{m\leq n<M}\ (l^{\nu'}_{n+1}+1)=\Sigma_{m\leq n<M}\ (p_0^{\nu'(0)+2}\cdots p_n^{\nu'(n)+2}).$$

We may assume that r is large enough to ensure that $r,m\geq m_0$, which implies that $p_{m_0}^{\nu'(m_0)+2}$ divides $p_0^{\nu(0)+2}\cdots p_r^{\nu(r)+2}$, which cannot be since $\nu(m_0)<\nu'(m_0)$. $\square$

It remains to prove Theorem 1.1 (b) for $\kappa\geq 3$.

Proof of Theorem 1.1 (b) for $\kappa\geq 2$. We set, for $\varepsilon\in\kappa$, $\varepsilon^+:=(\varepsilon+1)$ mod κ. We then set, for $\nu\in(\omega\backslash\{0\})^\omega$ and $j,k\in\omega$, $\beta_0^{\nu,k,j}:=0^{2+j+\Sigma_{i<k}\nu(i)}1^\infty$ if $j<\nu(k)$, $\beta_\varepsilon^{\nu,2k+1,j}:=\varepsilon^{2+j+\Sigma_{i<2k+1}\nu(i)}(\varepsilon^+)^\infty$ if $0<\varepsilon<\kappa$ and $j<\nu(2k+1)$, and $\beta_\varepsilon^{\nu,2k,j}:=\varepsilon^{2+j+\Sigma_{i<2k}\nu(i)}(\varepsilon^+)^\infty$ if $\kappa\leq\varepsilon<2\kappa-1$ and $j<\nu(2k)$. This allows us to define the (countable) set of vertices

$$\begin{aligned}X_\nu:=\{\varepsilon^\infty\mid\varepsilon\in 2\kappa-1\}\cup\{01^\infty\}\cup\{\beta_0^{\nu,k,j}\mid k\in\omega\wedge j<\nu(k)\}\cup\\ \{\beta_\varepsilon^{\nu,2k+1,j}\mid k\in\omega\wedge 0<\varepsilon<\kappa\wedge j<\nu(2k+1)\}\cup\{\beta_\varepsilon^{\nu,2k,j}\mid k\in\omega\wedge\kappa\leq\varepsilon<2\kappa-1\wedge j<\nu(2k)\}\end{aligned}$$

and the set of edges

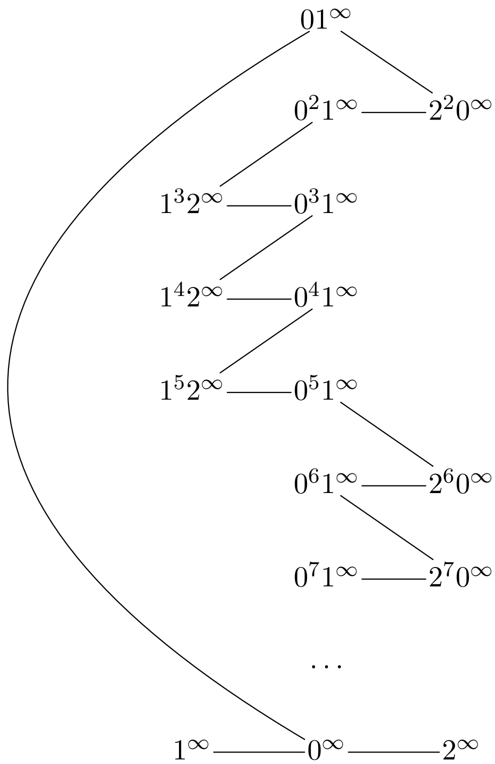
$$\begin{aligned}G_\nu:=s(\{(\varepsilon^\infty,\eta^\infty)\mid\varepsilon\neq\eta\in\kappa\}\cup\{(\varepsilon^\infty,\eta^\infty)\mid\varepsilon\neq\eta\in\{0\}\cup[\kappa,2\kappa-1)\}\cup\{(0^\infty,01^\infty)\}\cup\\ \{(01^\infty,\beta_\varepsilon^{\nu,0,0})\mid\kappa\leq\varepsilon<2\kappa-1\}\cup\\ \{(\beta_\varepsilon^{\nu,2k,j},\beta_\eta^{\nu,2k,j})\mid k\in\omega\wedge\varepsilon\neq\eta\in\{0\}\cup[\kappa,2\kappa-1)\wedge j<\nu(2k)\}\cup\\ \{(\beta_0^{\nu,2k,\nu(2k)-1+j},\beta_\varepsilon^{\nu,2k+1,j})\mid k\in\omega\wedge 0\neq\varepsilon\in\kappa\wedge j<\nu(2k+1)\}\cup\\ \{(\beta_\varepsilon^{\nu,2k+1,j},\beta_\eta^{\nu,2k+1,j})\mid k\in\omega\wedge\varepsilon\neq\eta\in\kappa\wedge j<\nu(2k+1)\}\cup\\ \{(\beta_0^{\nu,2k+1,\nu(2k+1)-1+j},\beta_\varepsilon^{\nu,2k+2,j})\mid k\in\omega\wedge\varepsilon\in[\kappa,2\kappa-1)\wedge j<\nu(2k+2)\}).\end{aligned}$$

Note that X_ν is a closed subspace of κ^ω, and thus a 0DMC space, with Cantor-Bendixson rank two. Also, the set G_ν is closed graph on X_ν. If $c:X_\nu\to\kappa$ is a continuous coloring of (X_ν,G_ν), then, for example, $c(0^\infty)=0$. This implies that $c(\beta_0^{\nu,k,j})=0$ if k is big enough. Inductively, this implies that $c(\beta_0^{\nu,k,j})=0$ for each $k\in\omega$ since 0^n1^∞ and $0^{n+1}1^\infty$ have $\kappa-1$ common G_ν-neighbors which are all pairwise G_ν-related. Thus $c(01^\infty)=0$. In particular, $c(01^\infty)=c(0^\infty)=0$, contradicting $(0^\infty,01^\infty)\in G_\nu$. This shows that $(X_\nu,G_\nu)\in\mathcal{C}_\kappa$. For the minimality, it is enough to see that $(X_\nu,G_\nu)\preceq^i_c(X,G)$ if $X\subseteq X_\nu$ and $G\subseteq G_\nu$ is a graph on X such that $(X,G)\in\mathcal{C}_\kappa$, by compactness. The previous discussion shows that

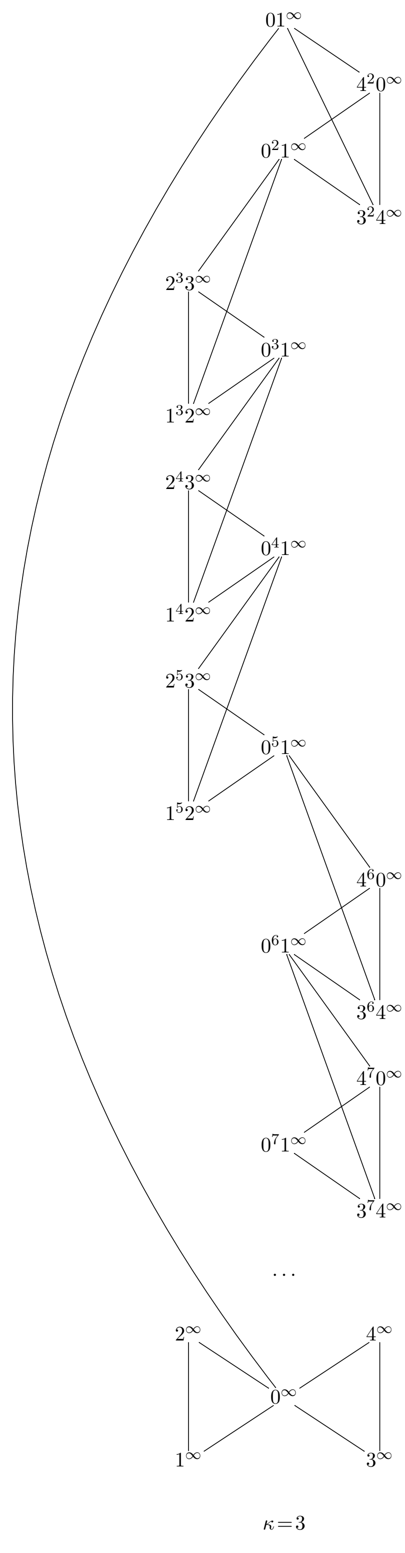
$$G\supseteq G_\nu\backslash s(\{(\varepsilon^\infty,\eta^\infty)\mid\varepsilon\neq\eta\in\kappa\}\cup\{(\varepsilon^\infty,\eta^\infty)\mid\varepsilon\neq\eta\in\{0\}\cup[\kappa,2\kappa-1)\}).$$

Indeed, if one edge e in the difference is not in G, then we may assume that $e\notin s(\{(0^\infty,01^\infty)\})$, and we can give the same color to the two vertices ($\varepsilon^{n+2}(\varepsilon^+)^\infty$ with $\varepsilon\neq 0$ for one of them) of e and ensure that $c(0^{n+1}1^\infty)\neq 0$. Thus $G=G_\nu$ since G is closed. This implies that $X=X_\nu$, and $(X_\nu,G_\nu)\preceq^i_c(X,G)$.

For instance, for $\kappa=2,3$ and $\nu\in N_{132}$, this gives the following pictures.



$\kappa=2$



$\kappa=3$

Let us prove that $(X_\nu, G_\nu) \not\preceq_c^i (X_{\nu'}, G_{\nu'})$ if $(\nu,\nu')\notin\mathbb{E}_t$, where $\mathbb{E}_t$ is the equivalence relation on $(\omega\backslash\{0\})^\omega$ (introduced for example in [Do-J-K]) defined by

$$(\nu,\nu')\in\mathbb{E}_t \Leftrightarrow \exists l,m\in\omega\ \ \forall n\in\omega\ \ \nu(l+n)=\nu(m+n).$$

We argue by contradiction, which gives $h:X_\nu\rightarrow X_{\nu'}$. Note that 0^∞ is the only limit point in X_ν with $\kappa+1$ neighbors in G_ν, so that $h(0^\infty)=0^\infty$. The limit vertex 1^∞ is G_ν-related to ε^∞ if $1\neq\varepsilon<\kappa$ only, so that

$$h[\{\varepsilon^\infty\mid\varepsilon<\kappa\}],h[\{\varepsilon^\infty\mid\varepsilon\in\{0\}\cup[\kappa,2\kappa-1)\}]\in\big\{\{\varepsilon^\infty\mid\varepsilon<\kappa\},\{\varepsilon^\infty\mid\varepsilon\in\{0\}\cup[\kappa,2\kappa-1)\}\big\}.$$

Assume that $h(\varepsilon^\infty)=\eta_\varepsilon^\infty$. The continuity of h implies that

$$\begin{cases} h(\beta_0^{\nu,k,j})\in N_0,\\ h(\beta_\varepsilon^{\nu,2k+1,j})\in N_{\eta_\varepsilon} \text{ if } 0<\varepsilon<\kappa,\\ h(\beta_\varepsilon^{\nu,2k,j})\in N_{\eta_\varepsilon} \text{ if } \kappa\leq\varepsilon<2\kappa-1\end{cases}$$

if $k\geq k_0$ is big enough. Assume, for example, that $\eta_1\geq\kappa$, the case $\eta_1<\kappa$ being similar. Then $h(\beta_1^{\nu,2k_0+1,0})$ is of the form $\beta_{\eta_1}^{\nu',2K_0,J_0}$. Thus $h(\beta_0^{\nu,2k_0+1,0})=\beta_0^{\nu',2K_0,J_0}$. Then, for $0<\varepsilon<\kappa$ and inductively on $0<j<\nu(2k_0+1)$, $h(\beta_\varepsilon^{\nu,2k_0+1,j})=\beta_{\eta_\varepsilon}^{\nu',2K_0,J_0+j}$ and $h(\beta_0^{\nu,2k_0+1,j})=\beta_0^{\nu',2K_0,J_0+j}$. Note then that, for $\kappa\leq\varepsilon<2\kappa-1$, $h(\beta_\varepsilon^{\nu,2k_0+2,0})=\beta_{\eta_\varepsilon}^{\nu',2K_0+1,0}$ and $h(\beta_0^{\nu,2k_0+2,0})=\beta_0^{\nu',2K_0+1,0}$. Then, for $\kappa<\varepsilon<2\kappa-1$ and inductively on $0<j<\nu(2k_0+2)$, $h(\beta_\varepsilon^{\nu,2k_0+2,j})=\beta_{\eta_\varepsilon}^{\nu',2K_0+1,j}$ and $h(\beta_0^{\nu,2k_0+2,j})=\beta_0^{\nu',2K_0+1,j}$. Note then that, for $0<\varepsilon<\kappa$, $h(\beta_\varepsilon^{\nu,2k_0+3,0})=\beta_{\eta_\varepsilon}^{\nu',2K_0+2,0}$ and $h(\beta_0^{\nu,2k_0+3,0})=\beta_0^{\nu',2K_0+2,0}$. This implies that

$$2+\nu(2k_0+2)-1+\big(\Sigma_{i<2K_0+1}\ \nu'(i)\big)+1=2+\Sigma_{i<2K_0+2}\ \nu'(i),$$

so that $\nu(2k_0+2)=\nu'(2K_0+1)$. Then, for $0<\varepsilon<\kappa$ and inductively on $0<j<\nu(2k_0+3)$, $h(\beta_\varepsilon^{\nu,2k_0+3,j})=\beta_{\eta_\varepsilon}^{\nu',2K_0+2,j}$ and $h(\beta_0^{\nu,2k_0+3,j})=\beta_0^{\nu',2K_0+2,j}$. Note then that, for $\kappa\leq\varepsilon<2\kappa-1$, $h(\beta_\varepsilon^{\nu,2k_0+4,0})=\beta_{\eta_\varepsilon}^{\nu',2K_0+3,0}$ and $h(\beta_0^{\nu,2k_0+4,0})=\beta_0^{\nu',2K_0+3,0}$. This implies that

$$2+\nu(2k_0+3)-1+\big(\Sigma_{i<2K_0+2}\ \nu'(i)\big)+1=2+\Sigma_{i<2K_0+3}\ \nu'(i),$$

so that $\nu(2k_0+3)=\nu'(2K_0+2)$. Inductively, we get $\nu(2k_0+2+n)=\nu'(2K_0+1+n)$ for each $n\in\omega$, so that $(\nu,\nu')\in\mathbb{E}_t$.

Consider now the sequence $(p_n)_{n\in\omega}$ of prime numbers. We set, for $\alpha\in 2^\omega$,

$$S_\alpha:=\{p_0^{\alpha(0)+1}\cdots p_n^{\alpha(n)+1}\mid n\in\omega\},$$

so that $S_\alpha\subseteq\omega\backslash\{0\}$ is infinite, and $S_\alpha\cap S_\beta$ is finite if $\alpha\neq\beta$. Let $\nu_\alpha\in(\omega\backslash\{0\})^\omega$ be an injective enumeration of S_α. Then $(\nu_\alpha,\nu_\beta)\notin\mathbb{E}_t$ if $\alpha\neq\beta$, so that $(X_{\nu_\alpha},G_{\nu_\alpha})\not\preceq_c^i(X_{\nu_\beta},G_{\nu_\beta})$. So $\big((X_{\nu_\alpha},G_{\nu_\alpha})\big)_{\alpha\in 2^\omega}$ is an antichain of size $2^{\aleph_0}$ made of minimal elements of $(\mathcal{C}_\kappa,\preceq_c^i)$, proving that any basis for this class must have size $2^{\aleph_0}$. $\square$

Remark. This proof improves the previous proof of Theorem 1.1 (b) for $\kappa=2$, in the sense that the Σ_ν's mentioned at the beginning of this section had Cantor-Bendixson rank three, while the X_ν's here have Cantor-Bendixson rank two.

6 Some analytic complete sets

Notation. The space $\mathcal{K}(X)$ of compact subsets of a metrizable compact space X, equipped with the Vietoris topology, is a metrizable compact space (see Theorem 4.22 in [K1]). Let $\kappa<\aleph_0$ (see the remark just before Theorem 1.1). We denote by $\mathfrak{C}_\kappa$ the set of closed graphs on 2^ω having no continuous κ-coloring, and code $\mathcal{C}_\kappa$ by

$$C_\kappa:=\{(K,G)\in\mathcal{K}(2^\omega)\times\mathcal{K}(2^\omega\times 2^\omega)\mid G\text{ is a graph on }K\text{ having no continuous }\kappa\text{-coloring}\}.$$

Note that $\mathfrak{C}_0$ is simply the set of closed graphs on 2^ω. We then set

$$Q^{\mathfrak{C}}_\kappa:=\{(G,H)\in\mathfrak{C}^2_\kappa\mid(2^\omega,G)\preceq^i_c(2^\omega,H)\},$$

$E^{\mathfrak{C}}_\kappa:=\{(G,H)\in\mathfrak{C}^2_\kappa\mid(2^\omega,G)\equiv^i_c(2^\omega,H)\}=i(Q^{\mathfrak{C}}_\kappa)$ (where $i(Q):=Q\cap Q^{-1}$),

$$Q^C_\kappa:=\big\{\big((K,G),(L,H)\big)\in C^2_\kappa\mid(K,G)\preceq^i_c(L,H)\big\}$$

and $E^C_\kappa:=i(Q^C_\kappa)$. Note that $Q^{\mathfrak{C}}_\kappa$, Q^C_κ are quasi-orders, while $E^{\mathfrak{C}}_\kappa$, E^C_κ are equivalence relations.

Now let $\kappa\leq 3$ (see Theorem 1.12.(b) in [L]), and $\mathcal{H}(2^\omega)$ be the set of homeomorphisms of 2^ω. We equip $\mathcal{H}(2^\omega)$ with the topology whose basic open sets are of the form

$$O_{U_1,\ldots,U_n,V_1,\ldots,V_n}:=\{f\in\mathcal{H}(2^\omega)\mid\forall 1\leq i\leq n\ \ f[U_i]=V_i\},$$

where n is a natural number and U_i,V_i are clopen subsets of 2^ω. By Section 2 in [I-Me], this defines a structure of Polish group on $\mathcal{H}(2^\omega)$. A compatible complete distance is given by

$$d(f,g):=\sup_{\alpha\in 2^\omega}\ d_{2^\omega}\big(f(\alpha),g(\alpha)\big)+\sup_{\alpha\in 2^\omega}\ d_{2^\omega}\big(f^{-1}(\alpha),g^{-1}(\alpha)\big).$$

We denote by $\mathfrak{H}_\kappa$ the set of homeomorphisms of 2^ω whose induced graph has no continuous κ-coloring and, as in the introduction of [L], code $\mathcal{H}_\kappa$ by

$$H_\kappa:=\{(K,f)\in\mathcal{K}(2^\omega)\times\mathcal{H}(2^\omega)\mid f[K1]=K\wedge(K,G_{f_{|K}})\text{ has no continuous }\kappa\text{-coloring}\}.$$

Note that $\mathfrak{H}_0=\mathcal{H}(2^\omega)$. We then set $Q^{\mathfrak{H}}_\kappa:=\{(f,g)\in\mathfrak{H}^2_\kappa\mid(2^\omega,G_f)\preceq^i_c(2^\omega,G_g)\}$, $E^{\mathfrak{H}}_\kappa:=i(Q^{\mathfrak{H}}_\kappa)$, $Q^H_\kappa:=\big\{\big((K,f),(L,g)\big)\in H^2_\kappa\mid(K,G_f)\preceq^i_c(L,G_g)\big\}$ and $E^H_\kappa:=i(Q^H_\kappa)$.

Now let $\kappa\leq\aleph_0$ and D be a countable dense subset of 2^ω. We identify $\mathcal{P}(D\times D)$ with $2^{D\times D}$ (equipped with the product topology of the discrete topology on 2). We denote by $\mathfrak{D}_\kappa$ the set of graphs G on D such that $(2^\omega,G)$ has no continuous κ-coloring and set

$$D_\kappa:=\{(K,G)\in\mathcal{K}(2^\omega)\times\mathcal{P}(D\times D)\mid G\text{ is a graph on }K\text{ having no continuous }\kappa\text{-coloring}\}.$$

Note that $\mathfrak{D}_0$ is simply the set of graphs on D. We then set

$$Q^{\mathfrak{D}}_\kappa:=\{(G,H)\in\mathfrak{D}^2_\kappa\mid(2^\omega,G)\preceq^i_c(2^\omega,H)\},$$

$E^{\mathfrak{D}}_\kappa:=i(Q^{\mathfrak{D}}_\kappa)$, $Q^D_\kappa:=\big\{\big((K,G),(L,H)\big)\in D^2_\kappa\mid(K,G)\preceq^i_c(L,H)\big\}$ and $E^D_\kappa:=i(Q^D_\kappa)$.

Theorem 6.1 *The spaces $\mathfrak{C}_\kappa$, C_κ, $\mathfrak{H}_\kappa$, H_κ, $\mathfrak{D}_\kappa$ and D_κ are Polish, and FCO is Borel reducible to the analytic relations $Q^{\mathfrak{C}}_\kappa$, $E^{\mathfrak{C}}_\kappa$, Q^C_κ, E^C_κ, $Q^{\mathfrak{H}}_\kappa$, $E^{\mathfrak{H}}_\kappa$, Q^H_κ, E^H_κ, $Q^{\mathfrak{D}}_\kappa$, $E^{\mathfrak{D}}_\kappa$, Q^D_κ and E^D_κ. In particular, these relations are analytic complete as sets.*

Proof. Let $G\in\mathcal{K}(2^\omega\times 2^\omega)$. Note that $G\in\mathfrak{C}_\kappa$ if and only if

$$G=G^{-1}\wedge G\cap\Delta(2^\omega)=\emptyset\wedge\forall(C_i)_{i<\kappa}\in\big(\mathbf{\Delta}^0_1(2^\omega)\big)^\kappa$$
$$(2^\omega\not\subseteq\textstyle\bigcup_{i<\kappa}C_i)\vee(\exists i\neq j<\kappa\ \ C_i\cap C_j\neq\emptyset)\vee(G\cap(\bigcup_{i<\kappa}C_i^2)\neq\emptyset).$$

Note that, by continuity of the map $(x,y)\mapsto(y,x)$ and 4.29 in [K1], the condition "$G=G^{-1}$" is closed. By definition of the Vietoris topology, the condition "$G\cap\Delta(2^\omega)=\emptyset$" is open, while the last condition is closed. Thus $\mathfrak{C}_\kappa$ is a difference of two open sets, and thus $\mathbf{\Pi}^0_2$ in $\mathcal{K}(2^\omega\times 2^\omega)$, and Polish. Note that $C_\kappa=\{(K,G)\in\mathcal{K}(2^\omega)\times\mathcal{K}(2^\omega\times 2^\omega)\mid G\in\mathfrak{C}_\kappa\wedge G\subseteq K^2\}$ by Theorem 2.2.1 in [E]. By 4.29 in [K1], C_κ is also Polish. By Theorem 12.5 in [L], the set $\mathfrak{H}_\kappa$ is $\mathbf{\Pi}^0_2$ in $\mathcal{H}(2^\omega)$, and Polish. By Theorem 1.14 in [L], the set H_κ is $\mathbf{\Pi}^0_2$ in $\mathcal{K}(2^\omega)\times\mathcal{H}(2^\omega)$, and Polish. Let $G\in\mathcal{P}(D\times D)$. Note that $G\in\mathfrak{D}_\kappa$ if and only if the formula above holds. We enumerate $D:=\{d_n\mid n\in\omega\}$ injectively. The condition "$G=G^{-1}$" can be written "$\forall m,n\in\omega\ \ (d_m,d_n)\notin G\vee(d_n,d_m)\in G$", which is a closed condition. The condition "$G\cap\Delta(2^\omega)=\emptyset$" can be written "$\forall n\in\omega\ \ (d_n,d_n)\notin G$", which is a closed condition. For the last condition, note that $\mathbf{\Delta}^0_1(2^\omega)$ is countable. If κ is finite, then the condition "$G\cap(\bigcup_{i<\kappa}C_i^2)\neq\emptyset$" can be written "$\exists m,n\in\omega\ \ \exists i<\kappa\ \ (q_m,q_n)\in G\cap C_i^2$", so that $\mathfrak{D}_\kappa$ is $\mathbf{\Pi}^0_2$ in $\mathcal{P}(D\times D)$, and Polish. If $\kappa=\aleph_0$, then by compactness the last condition can be written

$$\forall n\in\omega\ \ \forall(C_i)_{i<n}\in\big(\mathbf{\Delta}^0_1(2^\omega)\big)^n\ \ (2^\omega\not\subseteq\bigcup_{i<n}C_i)\vee(\exists i\neq j<n\ \ C_i\cap C_j\neq\emptyset)\vee(G\cap(\bigcup_{i<n}C_i^2)\neq\emptyset),$$

so that $\mathfrak{D}_\kappa$ is Polish again. Note that $D_\kappa=\{(K,G)\in\mathcal{K}(2^\omega)\times\mathcal{P}(D\times D)\mid G\in\mathfrak{D}_\kappa\wedge G\subseteq K^2\}$ by Theorem 2.2.1 in [E]. The condition "$G\subseteq K^2$" is "$\forall m,n\in\omega\ \ (d_m,d_n)\notin G\vee d_m,d_n\in K$", which is a closed condition, so that D_κ is Polish.

Recall that

$$(2^\omega,G)\preceq^i_c(2^\omega,H)\Leftrightarrow\exists\varphi:2^\omega\to 2^\omega\text{ injective continuous with }G\subseteq(\varphi\times\varphi)^{-1}(H).$$

Note that $\varphi:2^\omega\to 2^\omega$ is injective if and only if $\varphi[O\cap U]=\varphi[O]\cap\varphi[U]$ whenever O,U are clopen subsets of 2^ω. By Lemma 12.4 in [L], and [K, 4.19, 4.29, 27.7],

$$\{(G,H)\in\mathcal{K}(2^\omega\times 2^\omega)^2\mid(2^\omega,G)\preceq^i_c(2^\omega,H)\}$$

is analytic, and thus $Q^{\mathfrak{C}}_\kappa$ and $E^{\mathfrak{C}}_\kappa$ are analytic.

If now $(K,G),(L,H)\in\mathcal{K}(2^\omega)\times\mathcal{K}(2^\omega\times 2^\omega)$, then, since K is a retract of 2^ω by 2.8 in [K1],

$$(K,G)\preceq^i_c(L,H)\Leftrightarrow\exists\psi\in\mathcal{C}(2^\omega,2^\omega)\ \ \psi[K1]\subseteq L\wedge\psi_{|K}\text{ is injective}\wedge(\psi\times\psi)[G]\subseteq H.$$

By Lemma 12.4 in [L] and 4.29 in [K1],

$$\big\{\big((K,G),(L,H)\big)\in\big(\mathcal{K}(2^\omega)\times\mathcal{K}(2^\omega\times 2^\omega)\big)^2\mid(K,G)\preceq^i_c(L,H)\big\},$$

Q^C_κ and E^C_κ are analytic.

Let $f,g\in\mathcal{H}(2^\omega)$. Then

$$(2^\omega,G_f)\preceq^i_c(2^\omega,G_g)\Leftrightarrow\exists\varphi:2^\omega\to 2^\omega\text{ injective continuous}$$
$$\forall\alpha\in 2^\omega\ \ f(\alpha)=\alpha\vee\varphi\big(f(\alpha)\big)=g\big(\varphi(\alpha)\big)\vee\varphi\big(f(\alpha)\big)=g^{-1}\big(\varphi(\alpha)\big).$$

This shows that $\{(f,g)\in\mathcal{H}(2^\omega)^2\mid(2^\omega,G_f)\preceq^i_c(2^\omega,G_g)\}$ and thus $Q^{\mathfrak{H}}_\kappa$ and $E^{\mathfrak{H}}_\kappa$ are analytic. The previous discussions show that Q^H_κ and E^H_κ are also analytic. If $G,H\in\mathcal{P}(D\times D)$, then the condition "$G\subseteq(\varphi\times\varphi)^{-1}(H)$" can be written

$$\text{“}\forall m,n\in\omega\ \ (d_m,d_n)\notin G\vee\big(\varphi(d_m),\varphi(d_n)\big)\in H\text{”},$$

which is a closed condition, proving that $\{(G,H)\in\big(\mathcal{P}(D\times D)\big)^2\mid(2^\omega,G)\preceq^i_c(2^\omega,H)\}$, $Q^{\mathfrak{D}}_\kappa$ and $E^{\mathfrak{D}}_\kappa$ are analytic. The previous discussions show that Q^D_κ and E^D_κ are also analytic.

We define a map $\mathfrak{g}:\mathbb{M}\to\mathcal{K}(2^\omega\times 2^\omega)$ by $\mathfrak{g}(f):=G_f$. Let O be an open subset of $2^\omega\times 2^\omega$, and $(C^0_n)_{n\in\omega}$, $(C^1_n)_{n\in\omega}$ be sequences of clopen subsets of 2^ω with the property that $O=\bigcup_{n\in\omega}(C^0_n\times C^1_n)$. If $f\in\mathbb{M}$ and $G_f\subseteq O$, then there is a finite subset F of ω with $G_f=s\big(\text{Graph}(f)\big)\subseteq\bigcup_{n\in F}(C^0_n\times C^1_n)$. Note then that

$$\bigcup_{n\in F}(C^0_n\times C^1_n)=\bigcup_{S\subseteq F}\Big(\big(\bigcap_{n\in S}C^0_n\cap\bigcap_{n\in F\setminus S}2^\omega\setminus C^0_n\big)\times\big(\bigcup_{n\in S}C^1_n\big)\Big).$$

Thus

$$\begin{aligned}\text{Graph}(f)\subseteq\textstyle\bigcup_{n\in F}(C^0_n\times C^1_n)&\Leftrightarrow\forall S\subseteq F\ f[\textstyle\bigcap_{n\in S}C^0_n\cap\bigcap_{n\in F\setminus S}2^\omega\setminus C^0_n]\subseteq\bigcup_{n\in S}C^1_n\\&\Leftrightarrow\forall S\subseteq F\ \exists R_n\in\boldsymbol{\Delta}^0_1(2^\omega)\\&\qquad f[\textstyle\bigcap_{n\in S}C^0_n\cap\bigcap_{n\in F\setminus S}2^\omega\setminus C^0_n]=R_n\subseteq\bigcup_{n\in S}C^1_n.\end{aligned}$$

This implies that $\{f\in\mathbb{M}\mid G_f\subseteq O\}$ is an open subset of $\mathbb{M}$ since

$$G_f\subseteq O\Leftrightarrow\exists F\subseteq\omega\text{ finite with }\text{Graph}(f)\subseteq\bigcap_{\varepsilon\in 2}\big(\bigcup_{n\in F}(C^\varepsilon_n\times C^{1-\varepsilon}_n)\big).$$

Now $G_f\cap O\neq\emptyset\Leftrightarrow\exists n\in\omega\ \ \exists\varepsilon\in 2\ \ C^\varepsilon_n\cap f^{-1}(C^{1-\varepsilon}_n)\neq\emptyset\Leftrightarrow\exists n\in\omega\ \ \exists\varepsilon\in 2\ \ \exists\alpha\in C^\varepsilon_n\ \ f(\alpha)\in C^{1-\varepsilon}_n$, so that $\{f\in\mathbb{M}\mid G_f\cap O\neq\emptyset\}$ is an open subset of $\mathbb{M}$. Thus $\mathfrak{g}$ is continuous.

By Lemma 7.11 in [L], if $f,g\in\mathbb{M}$, then $(f,g)\in FCO$ if and only if $(2^\omega,G_f)\preceq^i_c(2^\omega,G_g)$. As FCO is symmetric, $(f,g)\in FCO$ if and only if $(2^\omega,G_f)\equiv^i_c(2^\omega,G_g)$. We define $\phi_1(f):N_1\to N_1$ by $\phi_1(f)(1\alpha):=1f(\alpha)$, so that $\phi_1(f)$ is a homeomorphism with infinite orbits and $G_{\phi_1(f)}$ is a closed graph on N_1. We define a map $\mathfrak{g}^+:\mathbb{M}\to\mathfrak{C}_\kappa$ by

$$\mathfrak{g}^+(f):=G_{\phi_1(f)}\cup\{(0^{m+1}1^\infty,0^{n+1}1^\infty)\mid m\neq n\in\kappa+1\}.$$

Note that $G_{\phi_1(f)}=\{(1\alpha,1\beta)\mid(\alpha,\beta)\in G(f)\}$, so that $\mathfrak{g}^+$ is continuous by 4.29 (iv and vi) in [K1]. In order to prove that FCO is Borel reducible to $Q^{\mathfrak{C}}_\kappa$ and $E^{\mathfrak{C}}_\kappa$, it is enough to prove that if $f,g\in\mathbb{M}$, then $(2^\omega,G_f)\preceq^i_c(2^\omega,G_g)$ if and only if $\big(2^\omega,\mathfrak{g}^+(f)\big)\preceq^i_c\big(2^\omega,\mathfrak{g}^+(g)\big)$. So let $\varphi:2^\omega\to 2^\omega$ injective continuous with $G_f\subseteq(\varphi\times\varphi)^{-1}(G_g)$. We define $\Phi:2^\omega\to 2^\omega$ by $\Phi(0\alpha):=0\alpha$ and $\Phi(1\alpha):=1\varphi(\alpha)$, so that Φ is injective continuous.

Moreover, $\big(\Phi(0^{m+1}1^\infty),\Phi(0^{n+1}1^\infty)\big)=(0^{m+1}1^\infty,0^{n+1}1^\infty)\in\mathfrak{g}^+(g)$ and

$$\Big(\Phi(1\alpha),\Phi\big(\phi_1(f)(1\alpha)\big)\Big)=\Big(1\varphi(\alpha),1\varphi\big(f(\alpha)\big)\Big)=\big(1\beta,1g^{\pm 1}(\beta)\big)\in\mathfrak{g}^+(g),$$

showing that Φ is a witness for the fact that $\big(2^\omega,\mathfrak{g}^+(f)\big)\preceq^i_c\big(2^\omega,\mathfrak{g}^+(g)\big)$.

Conversely, let $\Phi:2^\omega\to 2^\omega$ injective continuous with $\mathfrak{g}^+(f)\subseteq(\Phi\times\Phi)^{-1}\big(\mathfrak{g}^+(g)\big)$. Note that the $\mathfrak{g}^+(f)$-connected component of N_0 has size $\kappa+1$, while each $\mathfrak{g}^+(f)$-connected component of N_1 is infinite. This implies that $\Phi(1\alpha)(0)=1$. We now define $\varphi:2^\omega\to 2^\omega$ by the formula

$$\varphi(\alpha):=\Phi(1\alpha)^-:=\big(\Phi(1\alpha)(1),\Phi(1\alpha)(2),\cdots\big),$$

so that φ is injective continuous. Moreover,

$$\Big(1\varphi(\alpha),1\varphi\big(f(\alpha)\big)\Big)=\Big(\Phi(1\alpha),\Phi\big(1f(\alpha)\big)\Big)=\Big(\Phi(1\alpha),\Phi\big(\phi_1(f)(1\alpha)\big)\Big)\in\mathfrak{g}^+(g),$$

so that $\varphi\big(f(\alpha)\big)=g^{\pm 1}\big(\varphi(\alpha)\big)$ and $\Big(\varphi(\alpha),\varphi\big(f(\alpha)\big)\Big)\in G_g$, showing that φ is a witness for the fact that $(2^\omega,G_f)\preceq^i_c(2^\omega,G_g)$. Thus FCO is Borel (in fact continuously) reducible to $Q^{\mathfrak{C}}_\kappa$ and $E^{\mathfrak{C}}_\kappa$. Now note that the map $i_C:\mathfrak{C}_\kappa\to C_\kappa$ defined by $i_C(G):=(2^\omega,G)$ is continuous, $Q^{\mathfrak{C}}_\kappa=(i_C\times i_C)^{-1}(Q^C_\kappa)$ and $E^{\mathfrak{C}}_\kappa=(i_C\times i_C)^{-1}(E^C_\kappa)$, so that FCO is also Borel (in fact continuously) reducible to Q^C_κ and E^C_κ.

Note then that $\mathfrak{H}_3\subseteq\mathfrak{H}_2\subseteq\mathfrak{H}_1\subseteq\mathfrak{H}_0=\mathcal{H}(2^\omega)$. We define $h_3:2^\omega\to 2^\omega$ by $h_3(0^\infty):=0^\infty$ and $h_3(0^{2n+\varepsilon}1\alpha):=0^{2n+1-\varepsilon}1\alpha$, so that $h_3\in\mathcal{H}(2^\omega)$ has orbits of size at most two and $F_{h_3}=\{0^\infty\}$ is not open in 2^ω. By Proposition 2.1, there is no continuous $\aleph_0$-coloring of G_{h_3}, so that $h_3\in\mathfrak{H}_3$. We define, for $f\in\mathbb{M}$, $\phi(f)\in\mathcal{H}(2^\omega)$ by $\phi(f)(0\alpha):=0h_3(\alpha)$ and $\phi(f)(1\alpha):=1f(\alpha)$. Note that $\phi(f)\in\mathfrak{H}_3$ and $\phi:\mathbb{M}\to\mathfrak{H}_3$ is continuous (consider the distance d). In order to prove that FCO is Borel reducible to $Q^{\mathfrak{H}}_3$ and $E^{\mathfrak{H}}_3$, it is enough to prove that if $f,g\in\mathbb{M}$, then $(2^\omega,G_f)\preceq^i_c(2^\omega,G_g)$ if and only if $(2^\omega,G_{\phi(f)})\preceq^i_c(2^\omega,G_{\phi(g)})$. We argue essentially as above, using the facts that the $\phi(f)$-orbit of 0α has size at most two like the h_3-orbit of α, while the $\phi(f)$-orbit of 1α is infinite like the f-orbit of α. Thus FCO is Borel (in fact continuously) reducible to $Q^{\mathfrak{H}}_3$ and $E^{\mathfrak{H}}_3$, and in fact to $Q^{\mathfrak{H}}_\kappa$ and $E^{\mathfrak{H}}_\kappa$ because of the inclusions above. Now note that the map $i_H:\mathfrak{H}_\kappa\to H_\kappa$ defined by $i_H(f):=(2^\omega,f)$ is continuous, $Q^{\mathfrak{H}}_\kappa=(i_H\times i_H)^{-1}(Q^H_\kappa)$ and $E^{\mathfrak{H}}_\kappa=(i_H\times i_H)^{-1}(E^H_\kappa)$, so that FCO is also Borel (in fact continuously) reducible to Q^H_κ and E^H_κ.

By Corollary 5.10 in [L], if $f,g\in\mathbb{M}$, then $(f,g)\in FCO$ if and only if

$$(\overline{\text{proj}[\mathbb{G}_f]},\mathbb{G}_f)\preceq^i_c(\overline{\text{proj}[\mathbb{G}_g]},\mathbb{G}_g)$$

(see Section 5 in [L] for the definition of the graph $\mathbb{G}_f$, whose vertices have degree at most one). This definition, as well as the notation before Theorem 13.2 in [L], show that $\text{proj}[\mathbb{G}_f]$ is contained in the copy $\mathcal{K}_{2^\infty}:=(2\cup\{c,a,\overline{a}\})^\omega$ of 2^ω. In fact, the definition of $\mathbb{G}_f$ shows that $\text{proj}[\mathbb{G}_f]$ is in fact contained in the closed nowhere dense subset $\{x\in\mathcal{K}_{2^\infty}\mid\forall m\in\omega\ \ x(m)=c\vee\forall n\geq m\ \ x(n)\neq c\}$ of $\mathcal{K}_{2^\infty}$. In particular, $\overline{\text{proj}[\mathbb{G}_f]}$ is nowhere dense in $\mathcal{K}_{2^\infty}$.

Claim 1. *(Ryll-Nardzewski) Let P,Q be closed nowhere dense subsets of 2^ω, and $\varphi:P\to Q$ be a continuous injection. Then there is a homeomorphism φ^* of 2^ω such that $\varphi^*(\alpha)=\varphi(\alpha)$ if $\alpha\in P$.*

Indeed, the compact subset $R:=\varphi[P]$ of Q is also closed and nowhere dense in 2^ω, and the map $\varphi':P\to R$ defined by $\varphi'(\alpha):=\varphi(\alpha)$ is a homeomorphism. The Ryll-Nardzewski theorem (see Corollary 2 in [Kn-R]) provides a homeomorphism φ^* of 2^ω extending φ', and thus having the desired property. $\diamond$

Claim 1 implies that if $f, g \in \mathbb{M}$, then $(f, g) \in FCO$ if and only if $(\mathcal{K}_{2^\infty}, \mathbb{G}_f) \preceq_c^i (\mathcal{K}_{2^\infty}, \mathbb{G}_g)$. Indeed, assume that $\varphi : \overline{\text{proj}[\mathbb{G}_f]} \rightarrow \overline{\text{proj}[\mathbb{G}_g]}$ is injective continuous and $\mathbb{G}_f \subseteq (\varphi \times \varphi)^{-1}(\mathbb{G}_g)$. Claim 1 provides $\varphi^* : \mathcal{K}_{2^\infty} \rightarrow \mathcal{K}_{2^\infty}$ injective continuous coinciding with φ on $\overline{\text{proj}[\mathbb{G}_f]}$, which is a witness for the fact that $(\mathcal{K}_{2^\infty}, \mathbb{G}_f) \preceq_c^i (\mathcal{K}_{2^\infty}, \mathbb{G}_g)$. Conversely, if $\Phi : \mathcal{K}_{2^\infty} \rightarrow \mathcal{K}_{2^\infty}$ is injective continuous and $\mathbb{G}_f$ is contained in $(\Phi \times \Phi)^{-1}(\mathbb{G}_g)$, then $\text{proj}[\mathbb{G}_f]$ and thus $\overline{\text{proj}[\mathbb{G}_f]}$ are contained in $\Phi^{-1}(\overline{\text{proj}[\mathbb{G}_g]})$, so that the map $\varphi := \Phi_{|\overline{\text{proj}[\mathbb{G}_f]}}$ is a witness for the fact that $(\overline{\text{proj}[\mathbb{G}_f]}, \mathbb{G}_f) \preceq_c^i (\overline{\text{proj}[\mathbb{G}_g]}, \mathbb{G}_g)$. As FCO is symmetric, $(f, g) \in FCO$ if and only if $(\mathcal{K}_{2^\infty}, \mathbb{G}_f) \equiv_c^i (\mathcal{K}_{2^\infty}, \mathbb{G}_g)$.

Let $i : \mathcal{K}_{2^\infty} \rightarrow N_1 \subseteq 2^\omega$ be a homeomorphism. The definition of $\mathbb{G}_f$ shows that it is contained in the countable dense subset $\mathbb{Q} := \{x \in \mathcal{K}_{2^\infty} \mid \exists l \in \omega\ \exists \varepsilon \in \{a, \overline{a}\}\ \forall k \geq l\ \ x(k) = \varepsilon\}$ of $\mathcal{K}_{2^\infty}$. In particular $Q := i[\mathbb{Q}]$ is a countable dense subset of N_1, as well as $N_1 \cap D$.

Claim 2. *(van Engelen) Let Q, D be countable dense subsets of 2^ω. Then there is $h \in \mathcal{H}(2^\omega)$ such that $h[Q] = D$.*

Indeed, we enumerate injectively $Q = \{q_i \mid i \in \omega\}$ and $D = \{d_i \mid i \in \omega\}$. Note that $\{q_0\}$ is a zero-dimensional space homeomorphic to any of its nonempty clopen subsets, $\{q_i\}$ (resp., $\{d_i\}$) is closed nowhere dense in Q (resp., D), and $\{q_i\} \approx \{q_0\} \approx \{d_i\}$ for each i. Theorem 3.2.6 in [vE] provides the desired homeomorphism. $\diamond$

Claim 2 provides $h \in \mathcal{H}(N_1)$ such that $h[Q] = N_1 \cap D$. We set $H := h \circ i$, so that $H : \mathcal{K}_{2^\infty} \rightarrow N_1$ is a homeomorphism. Recall that the **chromatic number** of a graph (X, G) is the smallest cardinal κ for which there is a κ-coloring of (X, G).

Claim 3. *There is a sequence $\big((F_n, G_n)\big)_{n \in \omega}$ made of finite connected graphs which are pairwise $\preceq^i$-incomparable, have pairwise different chromatic numbers, and whose vertices have degree at least two.*

Indeed, we use the **Kneser graphs** $K(n, k)$. Recall that, if $n, k \in \omega \setminus \{0\}$, then $K(n, k)$ has set of vertices $[n]^k$, and $A, B \in [n]^k$ are $K(n, k)$-related if $A \cap B = \emptyset$. If $n \geq 3k$, then $K(n, k)$ is finite connected and its vertices have degree at least two. Note that Cardinality$([n]^k) = \binom{n}{k}$ and, by Theorem 6.29 in [H-N], $K(n, k)$ has chromatic number $n - 2k + 2$ if $n \geq 2k$. Moreover, by Proposition 6.27 in [H-N], $K(n, k) \not\preceq^i K(n', k')$ if $2 \leq \frac{n'}{k'} < \frac{n}{k}$ (even without necessarily injectivity). All this implies that it is enough to construct a sequence $\big((n_p, k_p)\big)_{p \in \omega}$ of pairs of positive natural numbers satisfying the following.

(1) $3 \leq \frac{n_{p+1}}{k_{p+1}} < \frac{n_p}{k_p}$
(2) $\binom{n_p}{k_p} < \binom{n_{p+1}}{k_{p+1}}$
(3) $(n_p - 2k_p + 2)_{p \in \omega}$ is injective

We set $n_p := 3 \cdot 2^p + 1$ and $k_p := 2^p$, so that (1) and (3) are satisfied. For (2), note that

$$\binom{n_p}{k_p} = \frac{n_p!}{k_p!(n_p - k_p)!} = \frac{(3 \cdot 2^p + 1)!}{(2^p)!(2 \cdot 2^p + 1)!},$$

thus $\binom{n_p}{k_p} < \binom{n_{p+1}}{k_{p+1}} \Leftrightarrow \frac{(3 \cdot 2^p + 1)!}{(2^p)!(2 \cdot 2^p + 1)!} < \frac{(6 \cdot 2^p + 1)!}{(2 \cdot 2^p)!(4 \cdot 2^p + 1)!} \Leftrightarrow (3 \cdot 2^p + 1)! \cdot (4 \cdot 2^p + 1)! < (6 \cdot 2^p + 1)! \cdot (2^p)! \cdot (2 \cdot 2^p + 1)$, $\binom{n_p}{k_p} < \binom{n_{p+1}}{k_{p+1}} \Leftrightarrow (3 \cdot 2^p + 1)! < (6 \cdot 2^p + 1) \cdots (4 \cdot 2^p + 2) \cdot (2 \cdot 2^p + 1) \cdot (2^p)!$. This holds since there are $2 \cdot 2^p$ factors in $(6 \cdot 2^p + 1) \cdots (4 \cdot 2^p + 2) > (3 \cdot 2^p + 1) \cdots (2^p + 2)$ and $2 \cdot 2^p + 1 > 2^p + 1$. $\diamond$

We may assume that $F_n \subseteq N_{0^{n+1}1} \cap D$. Claim 3 allows us to define $\mathcal{G} : \mathbb{M} \to \mathfrak{D}_{\aleph_0}$, by

$$\mathcal{G}(f) := (H \times H)[\mathbb{G}_f] \cup \bigcup_{n \in \omega} G_n.$$

This definition is correct since $\big(2^\omega, \mathcal{G}(f)\big)$ has no continuous $\aleph_0$-coloring. Indeed, we argue by contradiction to see that. By compactness of 2^ω, this graph would have a continuous κ-coloring for some $\kappa \in \omega$, which is not the case because the G_n's have pairwise different chromatic numbers. The beginning of the proof of Theorem 13.2 in [L] shows that $\mathcal{G}$ is Borel. Let $f, g \in \mathbb{M}$. If $(f, g) \in FCO$, then $(\mathcal{K}_{2^\infty}, \mathbb{G}_f) \preceq^i_c (\mathcal{K}_{2^\infty}, \mathbb{G}_g)$, with φ as a witness. We define $\Phi : 2^\omega \to 2^\omega$ by $\Phi(0\alpha) := 0\alpha$ and $\Phi(1\alpha) := H\Big(\varphi\big(H^{-1}(1\alpha)\big)\Big)$, so that Φ witnesses $\big(2^\omega, \mathcal{G}(f)\big) \preceq^i_c \big(2^\omega, \mathcal{G}(g)\big)$. Conversely, assume that $\Phi : 2^\omega \to 2^\omega$ is injective continuous and $\mathcal{G}(f) \subseteq (\Phi \times \Phi)^{-1}(\mathcal{G}(g))$. The vertices in N_1 have degree at least two, while the other vertices of $\mathcal{G}(f)$ have degree at most one like those of $\mathbb{G}_f$. This implies that $\Phi \times \Phi$ sends $\bigcup_{n \in \omega} G_n$ into itself, by injectivity. As the G_n's are connected and pairwise $\preceq^i$-incomparable, $\Phi \times \Phi$ sends G_n into itself, and onto itself by finiteness. Therefore $\Phi \times \Phi$ sends $(H \times H)[\mathbb{G}_f]$ into $(H \times H)[\mathbb{G}_g]$, by injectivity. Thus Φ sends $\text{proj}\big[(H \times H)[\mathbb{G}_f]\big]$ into $\text{proj}\big[(H \times H)[\mathbb{G}_g]\big]$, and $\overline{\text{proj}\big[(H \times H)[\mathbb{G}_f]\big]}$ into $\overline{\text{proj}\big[(H \times H)[\mathbb{G}_g]\big]}$. As H is a homeomorphism,

$$\overline{\text{proj}\big[(H \times H)[\mathbb{G}_f]\big]} = H\big[\overline{\text{proj}[\mathbb{G}_f]}\big],$$

and similarly with g. This allows us to define $\phi : \overline{\text{proj}[\mathbb{G}_f]} \to \overline{\text{proj}[\mathbb{G}_g]}$ by $\phi(\alpha) := H^{-1}\Big(\Phi\big(H(\alpha)\big)\Big)$, and ϕ is a witness for the fact that $(\overline{\text{proj}[\mathbb{G}_f]}, \mathbb{G}_f) \preceq^i_c (\overline{\text{proj}[\mathbb{G}_g]}, \mathbb{G}_g)$. Thus $(f, g) \in FCO$ and $\mathcal{G}$ Borel reduces FCO to $Q^{\mathfrak{D}}_{\aleph_0}$ and $E^{\mathfrak{D}}_{\aleph_0}$. As $\mathfrak{D}_{\aleph_0} \subseteq \mathfrak{D}_\kappa$, this also holds for $Q^{\mathfrak{D}}_\kappa$ and $E^{\mathfrak{D}}_\kappa$ if $\kappa < \aleph_0$. Now note that the map $i_D : \mathfrak{D}_\kappa \to D_\kappa$ defined by $i_D(G) := (2^\omega, G)$ is continuous, $Q^{\mathfrak{D}}_\kappa = (i_D \times i_D)^{-1}(Q^D_\kappa)$ and $E^{\mathfrak{D}}_\kappa = (i_D \times i_D)^{-1}(E^D_\kappa)$, so that FCO is also Borel reducible to Q^D_κ and E^D_κ.

[De-GR-Ka-Kun-Kw] shows that FCO is analytic complete as a set. Thus our sets are Borel analytic complete (using pre-images by Borel functions). By [K2], our sets are analytic complete. $\square$

Question. [De-GR-Ka-Kun-Kw] shows that FCO is analytic complete as a set. On the other hand, Theorem 5 in [Ca-G] shows that the conjugacy relation on $\mathcal{H}(2^\omega)$ is Borel-bi-reducible with the most complicated of the orbit equivalence relations induced by a Borel action of the group of bijections of ω. Also, in [Lo-R] it is proved that the bi-homomorphism relation between countable graphs is analytic complete as an equivalence relation. So we can ask about the position of the equivalence relations mentioned in Theorem 6.1 among analytic equivalence relations, in particular

(1) $E^{\mathfrak{C}}_0 := \{(G, H) \in \mathcal{K}(2^\omega \times 2^\omega)^2 \mid G, H \text{ are graphs} \wedge (2^\omega, G) \equiv^i_c (2^\omega, H)\}$,

(2) $E^{\mathfrak{H}}_0 := \{(f, g) \in \mathcal{H}(2^\omega)^2 \mid (2^\omega, G_f) \equiv^i_c (2^\omega, G_g)\}$,

(3) $E^{\mathfrak{D}}_0 := \{(G, H) \in \mathcal{P}(D \times D)^2 \mid G, H \text{ are graphs} \wedge (2^\omega, G) \equiv^i_c (2^\omega, H)\}$, where D is a countable dense subset of 2^ω.

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