Descriptive Set Theory and ω-Powers of Finitary Languages

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Abstract. The ω-power of a finitary language L over a finite alphabet Σ is the language of infinite words over Σ defined by

$$L^\infty := \{w_0w_1 \ldots \in \Sigma^\omega \mid \forall i \in \omega \ \ w_i \in L\}.$$

The ω-powers appear very naturally in Theoretical Computer Science in the characterization of several classes of languages of infinite words accepted by various kinds of automata, like Büchi automata or Büchi pushdown automata. We survey some recent results about the links relating Descriptive Set Theory and ω-powers.

Keywords and phrases. Languages of finite or infinite words, context-free, one-counter automaton, ω-power, topological complexity, Borel class, complete set

1 Introduction

In the sixties, Büchi studied acceptance of infinite words by finite automata with the now called Büchi acceptance condition, in order to prove the decidability of the monadic second order theory of one successor over the integers. Since then there has been a lot of work on regular ω-languages, accepted by Büchi automata, or by some other variants of automata over infinite words, like Muller or Rabin automata, and by other finite machines, like pushdown automata, counter automata, Petri nets, Turing machines, $\ldots$, with various acceptance conditions, see [Tho90, Sta97a, PP04].

The class of regular ω-languages, those acccepted by Büchi automata, is the ω-Kleene closure of the family REG of regular finitary languages. The ω-**Kleene closure** of a class of languages of finite words over finite alphabets is the class of ω-languages of the form $\bigcup_{1\leq j\leq n} U_j \cdot V_j^\infty$, for some regular finitary languages U_j and V_j, $1 \leq j \leq n$, where for any finitary language $L \subseteq \Sigma^{<\omega}$ over the alphabet Σ, the ω-**power** L^∞ of L is the set of the infinite words constructible with L by concatenation, i.e.,

$$L^\infty := \{ \ w_0 w_1 \ldots \in \Sigma^\omega \mid \forall i \in \omega \ \ w_i \in L \ \}.$$

Note that we denote here L^∞ the ω-power associated with L, as in [Lec05, FL09], while it is often denoted L^ω in Theoretical Computer Science papers, as in [Sta97a, Fin01, Fin03a, FL07]. Here we reserved the notation L^ω to denote the Cartesian product of countably many copies of L since this will be often used in this paper.

Similarly, the operation of taking the ω-power of a finitary language appears in the characterization of the class of context-free ω-languages as the ω-Kleene closure of the family of context-free finitary languages (we refer the reader to [ABB96] for basic notions about context-free languages). And the class of ω-languages accepted by Büchi one-counter automata is also the ω-Kleene closure of the family of finitary languages accepted by one-counter automata. Therefore the operation $L \to L^\infty$ is a fundamental operation over finitary languages leading to ω-languages. The ω-powers of regular languages have been studied in [LT87, Sta97a].

During the last years, the ω-powers have been studied from the perspective of Descriptive Set Theory in a few papers [Fin01, Fin03a, Fin04, Lec05, DF07, FL07, FL09, FL20]. We mainly review these recent works in the present survey.

Since the set Σ^ω of infinite words over a finite alphabet Σ can be equipped with the usual Cantor topology, the question of the topological complexity of ω-powers of finitary languages, from the point of view of descriptive set theory, naturally arises and has been posed by Niwinski [Niw90], Simonnet [Sim92], and Staiger [Sta97a].

As the concatenation map, from L^ω onto L^∞, which associates to a given sequence $(w_i)_{i\in\omega}$ of finite words the concatenated word $w_0 w_1 \ldots$, is continuous, an ω-power is always an analytic set. It was proved in [Fin03a] that there exists a (context-free) language L such that L^∞ is analytic but not Borel. Amazingly, the language L is very simple to describe and it is accepted by a simple one-counter automaton. Louveau has proved independently that analytic-complete ω-powers exist, but the existence was proved in a non effective way (this is non-published work).

One of our first tasks was to study the position of ω-powers with respect to the Borel hierarchy (and beyond to the projective hierarchy). A characterization of ω-powers in the Borel classes $\mathbf{\Sigma}^0_1$, $\mathbf{\Pi}^0_1$ and $\mathbf{\Pi}^0_2$ has been given by Staiger in [Sta97b].

Concerning Borel ω-powers, it was proved that, for each integer $n \geq 1$, there exist some ω-powers of (context-free) languages which are $\mathbf{\Pi}_n^0$-complete Borel sets, [Fin01]. It was proved in [Fin04] that there exists a finitary language L such that L^∞ is a Borel set of infinite rank, and in [DF07] that there is a (context-free) language W such that W^∞ is Borel above $\mathbf{\Delta}_\omega^0$. We recently proved that there are complete ω-powers of one-counter languages, for every Borel class of finite rank, [FL20].

We proved in [FL07, FL09] a result which showed that ω-powers exhibit a great topological complexity: for each countable ordinal $\xi \geq 1$, there are $\mathbf{\Pi}_\xi^0$-complete ω-powers, and $\mathbf{\Sigma}_\xi^0$-complete ω-powers. This result has an effective aspect: for each recursive ordinal $\xi < \omega_1^{\mathrm{CK}}$, where ω_1^{CK} is the first non-recursive ordinal, there are recursive finitary languages P and S such that P^∞ is $\mathbf{\Pi}_\xi^0$-complete and S^∞ is $\mathbf{\Sigma}_\xi^0$-complete.

Many questions are still open about the topological complexity of ω-powers of languages in a given class like the class of context-free languages, one-counter languages, recursive languages, or more generally languages accepted by some kind of automata over finite words. We mention some of these open questions in this paper.

This article is organized as follows. Some basic notions of topology are recalled in Section 2. Notions of automata and formal language theory are recalled in Section 3, and ω-powers of finitary languages accepted by automata are studied in this section. The study of ω-powers of finitary languages in the classical setting of descriptive set theory forms Section 4. Finally, we provide in Section 5 some complexity results about some sets of finitary languages whose associated ω-power is in some class of sets.

2 Topology

When Σ is a finite alphabet, a nonempty **finite word** over Σ is a sequence $w = a_0 \ldots a_{l-1}$, where $a_i \in \Sigma$ for each $i < l$, and $l \geq 1$ is a natural number. The **length** of w is l, denoted by $|w|$. A word of length one is of the form (a). The **empty word** is denoted by λ and satisfies $|\lambda| = 0$. When w is a finite word over Σ, we write $w = w(0)w(1)\ldots w(l-1)$, and the prefix $w(0)w(1)\ldots w(i-1)$ of w of length i is denoted by $w|i$, for any $i \leq l$. We also write $u \subseteq v$ when the word u is a prefix of the finite word v. The set of finite words over Σ is denoted by $\Sigma^{<\omega}$, and Σ^+ is the set of nonempty finite words over Σ. A (finitary) **language** over Σ is a subset of $\Sigma^{<\omega}$. For $L \subseteq \Sigma^{<\omega}$, the **complement** $\Sigma^{<\omega} \setminus L$ of L (in $\Sigma^{<\omega}$) is denoted by L^-. We sometimes write a for $\{(a)\}$, for short.

The first infinite ordinal is ω. An ω-**word** over Σ is an ω-sequence $a_0 a_1 \ldots$, where $a_i \in \Sigma$ for each natural number i. When σ is an ω-word over Σ, the length of σ is $|\sigma| = \omega$, and we write $\sigma = \sigma(0)\sigma(1)\ldots$, and the prefix $\sigma(0)\sigma(1)\ldots\sigma(i-1)$ of σ of length i is denoted by $\sigma|i$, for any natural number i. We also write $u \subseteq \sigma$ when the finite word u is a prefix of the ω-word σ. The set of ω-words over Σ is denoted by Σ^ω. An ω-**language** over Σ is a subset of Σ^ω. For $A \subseteq \Sigma^\omega$, the complement $\Sigma^\omega \setminus A$ of A is denoted by A^-.

The usual **concatenation** product of two finite words u and v is denoted $u^\frown v$ (and sometimes just uv). This product is extended to the product of a finite word u and an ω-word σ: the infinite word $u^\frown\sigma$ is then the ω-word such that $(u^\frown\sigma)(k) = u(k)$ if $k < |u|$, and $(u^\frown\sigma)(k) = \sigma(k - |u|)$ if $k \geq |u|$.

If E is a set, $l \in \omega$ and $(e_i)_{i<l} \in E^l$, then $\frown_{i<l}\, e_i$ is the concatenation $e_0 \ldots e_{l-1}$. Similarly, $\frown_{i\in\omega}\, e_i$ is the concatenation $e_0 e_1 \ldots$ For $L \subseteq \Sigma^{<\omega}$, $L^\infty := \{\sigma = w_0 w_1 \ldots \in \Sigma^\omega \mid \forall i \in \omega \ \ w_i \in L\}$ is the ω-**power** of L.

We now recall some notions of topology, assuming the reader to be familiar with the basic notions, that can be found in [Mos80, Kec95, Sta97a, PP04]. The topological spaces in which we will work in this paper will be subspaces of Σ^ω, where Σ is either finite having at least two elements (like $2 := \{\mathbf{0}, \mathbf{1}\}$), or countably infinite. Note that here 2 is considered as an alphabet, and we will do it also for 3,4; sometimes, we will view it as a letter, and in this case we will denote it by **2**, like we just did it for **0**,**1**. The topology on Σ^ω is the product topology of the discrete topology on Σ. For $w \in \Sigma^{<\omega}$, the set defined by $N_w := \{\alpha \in \Sigma^\omega \mid w \subseteq \alpha\}$ is a basic clopen (i.e., closed and open) set of Σ^ω. The open subsets of Σ^ω are of the form $W^\frown\Sigma^\omega := \{w\sigma \mid w \in W \text{ and } \sigma \in \Sigma^\omega\}$, where $W \subseteq \Sigma^{<\omega}$. When Σ is finite, this topology is called the **Cantor topology** and Σ^ω is compact. When $\Sigma = \omega$, Σ^ω is the Baire space, which is homeomorphic to $\mathbb{P}_\infty := \{\alpha \in 2^\omega \mid \forall i \in \omega \ \exists j \geq i \ \ \alpha(j) = \mathbf{1}\}$, via the map defined on ω^ω by $h(\beta) := \mathbf{0}^{\beta(0)}\mathbf{1}\mathbf{0}^{\beta(1)}\mathbf{1}\ldots$ There is a natural metric on Σ^ω, the **prefix metric** defined as follows. For $\sigma \neq \tau \in \Sigma^\omega$, $d(\sigma, \tau) := 2^{-l_{pref(\sigma,\tau)}}$, where $l_{pref(\sigma,\tau)}$ is the first natural number n such that $\sigma(n) \neq \tau(n)$. The topology induced on Σ^ω by this metric is our topology.

We now define the Borel hierarchy.

Definition 1 *Let X be a topological space, and $n \geq 1$ be a natural number. The classes $\mathbf{\Sigma}^0_n(X)$ and $\mathbf{\Pi}^0_n(X)$ of the* **Borel hierarchy** *are inductively defined as follows:*

$\mathbf{\Sigma}^0_1(X)$ *is the class of open subsets of X.*

$\mathbf{\Pi}^0_1(X)$ *is the class of closed subsets of X.*

$\mathbf{\Sigma}^0_{n+1}(X)$ *is the class of countable unions of $\mathbf{\Pi}^0_n$-subsets of X.*

$\mathbf{\Pi}^0_{n+1}(X)$ *is the class of countable intersections of $\mathbf{\Sigma}^0_n$-subsets of X.*

The Borel hierarchy is also defined for the transfinite levels. Let $\xi \geq 2$ be a countable ordinal.

$\mathbf{\Sigma}^0_\xi(X)$ *is the class of countable unions of subsets of X in $\bigcup_{\gamma<\xi} \mathbf{\Pi}^0_\gamma$.*

$\mathbf{\Pi}^0_\xi(X)$ *is the class of countable intersections of subsets of X in $\bigcup_{\gamma<\xi} \mathbf{\Sigma}^0_\gamma$.*

Suppose now that $\xi \geq 1$ is a countable ordinal and $X \subseteq Y$, where X is equipped with the induced topology. Then $\mathbf{\Sigma}^0_\xi(X) = \{A \cap X \mid A \in \mathbf{\Sigma}^0_\xi(Y)\}$, and similarly for $\mathbf{\Pi}^0_\xi$, see [Kec95, Section 22.A]. Note that we defined the Borel classes $\mathbf{\Sigma}^0_\xi(X)$ and $\mathbf{\Pi}^0_\xi(X)$ mentioning the space X. However, when the context is clear, we will sometimes omit X and denote $\mathbf{\Sigma}^0_\xi(X)$ by $\mathbf{\Sigma}^0_\xi$ and similarly for the dual class. The Borel classes are closed under finite intersections and unions, and continuous preimages. Moreover, $\mathbf{\Sigma}^0_\xi$ is closed under countable unions, and $\mathbf{\Pi}^0_\xi$ under countable intersections. As usual, the ambiguous class $\mathbf{\Delta}^0_\xi$ is the class $\mathbf{\Sigma}^0_\xi \cap \mathbf{\Pi}^0_\xi$. The class of **Borel sets** is

$$\mathbf{\Delta}^1_1 := \bigcup_{1 \leq \xi < \omega_1} \mathbf{\Sigma}^0_\xi = \bigcup_{1 \leq \xi < \omega_1} \mathbf{\Pi}^0_\xi,$$

where ω_1 is the first uncountable ordinal. The **Borel hierarchy** is as follows:

$$\begin{array}{ccccccccc}
 & \mathbf{\Sigma}^0_1 = \text{open} & & \mathbf{\Sigma}^0_2 & \ldots & & \mathbf{\Sigma}^0_\omega & \ldots & \\
\mathbf{\Delta}^0_1 = \text{clopen} & & \mathbf{\Delta}^0_2 & & & \mathbf{\Delta}^0_\omega & & & \mathbf{\Delta}^1_1 \\
 & \mathbf{\Pi}^0_1 = \text{closed} & & \mathbf{\Pi}^0_2 & \ldots & & \mathbf{\Pi}^0_\omega & \ldots &
\end{array}$$

This picture means that any class is contained in every class at the right of it, and the inclusion is strict in any of the spaces Σ^ω. A subset of Σ^ω is a Borel set of **rank** ξ if it is in $\mathbf{\Sigma}^0_\xi \cup \mathbf{\Pi}^0_\xi$ but not in $\bigcup_{1 \leq \gamma < \xi} (\mathbf{\Sigma}^0_\gamma \cup \mathbf{\Pi}^0_\gamma)$.

We now define completeness with respect to reducibility by continuous functions. Let Y, Σ be finite alphabets, $A \subseteq Y^\omega$ and $C \subseteq \Sigma^\omega$. We say that A is **Wadge reducible** to C if there exists a continuous function $f : Y^\omega \to \Sigma^\omega$ such that $A = f^{-1}(C)$. Now let $\mathbf{\Gamma}$ be a class of sets closed under continuous pre-images like $\mathbf{\Sigma}^0_\xi$ or $\mathbf{\Pi}^0_\xi$. A subset C of Σ^ω is said to be $\mathbf{\Gamma}$-**hard** if, for any finite alphabet Y and any $A \subseteq Y^\omega$, $A \in \mathbf{\Gamma}$ implies that A is Wadge reducible to C. If moreover C is in $\mathbf{\Gamma}(\Sigma^\omega)$, then we say that C is $\mathbf{\Gamma}$-**complete**. The $\mathbf{\Sigma}^0_n$-complete sets and the $\mathbf{\Pi}^0_n$-complete sets are thoroughly characterized in [Sta86]. Recall that a subset of Σ^ω is $\mathbf{\Sigma}^0_\xi$ (respectively $\mathbf{\Pi}^0_\xi$)-complete if and only if it is in $\mathbf{\Sigma}^0_\xi$ but not in $\mathbf{\Pi}^0_\xi$ (respectively in $\mathbf{\Pi}^0_\xi$ but not in $\mathbf{\Sigma}^0_\xi$), and that such sets exist (see [Kec95]). For example, the singletons of 2^ω are $\mathbf{\Pi}^0_1$-complete. The set $\mathbb{P}_\infty$ defined at the beginning of the present section is a well known example of a $\mathbf{\Pi}^0_2$-complete set. We say that $\mathbf{\Gamma}$ is a **Wadge class** if there is a $\mathbf{\Gamma}$-complete set. The **Wadge hierarchy** of Borel sets given by the inclusion of these classes is a great refinement of the Borel hierarchy of the classes $\mathbf{\Sigma}^0_\xi$ and $\mathbf{\Pi}^0_\xi$. Among the new classes appearing in this hierarchy, we can mention the classes of transfinite differences of $\mathbf{\Sigma}^0_\xi$ sets. If η is a countable ordinal and $(A_\theta)_{\theta<\eta}$ is an increasing sequence of subsets of some set X, then we set

$$D_\eta\big((A_\theta)_{\theta<\eta}\big) := \{x \in X \mid \exists \theta < \eta \quad x \in A_\theta \setminus \bigcup_{\theta' < \theta} A_{\theta'} \text{ and the parity of } \theta \text{ is opposite to that of } \eta\}.$$

If moreover $\xi \geq 1$ is a countable ordinal, then we set $D_\eta(\mathbf{\Sigma}^0_\xi) := \big\{D_\eta\big((A_\theta)_{\theta<\eta}\big) \mid \forall \theta < \eta \;\; A_\theta \in \mathbf{\Sigma}^0_\xi\big\}$.

The class $\check{\mathbf{\Gamma}} := \{\neg A \mid A \in \mathbf{\Gamma}\}$ is the class of the complements of the sets in $\mathbf{\Gamma}$, and is called the **dual class** of $\mathbf{\Gamma}$. In particular, $\check{\mathbf{\Sigma}}^0_\xi = \mathbf{\Pi}^0_\xi$ and $\check{\mathbf{\Pi}}^0_\xi = \mathbf{\Sigma}^0_\xi$.

There are some subsets of the topological space Σ^ω which are not Borel sets. In particular, there is another hierarchy beyond the Borel hierarchy, called the projective hierarchy. The first class of the projective hierarchy is the class $\mathbf{\Sigma}^1_1$ of analytic sets. A subset A of Σ^ω is **analytic** if we can find a finite alphabet Y and a Borel subset B of $(\Sigma \times Y)^\omega$ such that $x \in A \Leftrightarrow \exists y \in Y^\omega \;\; (x, y) \in B$, where $(x, y) \in (\Sigma \times Y)^\omega$ means that $(x, y)(i) = \big(x(i), y(i)\big)$ for each natural number i. A subset of Σ^ω is analytic if it is empty, or the image of the Baire space by a continuous map. The class $\mathbf{\Sigma}^1_1$ of analytic sets contains the class of Borel sets in any of the spaces Σ^ω. Note that $\mathbf{\Delta}^1_1 = \mathbf{\Sigma}^1_1 \cap \mathbf{\Pi}^1_1$, where $\mathbf{\Pi}^1_1 := \check{\mathbf{\Sigma}}^1_1$ is the class of **co-analytic** sets, i.e., of complements of analytic sets. Similarly, the class of projections of $\mathbf{\Pi}^1_1$ sets is denoted $\mathbf{\Sigma}^1_2$.

The ω-power of a finitary language L is always an analytic set. Indeed, if L is finite and has n elements, then L^∞ is the continuous image of the compact set $\{\mathbf{0}, \mathbf{1}, \ldots, \mathbf{n-1}\}^\omega$. If L is infinite, then there is a bijection between L and ω, and L^∞ is the continuous image of the Baire space ω^ω, [Sim92].

3 Complexity of ω-powers of languages accepted by automata

3.1 Automata

We assume the reader to be familiar with formal languages, see for example [HMU01, Tho90].

We first recall some of the definitions and results concerning automata, pushdown automata, regular and context-free languages, as presented in [ABB96, CG77, Sta97a].

Definition 2 *A* **pushdown automaton** *is a 7-tuple* $\mathcal{A} = (Q, \Sigma, \Gamma, q_0, Z_0, \delta, F)$, *where* Q *is a finite set of states,* Σ *is a finite input alphabet,* Γ *is a finite pushdown alphabet,* $q_0 \in Q$ *is the initial state,* $Z_0 \in \Gamma$ *is the start symbol which is the bottom symbol and always remains at the bottom of the pushdown stack,* δ *is a map from* $Q \times (\Sigma \cup \{\lambda\}) \times \Gamma$ *into the set of finite subsets of* $Q \times \Gamma^{<\omega}$, *and* $F \subseteq Q$ *is the set of final states. The automaton* $\mathcal{A}$ *is said to be* **deterministic** *if* δ *is a map from* $Q \times (\Sigma \cup \{\lambda\}) \times \Gamma$ *into the set of subsets of cardinal one, i.e., singletons, of* $Q \times \Gamma^{<\omega}$. *The automaton* $\mathcal{A}$ *is said to be* **real-time** *if there is no* λ*-transition, i.e., if* δ *is a map from* $Q \times \Sigma \times \Gamma$ *into the set of finite subsets of* $Q \times \Gamma^{<\omega}$.

If $\gamma \in \Gamma^+$ *describes the pushdown stack content, then the leftmost symbol will be assumed to be on the "top" of the stack. A* **configuration** *of the pushdown automaton* $\mathcal{A}$ *is a pair* (q, γ), *where* $q \in Q$ *and* $\gamma \in \Gamma^{<\omega}$. *For* $a \in \Sigma \cup \{\lambda\}$, $\gamma, \beta \in \Gamma^{<\omega}$ *and* $Z \in \Gamma$, *if* (p, β) *is in* $\delta(q, a, Z)$, *then we write* $a : (q, Z\gamma) \mapsto_{\mathcal{A}} (p, \beta\gamma)$.

Let $w = a_0 \ldots a_{l-1}$ *be a finite word over* Σ. *A sequence of configurations* $r = (q_i, \gamma_i)_{i<N}$ *is called a* **run of** $\mathcal{A}$ **on** w **starting in the configuration** (p, γ) *if*

(1) $(q_0, \gamma_0) = (p, \gamma)$,

(2) for each $i < N-1$, *there exists* $b_i \in \Sigma \cup \{\lambda\}$ *satisfying* $b_i : (q_i, \gamma_i) \mapsto_{\mathcal{A}} (q_{i+1}, \gamma_{i+1})$ *such that* $a_0 \ldots a_{l-1} = b_0 \ldots b_{N-2}$.

A run r *of* $\mathcal{A}$ *on* w *starting in configuration* (q_0, Z_0) *will be simply called a* **run of** $\mathcal{A}$ **on** w. *The run is* **accepting** *if it ends in a final state.*

The language $L(\mathcal{A})$ **accepted** *by* $\mathcal{A}$ *is the set of words admitting an accepting run by* $\mathcal{A}$. *A* **context-free language** *is a finitary language which is accepted by a pushdown automaton. We denote by* CFL *the class of context-free languages.*

If we omit the pushdown stack in the definition of a pushdown automaton, we get the notion of a (finite state) automaton. Note that every finite state automaton is equivalent to a deterministic real-time finite state automaton. A **regular language** *is a finitary language which is accepted by a (finite state) automaton. We denote by* REG *the class of regular languages.*

A **one-counter automaton** *is a pushdown automaton with a pushdown alphabet of the form* $\Gamma = \{Z_0, z\}$, *where* Z_0 *is the bottom symbol and always remains at the bottom of the pushdown stack. A* **one-counter language** *is a (finitary) language which is accepted by a one-counter automaton.*

Definition 3 *Let* Σ, Γ *be finite alphabets.*

(a) A (Σ, Γ)-**substitution** *is a map* $f : \Sigma \to 2^{\Gamma^{<\omega}}$.

(b) We extend this map to $\Sigma^{<\omega}$ *be setting* $f(\frown_{i<l} a_i) := \{\frown_{i<l} w_i \mid \forall i < l \;\; w_i \in f(a_i)\}$, *where* $l \in \omega$ *and* $a_0, \cdots, a_{l-1} \in \Sigma$.

(c) We further extend this map to $2^{\Sigma^{<\omega}}$ *by setting* $f(L) := \bigcup_{w \in L} f(w)$.

(d) Let f *be a* (Σ, Γ)*-substitution, and* $\mathcal{F}$ *be a family of languages. If the language* $f(a)$ *belongs to* $\mathcal{F}$ *for each* $a \in \Sigma$, *then the substitution* f *is called a* $\mathcal{F}$-**substitution**.

(e) We then define the operation $\Box$ *on families of languages. Let* $\mathcal{E}$, $\mathcal{F}$ *be families of (finitary) languages. Then* $\mathcal{E} \Box \mathcal{F} := \{f(L) \mid L \in \mathcal{E}$ *and* f *is a* $\mathcal{F}$*-substitution*$\}$.

The operation of substitution gives rise to an infinite hierarchy of context-free finitary languages defined as follows.

Definition 4 *Let $OCL(0) = REG$ be the class of regular languages, $OCL(1) = OCL$ be the class of one-counter languages, and $OCL(k+1) = OCL(k) \,\square\, OCL$, for $k \geq 1$.*

It is well known that the hierarchy given by the families of languages $OCL(k)$ is strictly increasing. And there is a characterization of these languages in terms of automata.

Proposition 5 ([ABB96]) *A language L is in $OCL(k)$ if and only if L is recognized by a pushdown automaton such that, during any computation, the words in the pushdown stack remain in a language of the form $(z_{k-1})^{<\omega} \ldots (z_0)^{<\omega} Z_0$, where $\{Z_0, z_0, \ldots, z_{k-1}\}$ is the pushdown alphabet. Such an automaton is called a k-***iterated counter automaton***. The union $ICL := \bigcup_{k \geq 1} OCL(k)$ is called the family of* **iterated counter languages***, which is the closure under substitution of the family OCL.*

3.2 Π^0_n-complete and Σ^0_n-complete ω-powers

Wadge first gave a description of the Wadge hierarchy of Borel sets, see [Wad83]. Duparc got in [Dup01] a new proof of Wadge's results in the case of Borel sets of finite rank, and he gave a normal form of Borel sets of finite rank, i.e., an inductive construction of a Borel set of every given degree. His proof relies on set theoretic operations which are the counterpart of arithmetical operations over ordinals needed to compute the Wadge degrees.

In fact J. Duparc studied the Wadge hierarchy via the study of the conciliating hierarchy. He introduced in [Dup01] the conciliating sets, which are sets of finite *or* infinite words over an alphabet Σ, i.e., subsets of $\Sigma^{\leq\omega} := \Sigma^{<\omega} \cup \Sigma^{\omega}$. In particular, the set theoretic operation of exponentiation, defined over concilating sets, has been very useful in the study of context-free ω-powers.

We first recall the following.

Definition 6 *Let Σ_A be a finite alphabet, $\twoheadleftarrow$ be a letter out of Σ_A, $\Sigma := \Sigma_A \cup \{\twoheadleftarrow\}$, and x be a finite or infinite word over the alphabet Σ. Then $x^{\twoheadleftarrow}$ is inductively defined as follows.*

- $\lambda^{\twoheadleftarrow} := \lambda$.

- *For a finite word $u \in \Sigma^{<\omega}$,* $\begin{cases} (ua)^{\twoheadleftarrow} := u^{\twoheadleftarrow} a \text{ if } a \in \Sigma_A, \\ (u \twoheadleftarrow)^{\twoheadleftarrow} := u^{\twoheadleftarrow} \text{ with its last letter removed if } |u^{\twoheadleftarrow}| > 0, \\ (u \twoheadleftarrow)^{\twoheadleftarrow} := \lambda \text{ if } |u^{\twoheadleftarrow}| = 0. \end{cases}$

- *For an infinite word σ, $\sigma^{\twoheadleftarrow} := lim_{n \in \omega} \, (\sigma|n)^{\twoheadleftarrow}$, where, given $(w_n) \in (\Sigma_A^{<\omega})^{\omega}$ and $w \in \Sigma_A^{<\omega}$,*

$$w \subseteq lim_{n \in \omega} \, w_n \Leftrightarrow \exists p \in \omega \ \ \forall n \geq p \ \ w_n||w| = w.$$

Remark. For $x \in \Sigma^{\leq\omega}$, $x^{\twoheadleftarrow}$ denotes the string x, once every $^{\twoheadleftarrow}$ occuring in x has been "evaluated" as the back space operation (the one familiar to your computer!), proceeding from left to right inside x. In other words, $x^{\twoheadleftarrow} = x$ from which every interval of the form "$a \twoheadleftarrow$ " ($a \in \Sigma_A$) is removed.

For example, if $x = (a \twoheadleftarrow)^n$ for some $n \geq 1$, $x = (a \twoheadleftarrow)^{\omega}$ or $x = (a \twoheadleftarrow\twoheadleftarrow)^{\omega}$ then $x^{\twoheadleftarrow} = \lambda$. If $x = (ab \twoheadleftarrow)^{\omega}$, then $x^{\twoheadleftarrow} = a^{\omega}$. If $x = bb(\twoheadleftarrow a)^{\omega}$, then $x^{\twoheadleftarrow} = b$.

We now can define the operation $A \mapsto A^{\sim}$ of exponentiation of conciliating sets.

Definition 7 *Let* Σ_A *be a finite alphabet,* $\twoheadleftarrow$ *be a letter out of* Σ_A*,* $\Sigma := \Sigma_A \cup \{\twoheadleftarrow\}$*, and* $A \subseteq \Sigma_A^{\leq\omega}$*. Then we set* $A^{\sim} := \{x \in \Sigma^{\leq\omega} \mid x^{\twoheadleftarrow} \in A\}$*.*

Roughly speaking, the operation $\sim$ is monotone with regard to the Wadge ordering and produces some sets of higher complexity.

The first author proved in [Fin01] that the class CFL_ω of context-free ω-languages, (i.e., those which are accepted by pushdown automata with a Büchi acceptance condition expressing that "some final state appears infinitely often during an infinite computation"), is closed under this operation $\sim$.

We now recall a slightly modified variant of the operation $\sim$, introduced in [Fin01], and which is particularly suitable to infer properties of ω-powers.

Definition 8 *Let* Σ_A *be a finite alphabet,* $\twoheadleftarrow$ *be a letter out of* Σ_A*,* $\Sigma := \Sigma_A \cup \{\twoheadleftarrow\}$*, and* $A \subseteq \Sigma_A^{\leq\omega}$*. Then we set* $A^{\approx} := \{x \in \Sigma^{\leq\omega} \mid x^{\twoheadleftarrow} \in A\}$*, where* $x^{\twoheadleftarrow}$ *is inductively defined as follows.*

- $\lambda^{\twoheadleftarrow} := \lambda$.
- *For a finite word* $u \in \Sigma^{<\omega}$, $\begin{cases} (ua)^{\twoheadleftarrow} := u^{\twoheadleftarrow}a \text{ if } a \in \Sigma_A, \\ (u \twoheadleftarrow)^{\twoheadleftarrow} := u^{\twoheadleftarrow} \text{ with its last letter removed if } |u^{\twoheadleftarrow}| > 0, \\ (u \twoheadleftarrow)^{\twoheadleftarrow} \text{ is undefined if } |u^{\twoheadleftarrow}| = 0. \end{cases}$
- *For an infinite word* σ, $\sigma^{\twoheadleftarrow} := lim_{n\in\omega}\ (\sigma|n)^{\twoheadleftarrow}$.

The only difference is that here $(u \twoheadleftarrow)^{\twoheadleftarrow}$ is undefined if $|u^{\twoheadleftarrow}| = 0$. It is easy to see that if $A \subseteq \Sigma_A^\omega$ is a Borel set such that $A \neq \Sigma_A^\omega$, i.e., $A^- \neq \emptyset$, then $A^{\approx}$ is Wadge equivalent to $A^{\sim}$ (see [Fin01]) and this implies the following result:

Theorem 9 *Let* Σ_A *be a finite alphabet, and* $n \geq 2$ *be a natural number. If* $A \subseteq \Sigma_A^\omega$ *is* $\boldsymbol{\Pi}_n^0$*-complete, then* $A^{\approx}$ *is* $\boldsymbol{\Pi}_{n+1}^0$*-complete.*

Notation. Let Σ_A be a finite alphabet, $\twoheadleftarrow$ be a letter out of Σ_A, and $\Sigma := \Sigma_A \cup \{\twoheadleftarrow\}$. The language L_3 over Σ is the context-free language generated by the context-free grammar with the following production rules:

$$S \to aS \twoheadleftarrow S \text{ with } a \in \Sigma_A,$$
$$S \to a \twoheadleftarrow S \text{ with } a \in \Sigma_A,$$
$$S \to \lambda$$

(see [HMU01] for the basic notions about grammars). This language L_3 corresponds to the words where every letter of Σ_A has been removed after using the backspace operation. It is easy to see that L_3 is a deterministic one-counter language, i.e., L_3 is accepted by a deterministic one-counter automaton. Moreover, for $a \in \Sigma_A$, the language L_3a is also accepted by a deterministic one-counter automaton.

We can now state the following result, which implies that the class of ω-powers is closed under the operation $A \to A^{\approx}$.

Lemma 10 (see [Fin01]) *Whenever* $A \subseteq \Sigma_A^\omega$*, the* ω*-language* $A^{\approx} \subseteq \Sigma^\omega$ *is obtained by substituting in* A *the language* L_3a *for each letter* $a \in \Sigma_A$*.*

An ω-word $\sigma \in A^{\approx}$ may be considered as an ω-word $\sigma^{\twoheadleftarrow} \in A$ to which we possibly add, before the first letter $\sigma^{\twoheadleftarrow}(0)$ of $\sigma^{\twoheadleftarrow}$ (respectively, between two consecutive letters $\sigma^{\twoheadleftarrow}(n)$ and $\sigma^{\twoheadleftarrow}(n+1)$ of $\sigma^{\twoheadleftarrow}$), a finite word belonging to the context-free (finitary) language L_3.

Corollary 11 *Whenever $A \subseteq \Sigma_A^{\omega}$ is an ω-power of a languageL_A, i.e., $A = L_A^{\infty}$, then $A^{\approx}$ is also an ω-power, i.e., there exists a (finitary) language E_A such that $A^{\approx} = E_A^{\infty}$. Moreover, if the language L_A is in the class $OCL(k)$ for some natural number k, then the language E_A can be found in the class $OCL(k+1)$.*

Proof. Let $h : \Sigma_A \to 2^{\Sigma^{<\omega}}$ be the substitution defined by $a \mapsto L_3 a$, where L_3 is the context-free language defined above. Then it is easy to see that now $A^{\approx}$ is obtained by substituting in A the language $L_3 a$ for each letter $a \in \Sigma_A$. Thus $E_A = h(L_A)$ satisfies the statement of the theorem. □

The following result was proved in [Fin01].

Theorem 12 *For each natural number $n \geq 1$, there is a context-free language P_n in the subclass of iterated counter languages such that P_n^{∞} is $\mathbf{\Pi}_n^0$-complete.*

Proof. Let $B_1 = \{\sigma \in \{\mathbf{0}, \mathbf{1}\}^{\omega} \mid \forall i \in \omega \;\; \sigma(i) = \mathbf{0}\} = \mathbf{0}^{\infty}$. B_1 is a $\mathbf{\Pi}_1^0$-complete set of the form P_1^{∞} where P_1 is the singleton containing only the word $(\mathbf{0})$. Note that $P_1 = \mathbf{0}$ is a regular language, hence in the class $OCL(0)$.

Let then $B_2 = \mathbb{P}_{\infty}$ be the well known $\mathbf{\Pi}_2^0$-complete regular ω-language. Note that $B_2 = (\mathbf{0}^{<\omega}\mathbf{1})^{\infty}$. Let $P_2 := \mathbf{0}^{<\omega}\mathbf{1}$. Then P_2 is a regular language, hence in the class $OCL(0)$.

We can now use iteratively Corollary 11 to end the proof □

Note that P_1 and P_2 are regular, hence accepted by some (real-time deterministic) finite automata (without any counter). On the other hand, the language P_3 is accepted by a one-counter automaton. Notice that the ω-powers of regular languages are regular ω-languages, and thus are boolean combination of $\mathbf{\Pi}_2^0$-sets, hence $\mathbf{\Delta}_3^0$-sets. Therefore there are no $\mathbf{\Pi}_3^0$-complete or $\mathbf{\Sigma}_3^0$-complete (or even higher in the Borel hierarchy) ω-powers of regular languages.

For the classes $\mathbf{\Sigma}_n^0$, we first give an example of a $\mathbf{\Sigma}_n^0$-complete ω-power for $n = 1, 2$. Consider the finitary language $S_1 := \{s \in 2^{<\omega} \mid 0 \subseteq s \text{ or } \exists k \in \omega \;\; 10^k 1 \subseteq s\}$ which is regular. Then the ω-power $S_1^{\infty} = 2^{\omega} \setminus \{10^{\infty}\}$ is open and not closed, and thus $\mathbf{\Sigma}_1^0$-complete.

Using another modification of the operation of exponentiation, we proved in [FL09] that there exists a one counter language $L \subseteq 2^{<\omega}$ such that L^{∞} is $\mathbf{\Sigma}_2^0$-complete. It is enough to find a finitary language $S_2 \subseteq 3^{<\omega}$, where $3 = \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$. We set, for $j \in 3$ and $s \in 3^{<\omega}$,

$$n_j(s) := \text{Cardinality}(\{i < |s| \mid s(i) = j\}),$$

$T := \{\alpha \in 3^{\leq\omega} \mid \forall l < 1 + |\alpha| \;\; n_{\mathbf{2}}(\alpha|l) \leq n_{\mathbf{1}}(\alpha|l)\}$. We inductively define, for $s \in T \cap 3^{<\omega}$, a "back space" sequence $s^{\hookleftarrow} \in 2^{<\omega}$ as follows:

$$s^{\hookleftarrow} := \begin{cases} \emptyset \text{ if } s = \emptyset, \\ t^{\hookleftarrow}\varepsilon \text{ if } s = t\varepsilon \text{ and } \varepsilon \in 2, \\ t^{\hookleftarrow}, \text{ except that its last } \mathbf{1} \text{ is replaced with } \mathbf{0}, \text{ if } s = t\mathbf{2}. \end{cases}$$

We then set $E\!:=\!\mathbf{0}\cup\big\{s\!\in\! T\cap 3^{<\omega}\!\setminus\!\{\emptyset\}\mid n_{\mathbf{2}}(s)\!=\!n_{\mathbf{1}}(s)\text{ and }\mathbf{1}\!\subseteq\!\big(s|(|s|\!-\!1)\big)^{\hookleftarrow}\big\}$, and

$$E^*\!:=\!\{{}^\frown{}_{i<l}\ s_i\!\in\! 3^{<\omega}\mid l\!\in\!\omega\text{ and }\forall i<l\ \ s_i\!\in\! E\}.$$

We put $S_2\!:=\!E\cup\big\{{}^\frown{}_{j\leq k}\ (c_j\mathbf{1})\!\in\! 3^{<\omega}\mid k\!\in\!\omega\text{ and }(k\!=\!0\Rightarrow c_0\!\neq\!\emptyset)\text{ and }\forall j\!\leq\! k\ \ c_j\!\in\! E^*\big\}$, and S_2^∞ is $\mathbf{\Sigma}^0_2$-complete. Note that S_2 is accepted by a one-counter automaton.

Finally, we recently proved in [FL20] the following result giving some complete ω-powers of a one-counter language, for any Borel class of finite rank.

Theorem 13 *Let $n\!\geq\!1$ be a natural number.*

(a) There is a finitary language P_n which is accepted by a one-counter automaton and such that the ω-power P_n^∞ is $\mathbf{\Pi}^0_n$-complete.

(b) There is a finitary language S_n which is accepted by a one-counter automaton and such that the ω-power S_n^∞ is $\mathbf{\Sigma}^0_n$-complete.

Moreover, for any given integer $n\geq 1$, one can effectively construct some one-counter automata accepting such finitary languages P_n and S_n (here a construction is effective if there is an algorithm allowing it).

3.3 Borel ω-powers of infinite rank

A first example of an ω-power which is a Borel set of infinite rank was obtained in [Fin04]. The idea was to iterate the operation $L\!\to\! L^{\approx}$, using an infinite number of erasers.

We can first iterate k times this operation $A\!\to\! A^{\approx}$. More precisely, we define, for a set $A\subseteq\Sigma^\omega$, where Σ is a finite alphabet.

- $A_k^{\approx.0}\!:=\!A$,
- $A_k^{\approx.1}\!:=\!A^{\approx}$,
- $A_k^{\approx.2}\!:=\!(A_k^{\approx.1})^{\approx}$,

. . . .

- $A_k^{\approx.(k)}\!:=\!(A_k^{\approx.(k-1)})^{\approx}$,

where we apply k times the operation $A\to A^{\approx}$ with different new letters $\twoheadleftarrow_k$, $\twoheadleftarrow_{k-1}$, $\ldots$,$\twoheadleftarrow_3$, $\twoheadleftarrow_2$, $\twoheadleftarrow_1$, in such a way that we successively have

$A_k^{\approx.0}\!=\!A\subseteq\Sigma^\omega$,
$A_k^{\approx.1}\!\subseteq\!(\Sigma\cup\{\twoheadleftarrow_k\})^\omega$,
$A_k^{\approx.2}\!\subseteq\!(\Sigma\cup\{\twoheadleftarrow_k,\twoheadleftarrow_{k-1}\})^\omega$,
. . . .
$A_k^{\approx.(k)}\!\subseteq\!(\Sigma\cup\{\twoheadleftarrow_k,\twoheadleftarrow_{k-1},\ldots,\twoheadleftarrow_1\})^\omega$.

and we set $\qquad A^{\approx.(k)}=A_k^{\approx.(k)}$.

Note that the choice of the erasers $\twoheadleftarrow_k$, $\twoheadleftarrow_{k-1}$, $\ldots$, $\twoheadleftarrow_2$, $\twoheadleftarrow_1$ in this precise order is important in the proof in [Fin04].

We can now describe the operation $A \to A^{\approx\cdot(k)}$ in a manner similar to the case of the operation $A \to A^{\approx}$, using the notion of a substitution.

Let $T_k \subseteq (\Sigma \cup \{\twoheadleftarrow_k, \twoheadleftarrow_{k-1}, \ldots, \twoheadleftarrow_1\})^{<\omega}$ be the language containing the finite words u over the alphabet $\Sigma \cup \{\twoheadleftarrow_k, \twoheadleftarrow_{k-1}, \ldots, \twoheadleftarrow_1\}$ such that one gets the empty word after applying to u the successive erasing operations with the erasers $\twoheadleftarrow_1$, $\twoheadleftarrow_2$, …, $\twoheadleftarrow_{k-1}$, $\twoheadleftarrow_k$. More precisely, $u \in T_k$ if when we start with u, we evaluate $\twoheadleftarrow_1$ as an eraser, and obtain $u_1 = u^{\twoheadleftarrow_1}$ (following Definition 8, i.e., every occurrence of a symbol $\twoheadleftarrow_1$ does erase a letter of Σ or an eraser $\twoheadleftarrow_i$ for $i > 1$). Then we start again with u_1, this time we evaluate $\twoheadleftarrow_2$ as an eraser, which yields $u_2 = u_1^{\twoheadleftarrow_2}$, and so on. When there is no more symbol $\twoheadleftarrow_i$ to be evaluated, then there remains $u_k \in \Sigma^{<\omega}$. By definition, $u \in T_k$ if and only if $u_k = \lambda$. It is easy to see that T_k is a context free language belonging to the subclass of iterated counter languages.

Now let h_k be the substitution $\Sigma \to 2^{((\Sigma\cup\{\twoheadleftarrow_k,\twoheadleftarrow_{k-1},\ldots,\twoheadleftarrow_1\})^{<\omega})}$ defined by $h_k(a) := L_k{}^\frown a$ for every letter $a \in \Sigma$. It holds that $A^{\approx\cdot(k)} = h_k(A)$, for every $A \subseteq \Sigma^\omega$.

We now set $\Sigma = \{\mathbf{0}, \mathbf{1}\}$. Consider now the ω-language $B_2 := (\mathbf{0}^{<\omega}\mathbf{1})^\infty = P_2^\infty$, where P_2 is the language $\mathbf{0}^{<\omega}\mathbf{1}$. B_2 is $\mathbf{\Pi}^0_2$-complete. Then, as in the proof of Theorem 12, $h_p(P_2^\infty) = (h_p(P_2))^\infty$ is a $\mathbf{\Pi}^0_{p+2}$-complete set, for each integer $p \geq 1$.

On the other hand, the languages T_k, for $k \geq 1$, form a sequence which is strictly increasing for the inclusion relation:

$$T_1 \subsetneq T_2 \subsetneq T_3 \subsetneq \ldots \subsetneq T_i \subsetneq T_{i+1} \ldots$$

In order to construct an ω-power which is Borel of infinite rank, the first idea is to substitute the language $\bigcup_{k\geq 1} L_k{}^\frown a$ to each letter $a \in \Sigma = \{\mathbf{0}, \mathbf{1}\}$ in the language P_2^∞. But this way we would get a language over the *infinite* alphabet $\Sigma \cup \{\twoheadleftarrow_1, \twoheadleftarrow_2, \twoheadleftarrow_3, \ldots\}$. In order to obtain a finitary language over a *finite* alphabet, every eraser $\twoheadleftarrow_j$ can be coded by a finite word $\alpha.\beta^j.\alpha$ over the alphabet $\{\alpha, \beta\}$, where α and β are two new letters.

One defines the substitution $\varphi_k : (\Sigma \cup \{\twoheadleftarrow_1, \ldots, \twoheadleftarrow_k\})^{<\omega} \to 2^{(\Sigma\cup\{\alpha,\beta\})^{<\omega}}$ by $\varphi_p(c) := \{c\}$ for each $c \in \Sigma$ and $\varphi_k(\twoheadleftarrow_j) = \{\alpha.\beta^j.\alpha\}$ for each integer $j \in [1, k]$. Now let $\mathcal{L} := \bigcup_{k\geq 1} \varphi_k(T_k)$, and $h : \Sigma \to 2^{((\Sigma\cup\{\alpha,\beta\})^{<\omega})}$ be the substitution defined by $h(a) := \mathcal{L}^\frown a$, for each $a \in \Sigma$.

Theorem 14 *Let $P_2 := \mathbf{0}^{<\omega}\mathbf{1}$. Then the ω-power $(h(P_2))^\infty \subseteq \{\mathbf{0}, \mathbf{1}, \alpha, \beta\}^\omega$ is a Borel set of infinite rank.*

The language $(h(P_2))$ is a simple recursive language but it is not context-free. Later, with a modification of the construction, and using a coding of an infinity of erasers previously defined in [Fin03b], Finkel and Duparc got a context-free language W such that W^∞ is a Borel set of infinite rank [DF07].

Theorem 15 *There exists a context-free finitary language $W \subseteq \Gamma^{<\omega}$, where Γ is a finite alphabet, such that W^∞ is a Borel set of infinite rank. Moreover W^∞ is above the class $\mathbf{\Delta}^0_\omega$.*

The coding of the infinity of erasers $\twoheadleftarrow_n$ is given by $\Phi(\twoheadleftarrow_n) = \alpha B^n C^n D^n E^n \beta$ with new letters $\alpha, B, C, D, E, \beta$. Actually the pushdown automaton constructed in order to accept the language W must be able to read the number n identifying the eraser four times.

The ω-power W^∞ is above the class $\mathbf{\Delta}^0_\omega$, i.e., it is not in the Borel class $\mathbf{\Delta}^0_\omega$. Note that the ω-power $(h(P_2))^\infty$ was actually also above the class $\mathbf{\Delta}^0_\omega$ but this was not shown in [Fin04]. We give the argument in this latter case, where the language $h(P_2)$ is simpler than W. This follows from the fact that $((h(P_2))^\infty)^\approx$ is Wadge equivalent to $(h(P_2))^\infty$, which is due to the precise way we ordered the erasers, as described above. On the other side the operation $A \to A^\approx$ is strictly increasing for the Wadge ordering inside $\mathbf{\Delta}^0_\omega$ (see [Dup01]). This implies that $(h(P_2))^\infty$, and also W^∞, are not in the class $\mathbf{\Delta}^0_\omega$.

Note that the language W is context-free but it cannot be accepted by a one-counter automaton.

3.4 Non-Borel ω-powers which are even $\mathbf{\Sigma}^1_1$-complete

A first example of language L such that L^∞ is not Borel, and even $\mathbf{\Sigma}^1_1$-complete, was obtained in [Fin03a]. It turned out that the language L may be described in a very simple way. Surprisingly it is actually accepted by a one-counter automaton. It was obtained via a coding of infinite labelled binary trees. We now recall the construction of this language L using the notion of a substitution.

Let d be a letter not in 2 and $D := \{\ u \cdot d \cdot v \mid u, v \in 2^{<\omega} \text{ and } |v| = 2|u| \text{ or } |v| = 2|u|+1\ \}$. It is easy to see that the language $D \subseteq (2 \cup \{d\})^{<\omega}$ is a context-free language accepted by a one-counter automaton.

Let $g : \Sigma \to 2^{(2 \cup \{d\})^{<\omega}}$ be the substitution defined by $g(a) = a \cdot D$. Since $W := \{\mathbf{0}\}^{<\omega} \cdot \mathbf{1}$ is a regular language, $L := g(W)$ is a context-free language and it is accepted by a one-counter automaton. Moreover, it is proved in [Fin03a] that $(g(W))^\infty$ is $\mathbf{\Sigma}^1_1$-complete, and thus non-Borel. This is done by reducing to this ω-language a well-known example of a $\mathbf{\Sigma}^1_1$-complete set: the set of infinite binary trees labelled in the alphabet 2 which have an infinite branch in the $\mathbf{\Pi}^0_2$-complete set W^∞.

4 Classical and effective complexity of the ω-powers

In [FL07], we prove that there are some ω-powers of any Borel rank. More precisely, Theorem 2 in [FL07] is as follows.

Theorem 16 *Let $\xi \geq 1$ be a countable ordinal.*

(a) There is a finitary language $P_\xi \subseteq 2^{<\omega}$ such that the ω-power P_ξ^∞ is $\mathbf{\Pi}^0_\xi$-complete.

(b) There is a finitary language $S_\xi \subseteq 2^{<\omega}$ such that the ω-power S_ξ^∞ is $\mathbf{\Sigma}^0_\xi$-complete.

In fact, we provide a general method proving this when $\xi \geq 3$. Examples of such finitary languages were given in Section 3.2 when $\xi \leq 2$.

We now turn to the general case. Let $\mathbf{\Gamma}$ be a class of sets of the form $\mathbf{\Sigma}^0_\xi$ or $\mathbf{\Pi}^0_\xi$, with $\xi \geq 3$. Fix a $\mathbf{\Gamma}$-complete set $B \subseteq 2^\omega$, so that $B \in \mathbf{\Pi}^0_{\xi+1}$. A result due to Kuratowski provides a closed subset C of ω^ω and a continuous bijection $f : C \to B$ with the property that f^{-1} is $\mathbf{\Sigma}^0_\xi$-measurable (i.e., $f[O]$ is a $\mathbf{\Sigma}^0_\xi$ subset of B if O is an open subset of C, see [Kur66]). This result is a level by level version of a result, due to Lusin and Souslin, asserting that every Borel subset of 2^ω is the image of a closed subset of ω^ω by a continuous bijection. By Proposition 11 in [Lec05], it is enough to find a finitary language $A \subseteq 4^{<\omega}$, where $4 := \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$, such that A^∞ is $\mathbf{\Gamma}$-complete.

The language A will be made of two pieces: $A = \mu \cup \pi$. The set π will code f, and π^∞ will look like B on some compact sets $K_{N,j}$. Outside this countable family of compact sets we will hide f, so that A^∞ will be the simple set μ^∞.

The Lusin-Souslin theorem has been used by Arnold in [Arn83] to prove that every Borel subset of Σ^ω, where Σ is a finite alphabet, is accepted by a non-ambiguous finitely branching transition system with a Büchi acceptance condition, and our first idea was to code the behaviour of such a transition system.

Definition 17 *A* **Büchi transition system** *is a 5-tuple $\mathcal{T} = (Q, \Sigma, q_0, \Delta, F)$, where Q is a (possibly infinite) countable set of states, Σ is a finite input alphabet, $q_0 \in Q$ is the initial state, $\Delta \subseteq Q \times \Sigma \times Q$ is the transition relation, and $F \subseteq Q$ is the set of final states.*

Let $\sigma = a_0 a_1 \ldots$ be an ω-word over Σ. An ω-sequence of states $r = (t_i)_{i \in \omega}$ is called a **run of $\mathcal{T}$ on** *σ if*

(1) $t_0 = q_0$,

(2) for each $i \in \omega$, $\big(t_i, \sigma(i), t_{i+1}\big) \in \Delta$.

The run r is said to be **accepting** *when $t_i \in F$ for infinitely many i's. The transition system $\mathcal{T}$ is said to be*

- *non-***ambiguous** *if each infinite word $\sigma \in \Sigma^\omega$ has at most one accepting run by $\mathcal{T}$,*

- **finitely branching** *if for each state $q \in Q$ and each $a \in \Sigma$, there are only finitely many states q' such that $(q, a, q') \in \Delta$.*

The ω-language accepted by $\mathcal{T}$ is

$$A(\mathcal{T}) := \{\sigma \in \Sigma^\omega \mid \textit{there exists an accepting run } r \textit{ of } \mathcal{T} \textit{ on } \sigma\}.$$

We will code the behaviour of a transition system coming from f.

- The set of states is $Q := \{(s,t) \in 2^{<\omega} \times 2^{<\omega} \mid |s| = |t|\}$, which is countably infinite. We enumerate Q as follows. We start with $q_0 := (\emptyset, \emptyset)$. Then we put the sequences of length 1 of elements of 2×2, in the lexicographical ordering: $q_1 := (0,0)$, $q_2 := (0,1)$, $q_3 := (1,0)$, $q_4 := (1,1)$. Then we put the 16 sequences of length 2: $q_5 := (0^2, 0^2)$, $q_6 := (0^2, 01)$, ... And so on.

We will sometimes use the coordinates of $q_n := (q_n^0, q_n^1)$. We put $M_j := \Sigma_{i<j}\ 4^{i+1}$. Note that the sequence $(M_j)_{j \in \omega}$ is strictly increasing, and that q_{M_j} is the last sequence of length j of elements of 2×2. We define, for $N, j \in \omega$ with $N \leq M_j$, the compact set

$$K_{N,j} := \{\ \mathbf{2}^N (\frown_{i \in \omega}\ m_i\ \mathbf{2}^{M_{j+i+1}}\ \mathbf{3}\ \mathbf{2}^{M_{j+i+1}}) \in 4^\omega \mid \forall i \in \omega\ \ m_i \in 2\ \}.$$

- The input alphabet is 2.

- The initial state is $q_0 := (\emptyset, \emptyset)$.

- If $m \in 2$ and $n, p \in \omega$, then we write $n \xrightarrow{m} p$ if $q_n^0 \subseteq q_p^0$ and $q_p^1 = q_n^1 m$. As f is continuous on C, the graph Graph(f) of f is a closed subset of $C \times 2^\omega$. As C is a closed subset of $\mathbb{P}_\infty$, Graph(f) is also a closed subset of $\mathbb{P}_\infty \times 2^\omega$. So there is a closed subset P of $2^\omega \times 2^\omega$ with the property that

$$\text{Graph}(f) = P \cap (\mathbb{P}_\infty \times 2^\omega).$$

We identify $2^\omega \times 2^\omega$ with $(2 \times 2)^\omega$, i.e., we view (β, α) as $\big(\beta(0), \alpha(0)\big), \big(\beta(1), \alpha(1)\big)$,...

By Proposition 2.4 in [Kec95], there is $R \subseteq (2\times 2)^{<\omega}$, closed under initial segments, such that

$$P = \{(\beta,\alpha) \in 2^\omega \times 2^\omega \mid \forall k \in \omega \ \ (\beta,\alpha)|k \in R\};$$

note that R is a tree whose infinite branches form the set P. In particular, we get

$$(\beta,\alpha) \in \text{Graph}(f) \Leftrightarrow \beta \in \mathbb{P}_\infty \text{ and } \forall k \in \omega \ \ (\beta,\alpha)|k \in R.$$

The transition relation $\Delta \subseteq Q \times 2 \times Q$ is given by $(q_n, m, q_p) \in \Delta \Leftrightarrow n \xrightarrow{m} p$, for $m \in 2$ and $n, p \in \omega$.

- The set of final states is $F := \{(t,s) \in R \mid t \neq \emptyset \text{ and } t(|t|-1) = 1\}$. Note that F is simply the set of pairs $(t,s) \in R$ such that the last letter of t is a 1.

Recall that a run of $\mathcal{T}$ is said to be Büchi accepting if final states occur infinitely often during this run. Then the set of ω-words over the alphabet 2 which are accepted by the transition system $\mathcal{T}$ from the initial state q_0 with Büchi acceptance condition is exactly the Borel set B.

We are now ready to define the finitary language π. We set

$$\pi := \left\{ \begin{array}{l} s \in 4^{<\omega} \mid \exists j, l \in \omega \ \ \exists (m_i)_{i \leq l} \in 2^{l+1} \ \ \exists (n_i)_{i \leq l}, (p_i)_{i \leq l}, (r_i)_{i \leq l} \in \omega^{l+1} \\ \\ \qquad\qquad\qquad\qquad n_0 \leq M_j \\ \qquad\qquad\qquad\qquad\quad \text{and} \\ \qquad\qquad\qquad\qquad \forall i \leq l \ \ n_i \xrightarrow{m_i} pi \text{ and } p_i + r_i = M_{j+i+1} \\ \qquad\qquad\qquad\qquad\quad \text{and} \\ \qquad\qquad\qquad\qquad \forall i < l \ \ p_i = n_{i+1} \\ \qquad\qquad\qquad\qquad\quad \text{and} \\ \qquad\qquad\qquad\qquad q_{p_l} \in F \\ \qquad\qquad\qquad\qquad\quad \text{and} \\ \qquad\qquad\qquad\qquad s = {}^\frown_{i \leq l} \ \mathbf{2}^{n_i} \ m_i \ \mathbf{2}^{p_i} \ \mathbf{2}^{r_i} \ \mathbf{3} \ \mathbf{2}^{r_i} \end{array} \right\}.$$

We are also ready to define μ. The idea is that an infinite sequence containing a word in μ cannot be in the union of the $K_{N,j}$'s. We set

$$\mu_0 := \left\{ \begin{array}{l} s \in 4^{<\omega} \mid \exists l \in \omega \ \ \exists (m_i)_{i \leq l+1} \in 2^{l+2} \ \ \exists N \in \omega \ \ \exists (P_i)_{i \leq l+1}, (R_i)_{i \leq l+1} \in \omega^{l+2} \\ \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall i \leq l+1 \ \ \exists j \in \omega \ \ P_i = M_j \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{and} \\ \qquad\qquad\qquad\qquad\qquad\qquad P_l \neq R_l \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{and} \\ \qquad\qquad\qquad\qquad\qquad\qquad s = \mathbf{2}^N ({}^\frown_{i \leq l+1} \ m_i \ \mathbf{2}^{P_i} \ \mathbf{3} \ \mathbf{2}^{R_i}) \end{array} \right\},$$

$$\mu_1 := \left\{ \begin{array}{l} s \in 4^{<\omega} \mid \exists l \in \omega \ \ \exists (m_i)_{i \leq l+1} \in 2^{l+2} \ \ \exists N \in \omega \ \ \exists (P_i)_{i \leq l+1}, (R_i)_{i \leq l+1} \in \omega^{l+2} \\ \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall i \leq l+1 \ \ \exists j \in \omega \ \ P_i = M_j \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{and} \\ \qquad\qquad\qquad\qquad\qquad\qquad \exists j \in \omega \ \ (P_l = M_j \text{ and } P_{l+1} \neq M_{j+1}) \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{and} \\ \qquad\qquad\qquad\qquad\qquad\qquad s = \mathbf{2}^N ({}^\frown_{i \leq l+1} \ m_i \ \mathbf{2}^{P_i} \ \mathbf{3} \ \mathbf{2}^{R_i}) \end{array} \right\},$$

and $\mu := \mu_0 \cup \mu_1$. Recall that $A = \mu \cup \pi$.

We just described how to get the finitary languages in the statement of Theorem 16. For the other Borel classes $\mathbf{\Delta}^0_\xi$, only $\mathbf{\Delta}^0_1$ is a Wadge class, and $A := \{s \in 2^{<\omega} \mid 0 \subseteq s \text{ or } 1^2 \subseteq s\}$ has the property that $A^\infty = 2^\omega \setminus N_{10}$ is $\mathbf{\Delta}^0_1$-complete (see [FL09]). In [FL09], we provide some complete sets for some other Wadge classes of Borel sets, in fact some dual classes of classes of differences of $\mathbf{\Sigma}^0_\xi$ sets (see also [Lec05]). It is worth noting that Theorem 16 may seem to indicate that ω-powers can be arbitrarily complex, but its proof uses closure properties of the classes of the Borel hierarchy that are not shared by all the Wadge classes of Borel sets, such as the closure by finite unions. The extension of Theorem 16 to all Wadge classes of Borel sets is an open problem.

An important result in [FL09] shows that Theorem 16 is as effective as it can be, in the context of effective descriptive set theory. In order to state it, we must recall some notions about this theory. Effective descriptive set theory is based on the notion of a recursive function. A function from ω^k to ω is said to be **recursive** if it is total and computable. By extension, a relation is called **recursive** if its characteristic function is recursive.

Definition 18 *A* **recursive presentation** *of a topological space X is a pair $\big((x_n)_{n\in\omega}, d\big)$ such that*

1. *$(x_n)_{n\in\omega}$ is dense in X,*
2. *d is a compatible complete distance on X such that the following relations P and Q are recursive:*

$$P(i,j,m,k) \iff d(x_i,x_j) \leq \frac{m}{k+1},$$
$$Q(i,j,m,k) \iff d(x_i,x_j) < \frac{m}{k+1}.$$

A topological space X is **recursively presented** *if it is given with a recursive presentation of it.*

Note that every recursively presented space is **Polish** (i.e., separable and completely metrizable). For example, one can check that the spaces ω and Σ^ω have a recursive presentation. Moreover, a product of two recursively presented spaces has a recursive presentation.

Note that the formula $(p,q) \mapsto 2^p(2q+1)-1$ defines a recursive bijection $\omega^2 \to \omega$. One can check that the coordinates of the inverse map are also recursive. They will be denoted $n \mapsto (n)_0$ and $n \mapsto (n)_1$ in the sequel. These maps will help us to define some of the basic effective classes.

Definition 19 *Let $\big((x_n)_{n\in\omega}, d\big)$ be a recursive presentation of a topological space X.*

1. *We fix a countable basis of X: $B(X,n)$ is the open ball $B_d(x_{(n)_0}, \frac{((n)_1)_0}{((n)_1)_1+1})$.*
2. *A subset S of X is* **semirecursive***, or* **effectively open** *(denoted $S \in \Sigma^0_1$) if*

$$S = \bigcup_{n\in\omega} B\big(X, f(n)\big),$$

   for some recursive function f.

3. *If $n \geq 1$ is a natural number, then Π^0_n is the class of complements of Σ^0_n sets. We say that $B \in \Sigma^0_{n+1}$ if there is $C \in \Pi^0_n(\omega \times X)$ such that $B = \exists^\omega C := \{x \in X \mid \exists i \in \omega\ (i,x) \in C\}$. We also set $\Delta^0_n := \Sigma^0_n \cap \Pi^0_n$.*

4. *A subset S of X is* **effectively analytic** *(denoted $S\in\Sigma^1_1$) if there is a Π^0_1 subset C of $X\times\omega^\omega$ such that $S=proj_X[C]:=\{x\in X\mid\exists\alpha\in\omega^\omega\ \ (x,\alpha)\in C\}$. A subset S of X is* **effectively co-analytic** *(denoted $S\in\Pi^1_1$) if its complement $\neg S$ is effectively analytic, and* **effectively Borel** *if it is in Σ^1_1 and Π^1_1 (denoted $S\in\Delta^1_1$). We also set $\Sigma^1_2:=\{\exists^{\omega^\omega}C\mid C\in\Pi^1_1\}$, $\Pi^1_2:=\check{\Sigma}^1_2$ and $\Delta^1_2:=\Sigma^1_2\cap\Pi^1_2$.*

5. *We will consider the* **relativized classes***: if Y is a recursively presented space and $y\in Y$, then we say that $A\subseteq X$ is in $\Sigma^1_1(y)$ if there is $S\in\Sigma^1_1(Y\times X)$ such that*

$$A=S_y:=\{x\in X\mid(y,x)\in S\}.$$

The class $\Pi^1_1(y)$ is defined similarly. We also set $\Delta^1_1(y):=\Sigma^1_1(y)\cap\Pi^1_1(y)$.

6. *Let $\gamma\in\omega^\omega$. We say that $\gamma\in\Sigma^0_1$ if $\{k\in\omega\mid\gamma\in B(\omega^\omega,k)\}\in\Sigma^0_1(\omega)$. A countable ordinal ξ is a* **recursive ordinal** *if there is $\gamma\in\Sigma^0_1$ coding a well-ordering on ω of order type ξ.*

7. *There is a* **good parametrization** *in Σ^0_1 for $\mathbf{\Sigma}^0_1$ (see 3E.2, 3F.6 and 3H.1 in [Mos80]). This means that there is a system of sets $G^{\Sigma^0_1,Y}\in\Sigma^0_1(\omega^\omega\times Y)$ such that, for each recursively presented space Y and for each $P\subseteq Y$,*

$$\begin{aligned}P\in\mathbf{\Sigma}^0_1\ &\Leftrightarrow\ \exists\gamma\in\omega^\omega\ \ P=G^{\Sigma^0_1,Y}_\gamma,\\ P\in\Sigma^0_1\ &\Leftrightarrow\ \exists\gamma\in\Sigma^0_1\ \ P=G^{\Sigma^0_1,Y}_\gamma.\end{aligned}$$

Moreover, if Z is a recursively presented space of type at most 1 (i.e., a finite product of spaces equal to ω, ω^ω or 2^ω), and Y is a recursively presented space, then there is $S^{Z,Y}_{\Sigma^0_1}:\omega^\omega\times Z\to\omega^\omega$ recursive such that $(\gamma,z,y)\in G^{\Sigma^0_1,Z\times Y}\ \Leftrightarrow\ \big(S^{Z,Y}_{\Sigma^0_1}(\gamma,z),y\big)\in G^{\Sigma^0_1,Y}$ (here, by $S^{Z,Y}_{\Sigma^0_1}$ recursive we mean that the relation defined by $R(\gamma,z,k)\Leftrightarrow S^{Z,Y}_{\Sigma^0_1}(\gamma,z)\in B(\omega^\omega,k)$ defines a Σ^0_1 subset of $\omega^\omega\times Z\times\omega$).

8. *We can code the partial recursive functions. Let Y be a recursively presented space, $f:X\to Y$ be a partial function, $D\subseteq Domain(f)$ and $P\subseteq X\times\omega$. Then P* **computes** *f on D if*

$$x\in D\ \Rightarrow\ \forall k\in\omega\ \big(f(x)\in B(Y,k)\ \Leftrightarrow\ (x,k)\in P\big).$$

If P is in Σ^0_1 and computes f on D, then we say that f is **recursive on** *D. This means that $f^{-1}\big(B(Y,k)\big)\in\Sigma^0_1$, uniformly in k.*

We now define a partial function $U:\omega^\omega\times X\to Y$ by

$$U(\gamma,x)\!\downarrow\ \Leftrightarrow\ U(\gamma,x)\text{is defined}\Leftrightarrow\exists y\in Y\ \forall k\in\omega\ \ \big(y\in B(Y,k)\Leftrightarrow(\gamma,x,k)\in G^{\Sigma^0_1,X\times\omega}\big),$$

$$U(\gamma,x)\ \ :=\ \ \text{the unique } y\in Y \text{ such that } \forall k\in\omega\ \ \big(y\in B(Y,k)\Leftrightarrow(\gamma,x,k)\in G^{\Sigma^0_1,X\times\omega}\big).$$

Now let $\gamma\in\omega^\omega$. The function $\{\gamma\}^{X,Y}:X\to Y$ is defined by $\{\gamma\}^{X,Y}(x):=U(\gamma,x)$. Then a partial function $f:X\to Y$ is recursive on its domain if and only if there is $\gamma\in\Sigma^0_1$ such that $f(x)=\{\gamma\}^{X,Y}(x)$ when $f(x)$ is defined. More generally, the functions of the form $\{\gamma\}^{X,Y}$ are the partial continuous functions from a subset of X into Y. In order to simplify the notation, we will write $\{\gamma\}$ instead of $\{\gamma\}^{X,Y}$ when $Y=\omega^\omega$.

9. *We now define, by induction on the countable ordinal $\xi \geq 1$, the set BC_ξ of Borel codes for $\boldsymbol{\Sigma}^0_\xi$ as follows. If $\gamma \in \omega^\omega$, then we define $\gamma^* \in \omega^\omega$ by $\gamma^*(i) := \gamma(i+1)$. We set*

$$\begin{aligned} BC_1 &:= \{\, \gamma \in \omega^\omega \mid \gamma(0) = 0 \,\}, \\ BC_\xi &:= \left\{\, \gamma \in \omega^\omega \mid \gamma(0) = 1 \text{ and } \forall i \in \omega \ \ \{\gamma^*\}(i)\!\downarrow \text{ and } \{\gamma^*\}(i) \in \textstyle\bigcup_{1 \leq \eta < \xi} \ BC_\eta \,\right\} \text{ if } \xi \geq 2. \end{aligned}$$

The set of Borel codes is $BC := \bigcup_{1 \leq \xi < \omega_1} \ BC_\xi$. We also set $BC^* := \bigcup_{2 \leq \xi < \omega_1} \uparrow \ BC_\xi$. We define $\rho^X : BC \to \boldsymbol{\Delta}^1_1(X)$ by induction:

$$\rho^X(\gamma) := \begin{cases} \bigcup_{i \in \omega} \ B\big(X, \gamma^*(i)\big) \ \text{ if } \ \gamma \in BC_1, \\ \\ \bigcup_{i \in \omega} \ X \setminus \rho^X\big(\{\gamma^*\}(i)\big) \ \text{ if } \ \gamma \in BC^*. \end{cases}$$

Clearly, $\rho^X[BC_\xi] = \boldsymbol{\Sigma}^0_\xi(X)$, by induction on ξ.

10. *We can now define the* **hyperarithmetical hierarchy**. *Let $\xi \geq 1$ be a countable ordinal. Then*

$$\begin{aligned} \Sigma^0_\xi(X) &= \{\rho^X(\gamma) \mid \gamma \in \Sigma^0_1 \cap BC_\xi\}, \\ \\ \Pi^0_\xi(X) &= \check{\Sigma}^0_\xi(X), \\ \\ \Delta^0_\xi(X) &= \Sigma^0_\xi(X) \cap \Pi^0_\xi(X). \end{aligned}$$

This definition is compatible with the item 3.

The crucial link between the effective classes and the classical corresponding classes is as follows: the class of analytic (resp., co-analytic, Borel) sets is equal to $\bigcup_{\alpha \in \omega^\omega} \ \Sigma^1_1(\alpha)$ (resp., $\bigcup_{\alpha \in \omega^\omega} \ \Pi^1_1(\alpha)$, $\bigcup_{\alpha \in \omega^\omega} \ \Delta^1_1(\alpha)$). This allows to use effective descriptive set theory to prove results of classical type.

Theorem 20 *Let $\xi \geq 1$ be a recursive ordinal.*

(a) There is a finitary language $P_\xi \subseteq 2^{<\omega}$, that can be coded by a Δ^0_1 subset of ω, such that the ω-power P_ξ^∞ is in the effective class Π^0_ξ but not in $\boldsymbol{\Sigma}^0_\xi$.

(b) There is a finitary language $S_\xi \subseteq 2^{<\omega}$, that can be coded by a Δ^0_1 subset of ω, such that the ω-power S_ξ^∞ is in the effective class Σ^0_ξ but not in $\boldsymbol{\Pi}^0_\xi$.

5 Complexity of some sets of finitary languages related to the ω-powers

In [Lec05], the following question is raised. What is the topological complexity of the set of finitary languages whose associated ω-power is of a given level of complexity?

This question arises naturally when we look at the characterizations of closed, $\boldsymbol{\Pi}^0_2$ and open ω-powers obtained in [Sta97b] (see Corollary 14 and Lemmas 25, 26). This leads to set, for a class of sets $\boldsymbol{\Gamma}$, $\mathcal{L}_{\boldsymbol{\Gamma}} := \{L \subseteq 2^{<\omega} \mid L^\infty \in \boldsymbol{\Gamma}\}$. It is proved in [Lec05] (see Theorem 4) that $\mathcal{L}_{\{\emptyset\}}$ is $\boldsymbol{\Pi}^0_1$-complete, $\mathcal{L}_{\check{\{\emptyset\}}}$ is $\boldsymbol{\Sigma}^0_1$-complete, and

Theorem 21 *The set $\mathcal{L}_{\boldsymbol{\Delta}^0_1}$ is $\boldsymbol{\Sigma}^0_2$-complete.*

For the next classes of the Borel hierarchy, it is proved in [Lec05] that $\mathcal{L}_{\mathbf{\Sigma}^0_\xi}$ are $\mathcal{L}_{\mathbf{\Pi}^0_\xi}$ are $\mathbf{\Sigma}^1_2$ (see Proposition 16). A consequence of Theorem 20 is that these sets are $\mathbf{\Pi}^1_1$-hard if $\xi \geq 3$ (see Corollary 6.4 in [FL09]). It is proved in [Fin10] that for every integer $k \geq 2$ (respectively, $k \geq 3$) the set $\mathcal{L}_{\mathbf{\Pi}^0_{k+1}}$ (respectively, $\mathcal{L}_{\mathbf{\Sigma}^0_{k+1}}$) is "more complex" than the set $\mathcal{L}_{\mathbf{\Pi}^0_k}$ (respectively, $\mathcal{L}_{\mathbf{\Sigma}^0_k}$), with respect to the Wadge reducibility. The following result is proved in [Lec05, Fin10].

Theorem 22 *The set $\mathcal{L}_{\mathbf{\Delta}^1_1}$ is in $\mathbf{\Sigma}^1_2 \setminus \mathbf{\Pi}^0_2$.*

Along similar lines, some other results of effective nature are available in [Lec05, FL09]. For instance, we set $\mathcal{L}_\Delta := \{L \subseteq 2^{<\omega} \mid L^\infty \in \Delta^1_1(L)\}$. The following is proved in [Lec05] and [FL09].

Theorem 23 *The following sets are co-analytic and not Borel.*

(a) $\mathcal{L}_\Delta$,

(b) $\mathcal{L}_{\mathbf{\Sigma}^0_\xi} \cap \mathcal{L}_\Delta$ ($\mathbf{\Pi}^1_1$-complete if $\xi \geq 3$),

(c) $\mathcal{L}_{\mathbf{\Pi}^0_\xi} \cap \mathcal{L}_\Delta$ if $\xi \geq 2$ ($\mathbf{\Pi}^1_1$-complete if $\xi \geq 3$).

There is a very natural subset of $\mathcal{L}_{\mathbf{\Pi}^0_1}$, namely the set of finitely generated ω-powers. If we set $\mathbf{\Gamma}_f := \{L^\infty \mid L \text{ is finite}\}$, then this is $\mathcal{L}_{\mathbf{\Gamma}_f}$. We can decompose $\mathbf{\Gamma}_f$ with respect to the cardinality, setting, for $p \in \omega$, $\mathbf{\Gamma}_p := \{L^\infty \mid \text{Cardinality}(L) = p\}$, so that $\mathbf{\Gamma}_f = \bigcup_{p\in\omega} \mathbf{\Gamma}_p$. Note that $\mathbf{\Gamma}_0 = \mathcal{L}_{\{\emptyset\}}$, and we can prove that $\mathbf{\Gamma}_1$ is $\mathbf{\Pi}^0_1$-complete (see Proposition 6 in [Lec05]). The complexity of $\mathbf{\Gamma}_2$ is very surprising since it is not clear at all on its definition (see Corollary 10 in [Lec05]).

Theorem 24 *The set $\mathbf{\Gamma}_2$ is $\check{D}_\omega(\mathbf{\Sigma}^0_1)$-complete.*

6 Open questions

It is still open to determine all the infinite Borel ranks of the ω-powers of context-free languages. However the results of [Fin06] suggest that the ω-powers of context-free languages or even of languages accepted by one-counter automata exhibit also a great topological complexity. Indeed, there are ω-languages accepted by Büchi one-counter automata of every Borel rank (and even of every Wadge degree) of an effective analytic set.

In particular, for each recursive ordinal $\xi < \omega_1^{\text{CK}}$, there are some ω-languages P_ξ and S_ξ in the class Δ^1_1 such that P_ξ is $\mathbf{\Pi}^0_\xi$-complete and S_ξ is $\mathbf{\Sigma}^0_\xi$-complete. But effective analytic sets are much more complicated than Δ^1_1 sets: Kechris, Marker and Sami proved in [KMS89] that the supremum of the set of Borel ranks of (effective) Σ^1_1 sets is the ordinal γ^1_2. This ordinal is proved to be strictly greater than the ordinal δ^1_2 which is the first non Δ^1_2 ordinal. In particular, the ordinal γ^1_2 is strictly greater than the ordinal ω_1^{CK} (note that the exact value of the ordinal γ^1_2 may depend on axioms of set theory).

Moreover each ω-language $L \subseteq \Sigma^\omega$ accepted by a Büchi one-counter automaton is of the form $L = \bigcup_{1\leq j\leq n} U_j \cdot V_j^\infty$, for some one-counter finitary languages U_j and V_j, $1 \leq j \leq n$.

Therefore it seems plausible that there exist complete ω-powers of a one-counter language, for each Borel class of recursive rank, and we can even conjecture that there exist some ω-powers of languages accepted by one-counter automata which have Borel ranks up to the ordinal γ^1_2, although these languages are located at the very low level in the complexity hierarchy of finitary languages.

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