Descriptive complexity of countable unions of Borel rectangles

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Abstract. We give, for each countable ordinal $\xi \geq 1$, an example of a $\Delta^0_\xi$ countable union of Borel rectangles that cannot be decomposed into countably many $\Pi^0_\xi$ rectangles. In fact, we provide a graph of a partial injection with disjoint domain and range, which is a difference of two closed sets, and which has no $\Delta^0_\xi$-measurable countable coloring.

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1 Introduction

In this paper, we work in products of two Polish spaces. One of our goals is to give an answer to the following simple question. Assume that a countable union of Borel rectangles has low Borel rank. Is there a decomposition of this union into countably many rectangles of low Borel rank? In other words, is there a map \( r : \omega_1 \setminus \{0\} \to \omega_1 \setminus \{0\} \) such that \( \Pi_0^0 \cap (\Delta^1 \times \Delta^1) \subseteq (\Pi_0^0(\xi) \times \Pi_0^0(\xi)) \) for each \( \xi \in \omega_1 \setminus \{0\} \)?

By Theorem 3.6 in [Lo], a Borel set with open vertical sections is of the form \( (\Delta^1 \times \Sigma^0_1) \sigma \). This leads to a similar problem: is there a map \( s : \omega_1 \setminus \{0\} \to \omega_1 \setminus \{0\} \) such that, for each \( \xi \in \omega_1 \setminus \{0\} \), \( \Pi_0^0 \cap (\Delta^1 \times \Sigma^0_1) \sigma \subseteq (\Pi_0^0(\xi) \times \Sigma^0_1) \sigma \)?

The answer to these questions is negative:

**Theorem 1.1** Let \( 1 \leq \xi < \omega_1 \). Then there exists a partial map \( f : \omega^\omega \to \omega^\omega \) such that the complement \(-\text{Gr}(f)\) of the graph of \( f \) is \( \Pi_2^0 \) but not \( (\Sigma^0_\xi \times \Delta^1_1) \sigma \).

In fact, we prove a result related to \( \Delta^0_\xi \)-measurable countable colorings. A study of such colorings is made in [L-Z]. It was motivated by the \( G_0 \)-dichotomy (see Theorem 6.3 in [K-S-T]). More precisely, let \( B \) be a Borel binary relation having a Borel countable coloring (i.e., a Borel map \( c : X \to \omega \) such that \( c(x) \neq c(y) \) if \( (x, y) \in B \)). Is there a relation between the Borel class of \( B \) and that of the coloring? In other words, is there a map \( k : \omega_1 \setminus \{0\} \to \omega_1 \setminus \{0\} \) such that any \( \Pi_0^0 \) binary relation having a Borel countable coloring has in fact a \( \Delta^0_\xi \)-measurable countable coloring, for each \( \xi \in \omega_1 \setminus \{0\} \)? Here again, the answer is negative:

**Theorem 1.2** Let \( 1 \leq \xi < \omega_1 \). Then there exists a partial injection with disjoint domain and range \( i : \omega^\omega \to \omega^\omega \) whose graph is the difference of two closed sets, and has no \( \Delta^0_\xi \)-measurable countable coloring.

These two results are consequences of Theorem 4 in [Má] and its proof. This latter can also be used positively, to produce examples of graphs of fixed point free partial injections having reasonable chances to characterize the analytic binary relations without \( \Delta^0_\xi \)-measurable countable coloring. We will see in Section 4 that such a characterization indeed holds when \( \xi = 3 \), and give an example much simpler than the one in [L-Z]. In Section 2, we give a proof of Theorem 4 in [Má], in \( \omega^\omega \) instead of \( 2^\omega \), and also prove some additional properties needed for the construction of our partial maps. In Section 3, we prove Theorems 1.1 and 1.2. At the end of Section 4, we show that Theorem 1.2 is optimal in terms of descriptive complexity of the graph, and also give a positive result concerning the first two problems in the case of finite unions of rectangles.

2 Mátrai sets

Before proving our version of Theorem 4 in [Má], we need some notation, definition, and a few basic facts. The maps with closed graph will be of particular interest for us.
Lemma 2.1 Let \((X_i)_{i \in \omega}, (Y_i)_{i \in \omega}\) be sequences of metrizable spaces, and, for each \(i \in \omega\), \(f_i : X_i \to Y_i\) be a partial map whose graph is a closed subset of \(X_i \times Y_i\). Then the graph of the partial map \(f := \Pi_{i \in \omega} f_i : \Pi_{i \in \omega} X_i \to \Pi_{i \in \omega} Y_i\) is closed.

**Proof.** Let \((x^j)_{j \in \omega}\) be a sequence of elements of \(\Pi_{i \in \omega} X_i\) converging to \(x := (x_i)_{i \in \omega}\) such that \((f(x^j))_{j \in \omega}\) converges to \(y := (y_i)_{i \in \omega} \in \Pi_{i \in \omega} Y_i\). Then \(y_i = f_i(x_i)\), since \(\text{Gr}(f_i)\) is closed, for each \(i \in \omega\). This implies that \(y = f(x)\) and the proof is finished. \(\square\)

**Notation.** Let \(X\) be a set and \(\mathcal{F}\) be a family of subsets of \(X\). Then the symbol \(\langle \mathcal{F} \rangle\) denotes the smallest topology on \(X\) containing \(\mathcal{F}\).

The next two lemmas can be found in [K] (see Lemmas 13.2 and 13.3).

Lemma 2.2 Let \((X, \sigma)\) be a Polish space and \(F\) be a \(\sigma\)-closed subset of \(X\). Then the topology \(\sigma_F := \langle \sigma \cup \{F\} \rangle\) is Polish and \(F\) is \(\sigma_F\)-clopen.

Lemma 2.3 Let \((\sigma_n)_{n \in \omega}\) be a sequence of Polish topologies on \(X\). Then the topology \(\langle \bigcup_{n \in \omega} \sigma_n \rangle\) is Polish.

Lemma 2.4 Let \((H_n)_{n \in \omega}\) be a disjoint family of sets in a zero-dimensional Polish space \((X, \sigma)\) and \((\sigma_n)_{n \in \omega}\) be a sequence of topologies on \(X\) such that
\[
\sigma_0 = \sigma, \; H_0 \text{ is } \sigma_0\text{-closed,}
\]
\[
\sigma_{n+1} = (\sigma_n \cup \{H_n\}), \; H_{n+1} \text{ is } \sigma_{n+1}\text{-closed for every } n \in \omega.
\]
Then the topology \(\sigma_\infty = \langle \bigcup_{n \in \omega} \sigma_n \rangle\) satisfies the following properties:
\[
\begin{align*}
& (a) \; \sigma_\infty \text{ is zero-dimensional Polish,} \\
& (b) \; \sigma_\infty|X \setminus \bigcup_{n \in \omega} H_n = \sigma|X \setminus \bigcup_{n \in \omega} H_n, \\
\end{align*}
\]
and, for every \(n \in \omega\),
\[
\begin{align*}
& (c) \; \sigma_\infty|H_n = \sigma|H_n, \\
& (d) \; H_n \text{ is } \sigma_\infty\text{-clopen.}
\end{align*}
\]

**Proof.** Using Lemma 2.2 we see that each topology \(\sigma_n\) is Polish. Then the topology \(\sigma_\infty\) is Polish by Lemma 2.3. Now observe that the following claim holds.

**Claim.** A set \(G \subseteq X\) is \(\sigma_\infty\)-open if and only if \(G\) can be written as \(G = G' \cup \bigcup_{n \in \omega} G_n \cap H_n\), where \(G', G_n\) are \(\sigma\)-open.

Note that \(H_n \in \Sigma^0_1(\sigma_{n+1}) \subseteq \Sigma^0_1(\sigma_\infty)\) and \(H_n \in \Pi^0_3(\sigma_n) \subseteq \Pi^0_3(\sigma_\infty)\), thus \(H_n\) is \(\sigma_\infty\)-clopen. Thus (d) is satisfied. Let \(\mathcal{B}\) be a basis for \(\sigma\) made of \(\sigma\)-clopen sets. Then the family
\[
\mathcal{B} \cup \{G \cap H_n \mid G \in \mathcal{B} \wedge n \in \omega\}
\]
is made of \(\sigma_\infty\)-clopen sets and form a basis for \(\sigma_\infty\) by the claim. This gives (a).

Let \(G \in \Sigma^0_1(\sigma_\infty)\). By the claim, we find \(\sigma\)-open sets \(G', G_n\) such that \(G = G' \cup \bigcup_{n \in \omega} G_n \cap H_n\). Then \(G \cap (X \setminus \bigcup_{n \in \omega} H_n) = G' \cap (X \setminus \bigcup_{n \in \omega} H_n)\). This implies (b). Moreover, \(G \cap H_n = G_n \cap H_n\), and (c) holds. \(\square\)
Notation. The symbol $\tau$ denotes the product topology on $\omega^\omega$.

**Definition 2.5** We say that a partial map $f : \omega^\omega \to \omega^\omega$ is **nice** if $\text{Gr}(f)$ is a $(\tau \times \tau)$-closed subset of $\omega^\omega \times \omega^\omega$.

The construction of $P_\xi$ and $\tau_\xi$, and the verification of the properties $(1)_\xi,(3)_\xi$ from the next lemma, can be found in [Má], up to minor modifications.

**Lemma 2.6** Let $1 \leq \xi < \omega_1$. Then there are $P_\xi \subseteq \omega^\omega$, and a topology $\tau_\xi$ on $\omega^\omega$ such that

1. $\tau_\xi$ is zero-dimensional perfect Polish and $\tau \subseteq \tau_\xi \subseteq \Sigma^0_2(\tau)$,
2. $P_\xi$ is a nonempty $\tau_\xi$-closed nowhere dense set,
3. if $S \in \Sigma^0_2(\omega^\omega, \tau)$ is $\tau_\xi$-nonmeager in $P_\xi$, then $S$ is $\tau_\xi$-nonmeager in $\omega^\omega$,
4. if $U$ is a nonempty $\tau_\xi$-$\tau$-open subset of $P_\xi$, then we can find a $\tau_\xi$-dense $G_\delta$ subset $G$ of $U$, and a nice $(\tau_\xi, \tau)$-homeomorphism $\varphi_{\xi, G}$ from $G$ onto $\omega^\omega$,
5. if $V$ is a nonempty $\tau_\xi$-$\tau$-open subset of $\omega^\omega$, then we can find a $\tau_\xi$-dense $G_\delta$ subset $H$ of $V$, and a nice $(\tau_\xi, \tau)$-homeomorphism $\psi_{\xi, H}$ from $H$ onto $\omega^\omega$,
6. if $U$ is a nonempty $\tau_\xi$-$\tau$-open subset of $P_\xi$ and $W$ is a nonempty open subset of $\omega^\omega$, then we can find a $\tau_\xi$-dense $G_\delta$ subset $G$ of $U$, and a nice $(\tau_\xi, \tau_\xi)$-homeomorphism $\varphi_{\xi, G, K}$ from $G$ onto $K$.

7. if $V, W$ are nonempty $\tau_\xi$-$\tau$-open subsets of $\omega^\omega$, then we can find a $\tau_\xi$-dense $G_\delta$ subset $H$ of $V \setminus P_\xi$, a $\tau_\xi$-dense $G_\delta$ subset $L$ of $W \setminus P_\xi$, and a nice $(\tau_\xi, \tau_\xi)$-homeomorphism $\psi_{\xi, H, L}$ from $H$ onto $L$.

**Proof.** We proceed by induction on $\xi$.

The case $\xi = 1$

We set $P_1 := \{ \alpha \in \omega^\omega \mid \forall n \in \omega \ \alpha(2n) = 0 \}$ and $\tau_1 := \tau$. The properties $(1)_1,(3)_1$ are clearly satisfied.

(4) Note that $(P_1, \tau_1)$ is homeomorphic to $(\omega^\omega, \tau)$. As any nonempty open subset of $(\omega^\omega, \tau)$ is homeomorphic to $(\omega^\omega, \tau)$, $(U, \tau_1)$ is homeomorphic to $(\omega^\omega, \tau)$. This gives $\varphi_{\xi, U}$, which is nice since $\omega^\omega$ is closed in itself. This shows that we can take $G := U$.

(5) As in (4) we see that $(V, \tau_1)$ is homeomorphic to $(\omega^\omega, \tau)$, and we can take $H := V$.

(6) Note that $U$ is the disjoint union of a sequence $(C_n)_{n \in \omega}$ of nonempty clopen subsets of $(P_1, \tau_1)$. Let $(U_{1,n})_{n \in \omega}$ be a partition of $W \setminus P_1$ into clopen subsets of $(\omega^\omega, \tau_1)$. As any nonempty open subset of $(P_1, \tau_1)$ or $(\omega^\omega, \tau_1)$ is homeomorphic to $(\omega^\omega, \tau)$, we can find homeomorphisms

\[ \varphi_0 : (C_0, \tau_1) \to (U_0, \tau_1) \]

and

\[ \varphi_1 : (\bigcup_{n>0} C_n, \tau_1) \to (U_1, \tau_1) \].

As $C_0$ and $U_{1,0}$ are $\tau$-closed, $\varphi_0$ and $\varphi_1$ are nice. This shows that the gluing of $\varphi_0$ and $\varphi_1$ is a nice homeomorphism from $(U, \tau_1)$ onto $(W \setminus P_1, \tau_1)$. Thus we can take $G := U$ and $K := W \setminus P_1$.

(7) As in (6) we write $V \setminus P_1$ as the disjoint union of a sequence of nonempty clopen subsets of $(\omega^\omega, \tau_1)$, and similarly for $W \setminus P_1$. Since these clopen sets are homeomorphic to $(\omega^\omega, \tau_1)$, we can take $H := V \setminus P_1$ and $L := W \setminus P_1$. 

4
The induction step

We assume that $1 < \xi < \omega_1$ and that the assertion holds for each ordinal $\theta < \xi$. We fix a sequence of ordinals $(\xi_n)_{n\in\omega}$ containing each ordinal in $\xi \setminus \{0\}$ infinitely many times. We set

\[ P_\xi = \omega^\omega \times (\Pi_{i<\omega} \neg P_{\xi_i}), \]
\[ \tau_\xi = \tau \times (\Pi_{i<\omega} \tau_{\xi_i}), \]
\[ U_{\xi,n} = \omega^\omega \times (\Pi_{i<n} \neg P_{\xi_i}) \times P_{\xi_n} \times (\omega^\omega)^\omega \quad (n \in \omega). \]

The family $\{U_{\xi,n} \mid n \in \omega\}$ is disjoint. We set $\sigma_0 = \tau_\xi^\omega$ and $\sigma_{n+1} = \langle \sigma_n \cup \{U_{\xi,n}\} \rangle$. It is easy to check that $U_{\xi,n} \in \Pi_0^1(\sigma_n)$. Applying Lemma 2.4 we get a topology $\tau_\xi := \sigma_\infty$ such that

(a) $\tau_\xi$ is zero-dimensional Polish,
(b) $\tau_\xi|_{P_\xi} = \tau_\xi^\omega|_{P_\xi}$,
and, for every $n \in \omega$,
(c) $\tau_\xi|_{U_{\xi,n}} = \tau_\xi^\omega|_{U_{\xi,n}}$,
(d) $U_{\xi,n}$ is $\tau_\xi$-clopen.

We defined the topology $\tau_\xi$ on $(\omega^\omega)^\omega$ instead of $\omega^\omega$. However, since the spaces $((\omega^\omega)^\omega, \tau_\omega)$ and $(\omega^\omega, \tau)$ are homeomorphic we can replace the latter space by the former one in the proof. Since there is no danger of confusion we will write $\tau$ instead of $\tau_\omega$ to simplify the notation.

(1) Clearly, $\tau \subseteq \tau_\xi$. Note that $U_{\xi,n} \in \Sigma^0_\xi(\tau)$ for every $n \in \omega$ and $\tau_\xi \subseteq \Sigma^0_\xi(\tau)$, so that $\tau_\xi \subseteq \Sigma^0_\xi(\tau)$. Moreover, $(\omega^\omega, \tau_\xi)$ is clearly perfect.

(2) As $U_{\xi,n}$ is $\tau_\xi$-clopen, $P_\xi$ is $\tau_\xi$-closed. Note that $\tau_\xi|_{P_\xi} = \tau_\xi^\omega|_{P_\xi}$ and $P_\xi$ contains no nonempty basic $\tau_\xi^\omega$-open set. This implies that $P_\xi$ is $\tau_\xi$-nowhere dense.

(3) Let $S \in \Sigma^0_\xi(\tau)$ be $\tau_\xi$-nonmeager in $P_\xi$. We may assume that $S \in \Pi^0_\theta(\tau)$ for some $\theta < \xi$. As $\tau_\xi|_{P_\xi} = \tau_\xi^\omega|_{P_\xi}$ and $S$ has the Baire property with respect to the topology $\tau_\xi^\omega$ there exists a $\tau_\xi^\omega$-open set $V$ such that $S$ is $\tau_\xi^\omega$-comeager in $P_\xi \cap V$. Moreover, we may assume that $V$ has the following form:

\[ V = \tilde{V} \times (\Pi_{i\leq k} V_i) \times (\omega^\omega)^\omega, \]

where $\tilde{V} \in \tau$, $V_i \in \tau_\xi$, and $V_i \subseteq \neg P_{\xi_i}$ for each $i \leq k$. The set $V^* = \tilde{V} \times (\Pi_{i\leq k} V_i) \times (\Pi_{i\geq k} \neg P_{\xi_i})$ is $\tau_\xi^\omega$-comeager in $V$ since $\neg P_{\xi_i}$ is $\tau_\xi$-comeager in $\omega^\omega$ for every $i \in \omega$. As $P_\xi \cap V = V^*$, $S$ is $\tau_\xi^\omega$-comeager in $V^*$. Let $p \in \omega$ be such that $p > k$ and $\xi_p \geq \theta$. Define

\[ \tau^\circ = \tau \times (\Pi_{i\neq p} \tau_{\xi_i}), \]
\[ Z = \tilde{V} \times V_0 \times \cdots \times V_k \times \neg P_{\xi_{k+1}} \times \cdots \times \neg P_{\xi_{p-1}} \times (\omega^\omega)^\omega, \]
\[ \tau^\circ = \tau \times (\Pi_{i<p} \tau_{\xi_i}) \times \times (\Pi_{i>p} \tau_{\xi_i}). \]

For $\alpha \in \omega^\omega$ define a set $(-S)_\alpha$ by

\[ (-S)_\alpha := \{(\bar{y}, y_0, y_1, \ldots, y_{p-1}, y_{p+1}, \ldots) \in \omega^\omega \mid (\bar{y}, y_0, y_1, \ldots, y_{p-1}, \alpha, y_{p+1}, \ldots) \in -S\}. \]
Denote $S^* := \{ \alpha \in \omega^\omega \mid (\neg S)_\alpha \text{ is } \tau^*-\text{nonmeager in } Z \}$. Note that $\neg S \in \Sigma_0^\omega(\tau) \subseteq \Sigma_0^\omega(\tau^\omega)$. By the Montgomery theorem (see 22.D in [K]), $S^* \in \Sigma_0^\omega(\tau) \subseteq \Sigma_0^\omega(\tau^\omega)$. By the Kuratowski-Ulam theorem, $S^*$ is $\tau_{\xi_p}$-meager in $-P_{\xi_p}$. Using the induction hypothesis, Condition (3) implies that $S^*$ is $\tau_{\xi_p}$-meager in $-P_{\xi_p}$. Using the Kuratowski-Ulam theorem again, we see that $S$ is $\tau_{\xi}$-comeager in the $\tau_{\xi}$-open set

$$W = \tilde{V} \times V_0 \times \cdots \times V_k \times -P_{\xi_{k+1}} \times \cdots \times -P_{\xi_{p-1}} \times P_{\xi_p} \times (\omega^\omega)^\omega.$$  

As $W \subseteq U_{\xi_p}$, $\tau_{\xi_p}|W = \tau_{\xi}^\omega|W$ by (c), and consequently $S$ is $\tau_{\xi}$-comeager in $W$. Thus $S$ is $\tau_{\xi}$-nonmeager in $(\omega^\omega)^\omega$ since $W$ is $\tau_{\xi}$-open.

(4) We first construct a $\tau_{\xi}$-dense open subset of $U$, which is the disjoint union of sets of the form

$$U^n := (W^n \times (\Pi_{i<k_n} W_i^n) \times (\omega^\omega)^\omega) \cap P_{\xi} = W^n \times (\Pi_{i<k_n} W_i^n \setminus P_{\xi_i}) \times (\Pi_{i \geq k_n} -P_{\xi_i}),$$

where $W^n$ is a nonempty $\tau$-clopen set and $W_i^n$ is a nonempty $\tau_{\xi_i}$-clopen set. In order to do this, we fix an injective $\tau_{\xi}$-dense sequence $(x_n)_{n \in \omega}$ of $U$, which is possible since $(P_{\xi}, \tau_{\xi})$ is nonempty and perfect. We first choose $W^0$ and the $W^0_i$s in such a way that $U^0$ is a proper $\tau_{\xi}$-clopen neighborhood of $x_0$ in $U$, which is possible since $\tau_{\xi|P_{\xi}} = \tau_{\xi}^\omega|P_{\xi}$. For the induction step, we choose $p_n$ minimal such that $x_{p_n} \notin \bigcup_{q \leq n} U^q$. Then we choose $W^{n+1}$ and the $W^{n+1}_i$s in such a way that $U^{n+1}$ is a proper $\tau_{\xi}$-clopen neighborhood of $x_{p_{n+1}}$ in $U \setminus (\bigcup_{q \leq n} U^q)$.

There is a nice $(\tau, \tau)$-homeomorphism $\psi_n$ from $W^n$ onto $N_n := \{ \alpha \in \omega^\omega \mid \alpha(0) = n \}$. The induction assumption gives,

- for $i < k_n$, a $\tau_{\xi_i}$-dense $G_\delta$ subset $G^n_i$ of $W^n \setminus P_{\xi_i}$, and a nice $(\tau_{\xi_i}, \tau)$-homeomorphism $\psi_{\xi_i,G^n_i}$ of $G^n_i$ onto $\omega^\omega$,

- for $i \geq k_n$, a $\tau_{\xi_i}$-dense $G_\delta$ subset $G^n_i$ of $-P_{\xi_i}$, and a nice $(\tau_{\xi_i}, \tau)$-homeomorphism $\psi_{\xi_i,G^n_i}$ of $G^n_i$ onto $\omega^\omega$.

By Lemma 2.1, the map $\psi_n \times (\Pi_{i \in \omega} \psi_{\xi_i,G^n_i})$ is a nice $(\tau_{\xi}, \tau)$-homeomorphism from

$$W^n \times (\Pi_{i \in \omega} G^n_i)$$

onto $N_n \times (\omega^\omega)^\omega$. If we set $G := \bigcup_{n \in \omega} \left( W^n \times (\Pi_{i \in \omega} G^n_i) \right)$, then we get a nice $(\tau_{\xi}, \tau)$-homeomorphism from $G$ onto $\omega^\omega$. We are done since $\tau_{\xi|P_{\xi}} = \tau_{\xi}^\omega|P_{\xi}$.

(5) We essentially argue as in (4) $\xi$. As $P_{\xi}$ is $\tau_{\xi}$-closed nowhere dense, we may assume that

$$V \subseteq -P_{\xi} = \bigcup_{n \in \omega} U_{\xi,n}.$$

We first construct a $\tau_{\xi}$-dense open subset of $V \cap U_{\xi,n}$, which is the disjoint union of sets of the form $V^{n,p} := W^{n,p} \times (\Pi_{i < n} W_i^{n,p} \setminus P_{\xi_i}) \times (W_i^{n,p} \cap P_{\xi_i}) \times (\Pi_{i \geq k_n} W_i^{n,p})\times(\omega^\omega)^\omega$, where $W^{n,p}$ is a nonempty $\tau$-clopen set and $W_i^{n,p}$ is a nonempty $\tau_{\xi_i}$-clopen set. This is possible since $\tau_{\xi|U_{\xi,n}} = \tau_{\xi}^\omega|U_{\xi,n}$. We are done since $U_{\xi,n}$ is $\tau_{\xi}$-clopen.
(6) As in (4) we construct a \(\tau_\xi\)-dense open subset of \(U\), which is the disjoint union of sets of the form \(U_n := (W^n \times (\Pi_{i<k_0} W^n_{\xi})) \times (\omega^\omega)^\omega) \cap P_\xi = W^n \times (\Pi_{i<k_0} W^n_{\xi}) \times (\Pi_{k_0<P_{\xi}} - P_{\xi})\), where \(W^n\) is a nonempty \(\tau\)-clopen set and \(W^n_i\) is a nonempty \(\tau_\xi\)-clopen set. Recall also that
\[
U_{\xi,n} = \omega^\omega \times (\Pi_{i<n\neq P_{\xi}}) \times (\omega^\omega)^\omega.
\]
We also construct a \(\tau_\xi\)-dense open subset of \(W\), which is the disjoint union of sets of the form
\[
\pi^n := Z^n \times (\Pi_{i<l_0} Z^n_{\xi}) \times (Z^n_{\xi} \cap P_{\xi}) \times (\Pi_{l_0<i<k_0} Z^n_{\xi}) \times (\omega^\omega)^\omega \subseteq U_{\xi,n},
\]
where \(Z^n\) is a nonempty \(\tau\)-clopen set and \(Z^n_\xi\) is a nonempty \(\tau_\xi\)-clopen set. Let \((W^{0,p})_{p\in\omega}\) (respectively, \((Z^{0,p})_{p\in\omega}\)) be a partition of \(W^0\) (respectively, \(Z^0\)) into nonempty \(\tau\)-clopen sets. Using the facts that \(\tau_\xi|_{P_{\xi}} = \tau_\xi\big|_{P_{\xi}}\) and \(\tau_{\xi_0} = \tau_{\xi_0}\big|_{U_{\xi_0}}\), we will build
- a nice \((\tau_\xi, \tau_\xi)\)-homeomorphism from a dense \(G_\delta\) subset \(G^{0,p}\) of
\[
U^{0,p} := W^{0,p} \times (\Pi_{i<k_0} W^n_{\xi} \times (\Pi_{i<k_0} P_{\xi})) \times (\Pi_{k_0<P_{\xi} - P_{\xi}})
\]
on to a dense \(G_\delta\) subset \(K^{0,p}\) of \(U^{0,p}\). Then, using the fact that the \(U^{0,p}\)'s are \(\tau\)-clopen, the gluing of these \((\tau_\xi, \tau_\xi)\)-homeomorphisms will be a nice \((\tau_\xi, \tau_\xi)\)-homeomorphism \(\varphi_0\) from
\[
G^0 := \bigcup_{p\in\omega} G^{0,p} \subseteq U^0
\]
on to \(K^0 := \bigcup_{p\in\omega} K^{0,p}\) and \(U^0\).
- a nice \((\tau_\xi, \tau_\xi)\)-homeomorphism from a dense \(G_\delta\) subset \(G^{1,p}\) of \(U^{p+1}\) onto a dense \(G_\delta\) subset \(K^{1,p}\) of \(\pi^p\). Then the gluing of these \((\tau_\xi, \tau_\xi)\)-homeomorphisms will be a nice \((\tau_\xi, \tau_\xi)\)-homeomorphism \(\varphi_1\) from \(G^1 := \bigcup_{p\in\omega} G^{1,p} \subseteq \bigcup_{p>0} U^p\) onto \(K^1 := \bigcup_{p\in\omega} K^{1,p}\). The gluing of these two \((\tau_\xi, \tau_\xi)\)-homeomorphisms will be a nice \((\tau_\xi, \tau_\xi)\)-homeomorphism from \(G := G^0 \cup G^1\) onto \(K := K^0 \cup K^1\). The set \(G^{0,p}\) (respectively, \(K^{0,p}\)) will be of the form
\[
W^{0,p} \times (\Pi_{i\in\omega} G^{0,p})
\]
(respectively, \(Z^{p+1} \times (\Pi_{i\in\omega} K^{0,p})\)). Note first that there is a \((\tau, \tau)\)-homeomorphism \(\psi_p\) from \(W^{0,p}\) onto \(Z^{p+1}\). Then we build a permutation \(i \mapsto j_i\) of the coordinates (with inverse \(q \mapsto J_q\)). This permutation is constructed in such a way that \(\xi_j = \xi_i\), which will be possible since \((\xi_n)_{n\in\omega}\) contains each ordinal in \(\xi\{0\}\) infinitely many times. If \(i < m_{p+1}\) (respectively, \(q < k_0\)), then we choose \(j_i \geq k_0\) (respectively, \(J_q \geq m_{p+1}\)), ensuring injectivity. For a remaining coordinate \(q \notin \{0, ..., k_0-1\} \cup \{j_i \mid l < m_{p+1}\}\), we choose \(J_q \notin \{0, ..., m_{p+1}-1\} \cup \{j_i \mid l < k_0\}\), ensuring that the map \(q \mapsto J_q\) is a bijection from \(\neg(\{0, ..., k_0-1\} \cup \{j_i \mid l < m_{p+1}\})\) onto \(\neg(\{0, ..., m_{p+1}-1\} \cup \{j_i \mid l < k_0\})\). Then, using the induction assumption, we build our homeomorphism coordinate by coordinate, which means that \(G^0_i\) will be homeomorphic to \(K^{0,p}\). The induction assumption gives
- for \(i < l_{p+1}\), a \(\tau_{\xi_i}\)-dense \(G_\delta\) subset \(G^{0,p}_{j_i}\) of \(\neg P_{\xi_i}\), a \(\tau_{\xi_i}\)-dense \(G_\delta\) subset \(K^{0,p}_{j_i}\) of \(Z^{p+1}_{\xi_i}\) \(P_{\xi_i}\), and a nice \((\tau_{\xi_i}, \tau_{\xi_i})\)-homeomorphism \(\psi_{\xi_i,G^{0,p}_{j_i},K^{0,p}_{j_i}}\) from \(G^{0,p}_{j_i}\) onto \(K^{0,p}_{j_i}\).
- a $\tau_{j\xi_{p+1}}$-dense $G_\delta$ subset $G_{j\xi_{p+1}}^p$ of $-P_{j\xi_{p+1}}$, a $\tau_{\xi_{p+1}}$-dense $G_\delta$ subset $K_{j\xi_{p+1}}^p$ of $P_{j\xi_{p+1}}$, and a nice $(\tau_{j\xi_{p+1}}, \tau_{\xi_{p+1}})$-homeomorphism $\varphi^{-1}_{j\xi_{p+1}}$ from $G_{j\xi_{p+1}}^p$ onto $K_{j\xi_{p+1}}^p$

- for $l_{p+1} < i < m_{p+1}$, a $\tau_{\xi_{i}}$-dense $G_\delta$ subset $G_{i}^p$ of $-P_{\xi_{i}}$, a $\tau_{\xi_{i}}$-dense $G_\delta$ subset $K_{i}^p$ of $Z_{p+1}^1 \setminus P_{\xi_{i}}$, and a nice $(\tau_{\xi_{i}}, \tau_{\xi_{i}})$-homeomorphism $\psi_{i,G_{i}^p,K_{i}^p}$ from $G_{i}^p$ onto $K_{i}^p$

- for $q < k_0$, a $\tau_{\xi_{q}}$-dense $G_\delta$ subset $G_{q}^p$ of $W_{q}^0 \setminus P_{\xi_{q}}$, a $\tau_{\xi_{j_{q}}}^p$-dense $G_\delta$ subset $K_{j_{q}}^p$ of $-P_{\xi_{j_{q}}}$, and a nice $(\tau_{\xi_{q}}, \tau_{\xi_{j_{q}}})$-homeomorphism $\psi_{\xi_{q},G_{q}^{p},K_{j_{q}}^{p}}$ from $G_{q}^{p}$ onto $K_{j_{q}}^{p}$

By Lemma 2.1, the product $\varphi^{p}$ of $\psi_{p}$ with these nice homeomorphisms is a nice $(\tau_{\xi}, \tau_{\xi})$-homeomorphism from $G^{0,p} := W^{0,p} \times (\Pi_{\xi} G_{\xi}^{p})$ onto $K^{0,p} := Z^{p+1} \times (\Pi_{\xi} K_{\xi}^{p})$, as well as a $(\tau_{\xi}, \tau_{\xi})$-homeomorphism since $\tau_{\xi}|_{P_{\xi}} = \tau_{\xi}$ and $\tau_{\xi}|_{U_{\xi_{j+1}}^p} = \tau_{\xi}|_{U_{\xi_{j+1}}^p}$. As $G^{0,p}$ is the sum of the $G^{0,p}$'s, $G$ is a $\tau_{\xi}$-dense $G_\delta$ subset of $U^{0}$. Similarly, $K^{0}$ is a $\tau_{\xi}$-dense $G_\delta$ subset of $\bigcup p > 0 \pi^{p}$. Moreover, the gluing $\varphi^{0}$ of the $\varphi^{p}$'s is a $(\tau_{\xi}, \tau_{\xi})$-homeomorphism from $G^{0}$ onto $K^{0}$.

The construction of $\varphi^{1}$ is similar.

(7) $\xi$ We argue as in (6) $\xi$. $\square$

**Lemma 2.7** Let $1 \leq \xi < \omega_1$. Then there are disjoint families $F_\xi$, $G_\xi$ of subsets of $\omega^{\omega}$ and a topology $T_\xi$ on $\omega^{\omega}$ such that

(a) $T_\xi$ is zero-dimensional perfect Polish and $\tau \subseteq T_\xi \subseteq \Sigma^0_1(\tau)$,

(b) $F_\xi$ is $T_\xi$-dense, i.e., for any nonempty $T_\xi$-open set $V$, there is $F \in F_\xi$ with $F \subseteq V$,

and, for every $F \in F_\xi$,

(c) $\forall_{\xi} F$ is nonempty, $T_\xi$-nowhere dense, and in $\Pi^0_2(T_\xi)$,

(d) $\forall_{\xi}$ if $S \in \Sigma^0_1(\tau)$ is $T_\xi$-nonmeager in $F$, then $S$ is $T_\xi$-nonmeager in $\omega^{\omega}$,

(e) $\exists_{\xi}$ there is a nice $(T_\xi, \tau)$-homeomorphism $\varphi_F$ from $F$ onto $\omega^{\omega}$,

(f) $\forall_{\xi}$ for any nonempty $T_\xi$-open sets $V, V'$, there are disjoint $G, G' \in G_\xi$ with $G \subseteq V, G' \subseteq V'$, and there is a nice $(T_\xi, T_\xi)$-homeomorphism $\varphi_G, G'$ from $G$ onto $G'$,

and, for every $G \in G_\xi$,

(g) $\forall_{\xi} G$ is nonempty, $T_\xi$-nowhere dense, and in $\Pi^0_2(T_\xi)$,

(h) $\forall_{\xi}$ if $S \in \Sigma^0_1(\tau)$ is $T_\xi$-nonmeager in $G$, then $S$ is $T_\xi$-nonmeager in $\omega^{\omega}$.

**Proof.** Let $P_\xi$ and $\tau_\xi$ be as in Lemma 2.6. We set $T_\xi = (\tau_\xi)^{\omega}$. Let $(U_n)_{n \in \omega}$ be a basis for the topology $T_\xi$ made of nonempty sets. For each $n \in \omega$, there is a finite sequence $(V^n_i)_{i < k_n}$ of nonempty $\tau_\xi$-open sets such that $(\Pi_{i < k_n} V^n_i) \times (\omega^{\omega})^{\omega} \subseteq U_n$. Moreover, the sequence $(k_n)_{n \in \omega}$ is chosen to be strictly increasing.
Lemma 2.6 provides

- for $i < k_n$, a $\tau_\xi$-dense $G_\delta$ subset $H^i_\xi$ of $V^i_\xi \setminus P_\xi$ and a nice $(\tau_\xi, \tau)$-homeomorphism

$$\psi_{\xi, H^i_\xi} : H^i_\xi \to \omega^\omega,$$

- a $\tau_\xi$-dense $G_\delta$ subset $G^0_{k_n}$ of $P_\xi$ and a nice $(\tau_\xi, \tau)$-homeomorphism $\varphi_{\xi, G^0_{k_n}} : G^0_{k_n} \to \omega^\omega$,

- for $i > k_n$, a $\tau_\xi$-dense $G_\delta$ subset $H^i_\xi$ of $\omega^\omega$ and a nice $(\tau_\xi, \tau)$-homeomorphism $\psi_{\xi, H^i_\xi} : H^i_\xi \to \omega^\omega$.

We then put $F_n := (\Pi_{i<k_n} H^i_\xi) \times G^0_{k_n} \times (\Pi_{i>k_n} H^i_\xi)$, so that $F_n \subseteq U_n$. We set $\mathcal{F}_\xi = \{ F_n \mid n \in \omega \}$. Then $\mathcal{F}_\xi$ is clearly a disjoint family and the properties (a) and (b) are obviously satisfied.

(c) As $P_\xi$ is $\tau_\xi$-nowhere dense, each $F_n$ is $T_\xi$-nowhere dense. Each $F_n$ is obviously also in $\Pi^0_2 (T_\xi)$.

(d) Let $n \in \omega$ and $S \in \Sigma^0_2(\tau)$ be $T_\xi$-nonmeager in $F_n$. We define

$$T^*_\xi = \Pi_{i\neq k_n} \tau_\xi | H^i_\xi, \quad T^*_\xi = (\Pi_{i<k_n} \tau_\xi | H^i_\xi) \times \tau \times (\Pi_{i>k_n} \tau_\xi | H^i_\xi).$$

If $\alpha \in \omega^\omega$, then we denote

$$S_{\alpha} := \{(y_0, \ldots, y_{k_n-1}, y_{k_n}, \ldots) \in \omega^\omega \mid (y_0, \ldots, y_{k_n-1}, \alpha, y_{k_n}, \ldots) \in S \}. $$

We set $S^* = \{ \alpha \in \omega^\omega \mid S_{\alpha} \text{ is } T^*_\xi \text{-nonmeager} \}$. By the Montgomery theorem, $S^* \in \Sigma^0_2(\tau)$ since $S \in \Sigma^0_2(\tilde{T}_\xi)$. The set $S^*$ is $\tau_\xi$-nonmeager in $G^0_{k_n}$ by the Kuratowski-Ulam theorem, in $P_\xi$ also, and thus $S^*$ is $\tau_\xi$-nonmeager in $\omega^\omega$. Using the Kuratowski-Ulam theorem again, we see that $S$ is $T^*_\xi$-nonmeager in $(\Pi_{i<k_n} H^i_\xi) \times \omega^\omega \times (\Pi_{i>k_n} H^i_\xi)$ and thus in $(\omega^\omega)^\omega$.

(e) We set $\varphi_F = (\Pi_{i<k_n} \psi_{\xi, H^i_\xi}) \times \varphi_{\xi, G^0_{k_n}} \times (\Pi_{i>k_n} \psi_{\xi, H^i_\xi})$. The map $\varphi_F$ is clearly a $(T^*_\xi, \tau)$-homeomorphism from $F$ onto $(\omega^\omega)^\omega$. It is nice by Lemma 2.1.

We now construct $G_\xi$. For each $m \in \omega$, there are finite sequences $(V^m_1)_{i<m_n}, (W^m_1)_{i<l_m}$ of nonempty $\tau_\xi$-open sets such that $(\Pi_{i<k_m} V^m_i) \times (\omega^\omega)^\omega \subseteq U(m)_0$ and $(\Pi_{i<l_m} W^m_i) \times (\omega^\omega)^\omega \subseteq U(m)_1$. Moreover, the sequences $(k_m)_{m \in \omega}$ and $(l_m)_{m \in \omega}$ are chosen to be strictly increasing and disjoint. Assume for example that $k_m < l_m$. Lemma 2.6 provides

- for $i < k_m$, a $\tau_\xi$-dense $G_\delta$ subset $H^m_i$ of $V^m_i \setminus P_\xi$, a $\tau_\xi$-dense $G_\delta$ subset $L^m_i$ of $W^m_i \setminus P_\xi$, and a nice $(\tau_\xi, \tau_\xi)$-homeomorphism $\psi_{\xi, H^m_i, L^m_i}$,

- a $\tau_\xi$-dense $G_\delta$ subset $G^m_{k_m}$ of $P_\xi$, a $\tau_\xi$-dense $G_\delta$ subset $K^m_{k_m}$ of $W^m_i \setminus P_\xi$, and a nice $(\tau_\xi, \tau_\xi)$-homeomorphism $\varphi_{\xi, G^m_{k_m}, K^m_{k_m}}$,

- for $k_m < i < l_m$, a $\tau_\xi$-dense $G_\delta$ subset $H^m_i$ of $\neg P_\xi$, a $\tau_\xi$-dense $G_\delta$ subset $L^m_i$ of $W^m_i \setminus P_\xi$, and a nice $(\tau_\xi, \tau_\xi)$-homeomorphism $\psi_{\xi, H^m_i, L^m_i}$,

- a $\tau_\xi$-dense $G_\delta$ subset $K^m_{l_m}$ of $\neg P_\xi$, a $\tau_\xi$-dense $G_\delta$ subset $G^m_{l_m}$ of $P_\xi$, and a nice $(\tau_\xi, \tau_\xi)$-homeomorphism $\varphi^{-1}_{\xi, G^m_{l_m}, K^m_{l_m}}$.
- for \( i > l_m \), a \( \tau_\xi \)-dense \( G_\delta \) subset \( H^m_i \) of \( \neg P_\xi \), a \( \tau_\xi \)-dense \( G_\delta \) subset \( L^m_i \) of \( \neg P_\xi \), and a nice \((\tau_\xi, \tau_\xi)\)-homeomorphism \( \psi_\xi, H^m_i, L^m_i \).

We then put
\[
\begin{align*}
F'_m := (\Pi_{i<k_m} H^m_i) \times G^m_{k_m} \times (\Pi_{k_m<i<l_m} H^m_i) \times K^m_{l_m} \times (\Pi_{l_m<i} H^m_i), \\
G_m := (\Pi_{i<k_m} L^m_i) \times K^m_{k_m} \times (\Pi_{k_m<i<l_m} L^m_i) \times G^m_{l_m} \times (\Pi_{l_m<i} L^m_i),
\end{align*}
\]
so that \( F'_m \times G_m \subseteq U(m)_0 \times U(m)_1 \). We set \( G_\xi = \{ F'_m \mid m \in \omega \} \cup \{ G_m \mid m \in \omega \} \). Then \( G_\xi \) is clearly a disjoint family.

\( \text{(f)} \xi \) The map \( \varphi_{F'_m, G_m} \) is by definition
\[
\begin{align*}
(\Pi_{i<k_m} \psi_\xi, H^m_i, L^m_i) \times \varphi_{\xi, G^m_{k_m}, K^m_{l_m}} \times (\Pi_{k_m<i<l_m} \psi_\xi, H^m_i, L^m_i) \times \varphi_{\xi, G^m_{l_m}, K^m_{l_m}} \times (\Pi_{l_m<i} \psi_\xi, H^m_i, L^m_i).
\end{align*}
\]
Note that \( \varphi_{F'_m, G_m} \) is clearly a \((T_\xi, T_\xi)\)-homeomorphism from \( F'_m \) onto \( G_m \). It is nice by Lemma 2.1.

\( \text{(g)} \xi \) We argue as in \((c)\xi\).

\( \text{(h)} \xi \) We argue as in \((d)\xi\).

\[ \square \]

3 Negative results

**Proof of Theorem 1.1.** We apply Lemma 2.7 to the ordinal \( \xi + 1 \), which gives a family \( F_{\xi+1} \) and a topology \( T_{\xi+1} \) satisfying \((a)_{\xi+1}-\text{(e)}_{\xi+1}\). Let \((U_n \times V_n)_{n \in \omega}\) be a sequence of nonempty sets such that

- \( U_n \in T_{\xi+1}, V_n \) is \( \tau \)-clopen,
- \( \{ U_n \times V_n \mid n \in \omega \} \) is a basis for the topology \( T_{\xi+1} \times \tau \).

For each \( n \in \omega \) we find \( F_n \in F_{\xi+1}, \{ F_q \mid q < n \} \) with \( F_n \subseteq U_n \). By the property \((e)_{\xi+1}\) of \( F_{\xi+1} \) we find, for each \( n \in \omega \), a nice \((T_{\xi+1}, \tau)\)-homeomorphism \( f_n \) from \( F_n \) onto \( V_n \). We define \( f: \bigcup_{n \in \omega} F_n \rightarrow \omega^\omega \) by \( f(x) := f_n(x) \) if \( x \in F_n \). As \( F_{\xi+1} \) is a disjoint family, \( f \) is well-defined. The graph of \( f \) is \( \Sigma^2_\xi(\tau \times \tau) \) since each \( \text{Gr}(f_n) \) is \((\tau \times \tau)\)-closed.

Suppose, towards a contradiction, that there exist, for \( n \in \omega \), \( C_n \in \Sigma^0_{\xi+1}(\tau) \) and \( D_n \in \Delta^1_{\xi+1}(\tau) \) such that \( \neg \text{Gr}(f) = \bigcup_{n \in \omega} C_n \times D_n \). By the Baire category theorem there is \( n_0 \in \omega \) such that \( C_{n_0} \) is \( T_{\xi+1} \)-nonmeager and \( D_{n_0} \) is \( \tau \)-nonmeager. As \( C_{n_0} \) has the Baire property, we find a nonempty \( T_{\xi+1} \)-open set \( O_1 \) such that \( C_{n_0} \) is \( T_{\xi+1} \)-comeager in \( O_1 \). Similarly, we find a \( \tau \)-open set \( O_2 \) such that \( D_{n_0} \) is \( \tau \)-comeager in \( O_2 \).

Let \( n \in \omega \) and \( F_n \subseteq O_1 \). Suppose that \( C_{n_0} \) is not \( T_{\xi+1} \)-comeager in \( F_n \). Then \( O_1 \setminus C_{n_0} \) is \( T_{\xi+1} \)-nonmeager in \( F_n \). Note that \( O_1 \in \Sigma^0_{\xi+1}(\tau) \) and \( C_{n_0} \in \Sigma^0_{\xi}(\tau) \). Therefore \( O_1 \setminus C_{n_0} \in \Sigma^0_{\xi+1}(\tau) \). Thus \( O_1 \setminus C_{n_0} \) is \( T_{\xi+1} \)-nonmeager in \( \omega^\omega \) by \((d)_{\xi+1}\). Consequently, \( O_1 \setminus C_{n_0} \) is \( T_{\xi+1} \)-nonmeager in \( O_1 \), a contradiction. Thus \( C_{n_0} \) is \( T_{\xi+1} \)-comeager in \( F_n \) for any \( n \in \omega \) with \( F_n \subseteq O_1 \).
Find \( n \in \omega \) such that \( \text{Gr}(f_n) \subseteq O_1 \times O_2 \). Then \( C_{n_0} \) is \( T_{\xi+1} \)-comeager in \( F_n \) and \( D_{n_0} \) is \( \tau \)-comeager in \( V_n \). As \( f_n \) is a \( (T_{\xi+1}, \tau) \)-homeomorphism, \( f_n^{-1}((V_n \cap D_{n_0}) \cap F_n \cap C_{n_0}) \) is \( T_{\xi+1} \)-comeager in \( F_n \). As \( n \in \Pi_2^0(T_{\xi+1}) \) there exists \( \alpha \in f_n^{-1}(V_n \cap D_{n_0}) \cap F_n \cap C_{n_0} \). This implies that \( (\alpha, f_n(\alpha)) \in C_{n_0} \times D_{n_0} \), a contradiction. \( \square \)

**Proof of Theorem 1.2.** Apply Lemma 2.7 to the ordinal \( \xi + 1 \), which gives a family \( G_{\xi+1} \) and a topology \( T_{\xi+1} \) satisfying (a)\((\xi+1)-(h)_{\xi+1} \). Let \( \mathcal{U} = \{U_n \mid n \in \omega\} \) be a basis for the space \((\omega^\omega, T_{\xi+1})\) made of nonempty sets. For each \( n \in \omega \) we find \( T_{\xi+1} \)-open sets \( V_n, W_n \) such that

\[
V_n \times W_n \subseteq B_{\tau \times \tau}(\Delta(\omega^\omega), 2^{-n}) \cap (U_n \times U_n) \setminus \Delta(\omega^\omega)
\]

(we use the standard metric on \((\omega^\omega, T_{\xi+1})\)).

By the properties (0)\(_{\xi+1}\) and (g)\(_{\xi+1}\) of \( G_{\xi+1} \) we find, for each \( n \in \omega \), sets \( F_n \) and \( H_n \) from \( G_{\xi+1} \) such that

\[
(*) \quad F_n \subseteq V_n \setminus \left( \bigcup_{j < n} F_j \cup H_j \right) \land H_n \subseteq W_n \setminus \left( \bigcup_{j < n} F_j \cup H_j \right).
\]

Moreover, there is a nice \((T_{\xi+1}, T_{\xi+1})\)-homeomorphism \( f_n \) from \( F_n \) onto \( H_n \). We set

\[
\mathcal{G} = \bigcup \{\text{Gr}(f_n) \mid n \in \omega\}.
\]

Now we check the desired properties.

As \( \tau \subseteq T_{\xi+1} \), \( \overline{\tau \times \tau} = \mathcal{G} \cup \Delta(\omega^\omega) \), by construction. Thus \( \mathcal{G} \) is a difference of two \((\tau \times \tau)\)-closed sets.

As each \( f_n \) is a \((T_{\xi+1}, T_{\xi+1})\)-homeomorphism, the property (*) implies that \( f \) is a partial injection with disjoint domain and range. In order to see that \( \mathcal{G} \) has no \( \Delta^0_2 \)-measurable countable coloring, we proceed by contradiction. Suppose that there are \( \mathcal{G} \)-discrete sets \( C_n \in \Delta^0_2(\tau) \) (a set \( C \) is \( \mathcal{G} \)-discrete if \( C^2 \cap \mathcal{G} = \emptyset \)), for \( n \in \omega \), such that \( \Delta(\omega^\omega) \subseteq \bigcup_{n \in \omega} C_n^2 \). By the Baire theorem there exists \( n_0 \in \omega \) such that \( C_{n_0} \) is \( T_{\xi+1} \)-nonmeager. As \( C_{n_0} \) has the Baire property, we find a nonempty \( T_{\xi+1} \)-open set \( O \) such that \( C_{n_0} \cap O \) is \( T_{\xi+1} \)-comeager in \( O \).

Let \( F \in \mathcal{G}_{\xi+1} \) with \( F \subseteq O \). Suppose that \( C_{n_0} \) is not \( T_{\xi+1} \)-comeager in \( F \). Then \( O \setminus C_{n_0} \) is \( T_{\xi+1} \)-nonmeager in \( F \). Note that \( O \in \Sigma^0_{\xi+1}(\tau) \) and \( C_{n_0} \in \Delta^0_2(\tau) \). Therefore \( O \setminus C_{n_0} \in \Sigma^0_{\xi+1}(\tau) \). Thus \( O \setminus C_{n_0} \) is \( T_{\xi+1} \)-nonmeager in \( \omega^\omega \) by (h)\(_{\xi+1} \). Consequently, \( O \setminus C_{n_0} \) is \( T_{\xi+1} \)-nonmeager in \( O \), a contradiction. Thus \( C_{n_0} \) is \( T_{\xi+1} \)-comeager in \( F \) for any \( F \in \mathcal{G}_{\xi+1} \) with \( F \subseteq O \).

Find \( n \in \omega \) such that \( \text{Gr}(f_n) \subseteq O^2 \). Then \( C_{n_0} \) is \( T_{\xi+1} \)-comeager in \( F_n \) and in \( H_n \). As \( f_n \) is a \((T_{\xi+1}, T_{\xi+1})\)-homeomorphism, \( f_n^{-1}(H_n \cap C_{n_0}) \) is \( T_{\xi+1} \)-comeager in \( F_n \in \Pi_2^0(T_{\xi+1}) \). Thus there exists \( \alpha \in f_n^{-1}(H_n \cap C_{n_0}) \cap F_n \cap C_{n_0} \). This implies that \( (\alpha, f_n(\alpha)) \in C_{n_0}^2 \), a contradiction. \( \square \)

### 4 Positive results

(A) \( \Delta^0_2 \)-measurable countable colorings

In [L-Z], the following conjecture is made.
**Conjecture** Let $1 \leq \xi < \omega_1$. Then there are
- a $0$-dimensional Polish space $X_\xi$,
- an analytic relation $\Lambda_\xi$ on $X_\xi$

such that for any ($0$-dimensional if $\xi = 1$) Polish space $X$, and for any analytic relation $A$ on $X$, exactly one of the following holds:

(a) there is a $\Delta^0_\xi$-measurable countable coloring of $A$ (i.e., a $\Delta^0_\xi$-measurable map $c : X \to \omega$ such that $A \subseteq (c \times c)^{-1}(\neq)$),

(b) there is a continuous map $f : \mathbb{X}_\xi \to X$ such that $A_\xi \subseteq (f \times f)^{-1}(A)$.

This would be a $\Delta^0_\xi$-measurable version of the $\mathcal{G}_0$-dichotomy in [K-S-T]. This conjecture is proved for $\xi \leq 3$ in [L-Z]. Our goals here are the following. We want to give

- a reasonable candidate for $A_\xi$ in the general case,
- an example for $\xi = 3$ that is much simpler than the one in [L-Z].

We set $\Pi^0_\xi := \Delta^0_\xi$. The following result is proved in [Má] (see Theorem 4 and Lemma 13.(i)).

**Theorem 4.1** (Mátrai) Let $1 \leq \xi < \omega_1$. There are a true $\Pi^0_\xi$ subset $P_\xi$ of $2^\omega$, and a Polish topology $\tau_\xi$ on $2^\omega$ such that

1. $\tau_\xi$ is finer than the usual topology $\tau'$ on $2^\omega$,
2. $P_\xi$ is $\tau_\xi$-closed and $\tau_\xi$-nowhere dense,
3. if $G$ is a basic $\tau_\xi$-open set meeting $P_\xi$, and $D \in \Pi^0_\xi \cap 2^\omega$, $\tau'$ is such that $D \cap P_\xi \cap G$ is comeager in $(P_\xi \cap G, \tau_\xi | (P_\xi \cap G))$, then there is a $\tau_\xi$-open set $G'$ such that $P_\xi \cap G' = P_\xi \cap G$ and $D \cap G'$ is comeager in $(G', \tau_\xi | (G'))$.

**Notation.** In the sequel $1 \leq \xi < \omega_1$. Fix, for each $\xi$, an increasing sequence $(\eta_n)_{n \in \omega}$ of elements of $\xi$ (different from $0$ if $\xi \geq 2$) such that $\sup_{n \in \omega}(\eta_n + 1) = \xi$.

- Let $< ..., > : \omega^2 \to \omega$ be a bijection, defined for example by $< n, p > := (\Sigma_{k \leq n+p} k) + p$, whose inverse bijection is $q \mapsto ((q)_0, (q)_1)$.

- If $u \in 2^{< \omega}$ and $n \in \omega$, then we define $(u)_n \in 2^{< \omega}$ by $(u)_n(p) := u(< n, p >)$ if $< n, p > < |u|$.

- Let $(t_n)_{n \in \omega}$ be a dense sequence in $\omega^{< \omega}$ with $|t_n| = n$. For example, let $(p_n)_{n \in \omega}$ be the sequence of prime numbers, and $I : \omega^{< \omega} \to \omega$ defined by $I(\emptyset) := 1$, and $I(s) := p_0^{s(0)+1} ... p_{|s|-1}^{s(|s|-1)+1}$ if $s \neq \emptyset$. Note that $I$ is one-to-one, so that there is an increasing bijection $i : I[\omega^{< \omega}] \to \omega$. Set $\psi := (i \circ I)^{-1} : \omega \to \omega^{< \omega}$, so that $\psi$ is a bijection. Note that $|\psi(n)| \leq n$ if $n \in \omega$. Indeed,

$$I(\psi(n)[0]) < I(\psi(n)[1]) < ... < I(\psi(n)),$$

so that $(b \circ I)(\psi(n)[0]) < (b \circ I)(\psi(n)[1]) < ... < (b \circ I)(\psi(n)) = n$. As $|\psi(n)| \leq n$, we can define $t_n := \psi(n)(0^{n-|\psi(n)|})$, and $(t_n)_{n \in \omega}$ is suitable.

- Theorem 4.1 gives $P_\xi$ and $\tau_\xi$. Let $Q_\xi := 2 \times P_\xi$, $T_\xi := \text{discrete } \times \tau_\xi$, and $T_\xi^{<} := \Pi_{i \in \omega} T_{\eta_i}$ if $\xi \geq 2$. 12
• $(W_{\xi,n})_{n\in\omega}$ is a sequence of nonempty $T_\xi$-open sets.

• $S_i := Q_{\eta_i} \cup \bigcup_{n\in\omega} W_{\eta_i,n}$ (for $i \in \omega$), and $S := \Pi_{i\in\omega} S_i$, so that $S \in \Pi_2^0(T_\xi^\omega)$ is a Polish space.

• If $\xi \geq 2$, then we set

$$\mathbb{K}_\xi := \bigcup_{n\in\omega} \left\{ (\alpha, \beta) \in 2^{\omega} \times 2^{\omega} \mid (\forall i < n) \ (\alpha)_i = (\beta)_i \in W_{\eta_i,t_n(i)} \land \right.$$  

$$\left. (\exists \gamma \in \Pi_{\eta_n} \ ((\alpha)_n, (\beta)_n) = (0\gamma, 1\gamma)) \land (\forall i > n) \ (\alpha)_i = (\beta)_i \right\},$$

**Lemma 4.2** Let $2 \leq \xi \leq \omega_1$. We assume that $Q_{\eta_i} \subseteq \bigcup_{n\in\omega} W_{\eta_i,n}^{T_{\eta_i}}$ for each $i \in \omega$. Then any $\mathbb{K}_\xi$-discrete $\Sigma_\xi^0$ subset $C$ of $(S, \tau')$ is $T_\xi^\omega$-meager in $S$.

**Proof.** We may assume that $C$ is $\Pi_2^{0,\leq \xi}$. We argue by contradiction. This gives $n \in \omega$ with $C \in \Pi_{\eta_n}^0$, a basic $T_\xi^\omega$-open set $O$ such that $C \cap O$ is $T_\xi^\omega$-comeager in $O \cap S \neq \emptyset$, $l \geq n$, and a sequence $(O_i)_{i<l}$ with $O_i \in T_{\eta_i}$ and $O = \{ \alpha \in 2^{\omega} \mid (\forall i < l) \ (\alpha)_i \in O_i \}$. The assumption gives, for each $i < l$, $n_i \in \omega$ such that $O_i \cap W_{\eta_i,n_i} \neq \emptyset$. Let $m \geq l$ such that $t_m(i) = n_i$ for each $i < l$, and

$$U := \left\{ (\alpha) \in S \mid (\forall i < l) \ (\alpha)_i \in O_i \land (\forall i < m) \ (\alpha)_i \in W_{\eta_i,t_m(i)} \right\},$$

which is a nonempty $T_\xi^\omega$-open subset of $S$. In particular, $C \cap U$ is $T_\xi^\omega$-comeager in $U$. We set

$$V := \left\{ (\alpha)_{i \neq m} \in \bigcup_{i \neq m} S_i \mid (\forall i < l) \ (\alpha)_i \in O_i \land (\forall i < m) \ (\alpha)_i \in W_{\eta_i,t_m(i)} \right\},$$

so that, up to a permutation of coordinates, $U \equiv S_m \times V$. We also set

$$C' := \left\{ \alpha \in S_m \mid (C \cap (S_m \times V))_{\alpha} \text{ is } \Pi_{i \neq m} T_{\eta_i} \text{-comeager in } V \right\}.$$

By the Kuratowski-Ulam theorem, $C'$ is $T_{\eta_m}^\omega$-comeager in $S_m$ (see 8.41 in [K]). Write $C = D \cap S$, where $D \in \Pi_{\eta_m}^0(2^{\omega})$. Note that $C' := S_m \cap \left\{ \alpha \in 2^{\omega} \mid (D \cap (2^{\omega} \times V))_{\alpha} \text{ is } \Pi_{i \neq m} T_{\eta_i} \text{-comeager in } V \right\}$. As $m \geq n$ and $\Pi_{i \neq m} T_{\eta_i}$ is finer than the usual topology, $D \cap (2^{\omega} \times V) \in \Pi_{\eta_m}^0(2^{\omega}, \tau \times (\Pi_{i \neq m} T_{\eta_i})_{\mid V})$. By the Montgomery theorem, $C'$ is $\Pi_{\eta_m}^0(S_m, \tau')$ (see 22.22 in [K]).

The set $C'$ cannot be $T_{\eta_m}^\omega$-comeager both in $Q_{\eta_m} \cap N_0$ and in $Q_{\eta_m} \cap N_1$. Indeed, we argue by contradiction to see that. We set $h_0(\alpha) := \langle 1 - \alpha(0), \alpha(1), \alpha(2), \ldots \rangle$. As $h_0|_{Q_{\eta_m} \cap N_0}$ is a $T_{\eta_m}^\omega$-homeomorphism, $C' \cap h_0|_{Q_{\eta_m} \cap N_0}^{-1} (C' \cap Q_{\eta_m} \cap N_1)$ is $T_{\eta_m}^\omega$-comeager in $Q_{\eta_m} \cap N_0$, and if $\gamma$ is in it, then $1 \gamma \in C'$, which gives $\delta \in (C \cap U)_{\gamma} \cap (C \cap U)_{1 \gamma}$ and contradicts the $\mathbb{K}_\xi$-discreteness of $C$.

Assume for example that $C'$ is not $T_{\eta_m}^\omega$-comeager in $Q_{\eta_m} \cap N_0$. Then $-C'$ is $T_{\eta_m}^\omega$-non meager in $Q_{\eta_m}$. As $C'$ is $\Pi_{\eta_m}^0(S_m, \tau')$, there is a sequence $(C_j)_{j \in \omega}$ of $\Pi_{\eta_m}^0(2^{\omega})$ sets such that

$$S_m \setminus C' = \bigcup_{j \in \omega} C_j \cap S_m.$$

This gives $j \in \omega$ such that $C_j \cap Q_{\eta_m}$ is $T_{\eta_m}$-non meager in $Q_{\eta_m}$, and a basic $T_{\eta_m}$-open set $O$ such that $C_j \cap Q_{\eta_m} \cap O$ is $T_{\eta_m}$-comeager in $Q_{\eta_m} \cap O \neq \emptyset$.  

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The set $O$ is of the form $\{\varepsilon\} \times G$, where $\varepsilon \in 2$ and $G$ is a basic $\tau_{\eta m}$-open set. Let $S : N_\varepsilon \rightarrow 2^\omega$ be the map defined by $S(\varepsilon \alpha) := \alpha$. Note that $S$ is a $\tau' \tau'$ and $T_\varepsilon \tau_\xi$ homeomorphism. In particular, $E := \{\alpha \in 2^\omega \mid \varepsilon \alpha \in C_j\}$ is $\tau' \Pi^0_\eta \eta m$, and $E \cap P_{\eta m} \cap G$ is comeager in $(P_{\eta m} \cap G, \tau_{\eta m} | P_{\eta m} \cap G)$.

Theorem 4.1.(3) gives a $\tau_{\eta m}$-open set $G'$ such that $P_{\eta m} \cap G' = P_{\eta m} \cap G$ and $E \cap G'$ is comeager in $(G', \tau_{\eta m} | G')$. Now $O' := \{\varepsilon\} \times G'$ is a $T_{\eta m}$-comeager in $O'$. The assumption gives $n \in \omega$ such that $W_{\eta m, n} \cap O' \neq \emptyset$. Note that $C_j \cap W_{\eta m, n} \cap O'$ is $T_{\eta m}$-comeager in $W_{\eta m, n} \cap O'$, so that $\neg C'$ is $T_{\eta m}$-non meager in $S_m$, which is absurd. \qed

**Corollary 4.3** Let $2 \leq \xi \leq \omega_1$. We assume that $Q_\eta \subseteq \bigcap_{n \in \omega} W_{\eta, n}^{-T_{\eta}}$ for each $i \in \omega$. Then

(a) there is no $\Delta^0_1$-measurable map $c : 2^\omega \rightarrow \omega$ such that $K_\xi \subseteq (c \times c)^{-1}(\emptyset)$,

(b) if $X_\xi \in \Pi^0_1(2^\omega)$ and $K_\xi \subseteq X_\xi$, then there is no $\Delta^0_1$-measurable map $c : X_\xi \rightarrow \omega$ such that $K_\xi \subseteq (c \times c)^{-1}(\emptyset)$.

**Proof.** (a) We just have to apply Lemma 4.2.

(b) We argue by contradiction. This gives a partition $(C_k)_{k \in \omega}$ of $X_\xi$ into $K_\xi$-discrete $\Delta^0_1(X_\xi)$ sets.

We set $D_0 := 2^\omega \setminus X_\xi$, and choose $D_{k+1} \in \Sigma^0_0(2^\omega)$ such that $C_k = D_k \cup X_\xi$. Then $(D_k)_{k \in \omega}$ is a covering of $2^\omega$ into $K_\xi$-discrete $\Sigma^0_1$ sets. It remains to apply the reduction property of the class $\Sigma^0_1$ to contradict (a). \qed

**The case $\xi = 2$**

**Example.** Let $\alpha \mapsto \alpha^*$ be the shift map on $2^\omega$: $\alpha^*(j) := \alpha(j + 1)$. Then we set

$A_2 := \bigcup_{n \in \omega} \left\{ (\alpha, \beta) \in 2^\omega \times 2^\omega \mid (\forall i < n \ (\alpha)_i = (\beta)_i \land 0^{m(i)} 1 \subseteq (\alpha)_i^* \land ((\alpha)_n, (\beta)_n) = (0^\infty, 10^\infty) \land (\forall i > n \ (\alpha)_i = (\beta)_i) \right\}.$

**Theorem 4.4** $(2^\omega, A_2)$ satisfies the conjecture.

**Proof.** We set $P_1 := \{0^\infty\}$ and $\tau_1 := \tau'$, so that $P_1$ and $\tau_1$ satisfy the properties of Theorem 4.1. We also set $W_{1, n} := N_{0n+1} \cup N_{1n+1}$, so that $(W_{1, n})_{n \in \omega}$ is a sequence of nonempty $T_1$-open sets satisfying the assumption of Corollary 4.3, so that $A_2 = K_2$ satisfies its conclusions. In particular, (a) and (b) cannot hold simultaneously.

We define, for $(\varepsilon, n) \in 2 \times \omega$, $K^\varepsilon_n := \{\alpha \in 2^\omega \mid (\forall i < n \ 0^{m(i)} 1 \subseteq (\alpha)_i^* \land (\alpha)_n(0) = \varepsilon\}$, and also $C^\varepsilon_n := K^\varepsilon_n \setminus \bigcup_{k < k} K^0_k \cup K^1_k$, so that $C^\varepsilon_n$ is closed, the $C^\varepsilon_n$’s are pairwise disjoint, and $A_2 \subseteq \bigcup_{n \in \omega} C^0_n \times C^1_n$. We set, for each $p, q \in \omega$,

$O^p_q := \begin{cases} K^\varepsilon_n \setminus \bigcup_{k \leq q} K^0_k \cup K^1_k & \text{if } p = 2n + \varepsilon \leq 2q + 1, \\ 2^\omega \setminus \bigcup_{p' \leq 2q + 1} O^{p'}_q & \text{if } p = 2q + 2, \\ \emptyset & \text{if } p \geq 2q + 3, \end{cases}$

so that $(O^p_q)_{p,q} \subseteq 2^\omega$ is a covering of $2^\omega$ into clopen sets. Assume that $p = 2n + \varepsilon \neq p' = 2n' + \varepsilon' \leq 2q + 1$ and $\alpha \in O^p_q \cap O^{p'}_q$, so that $n, n' \leq q$. As $\alpha \in K^\varepsilon_n \cap K^{\varepsilon'}_{n'}$, $n \neq n'$ and for example $n < n'$, which is absurd. Thus $(O^p_q)_{p,q} \subseteq 2^\omega$ is a partition of $2^\omega$. \hfill 14
(a) Assume that \( q < n \). Note that \( C_n^0 \cup C_n^1 \) is contained in or disjoint from each set of the form \( K^q_n \) with \( k \leq q \). By disjonction, there is at most one couple \((\varepsilon, r)\) such that \( 2r + \varepsilon \leq 2q+1 \) and \( C_n^0 \cup C_n^1 \subseteq O^q_{q+\varepsilon} \). If it does not exist, then \( C_n^0 \cup C_n^1 \subseteq O^q_{2q+2} \).

(b) Assume that \( q \geq n \). Note that \( C_n^0 \subseteq K_n^q \). As \( q \geq n, p:=2n+\varepsilon \leq 2q+1 \). Thus \( C_n^0 \subseteq O^p_q \).

It remains to apply Proposition 4.6 in [L-Z] to see that (a) or (b) holds.

The case \( \xi = 3 \)

Example. Let \( (s_n)_{n \in \omega} \) be a dense sequence in \( 2^{<\omega} \) with \( |s_n| = n \). For example, let \( \phi : \omega \to 2^{<\omega} \) be a natural bijection. More specifically, \( \phi(0) := \emptyset \) is the sequence of length 0, \( \phi(1) := 0, \phi(2) := 1 \) are the sequences of length 1, and so on. Note that \( |\phi(n)| \leq n \) if \( n \in \omega \). Let \( n \in \omega \). As \( |\phi(n)| \leq n \), we can define \( s_n := \phi(n)0^{n-|\phi(n)|} \).

We set \( P_2 := \{ \alpha \in 2^{\omega} \mid \forall p \in \omega \exists q \geq p \big( \alpha(q) = 1 \big) \} \), and

\[
A_3 := \bigcup_{n \in \omega} \left\{ (\alpha, \beta) \in 2^{\omega} \times 2^{\omega} \mid \left( \forall i < n \ (\alpha)_i = (\beta)_i = s_{t_n(i)}^{10^\infty} \right) \land \left( \exists \gamma \in P_2 \ (\alpha)_i = (\beta)_i = 0_\gamma, 1_\gamma \right) \land \forall i > n \ (\alpha)_i = (\beta)_i \right\}.
\]

We will see that \( A_3 \), together with a suitable \( \Pi^0_2 \) subset \( X_3 \) of \( 2^{\omega} \), satisfies the conjecture. The topology \( \tau_2 \) makes the countably many singletons of \( \neg P_2 \) open. Then \( P_2 \) is a true \( \Pi^0_2 \) subset of \( 2^{\omega} \) (see 23.A in [K]), \( \tau_2 \) is Polish finer than \( \tau' \), \( P_2 \) is closed nowhere dense for \( \tau_2 \) since \( \tau_2 \) coincides with \( \tau' \) on \( P_2 \) and \( \neg P_2 \) is \( \tau'-\)dense, and 4.1.3 is satisfied since a basic \( \tau_2 \)-open set meeting \( P_2 \) is a basic \( \tau'-\)clopen set and \( P_2 \) is \( \tau'-\)comeager. Thus \( P_2 \) and \( \tau_2 \) satisfy the properties of Theorem 4.1. We set

\[ W_{2,n} := \{ s_n^{10^\infty} \}. \]

Then \( Q_2 \subseteq \bigcup_{n \in \omega} W_{2,n}^{\tau_2} \) since \( (s_n)_{n \in \omega} \) is dense. This shows that \( A_3 = X_3 \) satisfies the conclusions of Corollary 4.3. In particular, (a) and (b) cannot hold simultaneously. In order to prove that (a) or (b) holds, we simply indicate the modifications to make to Section 5 in [L-Z]. We just need to prove the right lemmas since the final construction is the same.

Lemma 4.5 (a) Let \( n \in \omega \) and \( i < n \). Then \( t_n(i) < n - i \).

(b) The map \( M : \{ s_{t_n(i)}^{10^\infty} \mid n \in \omega \land i < n \} \to \omega \), defined by \( M(\alpha) := \max \{ p \in \omega \mid \alpha(p) = 1 \} \), is one-to-one.

Proof. (a) Recall the map \( \psi \) defined after Theorem 4.1. It is enough to prove that \( \psi(n)(i) < n - i \) if \( i < |\psi(n)| \). We argue by induction on \( n \), and the result is clear for \( n = 0 \). We may assume that \( \psi(n)(i) = q + 1 \) for some natural number \( q \). We define \( t \in \omega^{<\omega} \) by \( t(i) := q \), and \( t(j) := \psi(n)(j) \) if \( j \neq i \). Let \( p \in \omega \) with \( \psi(p) = t \). Note that \( I(\psi(p)) < I(\psi(n)) \), so that \( p < n \). The induction assumption implies that \( q = \psi(p)(i) < n - i \), so that \( \psi(n)(i) = q + 1 \leq p - i < n - i \).

(b) Assume that \( M(\alpha) = M(\alpha') \). Let \( n, n', i, i' \) with \( \alpha = s_{t_n(i)}^{10^\infty} \) and \( \alpha' = s_{t_{n'}(i')}^{10^\infty} \). Then \( t_n(i) = s_{t_n(i)} = M(\alpha) = M(\alpha') = t_{n'}(i') \), so that \( \alpha = \alpha' \).

\[ \square \]

Notation. If \( \emptyset \neq u \in 2^{<\omega} \), then \( u^m := u(|u| - 1) \).

The following notion is technical but crucial.
We argue by contradiction. As

Proof. (a) This comes from Corollary 4.3.

(b) We say that

Lemma 4.8

(a) The set \( A_3 \) is a \( \Sigma^0_3 \) (and thus analytic) relation on \( X_3 \).

(b) \( (X_3, A_3) \) is not \( \Delta^0_3 \) \((\omega, -\Delta(\omega))\).

Proof. (a) \( A_3 \) is clearly a \( \Sigma^0_3 \) relation on \( 2^\omega \). So it is enough to see that it is a relation on \( X_3 \). Fix \((\alpha, \beta) \in A_3 \) (which defines a natural number \( n \)). Choose an infinite sequence \((p_k)_{k \in \omega} \) of natural numbers such that \((\alpha)_n(p_k) = (\beta)_n(p_k) = 1\). Then \( \alpha|\langle< n, p_k > + 1 \rangle \) and \( \beta|\langle< n, p_k > + 1 \rangle \) are placed, so that \( \alpha, \beta \in X_3 \).

(b) This comes from Corollary 4.3.

Lemma 4.8 Let \( n \in \omega, \alpha \in 2^\omega \) such that \((\alpha)_i = s_{t_n(i)} 10^{\infty} \) for each \( i < n \), and \( p > < n, 0 > \) such that \( \alpha|p \) is placed. Then \((p-1)_0 \geq n \).

Proof. We argue by contradiction. As \( p-1 \geq < n, 0 > \), \((p-1)_0 + (p-1)_1 \geq n + 0 = n \). Thus \((p-1)_1 \geq n -(p-1)_0 > 0 \). As \( \alpha|p \) is placed, \( \alpha(p-1) = 1 \). But

\[ \alpha(p-1) = \alpha(\langle(p-1)_0, (p-1)_1 >) = \alpha(p-1)_0 (\langle(p-1)_1 ) = (s_{t_n((p-1)_0} 10^{\infty}) (\langle(p-1)_1 )) \]

By Lemma 4.5.(a), we get \( (p-1)_1 < n -(p-1)_0 \), which is absurd.

Definition 4.9 Let \( u \in 2^{<\omega} \) and \( l \in \omega \).

(a) If \( u \) is placed, then we will consider

- the natural number \( l(u) := (|u^m|)_0 \)
- the sequence \( u_l(u) \in 2^{[u]} \backslash \{u\} \) defined by \( u_l(u)(m) := u(m) \) exactly when \( m \neq < l(u), 0 > \). Note that \( u_l(u) \) is placed, \( l(u_l(u)) = l(u) \) and \( u_l(u(l(u)) = u \)
- the digit \( \varepsilon (u) := u(< l(u), 0 >) \). Note that \( \varepsilon(u_l(u)) = 1 - \varepsilon (u) \).

(b) We say that \( u \) is \( l \)-placed if \( u \) is placed and \( l(u) = l \). We say that \( u \) is \((< l)\)-placed (resp., \((> l)\)-placed, \((> l)\)-placed) if there is \( l' \leq l \) (resp., \( l' < l \), \( l' > l \)) such that \( u \) is \( l' \)-placed.

When we consider the finite approximations of an element of \( A_3 \), we have to guess the natural number \( n \). We usually make some mistakes. In this case, we have to be able to come back to an earlier position. This is the role of the following predecessors.
Notation. Let $u \in 2^{<\omega}$. Note that $< \eta >$ is 0-placed with $\varepsilon(< \eta >) = \eta$ if $\eta \in 2$. This allows us to define

$$u^- := \begin{cases} \emptyset & \text{if } |u| \leq 1, \\ u|\max\{l < |u| \mid u|l \text{ is placed}\} & \text{if } |u| \geq 2, \end{cases}$$

and, for $l \in \omega$,

$$u^-_l := \begin{cases} \emptyset & \text{if } |u| \leq 1, \\ u|\max\{k < |u| \mid u|k \text{ is } (\leq l)\text{-placed}\} & \text{if } |u| \geq 2. \end{cases}$$

The following key lemma explains the relation between these predecessors and the placed sequences.

**Lemma 4.10** Let $l \in \omega$ and $u \in 2^{<\omega}$ be $l$-placed with $|u| \geq 2$.

(a) Assume that $u^-$ is $l$-placed. Then $\varepsilon(u^-) = \varepsilon(u)$. If moreover $(u^-)_l$ is $l$-placed, then the equality $(u^-)_l = (u^-)^l$ holds.

(b) $u^{-l}$ is $l$-placed if and only if $(u^-)_l$ is $l$-placed. In this case, $\varepsilon(u^{-l}) = \varepsilon(u)$ and the equality $(u^{-l})_l = (u^{-l})^l$ holds.

(c) Assume that $u^-$ or $(u^-)_l$ is $(< l)$-placed. Then $u^- = u^{-l} = (u^-)_l = (u^-)^l$.

(d) Assume that $u^-$ or $(u^-)_l$ is $(> l)$-placed. Then exactly one of those two sequences is $(> l)$-placed, and the other one is $l$-placed. If $u^-$ (resp., $(u^-)_l$) is $(> l)$-placed, then $u^{-l} = ((u^-)_l)^l$ (resp., $u^{-l} = u^-$) and $\varepsilon(u^{-l}) = \varepsilon(u)$ (resp., $\varepsilon((u^-)_l) = \varepsilon(u)$).

**Proof.** We first prove the following claim:

**Claim.** (i) Assume that $(|u^m|)_1 = 0$. Then $u^- = u^{-l} = (u^-)_l = (u^-)^l$ is $(< l)$-placed.

(ii) Assume that $(|u^m|)_1 > 0$. Then $u^-$ (resp., $u^{-l}$) is $(\geq l)$-placed and there is $j_0$ (resp., $j_1$) with $u^- = u|(< l(u^-), j_0 > +1)$ (resp., $u^{-l} = u|(< l, j_1 > +1)$).

**Proof.** (i) Note that $l \geq 1$ since $|u| \geq 2$. As $(|u^m|)_1 = 0$, $|u^m| = < (|u^m|)_0, (|u^m|)_1 > = < l(u), 0 >$ and the sequence $u^-$ is $(< l)$-placed, which implies that $u^- = u^{-l} = (u^-)_l = (u^-)^l$.

(ii) The last assertion about $j_0$ and $j_1$ comes from the first one. It is enough to see that $u^-$ is $(\geq l)$-placed since the proof for $u^-_l$ is similar. We argue by contradiction. Then $u|(< l, 0 > +1)$ is $l$-placed and $u|(< l, 0 > +1) \subseteq u|(< l, (|u^m|)_1 > +1) \subseteq u$, so that $u|(< l, 0 > +1) \subseteq u^-$. This implies that $l + 0 \leq l(u^-) + (|u^-| - 1)_1$, $(|u^-| - 1)_1 \geq l - l(u^-) > 0$ and $u^- (|u^-| - 1)_1 = 1$. But

$$u^- (|u^-| - 1)_1 = u^- (< l(u^-), (|u^-| - 1)_1 >) = u^- (l(u^-), (|u^-| - 1)_1 >) = (u^-)_l (|u^-| - 1)_1 = (s_t(u^-)|l(u^-))^{10\infty}(|u^-| - 1)_1.$$

Lemma 4.5.(a) implies that $(|u^-| - 1)_1 < l - l(u^-)$, which is absurd. \hfill \Box

(a) By the claim, $(|u^m|)_1 > 0$. Therefore $u|(< l, 0 > +1) \subseteq u|(< l, (|u^m|)_1 > +1) \subseteq u$ is $l$-placed, $u|(< l, 0 > +1) \subseteq u^- 0$ and $< l, 0 > = |u^-|$. Thus $\varepsilon(u^-) = (u^-)(< l, 0 >) = (u^-)(< l, 0 >)$.

Assume now that $(u^-)_l$ is $l$-placed. As $u|(< l, 0 > +1) \subseteq u^- \not\subseteq u$, we get

$$(u|(< l, 0 > +1))_l \subseteq (u^-)_l.$$
(b) Assume that \( u^{-l} \) is \( l \)-placed. By the claim, we get \((|u^m|)_1 > 0 \) and \( j_1 \) with
\[ u^{-l} = u((l, j_1 > +1)) \]
Thus \((u')^{-l} = u^l((l, j_1 > +1)) = (u^{-l})^l \) is \( l \)-placed, by Lemma 4.8. The equivalence comes from the fact that \((u')^l = u \). We argue as in (a) to see that \( \varepsilon(u^{-l}) = \varepsilon(u) \) if \( u^{-l} \) is \( l \)-placed.

(c) Assume first that \( u^- \) is \((< l)\)-placed. Then \((|u^m|)_1 = 0 \), by the claim, (ii). Now the claim, (i), gives the result. If \((u')^- \) is \((< l)\)-placed, then we apply this to \( u' \), using the facts that \( u' \) is \( l \)-placed and \((u')^l = u \).

(d) Assume first that \( u^- \) is \((> l)\)-placed. The claim, (i), implies that \((|u^m|)_1 > 0 \), and the claim, (ii), gives \( j_1 \) with \( u^{-l} = u((l, j_1 > +1)) \). Note that \( u^{-l} \nsubseteq u^- \), \((u^-)_l \subseteq s_{t(u^-)(l)}10^{\infty} \) and \( M(s_{t(u^-)(l)}10^{\infty}) < l(u^-) - l \), by Lemma 4.5.(a). Thus
\[ < l, M(s_{t(u^-)(l)}10^{\infty}) > \leq < l(u^-), 0 > \leq l(u^-), (|u^-| - 1)_1 > = |u^-| - 1 \]
and \((u^-)_l(M(s_{t(u^-)(l)}10^{\infty})) \) is defined. This shows that \( j_1 = M(s_{t(u^-)(l)}10^{\infty}) \).

Note that \((u')^l((< l, j_1 > +1)) \subseteq (u')^- \). The claim, (ii), shows that \((u')^- = u^l((< l, j_1 > +1)) \). We argue by contradiction to see that \((u')^- \) is not \((> l)\)-placed. The proof of the previous point shows that \( j_1 = M(s_{t(u^-)(l)}10^{\infty}) \). Lemma 4.5.(b) shows that \( s_{t(u^-)(l)}10^{\infty} = s_{t((u^-)(l))10^{\infty}} \). Thus
\[ (u^-)_l(0) = s_{t(u^-)(l)}10^{\infty}(0) = s_{t(1(u^-)(l))10^{\infty}}(0) = ((u^-)^l)_l(0), \]
\[ \varepsilon(u) = u(< l, 0 >) = (u^l)_l(0) = ((u^-)_l(0) = \varepsilon(u)), \]
which is absurd. This shows that \((u')^- = u^l((< l, j_1 > +1)) = (u^-)^l \) is \( l \)-placed, by Lemma 4.8, so that \( u^{-l} = ((u')^-)^l \). Moreover, \( \varepsilon(u^-) = (u^-)(< l, 0 >) = u(< l, 0 >) = \varepsilon(u) \).

Assume now that \((u')^- \) is \((> l)\)-placed. As \( u' \) is \( l \)-placed and \((u')^l = u \), the previous arguments show that \( u^- \) is \( l \)-placed. In particular, \( u^{-l} = u^- \).

**Theorem 4.11** \( (\mathbb{K}_3, \mathbb{A}_3) \) satisfies the conjecture.

**Proof.** We already noticed that it is enough to see that (a) or (b) holds. In Condition (5) in the proof of Theorem 5.1 in [L-Z], \( u^{-l} \) should be replaced with \( u^{-l}(u) \). We need to check that the map \( f \) defined there satisfies \( \mathbb{A}_3 \subseteq (f \times f)^{-1}(A) \). So let \( (\alpha, \beta) \in \mathbb{A}_3 \), which defines \( n \). Let \( (p_j)_{j \in \omega} \) be the infinite strictly increasing sequence of natural numbers \( p_j \geq 1 \) such that \( (p_j - 1)_0 = n, (p_j - 1)_1 > 0 \) and \( \alpha(p_j - 1) = 1 \). In particular, \( \alpha|p_j \) is \( n \)-placed and \( \varepsilon(\alpha|p_j) = 0 \). Note that \( (p_j)_{j \in \omega} \) is also the infinite strictly increasing sequence of natural numbers \( p_j \geq 1 \) such that \( (p_j - 1)_0 = n, (p_j - 1)_1 > 0 \) and \( \beta(p_j - 1) = 1 \) on one side, and a subsequence of both \( (p_k^\alpha)_{k \in \omega} \) and \( (p_k^\beta)_{k \in \omega} \) on the other side.

If moreover \( p \geq p_0 \) and \( \alpha|p \) is placed, then \( l(\alpha|p) \geq n \), by Lemma 4.8. In particular, if \( p \geq p_0 \) and \( \alpha|p \) is \((\leq n)\)-placed, then \( \alpha|p \) is \( n \)-placed. This proves that \( (p_j)_{j \in \omega} \) is the infinite strictly increasing sequence of integers \( p_j \geq p_0 \) such that \( \alpha|p_j \) is \((\leq n)\)-placed. Therefore \( (\alpha|p_{j+})^{-n} = \alpha|p_j \).
By Condition (3), \((U_{\alpha[p]}_{j} \in \omega)\) is a non-increasing sequence of nonempty clopen subsets of \(A \cap \Omega_{X}^{2}\) whose GH-diameter tend to 0. So we can define \(F(\alpha, \beta) \in A\) by \(\{F(\alpha, \beta)\} := \bigcap_{j \in \omega} U_{\alpha[p]_{j}}\). Note that \(F(\alpha, \beta) = \lim_{j \to \infty} (x_{\alpha[p]_{j}}, x_{\beta[p]_{j}}) = (f(\alpha), f(\beta)) \in A\), so that \(\mathcal{A}_{3} \subseteq (f \times f)^{-1}(A)\).

It remains, when \(k \geq 2\) (second case), to replace \(l-1\) with \(l(u^-)\).

\[\square\]

The general case

Here we just give, for each \(i \in \omega\), a sequence \((W_{\eta,n})_{n \in \omega}\) of nonempty \(T_{\eta}-\)open sets such that \(Q_{\eta} \subseteq \bigcup_{n \in \omega} W_{\eta,n}^{T_{\eta}}\). This will imply that \(\mathbb{K}_{\xi}\) has no \(\Delta_{\xi}^{0}\)-measurable countable coloring, by Corollary 4.3. We assume that \(\xi \geq 4\), so that we may assume that \(\eta \geq 3\). If \(\eta = \sup_{n \in \omega} (\theta_{n}+1) \geq 2\), then we set \(V_{\eta,n} := \{\alpha \in 2^{\omega} \mid \forall i < n \ (\alpha)_i \notin P_{\eta} \land (\alpha)_n \in P_{\eta}\}\). We set, for \(\eta \geq 3\),

\[W_{\eta,n} := \{\alpha \in 2^{\omega} \mid (\alpha)(0) = s_{n+1}(0) \land (\alpha^*)_n \in P_{\eta} \land \forall i < n \ (\alpha^*)_i \in \bigcup_{j < n-i} V_{\eta,j}\}.

Máté’s construction ensures that \(V_{\eta,n}\) is \(\tau_{\eta}\)-open, and that \(W_{\eta,n}\) is a nonempty \(T_{\eta}\)-open set. Let \(O\) be a basic \(T_{\eta}\)-open set meeting \(Q_{\eta}\). As \(T_{\eta} = \text{discrete} \times \tau_{\eta}\) and \(\tau_{\eta} \supseteq \{(\xi_{\epsilon} \in \omega) \tau_{\eta}\}_\eta\), we can find \(\epsilon \in 2\) and \((O_{\epsilon}, \eta) \subseteq \pi_{\eta} \tau_{\eta}\) such that \(O = \{\alpha \in 2^{\omega} \mid (\alpha)(0) = \epsilon \land \forall i < l \ (\alpha^*)_i \in O_{\epsilon}\}\). As \(P_{\eta}\) is \(\tau_{\eta}\)-closed nowhere dense and \(-P_{\eta} = \bigcup_{n \in \omega} V_{\eta,n}\), we can find \(n\) such that \(O_{l} \subseteq V_{\eta,n}\). We choose \(n > \max_{i < l} (n_{i} + i)\) such that \(s_{n+1}(0) = \epsilon\). Then \(W_{\eta,n}\) meets \(O\).

Our motivation to introduce these examples is that they induce a set \(\mathbb{K}_{3}\) satisfying the conjecture. This is the reason why we think that they are reasonable candidates for the general case.

(B) The small classes

In Section 3, we met \(D_{2}(\Pi_{0}^{0})\) graphs of fixed point free partial injections with a Borel countable (2)-coloring, but without \(\Delta_{0}^{0}\)-measurable countable coloring. Their complement are \(D_{2}(\Pi_{1}^{0})\) sets in \((\Delta_{1}^{0} \times \Sigma_{1}^{0})_{\sigma}\), but not in \((\Sigma_{\xi}^{0} \times \Sigma_{\xi}^{0})_{\sigma}\). However, a positive result holds for the simpler classes, which shows some optimality in our results.

Proposition 4.12 Let \(\Gamma \subseteq D_{2}(\Pi_{1}^{0})\) be a Wadge class (in zero-dimensional spaces), and \(A\) be a set in \(\Gamma \cap (\Delta_{1}^{0} \times \Sigma_{1}^{0})_{\sigma}\) (resp., \((\Delta_{1}^{0} \times \Delta_{1}^{0})_{\sigma}\)). Then \(A \in (\Gamma \times \Sigma_{1}^{0})_{\sigma}\) (resp., \((\Gamma \times \Gamma)_{\sigma}\)).

Proof. Let us do it for \((\Delta_{1}^{0} \times \Sigma_{1}^{0})_{\sigma}\), the other case being similar. The result is clear for \(\{\emptyset\}, \{\emptyset\}, \Delta_{1}^{0}, \Sigma_{1}^{0}\). If \(\Gamma = \Pi_{0}^{0}\), then we can write \(A = \bigcup_{n \in \omega} C_{n} \times D_{n}\), with \(C_{n} \in \Delta_{1}^{0}\) and \(D_{n} \in \Sigma_{1}^{0}\). We just have to note that \(A = \bigcup_{n \in \omega} C_{n} \times D_{n}\), then we can write \(A = \bigcup_{n \in \omega} C_{n} \times D_{n} = (C \cap D) \cup (O \setminus D)\), with \(C_{n} \in \Delta_{1}^{0}, O, D, \neg D, D_{n} \in \Sigma_{0}^{0}\). Note that \(A = (D \cap \bigcup_{n \in \omega} C_{n} \times D_{n}) \cup (O \setminus D)\). Finally, if \(\Gamma = D_{2}(\Pi_{0}^{0})\), then write \(A = \bigcup_{n \in \omega} C_{n} \times D_{n} = C \cap O\), with \(C_{n} \in \Delta_{1}^{0}, -C, O, D_{n} \in \Sigma_{1}^{0}\). Note that \(A = O \cap \bigcup_{n \in \omega} C_{n} \times D_{n}\).

\(\square\)

(C) The finite case

Proposition 4.13 Assume that \(\Gamma\) is closed under finite intersections and continuous pre-images, \(X, Y\) are topological spaces, \(\kappa\) is finite, and \(A \in \Gamma(X \times Y)\) is the union of \(\kappa\) rectangles. Then \(A\) is the union of at most \(2^{2^{\kappa}}\) rectangles whose sides are in \(\Gamma\).
Proof. Assume that $A = \bigcup_{n<\kappa} A_n \times B_n$. Let us prove that

$$A = \bigcup_{I \subseteq \kappa, (\bigcap_{n \in I} A_n) \setminus (\bigcap_{n \not\in I} A_n) \neq \emptyset} \left( \bigcap_{n \in I} A_n \right) \times \left( \bigcup_{n \in I} B_n \right).$$

So let $(x, y) \in A$, and let $I := \{ n < \kappa \mid x \in A_n \}$. Then $x \in (\bigcap_{n \in I} A_n) \setminus (\bigcap_{n \not\in I} A_n)$, and $(x, y)$ is in $(\bigcap_{n \in I} A_n) \times (\bigcup_{n \in I} B_n)$ since $(x, y) \in A_n \times B_n$ for some $n < \kappa$. The other inclusion is clear.

Assume now that $x \in (\bigcap_{n \in I} A_n) \setminus (\bigcap_{n \not\in I} A_n)$. Then $A_n \times B_n$ is in $\Gamma$. So we proved the following:

$A$ is the union of at most $2^n$ rectangles $A'_n \times B'_n$, where $A'_n$ is a finite intersection of some of the $A_n$'s, and $B'_n$ is a finite union of some of the $B_n$'s which is in $\Gamma$.

Applying this again, we see that $A$ is the union of at most $2^{2^n}$ rectangles $A''_n \times B''_n$, where $A''_n$ is a finite union of some of the $A'_n$'s which is in $\Gamma$, and $B''_n$ is a finite intersection of some of the $B'_n$'s. We are done since $\Gamma$ is closed under finite intersections. □

This proof also shows the following result:

**Proposition 4.14** Assume that $\Gamma$ is closed under continuous pre-images, $X, Y$ are topological spaces, $\kappa$ is finite, and $A \in \Gamma(X \times Y)$ is the union of $\kappa$ rectangles of the form $2^X \times \Sigma^0_1(Y)$. Then $A$ is the union of at most $2^{2^n}$ rectangles of the form $\Gamma(X) \times \Sigma^0_1(Y)$.

**Remarks.** (1) For colorings, Theorem 1.2 gives, for each $\xi$, a $D_2(\Pi^0_1)$ binary relation with a Borel finite (2-)coloring, but with no $\Delta^0_\xi$-measurable finite coloring.

(2) $\emptyset$ has a 1-coloring. An open binary relation having a finite coloring $c$ has also a $D_2(\Pi^0_1)$-measurable finite coloring (consider the differences of the $c^{-1}(\{n\})$'s, for $n$ in the range of the coloring). This leads to the following question:

**Question.** Can we build, for each $\xi$, a closed binary relation with a Borel finite coloring but no $\Delta^0_\xi$-measurable finite coloring?

**5 References**


