Descriptive complexity of countable unions of Borel rectangles

Dominique LECOMTE and Miroslav ZELENY[1]

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• Université Paris 6, Institut de Mathématiques de Jussieu, Projet Analyse Fonctionnelle
Couloir 16-26, 4ème étage, Case 247, 4, place Jussieu, 75 252 Paris Cedex 05, France
dominique.lecomte@upmc.fr

• Université de Picardie, I.U.T. de l'Oise, site de Creil,
13, allée de la faïencerie, 60 107 Creil, France

•[1] Charles University, Faculty of Mathematics and Physics, Department of Mathematical Analysis
Sokolovská 83, 186 75 Prague, Czech Republic
zeleny@karlin.mff.cuni.cz

Abstract. We give, for each countable ordinal $\xi \geq 1$, an example of a $\boldsymbol{\Delta}^0_2$ countable union of Borel rectangles that cannot be decomposed into countably many $\boldsymbol{\Pi}^0_\xi$ rectangles. In fact, we provide a graph of a partial injection with disjoint domain and range, which is a difference of two closed sets, and which has no $\boldsymbol{\Delta}^0_\xi$-measurable countable coloring.

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1 Introduction

In this paper, we work in products of two Polish spaces. One of our goals is to give an answer to the following simple question. Assume that a countable union of Borel rectangles has low Borel rank. Is there a decomposition of this union into countably many rectangles of low Borel rank? In other words, is there a map $r:\omega_1\setminus\{0\}\to\omega_1\setminus\{0\}$ such that $\mathbf{\Pi}^0_\xi\cap(\mathbf{\Delta}^1_1\times\mathbf{\Delta}^1_1)_\sigma\subseteq(\mathbf{\Pi}^0_{r(\xi)}\times\mathbf{\Pi}^0_{r(\xi)})_\sigma$ for each $\xi\in\omega_1\setminus\{0\}$?

By Theorem 3.6 in [Lo], a Borel set with open vertical sections is of the form $(\mathbf{\Delta}^1_1\times\mathbf{\Sigma}^0_1)_\sigma$. This leads to a similar problem: is there a map $s:\omega_1\setminus\{0\}\to\omega_1\setminus\{0\}$ such that, for each $\xi\in\omega_1\setminus\{0\}$, $\mathbf{\Pi}^0_\xi\cap(\mathbf{\Delta}^1_1\times\mathbf{\Sigma}^0_1)_\sigma\subseteq(\mathbf{\Pi}^0_{s(\xi)}\times\mathbf{\Sigma}^0_1)_\sigma$?

The answer to these questions is negative:

Theorem 1.1 *Let $1\leq\xi<\omega_1$. Then there exists a partial map $f:\omega^\omega\to\omega^\omega$ such that the complement $\neg Gr(f)$ of the graph of f is $\mathbf{\Pi}^0_2$ but not $(\mathbf{\Sigma}^0_\xi\times\mathbf{\Delta}^1_1)_\sigma$.*

In fact, we prove a result related to $\mathbf{\Delta}^0_\xi$-measurable countable colorings. A study of such colorings is made in [L-Z]. It was motivated by the $\mathbb{G}_0$-dichotomy (see Theorem 6.3 in [K-S-T]). More precisely, let B be a Borel binary relation having a Borel countable coloring (i.e., a Borel map $c:X\to\omega$ such that $c(x)\neq c(y)$ if $(x,y)\in B$). Is there a relation between the Borel class of B and that of the coloring? In other words, is there a map $k:\omega_1\setminus\{0\}\to\omega_1\setminus\{0\}$ such that any $\mathbf{\Pi}^0_\xi$ binary relation having a Borel countable coloring has in fact a $\mathbf{\Delta}^0_{k(\xi)}$-measurable countable coloring, for each $\xi\in\omega_1\setminus\{0\}$? Here again, the answer is negative:

Theorem 1.2 *Let $1\leq\xi<\omega_1$. Then there exists a partial injection with disjoint domain and range $i:\omega^\omega\to\omega^\omega$ whose graph is the difference of two closed sets, and has no $\mathbf{\Delta}^0_\xi$-measurable countable coloring.*

These two results are consequences of Theorem 4 in [Má] and its proof. This latter can also be used positively, to produce examples of graphs of fixed point free partial injections having reasonable chances to characterize the analytic binary relations without $\mathbf{\Delta}^0_\xi$-measurable countable coloring. We will see in Section 4 that such a characterization indeed holds when $\xi=3$, and give an example much simpler than the one in [L-Z]. In Section 2, we give a proof of Theorem 4 in [Má], in ω^ω instead of 2^ω, and also prove some additional properties needed for the construction of our partial maps. In Section 3, we prove Theorems 1.1 and 1.2. At the end of Section 4, we show that Theorem 1.2 is optimal in terms of descriptive complexity of the graph, and also give a positive result concerning the first two problems in the case of finite unions of rectangles.

2 Mátrai sets

Before proving our version of Theorem 4 in [Má], we need some notation, definition, and a few basic facts. The maps with closed graph will be of particular interest for us.

Lemma 2.1 *Let $(X_i)_{i\in\omega}$, $(Y_i)_{i\in\omega}$ be sequences of metrizable spaces, and, for each $i\in\omega$, $f_i : X_i \to Y_i$ be a partial map whose graph is a closed subset of $X_i \times Y_i$. Then the graph of the partial map $f := \Pi_{i\in\omega}\ f_i : \Pi_{i\in\omega}\ X_i \to \Pi_{i\in\omega}\ Y_i$ is closed.*

Proof. Let $(x^j)_{j\in\omega}$ be a sequence of elements of $\Pi_{i\in\omega}\ X_i$ converging to $x := (x_i)_{i\in\omega}$ such that $\big(f(x^j)\big)_{j\in\omega}$ converges to $y := (y_i)_{i\in\omega} \in \Pi_{i\in\omega}\ Y_i$. Then $y_i = f_i(x_i)$, since $\mathrm{Gr}(f_i)$ is closed, for each $i\in\omega$. This implies that $y = f(x)$ and the proof is finished. $\square$

Notation. Let X be a set and $\mathcal{F}$ be a family of subsets of X. Then the symbol $\langle\mathcal{F}\rangle$ denotes the smallest topology on X containing $\mathcal{F}$.

The next two lemmas can be found in [K] (see Lemmas 13.2 and 13.3).

Lemma 2.2 *Let (X,σ) be a Polish space and F be a σ-closed subset of X. Then the topology $\sigma_F := \langle\sigma \cup \{F\}\rangle$ is Polish and F is σ_F-clopen.*

Lemma 2.3 *Let $(\sigma_n)_{n\in\omega}$ be a sequence of Polish topologies on X. Then the topology $\langle\bigcup_{n\in\omega}\ \sigma_n\rangle$ is Polish.*

Lemma 2.4 *Let $(H_n)_{n\in\omega}$ be a disjoint family of sets in a zero-dimensional Polish space (X,σ) and $(\sigma_n)_{n\in\omega}$ be a sequence of topologies on X such that*

$\sigma_0 = \sigma$, H_0 is σ_0-closed,

$\sigma_{n+1} = \langle\sigma_n \cup \{H_n\}\rangle$, H_{n+1} is σ_{n+1}-closed for every $n\in\omega$.

Then the topology $\sigma_\infty = \langle\bigcup_{n\in\omega}\ \sigma_n\rangle$ satisfies the following properties:

(a) σ_∞ is zero-dimensional Polish,

(b) $\sigma_{\infty|X\setminus\bigcup_{n\in\omega} H_n} = \sigma_{|X\setminus\bigcup_{n\in\omega} H_n}$,

and, for every $n\in\omega$,

(c) $\sigma_{\infty|H_n} = \sigma_{|H_n}$,

(d) H_n is σ_∞-clopen.

Proof. Using Lemma 2.2 we see that each topology σ_n is Polish. Then the topology σ_∞ is Polish by Lemma 2.3. Now observe that the following claim holds.

Claim. *A set $G \subseteq X$ is σ_∞-open if and only if G can be written as $G = G' \cup (\bigcup_{n\in\omega} G_n \cap H_n)$, where G', G_n are σ-open.*

Note that $H_n \in \mathbf{\Sigma}^0_1(\sigma_{n+1}) \subseteq \mathbf{\Sigma}^0_1(\sigma_\infty)$ and $H_n \in \mathbf{\Pi}^0_1(\sigma_n) \subseteq \mathbf{\Pi}^0_1(\sigma_\infty)$, thus H_n is σ_∞-clopen. Thus (d) is satisfied. Let $\mathcal{B}$ be a basis for σ made of σ-clopen sets. Then the family

$$\mathcal{B} \cup \{G \cap H_n \mid G\in\mathcal{B}\ \wedge\ n\in\omega\}$$

is made of σ_∞-clopen sets and form a basis for σ_∞ by the claim. This gives (a).

Let $G \in \mathbf{\Sigma}^0_1(\sigma_\infty)$. By the claim, we find σ-open sets G', G_n such that $G = G' \cup (\bigcup_{n\in\omega}\ G_n \cap H_n)$. Then $G \cap (X\setminus\bigcup_{n\in\omega}\ H_n) = G' \cap (X\setminus\bigcup_{n\in\omega}\ H_n)$. This implies (b). Moreover, $G \cap H_n = G_n \cap H_n$, and (c) holds. $\square$

Notation. The symbol τ denotes the product topology on ω^ω.

Definition 2.5 *We say that a partial map $f:\omega^\omega\rightarrow\omega^\omega$ is* **nice** if $\mathrm{Gr}(f)$ is a $(\tau\times\tau)$-closed subset of $\omega^\omega\times\omega^\omega$.

The construction of P_ξ and τ_ξ, and the verification of the properties $(1)_\xi$-$(3)_\xi$ from the next lemma, can be found in [Má], up to minor modifications.

Lemma 2.6 *Let $1\leq\xi<\omega_1$. Then there are $P_\xi\subseteq\omega^\omega$, and a topology τ_ξ on ω^ω such that*

$(1)_\xi$ τ_ξ is zero-dimensional perfect Polish and $\tau\subseteq\tau_\xi\subseteq\mathbf{\Sigma}^0_\xi(\tau)$,

$(2)_\xi$ P_ξ is a nonempty τ_ξ-closed nowhere dense set,

$(3)_\xi$ if $S\in\mathbf{\Sigma}^0_\xi(\omega^\omega,\tau)$ is τ_ξ-nonmeager in P_ξ, then S is τ_ξ-nonmeager in ω^ω,

$(4)_\xi$ if U is a nonempty $\tau_{\xi|P_\xi}$-open subset of P_ξ, then we can find a τ_ξ-dense G_δ subset G of U, and a nice (τ_ξ,τ)-homeomorphism $\varphi_{\xi,G}$ from G onto ω^ω,

$(5)_\xi$ if V is a nonempty τ_ξ-open subset of ω^ω, then we can find a τ_ξ-dense G_δ subset H of V, and a nice (τ_ξ,τ)-homeomorphism $\psi_{\xi,H}$ from H onto ω^ω,

$(6)_\xi$ if U is a nonempty $\tau_{\xi|P_\xi}$-open subset of P_ξ and W is a nonempty open subset of ω^ω, then we can find a τ_ξ-dense G_δ subset G of U, a τ_ξ-dense G_δ subset K of $W\setminus P_\xi$, and a nice (τ_ξ,τ_ξ)-homeomorphism $\varphi_{\xi,G,K}$ from G onto K,

$(7)_\xi$ if V,W are nonempty τ_ξ-open subsets of ω^ω, then we can find a τ_ξ-dense G_δ subset H of $V\setminus P_\xi$, a τ_ξ-dense G_δ subset L of $W\setminus P_\xi$, and a nice (τ_ξ,τ_ξ)-homeomorphism $\psi_{\xi,H,L}$ from H onto L.

Proof. We proceed by induction on ξ.

The case $\xi=1$

We set $P_1:=\{\alpha\in\omega^\omega\mid\forall n\in\omega\ \ \alpha(2n)=0\}$ and $\tau_1:=\tau$. The properties $(1)_1$-$(3)_1$ are clearly satisfied.

$(4)_1$ Note that (P_1,τ_1) is homeomorphic to (ω^ω,τ). As any nonempty open subset of (ω^ω,τ) is homeomorphic to (ω^ω,τ), (U,τ_1) is homeomorphic to (ω^ω,τ). This gives $\varphi_{\xi,U}$, which is nice since ω^ω is closed in itself. This shows that we can take $G:=U$.

$(5)_1$ As in $(4)_1$ we see that (V,τ_1) is homeomorphic to (ω^ω,τ), and we can take $H:=V$.

$(6)_1$ Note that U is the disjoint union of a sequence $(C_n)_{n\in\omega}$ of nonempty clopen subsets of (P_1,τ_1). Let $(U_{1,n})_{n\in\omega}$ be a partition of $W\setminus P_1$ into clopen subsets of (ω^ω,τ_1). As any nonempty open subset of (P_1,τ_1) or (ω^ω,τ_1) is homeomorphic to (ω^ω,τ), we can find homeomorphisms

$$\varphi_0:(C_0,\tau_1)\rightarrow(\bigcup_{n>0}\ U_{1,n},\tau_1)$$

and $\varphi_1:(\bigcup_{n>0}\ C_n,\tau_1)\rightarrow(U_{1,0},\tau_1)$. As C_0 and $U_{1,0}$ are τ-closed, φ_0 and φ_1 are nice. This shows that the gluing of φ_0 and φ_1 is a nice homeomorphism from (U,τ_1) onto $(W\setminus P_1,\tau_1)$. Thus we can take $G:=U$ and $K:=W\setminus P_1$.

$(7)_1$ As in $(6)_1$ we write $V\setminus P_1$ as the disjoint union of a sequence of nonempty clopen subsets of (ω^ω,τ_1), and similarly for $W\setminus P_1$. Since these clopen sets are homeomorphic to (ω^ω,τ_1), we can take $H:=V\setminus P_1$ and $L:=W\setminus P_1$.

The induction step

We assume that $1<\xi<\omega_1$ and that the assertion holds for each ordinal $\theta<\xi$. We fix a sequence of ordinals $(\xi_n)_{n\in\omega}$ containing each ordinal in $\xi\setminus\{0\}$ infinitely many times. We set

$$\begin{aligned}&P_\xi=\omega^\omega\times(\Pi_{i\in\omega}\ \neg P_{\xi_i}),\\&\tau_\xi^<=\tau\times(\Pi_{i\in\omega}\ \tau_{\xi_i}),\\&U_{\xi,n}=\omega^\omega\times(\Pi_{i<n}\ \neg P_{\xi_i})\times P_{\xi_n}\times(\omega^\omega)^\omega\quad(n\in\omega).\end{aligned}$$

The family $\{U_{\xi,n}\mid n\in\omega\}$ is disjoint. We set $\sigma_0=\tau_\xi^<$ and $\sigma_{n+1}=\langle\sigma_n\cup\{U_{\xi,n}\}\rangle$. It is easy to check that $U_{\xi,n}\in\mathbf{\Pi}^0_1(\sigma_n)$. Applying Lemma 2.4 we get a topology $\tau_\xi:=\sigma_\infty$ such that

(a) τ_ξ is zero-dimensional Polish,

(b) $\tau_{\xi|P_\xi}=\tau^<_{\xi\ |P_\xi}$,

and, for every $n\in\omega$,

(c) $\tau_{\xi|U_{\xi,n}}=\tau^<_{\xi\ |U_{\xi,n}}$,

(d) $U_{\xi,n}$ is τ_ξ-clopen.

We defined the topology τ_ξ on $(\omega^\omega)^\omega$ instead of ω^ω. However, since the spaces $\big((\omega^\omega)^\omega,\tau^\omega\big)$ and (ω^ω,τ) are homeomorphic we can replace the latter space by the former one in the proof. Since there is no danger of confusion we will write τ instead of τ^ω to simplify the notation.

$(1)_\xi$ Clearly, $\tau\subseteq\tau_\xi$. Note that $U_{\xi,n}\in\mathbf{\Sigma}^0_\xi(\tau)$ for every $n\in\omega$ and $\tau_\xi^<\subseteq\mathbf{\Sigma}^0_\xi(\tau)$, so that $\tau_\xi\subseteq\mathbf{\Sigma}^0_\xi(\tau)$. Moreover, (ω^ω,τ_ξ) is clearly perfect.

$(2)_\xi$ As $U_{\xi,n}$ is τ_ξ-clopen, P_ξ is τ_ξ-closed. Note that $\tau_{\xi|P_\xi}=\tau^<_{\xi\ |P_\xi}$ and P_ξ contains no nonempty basic $\tau_\xi^<$-open set. This implies that P_ξ is τ_ξ-nowhere dense.

$(3)_\xi$ Let $S\in\mathbf{\Sigma}^0_\xi(\tau)$ be τ_ξ-nonmeager in P_ξ. We may assume that $S\in\mathbf{\Pi}^0_\theta(\tau)$ for some $\theta<\xi$. As $\tau_{\xi|P_\xi}=\tau^<_{\xi\ |P_\xi}$ and S has the Baire property with respect to the topology $\tau_\xi^<$ there exists a $\tau_\xi^<$-open set V such that S is $\tau_\xi^<$-comeager in $P_\xi\cap V$. Moreover, we may assume that V has the following form:

$$V=\tilde V\times(\Pi_{i\le k}\ V_i)\times(\omega^\omega)^\omega,$$

where $\tilde V\in\tau$, $V_i\in\tau_{\xi_i}$ and $V_i\subseteq\neg P_{\xi_i}$ for each $i\le k$. The set $V^*=\tilde V\times(\Pi_{i\le k}\ V_i)\times(\Pi_{i>k}\ \neg P_{\xi_i})$ is $\tau_\xi^<$-comeager in V since $\neg P_{\xi_i}$ is τ_{ξ_i}-comeager in ω^ω for every $i\in\omega$. As $P_\xi\cap V=V^*$, S is $\tau_\xi^<$-comeager in V^*. Let $p\in\omega$ be such that $p>k$ and $\xi_p\ge\theta$. Define

$$\begin{aligned}&\tau^*=\tau\times(\Pi_{i\neq p}\ \tau_{\xi_i}),\\&Z=\tilde V\times V_0\times\cdots\times V_k\times\neg P_{\xi_{k+1}}\times\cdots\times\neg P_{\xi_{p-1}}\times(\omega^\omega)^\omega,\\&\tau^\sharp=\tau\times(\Pi_{i<p}\ \tau_{\xi_i})\times\tau\times(\Pi_{i>p}\ \tau_{\xi_i}).\end{aligned}$$

For $\alpha\in\omega^\omega$ define a set $(\neg S)_\alpha$ by

$$(\neg S)_\alpha:=\{(\tilde y,y_0,y_1,\ldots,y_{p-1},y_{p+1},\ldots)\in\omega^\omega\mid(\tilde y,y_0,y_1,\ldots,y_{p-1},\alpha,y_{p+1},\ldots)\in\neg S\}.$$

Denote $S^*:=\{\alpha\in\omega^\omega \mid (\neg S)_\alpha$ is τ^*-nonmeager in $Z\}$. Note that $\neg S\in\mathbf{\Sigma}^0_\theta(\tau)\subseteq\mathbf{\Sigma}^0_\theta(\tau^\sharp)$. By the Montgomery theorem (see 22.D in [K]), $S^*\in\mathbf{\Sigma}^0_\theta(\tau)\subseteq\mathbf{\Sigma}^0_{\xi_p}(\tau)$. By the Kuratowski-Ulam theorem, S^* is τ_{ξ_p}-meager in $\neg P_{\xi_p}$. Using the induction hypothesis, Condition $(3)_{\xi_p}$ implies that S^* is τ_{ξ_p}-meager in P_{ξ_p}. Using the Kuratowski-Ulam theorem again, we see that S is $\tau_\xi^<$-comeager in the τ_ξ-open set

$$W=\tilde{V}\times V_0\times\cdots\times V_k\times\neg P_{\xi_{k+1}}\times\cdots\times\neg P_{\xi_{p-1}}\times P_{\xi_p}\times(\omega^\omega)^\omega.$$

As $W\subseteq U_{\xi,p}$, $\tau_{\xi|W}=\tau^<_{\xi\ |W}$ by (c), and consequently S is τ_ξ-comeager in W. Thus S is τ_ξ-nonmeager in $(\omega^\omega)^\omega$ since W is τ_ξ-open.

$(4)_\xi$ We first construct a τ_ξ-dense open subset of U, which is the disjoint union of sets of the form

$$U^n:=\big(W^n\times(\Pi_{i<k_n}\ W^n_i)\times(\omega^\omega)^\omega\big)\cap P_\xi=W^n\times(\Pi_{i<k_n}\ W^n_i\setminus P_{\xi_i})\times(\Pi_{i\geq k_n}\ \neg P_{\xi_i}),$$

where W^n is a nonempty τ-clopen set and W^n_i is a nonempty τ_{ξ_i}-clopen set. In order to do this, we fix an injective τ_ξ-dense sequence $(x_n)_{n\in\omega}$ of U, which is possible since (P_ξ,τ_ξ) is nonempty and perfect. We first choose W^0 and the W^0_i's in such a way that U^0 is a proper τ_ξ-clopen neighborhood of x_0 in U, which is possible since $\tau_{\xi|P_\xi}=\tau^<_{\xi\ |P_\xi}$. For the induction step, we choose p_n minimal such that $x_{p_n}\notin\bigcup_{q\leq n}\ U^q$. Then we choose W^{n+1} and the W^{n+1}_i's in such a way that U^{n+1} is a proper τ_ξ-clopen neighborhood of x_{p_n} in $U\setminus(\bigcup_{q\leq n}\ U^q)$.

There is a nice (τ,τ)-homeomorphism ψ_n from W^n onto $N_n:=\{\alpha\in\omega^\omega\mid\alpha(0)=n\}$. The induction assumption gives,

- for $i<k_n$, a τ_{ξ_i}-dense G_δ subset G^n_i of $W^n_i\setminus P_{\xi_i}$, and a nice (τ_{ξ_i},τ)-homeomorphism ψ_{ξ_i,G^n_i} of G^n_i onto ω^ω,

- for $i\geq k_n$, a τ_{ξ_i}-dense G_δ subset G^n_i of $\neg P_{\xi_i}$, and a nice (τ_{ξ_i},τ)-homeomorphism ψ_{ξ_i,G^n_i} of G^n_i onto ω^ω.

By Lemma 2.1, the map $\psi_n\times(\Pi_{i\in\omega}\ \psi_{\xi_i,G^n_i})$ is a nice $(\tau^<_\xi,\tau)$-homeomorphism from

$$W^n\times(\Pi_{i\in\omega}\ G^n_i)$$

onto $N_n\times(\omega^\omega)^\omega$. If we set $G:=\bigcup_{n\in\omega}\ \big(W^n\times(\Pi_{i\in\omega}\ G^n_i)\big)$, then we get a nice $(\tau^<_\xi,\tau)$-homeomorphism from G onto ω^ω. We are done since $\tau_{\xi|P_\xi}=\tau^<_{\xi\ |P_\xi}$.

$(5)_\xi$ We essentially argue as in $(4)_\xi$. As P_ξ is τ_ξ-closed nowhere dense, we may assume that

$$V\subseteq\neg P_\xi=\bigcup_{n\in\omega}\ U_{\xi,n}.$$

We first construct a τ_ξ-dense open subset of $V\cap U_{\xi,n}$, which is the disjoint union of sets of the form $V^{n,p}:=W^{n,p}\times(\Pi_{i<n}\ W^{n,p}_i\setminus P_{\xi_i})\times(W^{n,p}_n\cap P_{\xi_n})\times(\Pi_{n<i<k^p_n}\ W^{n,p}_i)\times(\omega^\omega)^\omega$, where $W^{n,p}$ is a nonempty τ-clopen set and $W^{n,p}_i$ is a nonempty τ_{ξ_i}-clopen set. This is possible since $\tau_{\xi|U_{\xi,n}}=\tau^<_{\xi\ |U_{\xi,n}}$. We are done since $U_{\xi,n}$ is τ_ξ-clopen.

$(6)_\xi$ As in $(4)_\xi$ we construct a τ_ξ-dense open subset of U, which is the disjoint union of sets of the form $U^n := \big(W^n \times (\Pi_{i<k_n} W_i^n) \times (\omega^\omega)^\omega\big) \cap P_\xi = W^n \times (\Pi_{i<k_n} W_i^n \setminus P_{\xi_i}) \times (\Pi_{i\geq k_n} \neg P_{\xi_i})$, where W^n is a nonempty τ-clopen set and W_i^n is a nonempty τ_{ξ_i}-clopen set. Recall also that

$$U_{\xi,n} = \omega^\omega \times (\Pi_{i<n} \neg P_{\xi_i}) \times P_{\xi_n} \times (\omega^\omega)^\omega.$$

We also construct a τ_ξ-dense open subset of W, which is the disjoint union of sets of the form

$$\pi^n := Z^n \times (\Pi_{i<l_n} Z_i^n \setminus P_{\xi_i}) \times (Z_{l_n}^n \cap P_{\xi_{l_n}}) \times (\Pi_{l_n<i<m_n} Z_i^n) \times (\omega^\omega)^\omega \subseteq U_{\xi,l_n},$$

where Z^n is a nonempty τ-clopen set and Z_i^n is a nonempty τ_{ξ_i}-clopen set. Let $(W^{0,p})_{p\in\omega}$ (respectively, $(Z^{0,p})_{p\in\omega}$) be a partition of W^0 (respectively, Z^0) into nonempty τ-clopen sets. Using the facts that $\tau_{\xi|P_\xi} = \tau^<_{\xi\ |P_\xi}$ and $\tau_{\xi|U_{\xi,n}} = \tau^<_{\xi\ |U_{\xi,n}}$, we will build

- a nice (τ_ξ, τ_ξ)-homeomorphism from a dense G_δ subset $G^{0,p}$ of

$$U^{0,p} := W^{0,p} \times (\Pi_{i<k_0} W_i^0 \setminus P_{\xi_i}) \times (\Pi_{i\geq k_0} \neg P_{\xi_i})$$

onto a dense G_δ subset $K^{0,p}$ of π^{p+1}. Then, using the fact that the $W^{0,p}$'s are τ-clopen, the gluing of these (τ_ξ, τ_ξ)-homeomorphisms will be a nice (τ_ξ, τ_ξ)-homeomorphism φ_0 from

$$G^0 := \bigcup_{p\in\omega} G^{0,p} \subseteq U^0$$

onto $K^0 := \bigcup_{p\in\omega} K^{0,p} \subseteq \bigcup_{p>0} \pi^p$.

- a nice (τ_ξ, τ_ξ)-homeomorphism from a dense G_δ subset $G^{1,p}$ of U^{p+1} onto a dense G_δ subset $K^{1,p}$ of $Z^{0,p} \times (\Pi_{i<l_0} Z_i^0 \setminus P_{\xi_i}) \times (Z_{l_0}^0 \cap P_{\xi_{l_0}}) \times (\Pi_{l_0<i<m_0} Z_i^0) \times (\omega^\omega)^\omega$. Then the gluing of these (τ_ξ, τ_ξ)-homeomorphisms will be a nice (τ_ξ, τ_ξ)-homeomorphism φ_1 from $G^1 := \bigcup_{p\in\omega} G^{1,p} \subseteq \bigcup_{p>0} U^p$ onto $K^1 := \bigcup_{p\in\omega} K^{1,p} \subseteq \pi^0$.

The gluing of these two (τ_ξ, τ_ξ)-homeomorphisms will be a nice (τ_ξ, τ_ξ)-homeomorphism from $G := G^0 \cup G^1$ onto $K := K^0 \cup K^1$. The set $G^{0,p}$ (respectively, $K^{0,p}$) will be of the form

$$W^{0,p} \times (\Pi_{i\in\omega} G_i^p)$$

(respectively, $Z^{p+1} \times (\Pi_{i\in\omega} K_i^p)$). Note first that there is a (τ, τ)-homeomorphism ψ_p from $W^{0,p}$ onto Z^{p+1}. Then we build a permutation $i \mapsto j_i$ of the coordinates (with inverse $q \mapsto J_q$). This permutation is constructed in such a way that $\xi_{j_i} = \xi_i$, which will be possible since $(\xi_n)_{n\in\omega}$ contains each ordinal in $\xi \setminus \{0\}$ infinitely many times. If $i < m_{p+1}$ (respectively, $q < k_0$), then we choose $j_i \geq k_0$ (respectively, $J_q \geq m_{p+1}$), ensuring injectivity. For a remaining coordinate $q \notin \{0, ..., k_0-1\} \cup \{j_l \mid l < m_{p+1}\}$, we choose $J_q \notin \{0, ..., m_{p+1}-1\} \cup \{J_l \mid l < k_0\}$, ensuring that the map $q \mapsto J_q$ is a bijection from $\neg(\{0, ..., k_0-1\} \cup \{j_l \mid l < m_{p+1}\})$ onto $\neg\big(\{0, ..., m_{p+1}-1\} \cup \{J_l \mid l < k_0\}\big)$. Then, using the induction assumption, we build our homeomorphism coordinate by coordinate, which means that $G_{j_i}^p$ will be homeomorphic to K_i^p. The induction assumption gives

- for $i < l_{p+1}$, a $\tau_{\xi_{j_i}}$-dense G_δ subset $G_{j_i}^p$ of $\neg P_{\xi_{j_i}}$, a τ_{ξ_i}-dense G_δ subset K_i^p of $Z_i^{p+1} \setminus P_{\xi_i}$, and a nice $(\tau_{\xi_i}, \tau_{\xi_i})$-homeomorphism $\psi_{\xi_i, G_{j_i}^p, K_i^p}$ from $G_{j_i}^p$ onto K_i^p.

- a $\tau_{\xi_{j_{l_{p+1}}}}$-dense G_δ subset $G^p_{j_{l_{p+1}}}$ of $\neg P_{\xi_{j_{l_{p+1}}}}$, a $\tau_{\xi_{l_{p+1}}}$-dense G_δ subset $K^p_{l_{p+1}}$ of $P_{\xi_{l_{p+1}}}$, and a nice $(\tau_{\xi_{l_{p+1}}}, \tau_{\xi_{l_{p+1}}})$-homeomorphism $\varphi^{-1}_{\xi_{l_{p+1}}, K^p_{l_{p+1}}, G^p_{j_{l_{p+1}}}}$ from $G^p_{j_{l_{p+1}}}$ onto $K^p_{l_{p+1}}$.

- for $l_{p+1} < i < m_{p+1}$, a $\tau_{\xi_{j_i}}$-dense G_δ subset $G^p_{j_i}$ of $\neg P_{\xi_{j_i}}$, a τ_{ξ_i}-dense G_δ subset K^p_i of $Z^{p+1}_i \setminus P_{\xi_i}$, and a nice $(\tau_{\xi_i}, \tau_{\xi_i})$-homeomorphism $\psi_{\xi_i, G^p_{j_i}, K^p_i}$ from $G^p_{j_i}$ onto K^p_i.

- for $q < k_0$, a τ_{ξ_q}-dense G_δ subset G^p_q of $W^0_q \setminus P_{\xi_q}$, a $\tau_{\xi_{J_q}}$-dense G_δ subset $K^p_{J_q}$ of $\neg P_{\xi_{J_q}}$, and a nice $(\tau_{\xi_q}, \tau_{\xi_q})$-homeomorphism $\psi_{\xi_q, G^p_q, K^p_{J_q}}$ from G^p_q onto $K^p_{J_q}$.

- for a remaining coordinate $q \notin \{0, ..., k_0 - 1\} \cup \{j_l \mid l < m_{p+1}\}$, a τ_{ξ_q}-dense G_δ subset G^p_q of $\neg P_{\xi_q}$, a $\tau_{\xi_{J_q}}$-dense G_δ subset $K^p_{J_q}$ of $\neg P_{\xi_{J_q}}$, and a nice $(\tau_{\xi_q}, \tau_{\xi_q})$-homeomorphism $\psi_{\xi_q, G^p_q, K^p_{J_q}}$ from G^p_q onto $K^p_{J_q}$.

By Lemma 2.1, the product φ^0_p of ψ_p with these nice homeomorphisms is a nice $(\tau^<_\xi, \tau^<_\xi)$-homeomorphism from $G^{0,p} := W^{0,p} \times (\Pi_{i \in \omega}\ G^p_i)$ onto $K^{0,p} := Z^{p+1} \times (\Pi_{i \in \omega}\ K^p_i)$, as well as a (τ_ξ, τ_ξ)-homeomorphism since $\tau_{\xi|P_\xi} = \tau^<_{\xi\ |P_\xi}$ and $\tau_{\xi|U_{\xi,l_{p+1}}} = \tau^<_{\xi\ |U_{\xi,l_{p+1}}}$. As G^0 is the sum of the $G^{0,p}$'s, G is a τ_ξ-dense G_δ subset of U^0. Similarly, K^0 is a τ_ξ-dense G_δ subset of $\bigcup_{p>0}\ \pi^p$. Moreover, the gluing φ^0 of the φ^0_p's is a (τ_ξ, τ_ξ)-homeomorphism from G^0 onto K^0.

The construction of φ^1 is similar.

$(7)_\xi$ We argue as in $(6)_\xi$. $\square$

Lemma 2.7 *Let $1 \leq \xi < \omega_1$. Then there are disjoint families $\mathcal{F}_\xi$, $\mathcal{G}_\xi$ of subsets of ω^ω and a topology T_ξ on ω^ω such that*

$(a)_\xi$ T_ξ is zero-dimensional perfect Polish and $\tau \subseteq T_\xi \subseteq \mathbf{\Sigma}^0_\xi(\tau)$,

$(b)_\xi$ $\mathcal{F}_\xi$ is T_ξ-dense, i.e., for any nonempty T_ξ-open set V, there is $F \in \mathcal{F}_\xi$ with $F \subseteq V$,

and, for every $F \in \mathcal{F}_\xi$,

$(c)_\xi$ F is nonempty, T_ξ-nowhere dense, and in $\mathbf{\Pi}^0_2(T_\xi)$,

$(d)_\xi$ if $S \in \mathbf{\Sigma}^0_\xi(\tau)$ is T_ξ-nonmeager in F, then S is T_ξ-nonmeager in ω^ω,

$(e)_\xi$ there is a nice (T_ξ, τ)-homeomorphism φ_F from F onto ω^ω,

$(f)_\xi$ for any nonempty T_ξ-open sets V, V', there are disjoint $G, G' \in \mathcal{G}_\xi$ with $G \subseteq V$, $G' \subseteq V'$, and there is a nice (T_ξ, T_ξ)-homeomorphism $\varphi_{G,G'}$ from G onto G',

and, for every $G \in \mathcal{G}_\xi$,

$(g)_\xi$ G is nonempty, T_ξ-nowhere dense, and in $\mathbf{\Pi}^0_2(T_\xi)$,

$(h)_\xi$ if $S \in \mathbf{\Sigma}^0_\xi(\tau)$ is T_ξ-nonmeager in G, then S is T_ξ-nonmeager in ω^ω.

Proof. Let P_ξ and τ_ξ be as in Lemma 2.6. We set $T_\xi = (\tau_\xi)^\omega$. Let $(U_n)_{n \in \omega}$ be a basis for the topology T_ξ made of nonempty sets. For each $n \in \omega$, there is a finite sequence $(V^n_i)_{i < k_n}$ of nonempty τ_ξ-open sets such that $(\Pi_{i < k_n}\ V^n_i) \times (\omega^\omega)^\omega \subseteq U_n$. Moreover, the sequence $(k_n)_{n \in \omega}$ is chosen to be strictly increasing.

Lemma 2.6 provides

- for $i<k_n$, a τ_ξ-dense G_δ subset H^n_i of $V^n_i\setminus P_\xi$ and a nice (τ_ξ,τ)-homeomorphism

$$\psi_{\xi,H^n_i}:H^n_i\to\omega^\omega,$$

- a τ_ξ-dense G_δ subset $G^n_{k_n}$ of P_ξ and a nice (τ_ξ,τ)-homeomorphism $\varphi_{\xi,G^n_{k_n}}:G^n_{k_n}\to\omega^\omega$,

- for $i>k_n$, a τ_ξ-dense G_δ subset H^n_i of ω^ω and a nice (τ_ξ,τ)-homeomorphism $\psi_{\xi,H^n_i}:H^n_i\to\omega^\omega$.

We then put $F_n:=(\Pi_{i<k_n}\ H^n_i)\times G^n_{k_n}\times(\Pi_{i>k_n}\ H^n_i)$, so that $F_n\subseteq U_n$. We set $\mathcal{F}_\xi=\{F_n\mid n\in\omega\}$. Then $\mathcal{F}_\xi$ is clearly a disjoint family and the properties (a)$_\xi$ and (b)$_\xi$ are obviously satisfied.

(c)$_\xi$ As P_ξ is τ_ξ-nowhere dense, each F_n is T_ξ-nowhere dense. Each F_n is obviously also in $\mathbf{\Pi}^0_2(T_\xi)$.

(d)$_\xi$ Let $n\in\omega$ and $S\in\mathbf{\Sigma}^0_\xi(\tau)$ be T_ξ-nonmeager in F_n. We define

$$\begin{aligned}&T^*_\xi=\Pi_{i\neq k_n}\ \tau_{\xi|H^n_i},\\&\tilde{T}_\xi=(\Pi_{i<k_n}\ \tau_{\xi|H^n_i})\times\tau\times(\Pi_{i>k_n}\ \tau_{\xi|H^n_i}).\end{aligned}$$

If $\alpha\in\omega^\omega$, then we denote

$$S_\alpha:=\{(y_0,\dots,y_{k_n-1},y_{k_n+1},\dots)\in\omega^\omega\mid(y_0,\dots,y_{k_n-1},\alpha,y_{k_n+1},\dots)\in S\}.$$

We set $S^*=\{\alpha\in\omega^\omega\mid S_\alpha\text{ is }T^*_\xi\text{-nonmeager}\}$. By the Montgomery theorem, $S^*\in\mathbf{\Sigma}^0_\xi(\tau)$ since $S\in\mathbf{\Sigma}^0_\xi(\tilde{T}_\xi)$. The set S^* is τ_ξ-nonmeager in $G^n_{k_n}$ by the Kuratowski-Ulam theorem, in P_ξ also, and thus S^* is τ_ξ-nonmeager in ω^ω. Using the Kuratowski-Ulam theorem again, we see that S is T_ξ-nonmeager in $(\Pi_{i<k_n}\ H^n_i)\times\omega^\omega\times(\Pi_{i>k_n}\ H^n_i)$, and thus in $(\omega^\omega)^\omega$.

(e)$_\xi$ We set $\varphi_F=(\Pi_{i<k_n}\ \psi_{\xi,H^n_i})\times\varphi_{\xi,G^n_{k_n}}\times(\Pi_{i>k_n}\ \psi_{\xi,H^n_i})$. The map φ_F is clearly a (T_ξ,τ)-homeomorphism from F onto $(\omega^\omega)^\omega$. It is nice by Lemma 2.1.

We now construct $\mathcal{G}_\xi$. For each $m\in\omega$, there are finite sequences $(V^m_i)_{i<k_m}$, $(W^m_i)_{i<l_m}$ of nonempty τ_ξ-open sets such that $(\Pi_{i<k_m}\ V^m_i)\times(\omega^\omega)^\omega\subseteq U_{(m)_0}$ and $(\Pi_{i<l_m}\ W^m_i)\times(\omega^\omega)^\omega\subseteq U_{(m)_1}$. Moreover, the sequences $(k_m)_{m\in\omega}$ and $(l_m)_{m\in\omega}$ are chosen to be strictly increasing and disjoint. Assume for example that $k_m<l_m$. Lemma 2.6 provides

- for $i<k_m$, a τ_ξ-dense G_δ subset H^m_i of $V^m_i\setminus P_\xi$, a τ_ξ-dense G_δ subset L^m_i of $W^m_i\setminus P_\xi$, and a nice (τ_ξ,τ_ξ)-homeomorphism ψ_{ξ,H^m_i,L^m_i},

- a τ_ξ-dense G_δ subset $G^m_{k_m}$ of P_ξ, a τ_ξ-dense G_δ subset $K^m_{k_m}$ of $W^m_i\setminus P_\xi$, and a nice (τ_ξ,τ_ξ)-homeomorphism $\varphi_{\xi,G^m_{k_m},K^m_{k_m}}$,

- for $k_m<i<l_m$, a τ_ξ-dense G_δ subset H^m_i of $\neg P_\xi$, a τ_ξ-dense G_δ subset L^m_i of $W^m_i\setminus P_\xi$, and a nice (τ_ξ,τ_ξ)-homeomorphism ψ_{ξ,H^m_i,L^m_i},

- a τ_ξ-dense G_δ subset $K^m_{l_m}$ of $\neg P_\xi$, a τ_ξ-dense G_δ subset $G^m_{l_m}$ of P_ξ, and a nice (τ_ξ,τ_ξ)-homeomorphism $\varphi^{-1}_{\xi,G^m_{l_m},K^m_{l_m}}$,

- for $i > l_m$, a τ_ξ-dense G_δ subset H_i^m of $\neg P_\xi$, a τ_ξ-dense G_δ subset L_i^m of $\neg P_\xi$, and a nice (τ_ξ, τ_ξ)-homeomorphism ψ_{ξ, H_i^m, L_i^m}.

We then put

$$\begin{array}{l} F'_m := (\Pi_{i<k_m}\ H_i^m) \times G_{k_m}^m \times (\Pi_{k_m<i<l_m}\ H_i^m) \times K_{l_m}^m \times (\Pi_{i>l_m}\ H_i^m),\\ G_m := (\Pi_{i<k_m}\ L_i^m) \times K_{k_m}^m \times (\Pi_{k_m<i<l_m}\ L_i^m) \times G_{l_m}^m \times (\Pi_{i>l_m}\ L_i^m), \end{array}$$

so that $F'_m \times G_m \subseteq U_{(m)_0} \times U_{(m)_1}$. We set $\mathcal{G}_\xi = \{F'_m \mid m \in \omega\} \cup \{G_m \mid m \in \omega\}$. Then $\mathcal{G}_\xi$ is clearly a disjoint family.

(f)$_\xi$ The map $\varphi_{F'_m, G_m}$ is by definition

$$(\Pi_{i<k_m}\ \psi_{\xi, H_i^m, L_i^m}) \times \varphi_{\xi, G_{k_m}^m, K_{k_m}^m} \times (\Pi_{k_m<i<l_m}\ \psi_{\xi, H_i^m, L_i^m}) \times \varphi^{-1}_{\xi, G_{l_m}^m, K_{l_m}^m} \times (\Pi_{i>l_m}\ \psi_{\xi, H_i^m, L_i^m}).$$

Note that $\varphi_{F'_m, G_m}$ is clearly a (T_ξ, T_ξ)-homeomorphism from F'_m onto G_m. It is nice by Lemma 2.1.

(g)$_\xi$ We argue as in (c)$_\xi$.

(h)$_\xi$ We argue as in (d)$_\xi$. $\square$

3 Negative results

Proof of Theorem 1.1. We apply Lemma 2.7 to the ordinal $\xi+1$, which gives a family $\mathcal{F}_{\xi+1}$ and a topology $T_{\xi+1}$ satisfying (a)$_{\xi+1}$-(e)$_{\xi+1}$. Let $(U_n \times V_n)_{n\in\omega}$ be a sequence of nonempty sets such that

- $U_n \in T_{\xi+1}$, V_n is τ-clopen,
- $\{U_n \times V_n \mid n \in \omega\}$ is a basis for the topology $T_{\xi+1} \times \tau$.

For each $n\in\omega$ we find $F_n \in \mathcal{F}_{\xi+1} \setminus \{F_q \mid q<n\}$ with $F_n \subseteq U_n$. By the property (e)$_{\xi+1}$ of $\mathcal{F}_{\xi+1}$ we find, for each $n \in \omega$, a nice $(T_{\xi+1}, \tau)$-homeomorphism f_n from F_n onto V_n. We define $f : \bigcup_{n\in\omega}\ F_n \to \omega^\omega$ by $f(x) := f_n(x)$ if $x \in F_n$. As $\mathcal{F}_{\xi+1}$ is a disjoint family, f is well-defined. The graph of f is $\mathbf{\Sigma}^0_2(\tau\times\tau)$ since each Gr(f_n) is $(\tau\times\tau)$-closed.

Suppose, towards a contradiction, that there exist, for $n\in\omega$, $C_n \in \mathbf{\Sigma}^0_\xi(\tau)$ and $D_n \in \mathbf{\Delta}^1_1(\tau)$ such that $\neg\text{Gr}(f) = \bigcup_{n\in\omega}\ C_n \times D_n$. By the Baire category theorem there is $n_0 \in \omega$ such that C_{n_0} is $T_{\xi+1}$-nonmeager and D_{n_0} is τ-nonmeager. As C_{n_0} has the Baire property, we find a nonempty $T_{\xi+1}$-open set O_1 such that C_{n_0} is $T_{\xi+1}$-comeager in O_1. Similarly, we find a τ-open set O_2 such that D_{n_0} is τ-comeager in O_2.

Let $n \in \omega$ and $F_n \subseteq O_1$. Suppose that C_{n_0} is not $T_{\xi+1}$-comeager in F_n. Then $O_1 \setminus C_{n_0}$ is $T_{\xi+1}$-nonmeager in F_n. Note that $O_1 \in \mathbf{\Sigma}^0_{\xi+1}(\tau)$ and $C_{n_0} \in \mathbf{\Sigma}^0_\xi(\tau)$. Therefore $O_1 \setminus C_{n_0} \in \mathbf{\Sigma}^0_{\xi+1}(\tau)$. Thus $O_1 \setminus C_{n_0}$ is $T_{\xi+1}$-nonmeager in ω^ω by (d)$_{\xi+1}$. Consequently, $O_1 \setminus C_{n_0}$ is $T_{\xi+1}$-nonmeager in O_1, a contradiction. Thus C_{n_0} is $T_{\xi+1}$-comeager in F_n for any $n \in \omega$ with $F_n \subseteq O_1$.

Find $n \in \omega$ such that $\mathrm{Gr}(f_n) \subseteq O_1 \times O_2$. Then C_{n_0} is $T_{\xi+1}$-comeager in F_n and D_{n_0} is τ-comeager in V_n. As f_n is a $(T_{\xi+1}, \tau)$-homeomorphism, $f_n^{-1}(V_n \cap D_{n_0})$ is $T_{\xi+1}$-comeager in F_n. As $F_n \in \mathbf{\Pi}^0_2(T_{\xi+1})$ there exists $\alpha \in f_n^{-1}(V_n \cap D_{n_0}) \cap F_n \cap C_{n_0}$. This implies that $\big(\alpha, f_n(\alpha)\big) \in C_{n_0} \times D_{n_0}$, a contradiction. $\square$

Proof of Theorem 1.2. Apply Lemma 2.7 to the ordinal $\xi + 1$, which gives a family $\mathcal{G}_{\xi+1}$ and a topology $T_{\xi+1}$ satisfying (a)$_{\xi+1}$-(h)$_{\xi+1}$. Let $\mathcal{U} = \{U_n \mid n \in \omega\}$ be a basis for the space $(\omega^\omega, T_{\xi+1})$ made of nonempty sets. For each $n \in \omega$ we find $T_{\xi+1}$-open sets V_n, W_n such that

$$V_n \times W_n \subseteq B_{\tau\times\tau}\big(\Delta(\omega^\omega), 2^{-n}\big) \cap (U_n \times U_n) \setminus \Delta(\omega^\omega)$$

(we use the standard metric on (ω^ω, τ)).

By the properties (f)$_{\xi+1}$ and (g)$_{\xi+1}$ of $\mathcal{G}_{\xi+1}$ we find, for each $n \in \omega$, sets F_n and H_n from $\mathcal{G}_{\xi+1}$ such that

$$(*) \quad F_n \subseteq V_n \setminus (\bigcup_{j<n} F_j \cup H_j) \ \wedge \ H_n \subseteq W_n \setminus \big(F_n \cup (\bigcup_{j<n} F_j \cup H_j)\big).$$

Moreover, there is a nice $(T_{\xi+1}, T_{\xi+1})$-homeomorphism f_n from F_n onto H_n. We set

$$\mathcal{G} = \bigcup \{\mathrm{Gr}(f_n) \mid n \in \omega\}.$$

Now we check the desired properties.

As $\tau \subseteq T_{\xi+1}$, $\overline{\mathcal{G}}^{\tau\times\tau} = \mathcal{G} \cup \Delta(\omega^\omega)$, by construction. Thus $\mathcal{G}$ is a difference of two $(\tau\times\tau)$-closed sets. As each f_n is a $(T_{\xi+1}, T_{\xi+1})$-homeomorphism, the property $(*)$ implies that f is a partial injection with disjoint domain and range. In order to see that $\mathcal{G}$ has no $\mathbf{\Delta}^0_\xi$-measurable countable coloring, we proceed by contradiction. Suppose that there are $\mathcal{G}$-discrete sets $C_n \in \mathbf{\Delta}^0_\xi(\tau)$ (a set C is $\mathcal{G}$-**discrete** if $C^2 \cap \mathcal{G} = \emptyset$), for $n \in \omega$, such that $\Delta(\omega^\omega) \subseteq \bigcup_{n\in\omega} C_n^2$. By the Baire theorem there exists $n_0 \in \omega$ such that C_{n_0} is $T_{\xi+1}$-nonmeager. As C_{n_0} has the Baire property, we find a nonempty $T_{\xi+1}$-open set O such that $C_{n_0} \cap O$ is $T_{\xi+1}$-comeager in O.

Let $F \in \mathcal{G}_{\xi+1}$ with $F \subseteq O$. Suppose that C_{n_0} is not $T_{\xi+1}$-comeager in F. Then $O \setminus C_{n_0}$ is $T_{\xi+1}$-nonmeager in F. Note that $O \in \mathbf{\Sigma}^0_{\xi+1}(\tau)$ and $C_{n_0} \in \mathbf{\Delta}^0_\xi(\tau)$. Therefore $O \setminus C_{n_0} \in \mathbf{\Sigma}^0_{\xi+1}(\tau)$. Thus $O \setminus C_{n_0}$ is $T_{\xi+1}$-nonmeager in ω^ω by (h)$_{\xi+1}$. Consequently, $O \setminus C_{n_0}$ is $T_{\xi+1}$-nonmeager in O, a contradiction. Thus C_{n_0} is $T_{\xi+1}$-comeager in F for any $F \in \mathcal{G}_{\xi+1}$ with $F \subseteq O$.

Find $n \in \omega$ such that $\mathrm{Gr}(f_n) \subseteq O^2$. Then C_{n_0} is $T_{\xi+1}$-comeager in F_n and in H_n. As f_n is a $(T_{\xi+1}, T_{\xi+1})$-homeomorphism, $f_n^{-1}(H_n \cap C_{n_0})$ is $T_{\xi+1}$-comeager in $F_n \in \mathbf{\Pi}^0_2(T_{\xi+1})$. Thus there exists $\alpha \in f_n^{-1}(H_n \cap C_{n_0}) \cap F_n \cap C_{n_0}$. This implies that $\big(\alpha, f_n(\alpha)\big) \in C_{n_0}^2$, a contradiction. $\square$

4 Positive results

(A) $\mathbf{\Delta}^0_\xi$-measurable countable colorings

In [L-Z], the following conjecture is made.

Conjecture *Let $1\leq\xi<\omega_1$. Then there are*

- a 0-dimensional Polish space $\mathbb{X}_\xi$,

- an analytic relation $\mathbb{A}_\xi$ on $\mathbb{X}_\xi$

such that for any (0-dimensional if $\xi=1$) Polish space X, and for any analytic relation A on X, exactly one of the following holds:

(a) there is a $\mathbf{\Delta}^0_\xi$-measurable countable coloring of A (i.e., a $\mathbf{\Delta}^0_\xi$-measurable map $c\colon X\to\omega$ such that $A\subseteq(c\times c)^{-1}(\neq)$),

(b) there is a continuous map $f\colon\mathbb{X}_\xi\to X$ such that $\mathbb{A}_\xi\subseteq(f\times f)^{-1}(A)$.

This would be a $\mathbf{\Delta}^0_\xi$-measurable version of the $\mathbb{G}_0$-dichotomy in [K-S-T]. This conjecture is proved for $\xi\leq 3$ in [L-Z]. Our goals here are the following. We want to give

- a reasonable candidate for $\mathbb{A}_\xi$ in the general case,

- an example for $\xi=3$ that is much simpler than the one in [L-Z].

We set $\mathbf{\Pi}^0_0:=\mathbf{\Delta}^0_1$. The following result is proved in [Má] (see Theorem 4 and Lemma 13.(i)).

Theorem 4.1 *(Mátrai) Let $1\leq\xi<\omega_1$. There are a true $\mathbf{\Pi}^0_\xi$ subset P_ξ of 2^ω, and a Polish topology τ_ξ on 2^ω such that*

$(1)_\xi$ τ_ξ is finer than the usual topology τ' on 2^ω,

$(2)_\xi$ P_ξ is τ_ξ-closed and τ_ξ-nowhere dense,

$(3)_\xi$ if G is a basic τ_ξ-open set meeting P_ξ, and $D\in\mathbf{\Pi}^0_{<\xi}(2^\omega,\tau')$ is such that $D\cap P_\xi\cap G$ is comeager in $(P_\xi\cap G,\tau_{\xi|P_\xi\cap G})$, then there is a τ_ξ-open set G' such that $P_\xi\cap G'=P_\xi\cap G$ and $D\cap G'$ is comeager in $(G',\tau_{\xi|G'})$.

Notation. In the sequel $1\leq\xi<\omega_1$. Fix, for each ξ, an increasing sequence $(\eta_n)_{n\in\omega}$ of elements of ξ (different from 0 if $\xi\geq 2$) such that $\sup_{n\in\omega}\ (\eta_n+1)=\xi$.

• Let $<.,.>:\omega^2\to\omega$ be a bijection, defined for example by $<n,p>:=(\Sigma_{k\leq n+p}\ k)+p$, whose inverse bijection is $q\mapsto\big((q)_0,(q)_1\big)$.

• If $u\in 2^{\leq\omega}$ and $n\in\omega$, then we define $(u)_n\in 2^{\leq\omega}$ by $(u)_n(p):=u(<n,p>)$ if $<n,p><|u|$.

• Let $(t_n)_{n\in\omega}$ be a dense sequence in $\omega^{<\omega}$ with $|t_n|=n$. For example, let $(p_n)_{n\in\omega}$ be the sequence of prime numbers, and $I:\omega^{<\omega}\to\omega$ defined by $I(\emptyset):=1$, and $I(s):=p_0^{s(0)+1}...p_{|s|-1}^{s(|s|-1)+1}$ if $s\neq\emptyset$. Note that I is one-to-one, so that there is an increasing bijection $i:I[\omega^{<\omega}]\to\omega$. Set $\psi:=(i\circ I)^{-1}:\omega\to\omega^{<\omega}$, so that ψ is a bijection. Note that $|\psi(n)|\leq n$ if $n\in\omega$. Indeed,

$$I\big(\psi(n)|0\big)<I\big(\psi(n)|1\big)<...<I\big(\psi(n)\big),$$

so that $(b\circ I)\big(\psi(n)|0\big)<(b\circ I)\big(\psi(n)|1\big)<...<(b\circ I)\big(\psi(n)\big)=n$. As $|\psi(n)|\leq n$, we can define $t_n:=\psi(n)0^{n-|\psi(n)|}$, and $(t_n)_{n\in\omega}$ is suitable.

• Theorem 4.1 gives P_ξ and τ_ξ. Let $Q_\xi:=2\times P_\xi$, $T_\xi:=\text{discrete}\times\tau_\xi$, and $T^<_\xi:=\Pi_{i\in\omega}\ T_{\eta_i}$ if $\xi\geq 2$.

• $(W_{\xi,n})_{n\in\omega}$ is a sequence of nonempty T_ξ-open sets.

• $S_i := Q_{\eta_i} \cup \bigcup_{n\in\omega} W_{\eta_i,n}$ (for $i\in\omega$), and $S := \Pi_{i\in\omega} S_i$, so that $S\in\mathbf{\Pi}^0_2(T^<_\xi)$ is a Polish space.

• If $\xi\geq 2$, then we set

$$\mathbb{K}_\xi := \bigcup_{n\in\omega} \Big\{(\alpha,\beta)\in 2^\omega\times 2^\omega \mid \big(\forall i<n\ \ (\alpha)_i=(\beta)_i\in W_{\eta_i,t_n(i)}\big) \wedge \\ \big(\exists\gamma\in P_{\eta_n}\ \ \big((\alpha)_n,(\beta)_n\big)=(0\gamma,1\gamma)\big) \wedge \big(\forall i>n\ \ (\alpha)_i=(\beta)_i\big)\Big\},$$

Lemma 4.2 *Let $2\leq\xi\leq\omega_1$. We assume that $Q_{\eta_i}\subseteq\overline{\bigcup_{n\in\omega} W_{\eta_i,n}}^{T_{\eta_i}}$ for each $i\in\omega$. Then any $\mathbb{K}_\xi$-discrete $\mathbf{\Sigma}^0_\xi$ subset C of (S,τ') is $T^<_\xi$-meager in S.*

Proof. We may assume that C is $\mathbf{\Pi}^0_{<\xi}$. We argue by contradiction. This gives $n\in\omega$ with $C\in\mathbf{\Pi}^0_{\eta_n}$, a basic $T^<_\xi$-open set O such that $C\cap O$ is $T^<_\xi$-comeager in $O\cap S\neq\emptyset$, $l\geq n$, and a sequence $(O_i)_{i<l}$ with $O_i\in T_{\eta_i}$ and $O=\{\alpha\in 2^\omega \mid \forall i<l\ \ (\alpha)_i\in O_i\}$. The assumption gives, for each $i<l$, $n_i\in\omega$ such that $O_i\cap W_{\eta_i,n_i}\neq\emptyset$. Let $m\geq l$ such that $t_m(i)=n_i$ for each $i<l$, and

$$U := \Big\{\alpha\in S \mid \forall i<l\ \ (\alpha)_i\in O_i\ \wedge\ \forall i<m\ \ (\alpha)_i\in W_{\eta_i,t_m(i)}\Big\},$$

which is a nonempty $T^<_\xi$-open subset of S. In particular, $C\cap U$ is $T^<_\xi$-comeager in U. We set

$$V := \Big\{(\alpha_i)_{i\neq m}\in\Pi_{i\neq m} S_i \mid \forall i<l\ \ \alpha_i\in O_i\ \wedge\ \forall i<m\ \ \alpha_i\in W_{\eta_i,t_m(i)}\Big\},$$

so that, up to a permutation of coordinates, $U\equiv S_m\times V$. We also set

$$C' := \Big\{\alpha\in S_m \mid \big(C\cap(S_m\times V)\big)_\alpha \text{ is } \Pi_{i\neq m} T_{\eta_i}\text{-comeager in } V\Big\}.$$

By the Kuratowski-Ulam theorem, C' is T_{η_m}-comeager in S_m (see 8.41 in [K]). Write $C=D\cap S$, where $D\in\mathbf{\Pi}^0_{\eta_n}(2^\omega)$. Note that $C' := S_m\cap\Big\{\alpha\in 2^\omega \mid \big(D\cap(2^\omega\times V)\big)_\alpha \text{ is } \Pi_{i\neq m} T_{\eta_i}\text{-comeager in } V\Big\}$. As $m\geq n$ and $\Pi_{i\neq m} T_{\eta_i}$ is finer than the usual topology, $D\cap(2^\omega\times V)\in\mathbf{\Pi}^0_{\eta_m}(2^\omega,\tau'\times(\Pi_{i\neq m} T_{\eta_i})_{|V})$. By the Montgomery theorem, C' is $\mathbf{\Pi}^0_{\eta_m}(S_m,\tau')$ (see 22.22 in [K]).

The set C' cannot be T_{η_m}-comeager both in $Q_{\eta_m}\cap N_0$ and in $Q_{\eta_m}\cap N_1$. Indeed, we argue by contradiction to see that. We set $h_0(\alpha) :=< 1-\alpha(0),\alpha(1),\alpha(2),... >$. As $h_{0|Q_{\eta_m}\cap N_0}$ is a T_{η_m}-homeomorphism, $C'\cap h^{-1}_{0|Q_{\eta_m}\cap N_0}(C'\cap Q_{\eta_m}\cap N_1)$ is T_{η_m}-comeager in $Q_{\eta_m}\cap N_0$, and if 0γ is in it, then $1\gamma\in C'$, which gives $\delta\in(C\cap U)_{0\gamma}\cap(C\cap U)_{1\gamma}$ and contradicts the $\mathbb{K}_\xi$-discreteness of C.

Assume for example that C' is not T_{η_m}-comeager in $Q_{\eta_m}\cap N_0$. Then $\neg C'$ is T_{η_m}-non meager in Q_{η_m}. As C' is $\mathbf{\Pi}^0_{\eta_m}(S_m,\tau')$, there is a sequence $(C_j)_{j\in\omega}$ of $\mathbf{\Pi}^0_{<\eta_m}(2^\omega)$ sets such that

$$S_m\setminus C' = \bigcup_{j\in\omega} C_j\cap S_m.$$

This gives $j\in\omega$ such that $C_j\cap Q_{\eta_m}$ is T_{η_m}-non meager in Q_{η_m}, and a basic T_{η_m}-open set O such that $C_j\cap Q_{\eta_m}\cap O$ is T_{η_m}-comeager in $Q_{\eta_m}\cap O\neq\emptyset$.

The set O is of the form $\{\varepsilon\}\times G$, where $\varepsilon\in 2$ and G is a basic τ_{η_m}-open set. Let $S:N_\varepsilon\to 2^\omega$ be the map defined by $S(\varepsilon\alpha):=\alpha$. Note that S is a τ'-τ' and T_ξ-τ_ξ homeomorphism. In particular, $E:=\{\alpha\in 2^\omega\mid \varepsilon\alpha\in C_j\}$ is τ'-$\mathbf{\Pi}^0_{<\eta_m}$ and $E\cap P_{\eta_m}\cap G$ is comeager in $(P_{\eta_m}\cap G,\tau_{\eta_m|P_{\eta_m}\cap G})$. Theorem 4.1.(3) gives a τ_{η_m}-open set G' such that $P_{\eta_m}\cap G'=P_{\eta_m}\cap G$ and $E\cap G'$ is comeager in $(G',\tau_{\eta_m|G'})$. Now $O':=\{\varepsilon\}\times G'$ is a T_{η_m}-open set such that $Q_{\eta_m}\cap O'=Q_{\eta_m}\cap O$ and $C_j\cap O'$ is T_{η_m}-comeager in O'. The assumption gives $n\in\omega$ such that $W_{\eta_m,n}\cap O'\neq\emptyset$. Note that $C_j\cap W_{\eta_m,n}\cap O'$ is T_{η_m}-comeager in $W_{\eta_m,n}\cap O'$, so that $\neg C'$ is T_{η_m}-non meager in S_m, which is absurd. $\square$

Corollary 4.3 *Let $2\leq\xi\leq\omega_1$. We assume that $Q_{\eta_i}\subseteq\overline{\bigcup_{n\in\omega}\ W_{\eta_i,n}}^{T_{\eta_i}}$ for each $i\in\omega$. Then*

(a) there is no $\mathbf{\Delta}^0_\xi$-measurable map $c:2^\omega\to\omega$ such that $\mathbb{K}_\xi\subseteq(c\times c)^{-1}(\neq)$,

(b) if $\mathbb{X}_\xi\in\mathbf{\Pi}^0_2(2^\omega)$ and $\mathbb{K}_\xi\subseteq\mathbb{X}_\xi^2$, then there is no $\mathbf{\Delta}^0_\xi$-measurable map $c:\mathbb{X}_\xi\to\omega$ such that $\mathbb{K}_\xi\subseteq(c\times c)^{-1}(\neq)$.

Proof. (a) We just have to apply Lemma 4.2.

(b) We argue by contradiction. This gives a partition $(C_k)_{k\in\omega}$ of $\mathbb{X}_\xi$ into $\mathbb{K}_\xi$-discrete $\mathbf{\Delta}^0_\xi(\mathbb{X}_\xi)$ sets. We set $D_0:=2^\omega\setminus\mathbb{X}_\xi$, and choose $D_{k+1}\in\mathbf{\Sigma}^0_\xi(2^\omega)$ such that $C_k=D_{k+1}\cap\mathbb{X}_\xi$. Then $(D_k)_{k\in\omega}$ is a covering of 2^ω into $\mathbb{K}_\xi$-discrete $\mathbf{\Sigma}^0_\xi$ sets. It remains to apply the reduction property of the class $\mathbf{\Sigma}^0_\xi$ to contradict (a). $\square$

The case $\xi=2$

Example. Let $\alpha\mapsto\alpha^*$ be the shift map on 2^ω: $\alpha^*(j):=\alpha(j+1)$. Then we set

$$\begin{aligned}\mathbb{A}_2:=\bigcup_{n\in\omega}\ \Big\{(\alpha,\beta)\in 2^\omega\times 2^\omega\mid\big(\forall i<n\ \ (\alpha)_i=(\beta)_i\ \wedge\ 0^{t_n(i)}1\subseteq(\alpha)_i^*\big)\ \wedge\\ \big((\alpha)_n,(\beta)_n\big)=(0^\infty,10^\infty)\ \wedge\ \big(\forall i>n\ \ (\alpha)_i=(\beta)_i\big)\Big\}.\end{aligned}$$

Theorem 4.4 $(2^\omega,\mathbb{A}_2)$ *satisfies the conjecture.*

Proof. We set $P_1:=\{0^\infty\}$ and $\tau_1:=\tau'$, so that P_1 and τ_1 satisfy the properties of Theorem 4.1. We also set $W_{1,n}:=N_{0^{n+1}1}\cup N_{10^n1}$, so that $(W_{1,n})_{n\in\omega}$ is a sequence of nonempty T_1-open sets satisfying the assumption of Corollary 4.3, so that $\mathbb{A}_2=\mathbb{K}_2$ satisfies its conclusions. In particular, (a) and (b) cannot hold simultaneously.

We define, for $(\varepsilon,n)\in 2\times\omega$, $K^\varepsilon_n:=\{\alpha\in 2^\omega\mid\forall i<n\ \ 0^{t_n(i)}1\subseteq(\alpha)^*_i\ \wedge\ (\alpha)_n(0)=\varepsilon\}$, and also $C^\varepsilon_n:=K^\varepsilon_n\setminus\big(\bigcup_{n<k}\ K^0_k\cup K^1_k\big)$, so that C^ε_n is closed, the C^ε_n's are pairwise disjoint, and $\mathbb{A}_2\subseteq\bigcup_{n\in\omega}\ C^0_n\times C^1_n$. We set, for each $p,q\in\omega$,

$$O^p_q:=\begin{cases}K^\varepsilon_n\setminus\big(\bigcup_{n<k\leq q}\ K^0_k\cup K^1_k\big)\text{ if }p=2n+\varepsilon\leq 2q+1,\\ 2^\omega\setminus(\bigcup_{p'\leq 2q+1}\ O^{p'}_q)\text{ if }p=2q+2,\\ \emptyset\text{ if }p\geq 2q+3,\end{cases}$$

so that $(O^p_q)_{p\in\omega}$ is a covering of 2^ω into clopen sets. Assume that $p=2n+\varepsilon\neq p'=2n'+\varepsilon'\leq 2q+1$ and $\alpha\in O^p_q\cap O^{p'}_q$, so that $n,n'\leq q$. As $\alpha\in K^\varepsilon_n\cap K^{\varepsilon'}_{n'}$, $n\neq n'$ and for example $n<n'$, which is absurd. Thus $(O^p_q)_{p\in\omega}$ is a partition of 2^ω.

(a) Assume that $q<n$. Note that $C^0_n\cup C^1_n$ is contained in or disjoint from each set of the form K^ε_k with $k\leq q$. By disjonction, there is at most one couple (ε,r) such that $2r{+}\varepsilon\leq 2q{+}1$ and $C^0_n\cup C^1_n\subseteq O^{2r+\varepsilon}_q$. If it does not exist, then $C^0_n\cup C^1_n\subseteq O^{2q+2}_q$.

(b) Assume that $q\geq n$. Note that $C^\varepsilon_n\subseteq K^\varepsilon_n$. As $q\geq n$, $p:=2n{+}\varepsilon\leq 2q{+}1$. Thus $C^\varepsilon_n\subseteq O^p_q$.

It remains to apply Proposition 4.6 in [L-Z] to see that (a) or (b) holds. $\square$

The case $\xi=3$

Example. Let $(s_n)_{n\in\omega}$ be a dense sequence in $2^{<\omega}$ with $|s_n|=n$. For example, let $\phi:\omega\rightarrow 2^{<\omega}$ be a natural bijection. More specifically, $\phi(0):=\emptyset$ is the sequence of length 0, $\phi(1):=0$, $\phi(2):=1$ are the sequences of length 1, and so on. Note that $|\phi(n)|\leq n$ if $n\in\omega$. Let $n\in\omega$. As $|\phi(n)|\leq n$, we can define $s_n:=\phi(n)0^{n-|\psi(n)|}$. We set $P_2:=\{\alpha\in 2^\omega\mid\forall p\in\omega\ \ \exists q\geq p\ \ \alpha(q)=1\}$, and

$$\mathbb{A}_3:=\bigcup_{n\in\omega}\ \Big\{(\alpha,\beta)\in 2^\omega\times 2^\omega\mid\big(\forall i<n\ \ (\alpha)_i=(\beta)_i=s_{t_n(i)}10^\infty\big)\ \wedge$$
$$\big(\exists\gamma\in P_2\ \ \big((\alpha)_n,(\beta)_n\big)=(0\gamma,1\gamma)\big)\ \wedge\ \forall i>n\ \ (\alpha)_i=(\beta)_i\Big\}.$$

We will see that $\mathbb{A}_3$, together with a suitable $\mathbf{\Pi}^0_2$ subset $\mathbb{X}_3$ of 2^ω, satisfies the conjecture. The topology τ_2 makes the countably many singletons of $\neg P_2$ open. Then P_2 is a true $\mathbf{\Pi}^0_2$ subset of 2^ω (see 23.A in [K]), τ_2 is Polish finer than τ', P_2 is closed nowhere dense for τ_2 since τ_2 coincides with τ' on P_2 and $\neg P_2$ is τ'-dense, and 4.1.(3) is satisfied since a basic τ_2-open set meeting P_2 is a basic τ'-clopen set and P_2 is τ'-comeager. Thus P_2 and τ_2 satisfy the properties of Theorem 4.1. We set

$$W_{2,n}:=\{s_n10^\infty\}.$$

Then $Q_2\subseteq\overline{\bigcup_{n\in\omega}\ W_{2,n}}^{\tau_2}$ since $(s_n)_{n\in\omega}$ is dense. This shows that $\mathbb{A}_3=\mathbb{K}_3$ satisfies the conclusions of Corollary 4.3. In particular, (a) and (b) cannot hold simultaneously. In order to prove that (a) or (b) holds, we simply indicate the modifications to make to Section 5 in [L-Z]. We just need to prove the right lemmas since the final construction is the same.

Lemma 4.5 *(a) Let $n\in\omega$ and $i<n$. Then $t_n(i)<n{-}i$.*

(b) The map $M:\{s_{t_n(i)}10^\infty\mid n\in\omega\ \wedge\ i<n\}\rightarrow\omega$, defined by $M(\alpha):=max\{p\in\omega\mid\alpha(p)=1\}$, is one-to-one.

Proof. (a) Recall the map ψ defined after Theorem 4.1. It is enough to prove that $\psi(n)(i)<n-i$ if $i<|\psi(n)|$. We argue by induction on n, and the result is clear for $n=0$. We may assume that $\psi(n)(i)=q+1$ for some natural number q. We define $t\in\omega^{<\omega}$ by $t(i):=q$, and $t(j):=\psi(n)(j)$ if $j\neq i$. Let $p\in\omega$ with $\psi(p)=t$. Note that $I\big(\psi(p)\big)<I\big(\psi(n)\big)$, so that $p<n$. The induction assumption implies that $q=\psi(p)(i)<p{-}i$, so that $\psi(n)(i)=q{+}1\leq p{-}i<n{-}i$.

(b) Assume that $M(\alpha)=M(\alpha')$. Let n,n',i,i' with $\alpha=s_{t_n(i)}10^\infty$ and $\alpha'=s_{t_{n'}(i')}10^\infty$. Then $t_n(i)=|s_{t_n(i)}|=M(\alpha)=M(\alpha')=t_{n'}(i')$, so that $\alpha=\alpha'$. $\square$

Notation. If $\emptyset\neq u\in 2^{<\omega}$, then $u^m:=u|(|u|{-}1)$.

The following notion is technical but crucial.

Definition 4.6 *We say that $u\in 2^{<\omega}$ is placed if*

(a) $u\neq\emptyset$,

(b) $\forall i<(|u^m|)_0\ \ (u)_i\subseteq s_{t_{(|u^m|)_0}(i)}10^\infty$,

(c) $u(|u^m|)=1$ if $(|u^m|)_1>0$.

We are now ready to define

$$\mathbb{X}_3:=\big\{\alpha\in 2^\omega\mid\forall n\in\omega\ \ \exists p\geq n\ \ \alpha|p\text{ is placed}\big\}.$$

Note that $\mathbb{X}_3$ is a $\mathbf{\Pi}^0_2$ subset of 2^ω. In particular, $\mathbb{X}_3$ is a 0-dimensional Polish space.

Lemma 4.7 *(a) The set $\mathbb{A}_3$ is a $\mathbf{\Sigma}^0_3$ (and thus analytic) relation on $\mathbb{X}_3$.*

(b) $(\mathbb{X}_3,\mathbb{A}_3)\not\preceq_{\mathbf{\Delta}^0_3}\big(\omega,\neg\Delta(\omega)\big)$.

Proof. (a) $\mathbb{A}_3$ is clearly a $\mathbf{\Sigma}^0_3$ relation on 2^ω. So it is enough to see that it is a relation on $\mathbb{X}_3$. Fix $(\alpha,\beta)\in\mathbb{A}_3$ (which defines a natural number n). Choose an infinite sequence $(p_k)_{k\in\omega}$ of natural numbers such that $(\alpha)_n(p_k)=(\beta)_n(p_k)=1$. Then $\alpha|(<n,p_k>+1)$ and $\beta|(<n,p_k>+1)$ are placed, so that $\alpha,\beta\in\mathbb{X}_3$.

(b) This comes from Corollary 4.3. $\square$

Lemma 4.8 *Let $n\in\omega$, $\alpha\in 2^\omega$ such that $(\alpha)_i=s_{t_n(i)}10^\infty$ for each $i<n$, and $p><n,0>$ such that $\alpha|p$ is placed. Then $(p-1)_0\geq n$.*

Proof. We argue by contradiction. As $p-1\geq<n,0>$, $(p-1)_0+(p-1)_1\geq n+0=n$. Thus $(p-1)_1\geq n-(p-1)_0>0$. As $\alpha|p$ is placed, $\alpha(p-1)=1$. But

$$\alpha(p-1)=\alpha(<(p-1)_0,(p-1)_1>)=(\alpha)_{(p-1)_0}\big((p-1)_1\big)=\big(s_{t_n((p-1)_0)}10^\infty\big)\big((p-1)_1\big).$$

By Lemma 4.5.(a), we get $(p-1)_1<n-(p-1)_0$, which is absurd. $\square$

Definition 4.9 *Let $u\in 2^{<\omega}$ and $l\in\omega$.*

(a) If u is placed, then we will consider

- *the natural number $l(u):=(|u^m|)_0$*
- *the sequence $u^{l(u)}\in 2^{|u|}\setminus\{u\}$ defined by $u^{l(u)}(m):=u(m)$ exactly when $m\neq<l(u),0>$. Note that $u^{l(u)}$ is placed, $l(u^{l(u)})=l(u)$ and $(u^{l(u)})^{l(u)}=u$*
- *the digit $\varepsilon(u):=u(<l(u),0>)$. Note that $\varepsilon(u^{l(u)})=1-\varepsilon(u)$.*

(b) We say that u is l-placed if u is placed and $l(u)=l$. We say that u is $(\leq l)$-placed (resp., $(<l)$-placed, $(>l)$-placed) if there is $l'\leq l$ (resp., $l'<l$, $l'>l$) such that u is l'-placed.

When we consider the finite approximations of an element of $\mathbb{A}_3$, we have to guess the natural number n. We usually make some mistakes. In this case, we have to be able to come back to an earlier position. This is the role of the following predecessors.

Notation. Let $u\in 2^{<\omega}$. Note that $<\eta>$ is 0-placed with $\varepsilon(<\eta>)=\eta$ if $\eta\in 2$. This allows us to define

$$u^-:=\begin{cases}\emptyset \text{ if } |u|\leq 1,\\ u|\text{max}\{l<|u| \mid u|l \text{ is placed}\} \text{ if } |u|\geq 2,\end{cases}$$

and, for $l\in\omega$,

$$u^{-l}:=\begin{cases}\emptyset \text{ if } |u|\leq 1,\\ u|\text{max}\{k<|u| \mid u|k \text{ is } (\leq l)\text{-placed}\} \text{ if } |u|\geq 2.\end{cases}$$

The following key lemma explains the relation between these predecessors and the placed sequences.

Lemma 4.10 *Let $l\in\omega$ and $u\in 2^{<\omega}$ be l-placed with $|u|\geq 2$.*

(a) Assume that u^- is l-placed. Then $\varepsilon(u^-)=\varepsilon(u)$. If moreover $(u^l)^-$ is l-placed, then the equality $(u^l)^-=(u^-)^l$ holds.

(b) u^{-l} is l-placed if and only if $(u^l)^{-l}$ is l-placed. In this case, $\varepsilon(u^{-l})=\varepsilon(u)$ and the equality $(u^l)^{-l}=(u^{-l})^l$ holds.

(c) Assume that u^- or $(u^l)^-$ is $(<l)$-placed. Then $u^-=u^{-l}=(u^l)^-=(u^l)^{-l}$.

(d) Assume that u^- or $(u^l)^-$ is $(>l)$-placed. Then exactly one of those two sequences is $(>l)$-placed, and the other one is l-placed. If u^- (resp., $(u^l)^-$) is $(>l)$-placed, then $u^{-l}=\big((u^l)^-\big)^l$ (resp., $u^{-l}=u^-$) and $\varepsilon(u^{-l})=\varepsilon(u)$ (resp., $\varepsilon\big((u^l)^{-l}\big)=\varepsilon(u^l)$).

Proof. We first prove the following claim:

Claim. *(i) Assume that $(|u^m|)_1=0$. Then $u^-=u^{-l}=(u^l)^-=(u^l)^{-l}$ is $(<l)$-placed.*

(ii) Assume that $(|u^m|)_1>0$. Then u^- (resp., u^{-l}) is $(\geq l)$-placed and there is j_0 (resp., j_1) with $u^-=u|(<l(u^-),j_0>+1)$ (resp., $u^{-l}=u|(<l,j_1>+1)$).

Proof. (i) Note that $l\geq 1$ since $|u|\geq 2$. As $(|u^m|)_1=0$, $|u^m|=<(|u^m|)_0,(|u^m|)_1>=<l(u),0>$ and the sequence u^- is $(<l)$-placed, which implies that $u^-=u^{-l}=(u^l)^-=(u^l)^{-l}$.

(ii) The last assertion about j_0 and j_1 comes from the first one. It is enough to see that u^- is $(\geq l)$-placed since the proof for u^{-l} is similar. We argue by contradiction. Then $u|(<l,0>+1)$ is l-placed and $u|(<l,0>+1)\subsetneqq u|(<l,(|u^m|)_1>+1)\subseteq u$, so that $u|(<l,0>+1)\subsetneqq u^-$. This implies that $l+0\leq l(u^-)+(|u^-|-1)_1$, $(|u^-|-1)_1\geq l-l(u^-)>0$ and $u^-(|u^-|-1)=1$. But

$$\begin{aligned}u^-(|u^-|-1)&=u^-(<l(u^-),(|u^-|-1)_1>)=u(<l(u^-),(|u^-|-1)_1>)\\&=(u)_{l(u^-)}\big((|u^-|-1)_1\big)=(s_{t_l(l(u^-))}10^\infty)\big((|u^-|-1)_1\big).\end{aligned}$$

Lemma 4.5.(a) implies that $(|u^-|-1)_1<l-l(u^-)$, which is absurd. $\diamond$

(a) By the claim, $(|u^m|)_1>0$. Therefore $u|(<l,0>+1)\subsetneqq u|(<l,(|u^m|)_1>+1)\subseteq u$ is l-placed, $u|(<l,0>+1)\subseteq u^-$ and $<l,0><|u^-|$. Thus $\varepsilon(u^-)=(u^-)(<l,0>)=u(<l,0>)=\varepsilon(u)$.

Assume now that $(u^l)^-$ is l-placed. As $u|(<l,0>+1)\subseteq u^-\subsetneqq u$, we get

$$\big(u|(<l,0>+1)\big)^l\subseteq (u^l)^-.$$

Thus $<l,0><|(u^l)^-|$. If $u^-=u|(<l,j_0>+1)$, then there is no $j_0<j<(|u^m|)_1$ such that $u(<l,j>)=1$, and $(u^l)^-=u^l|(<l,j_0>+1)=(u^-)^l$.

(b) Assume that u^{-l} is l-placed. By the claim, we get $(|u^m|)_1 > 0$ and j_1 with

$$u^{-l} = u|(< l, j_1 > +1).$$

Thus $(u^l)^{-l} = u^l|(< l, j_1 > +1) = (u^{-l})^l$ is l-placed, by Lemma 4.8. The equivalence comes from the fact that $(u^l)^l = u$. We argue as in (a) to see that $\varepsilon(u^{-l}) = \varepsilon(u)$ if u^{-l} is l-placed.

(c) Assume first that u^- is $(< l)$-placed. Then $(|u^m|)_1 = 0$, by the claim, (ii). Now the claim, (i), gives the result. If $(u^l)^-$ is $(< l)$-placed, then we apply this to u^l, using the facts that u^l is l-placed and $(u^l)^l = u$.

(d) Assume first that u^- is $(> l)$-placed. The claim, (i), implies that $(|u^m|)_1 > 0$, and the claim, (ii), gives j_1 with $u^{-l} = u|(< l, j_1 > +1)$. Note that $u^{-l} \subsetneqq u^-$, $(u^-)_l \subseteq s_{t_{l(u^-)}(l)} 10^\infty$ and $M(s_{t_{l(u^-)}(l)} 10^\infty) < l(u^-) - l$, by Lemma 4.5.(a). Thus

$$< l, M(s_{t_{l(u^-)}(l)} 10^\infty) > \leq < l(u^-), 0 > \leq < l(u^-), (|u^-| - 1)_1 > = |u^-| - 1$$

and $(u^-)_l\big(M(s_{t_{l(u^-)}(l)} 10^\infty)\big)$ is defined. This shows that $j_1 = M(s_{t_{l(u^-)}(l)} 10^\infty)$.

Note that $u^l|(< l, j_1 > +1) \subseteq (u^l)^-$. The claim, (ii), shows that $(u^l)^{-l} = u^l|(< l, j_1 > +1)$. We argue by contradiction to see that $(u^l)^-$ is not $(> l)$-placed. The proof of the previous point shows that $j_1 = M(s_{t_{l((u^l)^-)}(l)} 10^\infty)$. Lemma 4.5.(b) shows that $s_{t_{l(u^-)}(l)} 10^\infty = s_{t_{l((u^l)^-)}(l)} 10^\infty$. Thus $(u^-)_l(0) = (s_{t_{l(u^-)}(l)} 10^\infty)(0) = (s_{t_{l((u^l)^-)}(l)} 10^\infty)(0) = \big((u^l)^-\big)_l(0)$,

$$\varepsilon(u) = u(< l, 0 >) = (u)_l(0) = (u^-)_l(0) = \big((u^l)^-\big)_l(0) = \varepsilon(u^l),$$

which is absurd. This shows that $(u^l)^- = u^l|(< l, j_1 > +1) = (u^{-l})^l$ is l-placed, by Lemma 4.8, so that $u^{-l} = \big((u^l)^-\big)^l$. Moreover, $\varepsilon(u^{-l}) = (u^{-l})(< l, 0 >) = u(< l, 0 >) = \varepsilon(u)$.

Assume now that $(u^l)^-$ is $(> l)$-placed. As u^l is l-placed and $(u^l)^l = u$, the previous arguments show that u^- is l-placed. In particular, $u^{-l} = u^-$. □

Theorem 4.11 $(\mathbb{X}_3, \mathbb{A}_3)$ *satisfies the conjecture.*

Proof. We already noticed that it is enough to see that (a) or (b) holds. In Condition (5) in the proof of Theorem 5.1 in [L-Z], u^{-l} should be replaced with $u^{-l(u)}$. We need to check that the map f defined there satisfies $\mathbb{A}_3 \subseteq (f \times f)^{-1}(A)$. So let $(\alpha, \beta) \in \mathbb{A}_3$, which defines n. Let $(p_j)_{j\in\omega}$ be the infinite strictly increasing sequence of natural numbers $p_j \geq 1$ such that $(p_j - 1)_0 = n$, $(p_j - 1)_1 > 0$ and $\alpha(p_j - 1) = 1$. In particular, $\alpha|p_j$ is n-placed and $\varepsilon(\alpha|p_j) = 0$. Note that $(p_j)_{j\in\omega}$ is also the infinite strictly increasing sequence of natural numbers $p_j \geq 1$ such that $(p_j - 1)_0 = n$, $(p_j - 1)_1 > 0$ and $\beta(p_j - 1) = 1$ on one side, and a subsequence of both $(p^\alpha_k)_{k\in\omega}$ and $(p^\beta_k)_{k\in\omega}$ on the other side.

If moreover $p \geq p_0$ and $\alpha|p$ is placed, then $l(\alpha|p) \geq n$, by Lemma 4.8. In particular, if $p \geq p_0$ and $\alpha|p$ is $(\leq n)$-placed, then $\alpha|p$ is n-placed. This proves that $(p_j)_{j\in\omega}$ is the infinite strictly increasing sequence of integers $p_j \geq p_0$ such that $\alpha|p_j$ is $(\leq n)$-placed. Therefore $(\alpha|p_{j+1})^{-n} = \alpha|p_j$.

By Condition (3), $(U_{\alpha|p_j})_{j\in\omega}$ is a non-increasing sequence of nonempty clopen subsets of $A\cap\Omega_{X^2}$ whose GH-diameter tend to 0. So we can define $F(\alpha,\beta)\in A$ by $\{F(\alpha,\beta)\}:=\bigcap_{j\in\omega}\ U_{\alpha|p_j}$. Note that $F(\alpha,\beta)=\lim_{j\to\infty}\ (x_{\alpha|p_j},x_{\beta|p_j})=\big(f(\alpha),f(\beta)\big)\in A$, so that $\mathbb{A}_3\subseteq(f\times f)^{-1}(A)$.

It remains, when $k\geq 2$ (second case), to replace $l-1$ with $l(u^-)$. □

The general case

Here we just give, for each $i\in\omega$, a sequence $(W_{\eta_i,n})_{n\in\omega}$ of nonempty T_{η_i}-open sets such that $Q_{\eta_i}\subseteq\overline{\bigcup_{n\in\omega}\ W_{\eta_i,n}}^{T_{\eta_i}}$. This will imply that $\mathbb{K}_\xi$ has no $\mathbf{\Delta}^0_\xi$-measurable countable coloring, by Corollary 4.3. We assume that $\xi\geq 4$, so that we may assume that $\eta_i\geq 3$. If $\eta=\sup_{n\in\omega}\ (\theta_n{+}1)\geq 2$, then we set $V_{\eta,n}:=\{\alpha\in 2^\omega\mid\forall i<n\ \ (\alpha)_i\notin P_{\theta_i}\ \wedge\ (\alpha)_n\in P_{\theta_n}\}$. We set, for $\eta\geq 3$,

$$W_{\eta,n}:=\big\{\alpha\in 2^\omega\mid\alpha(0)=s_{n+1}(0)\ \wedge\ (\alpha^*)_n\in P_{\theta_n}\ \wedge\ \forall i<n\ \ (\alpha^*)_i\in\bigcup_{j<n-i}\ V_{\theta_i,j}\big\}.$$

Mátrai's construction ensures that $V_{\eta,n}$ is τ_η-open, and that $W_{\eta,n}$ is a nonempty T_η-open set. Let O be a basic T_η-open set meeting Q_η. As $T_\eta=\text{discrete}\times\tau_\eta$ and $\tau_{\eta|P_\eta}\equiv(\Pi_{i\in\omega}\ \tau_{\theta_i})_{|P_\eta}$, we can find $\varepsilon\in 2$ and $(O_i)_{i<l}\in\pi_{i<l}\ \tau_{\theta_i}$ such that $O=\{\alpha\in 2^\omega\mid\alpha(0)=\varepsilon\ \wedge\ \forall i<l\ \ (\alpha^*)_i\in O_i\}$. As P_{θ_i} is τ_{θ_i}-closed nowhere dense and $\neg P_{\theta_i}=\bigcup_{n\in\omega}\ V_{\theta_i,n}$, we can find n_i such that O_i meets V_{θ_i,n_i}. We choose $n>\max_{i<l}\ (n_i{+}i)$ such that $s_{n+1}(0)=\varepsilon$. Then $W_{\eta,n}$ meets O.

Our motivation to introduce these examples is that they induce a set $\mathbb{K}_3$ satisfying the conjecture. This is the reason why we think that they are reasonable candidates for the general case.

(B) The small classes

In Section 3, we met $D_2(\mathbf{\Pi}^0_1)$ graphs of fixed point free partial injections with a Borel countable (2-)coloring, but without $\mathbf{\Delta}^0_\xi$-measurable countable coloring. Their complement are $\check{D}_2(\mathbf{\Pi}^0_1)$ sets in $(\mathbf{\Delta}^1_1\times\mathbf{\Sigma}^0_1)_\sigma$, but not in $(\mathbf{\Sigma}^0_\xi\times\mathbf{\Sigma}^0_\xi)_\sigma$. However, a positive result holds for the simpler classes, which shows some optimality in our results.

Proposition 4.12 *Let $\mathbf{\Gamma}\subseteq D_2(\mathbf{\Pi}^0_1)$ be a Wadge class (in zero-dimensional spaces), and A be a set in $\mathbf{\Gamma}\cap(\mathbf{\Delta}^1_1\times\mathbf{\Sigma}^0_1)_\sigma$ (resp., $(\mathbf{\Delta}^1_1\times\mathbf{\Delta}^1_1)_\sigma$). Then $A\in(\mathbf{\Gamma}\times\mathbf{\Sigma}^0_1)_\sigma$ (resp., $(\mathbf{\Gamma}\times\mathbf{\Gamma})_\sigma$).*

Proof. Let us do it for $(\mathbf{\Delta}^1_1\times\mathbf{\Sigma}^0_1)_\sigma$, the other case being similar. The result is clear for $\{\emptyset\}$, $\check{\{\emptyset\}}$, $\mathbf{\Delta}^0_1$, $\mathbf{\Sigma}^0_1$. If $\mathbf{\Gamma}=\mathbf{\Pi}^0_1$, then we can write $A=\bigcup_{n\in\omega}\ C_n\times D_n$, with $C_n\in\mathbf{\Delta}^1_1$ and $D_n\in\mathbf{\Sigma}^0_1$. We just have to note that $A=\bigcup_{n\in\omega}\ \overline{C_n}{\times}D_n$. If $\mathbf{\Gamma}=\mathbf{\Pi}^0_1\oplus\mathbf{\Sigma}^0_1$, then we can write $A=\bigcup_{n\in\omega}\ C_n{\times}D_n=(C{\cap}D)\cup(O{\setminus}D)$, with $C_n\in\mathbf{\Delta}^1_1$, $\neg C,O,D,\neg D,D_n\in\mathbf{\Sigma}^0_1$. Note that $A=(D\cap\bigcup_{n\in\omega}\ \overline{C_n}\times D_n)\cup(O\setminus D)$. Finally, if $\mathbf{\Gamma}=D_2(\mathbf{\Pi}^0_1)$, then write $A=\bigcup_{n\in\omega}\ C_n\times D_n=C\cap O$, with $C_n\in\mathbf{\Delta}^1_1$, $\neg C,O,D_n\in\mathbf{\Sigma}^0_1$. Note that $A=O\cap\bigcup_{n\in\omega}\ \overline{C_n}\times D_n$. □

(C) The finite case

Proposition 4.13 *Assume that $\mathbf{\Gamma}$ is closed under finite intersections and continuous pre-images, X,Y are topological spaces, κ is finite, and $A\in\mathbf{\Gamma}(X{\times}Y)$ is the union of κ rectangles. Then A is the union of at most 2^{2^κ} rectangles whose sides are in $\mathbf{\Gamma}$.*

Proof. Assume that $A=\bigcup_{n<\kappa}\ A_n\times B_n$. Let us prove that

$$A=\bigcup_{I\subseteq\kappa,(\bigcap_{n\in I}\ A_n)\setminus(\bigcap_{n\notin I}\ A_n)\neq\emptyset}(\bigcap_{n\in I}\ A_n)\times(\bigcup_{n\in I}\ B_n).$$

So let $(x,y)\in A$, and let $I:=\{n<\kappa\mid x\in A_n\}$. Then $x\in(\bigcap_{n\in I}\ A_n)\setminus(\bigcap_{n\notin I}\ A_n)$, and (x,y) is in $(\bigcap_{n\in I}\ A_n)\times(\bigcup_{n\in I}\ B_n)$ since $(x,y)\in A_n\times B_n$ for some $n<\kappa$. The other inclusion is clear.

Assume now that $x\in(\bigcap_{n\in I}\ A_n)\setminus(\bigcap_{n\notin I}\ A_n)$. Then $\bigcup_{n\in I}\ B_n=A_x=f^{-1}(A)$, where the formula $f(y):=(x,y)$ defines $f:Y\rightarrow X\times Y$ continuous. This shows that $\bigcup_{n\in I}\ B_n$ is in $\mathbf{\Gamma}$. So we proved the following:

A is the union of at most 2^κ rectangles $A'_n\times B'_n$, where A'_n is a finite intersection of some of the A_n's, and B'_n is a finite union of some of the B_n's which is in $\mathbf{\Gamma}$.

Applying this again, we see that A is the union of at most 2^{2^κ} rectangles $A''_n\times B''_n$, where A''_n is a finite union of some of the A'_n's which is in $\mathbf{\Gamma}$, and B''_n is a finite intersection of some of the B'_n's. We are done since $\mathbf{\Gamma}$ is closed under finite intersections. $\square$

This proof also shows the following result:

Proposition 4.14 *Assume that $\mathbf{\Gamma}$ is closed under continuous pre-images, X,Y are topological spaces, κ is finite, and $A\in\mathbf{\Gamma}(X\times Y)$ is the union of κ rectangles of the form $2^X\times\mathbf{\Sigma}^0_1(Y)$. Then A is the union of at most 2^{2^κ} rectangles of the form $\mathbf{\Gamma}(X)\times\mathbf{\Sigma}^0_1(Y)$.*

Remarks. (1) For colorings, Theorem 1.2 gives, for each ξ, a $D_2(\mathbf{\Pi}^0_1)$ binary relation with a Borel finite (2-)coloring, but with no $\mathbf{\Delta}^0_\xi$-measurable finite coloring.

(2) $\emptyset$ has a 1-coloring. An open binary relation having a finite coloring c has also a $D_2(\mathbf{\Pi}^0_1)$-measurable finite coloring (consider the differences of the $\overline{c^{-1}(\{n\})}$'s, for n in the range of the coloring). This leads to the following question:

Question. Can we build, for each ξ, a closed binary relation with a Borel finite coloring but no $\mathbf{\Delta}^0_\xi$-measurable finite coloring?

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