TOPOLOGY, FORCING, AND GRAPH COLOURINGS

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Abstract. We introduce a family of forcing notions that are helpful in showing that certain graphs do not have countable $\mathbf{\Sigma}^0_\alpha$ colourings. We construct graphs that are "weakly minimal" for such colourings.

1. Introduction

One of the major results in descriptive set theory is the $\mathbb{G}_0$ *dichotomy*:

Theorem 1.1 (Kechris,Solecki,Todorčević, [KST99])**.** *There is a Borel directed graph $\mathbb{G}_0$ on Cantor space such that for any analytic directed graph G on a Polish space X, exactly one of the following holds:*

(1) *G has a countable Borel colouring;*
(2) *There is a continuous homomorphism from $\mathbb{G}_0$ to G.*

This result has found a large number of applications (for a survey, see for example [Mil12]). It is natural to ask for a level-by-level version of this result, with respect to the Borel hierarchy. This work was initiated in [LZ14a], where the authors prove the following.

Theorem 1.2 (Lecomte,Zelený, [LZ14a])**.** *Let $\alpha \in \{1, 2, 3\}$. There is a zero-dimensional Polish space $\mathbb{X}_\alpha$, and an analytic directed graph A_α on $\mathbb{X}_\alpha$, such that for any Polish space X (zero dimensional if $\alpha = 1$), for any analytic directed graph G on X, exactly one of the following holds:*

(1) *There is a countable $\mathbf{\Sigma}^0_\alpha$ colouring of G.*
(2) *There is a continuous graph homomorphism from A_α to G.*

The known proofs of Theorem 1.2 (in [LZ14a] and in [LZ14b]) for $\alpha = 3$ are quite technical, and thus difficult to generalise. In this paper we introduce a method for defining families of graphs without $\mathbf{\Sigma}^0_\alpha$ colourings. For these families we obtain a weak version of Theorem 1.2 (see Theorem 3.2 below), in which the graph homomorphisms are not continuous, but in a class which is sufficient to preclude such colourings. Such graphs are relatively easy to define (showing that they are not colourable is a different matter). However, for $\alpha = 3$, the graph defined in Definition 3.1 is not minimal for non-colourability (see Proposition 4.23). In two steps, we devise a family of graphs H_α (Definition 5.14) that does generalise the graphs from Theorem 1.2, therefore giving yet a new proof of Theorem 1.2 (see Theorem 5.31 and Theorem 6.1).

The definition of the graphs H_α involves a notion of *true stages*, also known as a *representation theorem*, for specific Σ^0_α subsets of Cantor space. The representation theorem for Borel sets of Debs and Saint Raymond [DSR07] was used several times in investigations of close topics (see for example [Lec13, Lec19, Lec20]). An effective version of this method was independently introduced by Montalbán [Mon14], articulating dynamic aspects of iterated priority arguments in computable structure theory, originally designed by Ash [Ash86]. Montalbán's method was used in effective descriptive set theory [DDW22, DGHTT24] and computability theory to analyse variants of Wadge reducibility [GQT], and their proof-theoretic strength as measured by reverse mathematics [DGHTT]. In this paper we develop a particular system of true stage relations suited for our purposes. In future work, we aim to present a more general framework using general systems of representations (see for example [Lec13]).

The main tool that we introduce, to prove that certain graphs do not have $\boldsymbol{\Sigma}^0_\alpha$ colourings, is a family of notions of forcing that all share an "untagging" property, inspired by Steel's method of forcing with tagged trees [Ste78]. One of our aims is to explain how this method can be also presented in the language of topology, using Baire category as a main tool. In Section 2 we give a detailed development of the simplest notion of forcing in our family, and explain how to view it topologically; we explain the connection to a result of Mátrai [MÓ4], that relates descriptive complexity and Baire category. In Section 3 we introduce the first family of graphs, K_α, and prove the weak dichotomy theorem Theorem 3.2. Toward defining the family of graphs H_α, in Section 4 we define an intermediate family (Definition 4.3), and present a more elaborate notion of forcing to show non-colourability of these graphs. The most complicated family of graphs, H_α, is defined in Section 5, where we first develop the true stage machinery required to define these graphs. In Section 6 we show the minimality of H_3.

In Section 7 we give a new proof of a result of Debs and Saint Raymond regarding separators of the iterated Fréchet filters and ideals, using yet another variant of our forcing with untagging. Finally, in Section 8 we list some open questions, including one regarding separating subsets of product spaces by countable unions of Borel rectangles. This is a family of results and problems closely related to graph colourings; we mention some of the background in that section.

2. Topology and forcing

2.1. **The space.** Let α be a computable ordinal. (We work with a computable ordinal since some of our results rely on lightface arguments; however, all results relativise to an oracle, so apply to all countable ordinals.)

For the purposes of the following definition, and the rest of the section, we identify α with a computable well-ordering of a computable subset of $\mathbb{N}$, for which the successor relation and the set of limit points are both computable (roughly, an "ordinal notation"). Note that this computable presentation gives, for every limit $\delta \leqslant \alpha$, uniformly, a computable increasing and cofinal sequence (δ_k) in δ.

Definition 2.1. We let $T_\alpha \subset \mathbb{N}^{<\omega}$ be a computable well-founded tree of rank α, which has a computable rank function, defined as follows:

- The root $\langle\rangle$ of the tree T_α has rank α;
- If $\delta \leqslant \alpha$ is a limit, with a chosen computable cofinal and increasing sequence (δ_k), and $\sigma \in T_\alpha$ has rank δ, then for all k, $\sigma\hat{\ }k \in T_\alpha$, and has rank δ_k.
- If $\beta < \alpha$ and $\sigma \in T_\alpha$ has rank $\beta+1$, then for all k, $\sigma\hat{\ }k \in T_\alpha$ and has rank β.

Note that T_α is not "saturated"; if $\sigma \in T_\alpha$ and $\mathrm{rk}(\sigma) > 1$, then not all $\beta < \mathrm{rk}(\sigma)$ are realised as ranks of children of σ (nodes of the form $\sigma\hat{\ }k$). Indeed, we will make use of the following fact: for all $\sigma \in T_\alpha$, for all $\beta < \mathrm{rk}(\sigma)$, there are only finitely many k such that $\mathrm{rk}(\sigma\hat{\ }k) < \beta$. Indeed, if $\mathrm{rk}(\sigma)$ is a successor, then there are no such k; if $\mathrm{rk}(\sigma) = \delta$ is a limit, then we use the fact that (δ_k) is inceasing and cofinal in δ.

We let $\mathscr{L}_\alpha$ denote the collection of leaves of T_α, which is a computable set (the leaves of T_α are the nodes of rank 0). Any computable bijection between $\mathscr{L}_\alpha$ and ω induces a computable isomorphism of the space $2^{\mathscr{L}_\alpha}$ and Cantor space $2^{\mathbb{N}}$.

Henceforth, we often suppress mention of the ordinal α in some subscripts.

Definition 2.2. Suppose that $x \in 2^{\mathscr{L}_\alpha}$. We define a $\{0,1\}$-valued labelling $T^x = T^x_\alpha$ of T_α, extending x, by transfinite recursion on the nodes of T_α, as follows:

- For each $\sigma \in \mathscr{L}_\alpha$, $T^x(\sigma) = x(\sigma)$;
- For each $\sigma \in T_\alpha \smallsetminus \mathscr{L}_\alpha$,

$$T^x(\sigma) = 0 \qquad \Longleftrightarrow \qquad (\exists k)\ T^x(\sigma\hat{\ }k) = 1.$$

The definition of $T^x(\sigma) = 0$ mimics an existential quantifier. By effective transfinite recursion, we obtain:

Lemma 2.3. *If $\sigma \in T_\alpha$ has rank β, then*

$$\left\{x \in 2^{\mathscr{L}_\alpha} \,:\, T^x(\sigma) = 0\right\}$$

is Σ^0_β, uniformly in β.

If $\alpha = 1$, then T_α consists of a root and the set of leaves (which all have height 1). The space $2^{\mathscr{L}_1}$ is identified with Cantor space in a straightforward way: $x \in 2^{\mathscr{L}_1}$ is identified with $k \mapsto x(\langle k\rangle)$. (We say that the *location* k for Cantor space, which is an element of $\mathbb{N}$, is identified with the *location* $\langle k\rangle$ for the space $2^{\mathscr{L}_1}$, which is an element of $\mathscr{L}_1$.) The value of T^x at the root records the fact if, considered as a subset of $\mathbb{N}$ via identification with its characteristic function, x is empty or not, with the value 0 indicating the latter.

If $\alpha = 2$, then the space $2^{\mathscr{L}_2}$ is naturally identified with $(2^{\mathbb{N}})^{\mathbb{N}}$, which in turn is identified with Cantor space using a pairing function. A location in $\mathbb{N}$ coding the pair (k,l) is identified with the leaf $\langle k,l\rangle \in \mathscr{L}_2$. The value of T^x on a rank 1 node $\langle k\rangle$ records whether the k^{th} column of x is empty or not; while the value of T^x at the root records whether x has an empty column or not.

This continues to higher ranks. When $\alpha = 3$, the space is naturally identified with $((2^{\mathbb{N}})^{\mathbb{N}})^{\mathbb{N}}$, and the value of T^x at the root records if, when considering x as built up of columns of columns, it has a column, each sub-column of which is nonempty; and so on. At limit levels δ, the space $2^{\mathscr{L}_\delta}$ is naturally identified with the product $\prod 2^{\mathscr{L}_{\delta_k}}$.

We remark that these "versions of Cantor space" were used in the past, in particularly for understanding the iterated Fréchet ideals and filters; see [DSR09, DM18].

2.2. **The notion of forcing, and the associated topology.**

Definition 2.4. We let $\mathbb{Q}$ be the collection of all partial functions $p\colon T_\alpha \to \{0,1\}$ satisfying:

(i) $\operatorname{dom} p$ is finite; and

(ii) If $\sigma \in T_\alpha \setminus \mathscr{L}_\alpha$, then $p(\sigma) = 0$ if and only if there is some k such that $p(\sigma\hat{\ }k) = 1$.

The set $\mathbb{Q}$ is partially ordered by reverse extension: for $p, q \in \mathbb{Q}$, $q \leqslant p$ if and only if $p \subseteq q$.

The partial ordering $\mathbb{Q}$ is called a *notion of forcing*, and its elements are called *forcing conditions*. If $q \leqslant p$ then we say that q *extends* p. We also say that two conditions p and q are *compatible* if they have a common extension in $\mathbb{Q}$.

Lemma 2.5. *Conditions $p, q \in \mathbb{Q}$ are compatible if and only if $p \cup q$ is a function, in which case $p \cup q \in \mathbb{Q}$.*

Proof. If r extends both p and q then $p \cup q \subseteq r$, implying that $p \cup q$ is a function. On the other hand, suppose that $s = p \cup q$ is a function. For all $\sigma \in T_\alpha \setminus \mathscr{L}_\alpha$, $s(\sigma) = 0$ if and only if $p(\sigma) = 0$ or $q(\sigma) = 0$ if and only if there is some k such that $p(\sigma\hat{\ }k) = 1$ or $q(\sigma\hat{\ }k) = 1$ if and only if there is some k such that $s(\sigma\hat{\ }k) = 1$; so $s \in \mathbb{Q}$, and extends both p and q. ☐

Viewed topologically, a condition $p \in \mathbb{Q}$ can be thought of as a code of a Δ^1_1 subset of our space $2^{\mathscr{L}_\alpha}$.

Definition 2.6. For $p \in \mathbb{Q}$ we let

$$[p] = \left\{x \in 2^{\mathscr{L}_\alpha} : T^x \supset p\right\}.$$

We let $\tau_{\mathbb{Q}}$ denote the topology on $2^{\mathscr{L}_\alpha}$ generated by the sets $[p]$ for $p \in \mathbb{Q}$.

Observe that if q extends p then $[q] \subseteq [p]$ (this explains writing $q \leqslant p$ when q extends p, rather than $p \leqslant q$). The converse holds, but this is not (yet) immediate, nor is it important.

We will shortly verify that the $\tau_{\mathbb{Q}}$-topology satisfies the Baire category theorem. To do so, we will show that we can view points in the space $2^{\mathscr{L}_\alpha}$ as filters on $\mathbb{Q}$. We recall the following definitions.

Definition 2.7.

(a) A *filter* on $\mathbb{Q}$ is a nonempty subset $G \subseteq \mathbb{Q}$ that is:
  - directed, i.e., for all $p, q \in G$ there is some $r \in G$ that extends both p and q; and
  - upwards closed, i.e., for all $p, q \in \mathbb{Q}$, if q extends p and $q \in G$ then $p \in G$.

(b) A set $D \subseteq \mathbb{Q}$ is called *dense* if every $p \in \mathbb{Q}$ has an extension in D.

(c) Let $\mathscr{D}$ be a collection of dense subsets of $\mathbb{Q}$. We say that a filter G of $\mathbb{Q}$ is *$\mathscr{D}$-generic* if G intersects every $D \in \mathscr{D}$.

Definition 2.8. We define some subsets of $\mathbb{Q}$:

- For $\sigma \in T_\alpha$, let D_σ be the set of $p \in \mathbb{Q}$ such that $\sigma \in \operatorname{dom} p$.
- For $p \in \mathbb{Q}$, let D_p be the set $\{q \in \mathbb{Q} : q \leqslant p,$ or p and q are incompatible$\}$.

We let $\mathscr{D}_{\text{point}}$ be the collection of all sets D_σ for $\sigma \in T_\alpha$ and D_p for $p \in \mathbb{Q}$.

Lemma 2.9. *Each set in $\mathscr{D}_{\text{point}}$ is dense.*

Proof. Let $\sigma \in T_\alpha$, and let $p \in \mathbb{Q}$; we find an extension of p in D_σ. If $\sigma \notin \operatorname{dom} p$, we extend p to p' by setting $p'(\sigma) = 0$, and further, if σ is not a leaf of T_α, then we set $p'(\sigma\hat{}k) = 1$ for some large k (so that $\operatorname{dom} p$ contains no extensions of $\sigma\hat{}k$). This choice of k ensures that $p' \in \mathbb{Q}$, so $p' \in D_\sigma$.

For D_p, the argument is quick. Let $q \in \mathbb{Q}$. If p and q are incompatible then $q \in D_p$. Otherwise, q and p have a common extension r; then r is an extension of q in D_p. □

Definition 2.10. For a filter $G \subseteq \mathbb{Q}$, we let

$$x_G = \left(\bigcup G\right)\restriction \mathscr{L}_\alpha.$$

Lemma 2.11. *If G is $\mathscr{D}_{\text{point}}$-generic, then $x_G \in 2^{\mathscr{L}_\alpha}$, and for all $p \in \mathbb{Q}$,*

$$x_G \in [p] \iff p \in G.$$

Proof. Suppose that G is a $\mathscr{D}_{\text{point}}$-generic filter. We let

$$S = \bigcup G.$$

Since G is directed, S is a function on T_α. Since $G \cap D_\sigma \neq \varnothing$ for all $\sigma \in T_\alpha$, S is a total function, i.e., $\operatorname{dom} S = T_\alpha$.

We claim that $S = T^{x_G}$. To do so, we check that S satisfies the definition of T^{x_G}. By definition of x_G, we have $S\restriction\mathscr{L}_\alpha = x_G$. Suppose that $\sigma \in T_\alpha \setminus \mathscr{L}_\alpha$. If $S(\sigma) = 0$, let $p \in G$ with $p(\sigma) = 0$; since $p \in \mathbb{Q}$, there is some k such that $p(\sigma\hat{}k) = 1$, so $S(\sigma\hat{}k) = 1$. On the other hand, if for some k we have $S(\sigma\hat{}k) = 1$, let $p \in G$ such that $p(\sigma\hat{}k) = 1$; since $p \in \mathbb{Q}$, $p(\sigma) = 0$, so $S(\sigma) = 0$.

Hence, if $p \in G$, then $p \subset S = T^{x_G}$, that is, $x_G \in [p]$. In the other direction, let $p \in \mathbb{Q}$, and suppose that $x_G \in [p]$, i.e., that $p \subseteq S$. For all $q \in G$, $q \subseteq S$, so $p \cup q$ is a function. By Lemma 2.5, p and q are compatible. That is, p is compatible with all $q \in G$. Since $G \cap D_p \neq \varnothing$, G must contain an extension of p. Since G is closed upwards in $\mathbb{Q}$, $p \in G$. □

Remark 2.12. In fact, the map $G \mapsto x_G$ is a *bijection* between the $\mathscr{D}_{\text{point}}$-generic filters and the space $2^{\mathscr{L}_\alpha}$, giving us some kind of Stone duality. We will however not need this fact. It is enough that "most" points in $2^{\mathscr{L}_\alpha}$ are x_G for a generic G, in a sense made precise below.

The following is often referred to as the *Rasiowa-Sikorski lemma*.

Lemma 2.13. *For any countable collection $\mathscr{D}$ of dense subsets of $\mathbb{Q}$, for all $p \in \mathbb{Q}$, there is a $\mathscr{D}$-generic filter G such that $p \in G$.*

Proof. Write $\mathscr{D} = \{D_n : n \in \mathbb{N}\}$ and define a decreasing sequence of conditions (p_n) as follows: $p_0 = p$; given p_n, p_{n+1} is some condition extending p_n such that $p_{n+1} \in D_n$, which exists since D_n is dense. Then $G = \{q : (\exists n)\ p_n \text{ extends } q\}$ is a $\mathscr{D}$-generic filter containing p. □

Applying Lemma 2.13 to $\mathscr{D}_{\text{point}}$, we get that for all $p \in \mathbb{Q}$, $[p] \neq \varnothing$. This implies:

Lemma 2.14.

(a) *A set $U \subseteq 2^{\mathscr{L}_\alpha}$ is $\tau_{\mathbb{Q}}$-dense and open if and only if there is a dense set $D \subseteq \mathbb{Q}$ such that*

$$U = \bigcup \{[p] : p \in D\}.$$

(b) *A set $A \subseteq 2^{\mathscr{L}_\alpha}$ is $\tau_{\mathbb{Q}}$-comeagre if and only if there is a countable collection $\mathscr{D} \supseteq \mathscr{D}_{\text{point}}$ of dense subsets of $\mathbb{Q}$ such that*

$$A \supseteq \{x_G : G \text{ is a } \mathscr{D}\text{-generic filter}\}.$$

Now Lemma 2.14 and Lemma 2.13 imply:

Proposition 2.15. *The topology $\tau_{\mathbb{Q}}$ satisfies the Baire category theorem: for all $p \in \mathbb{Q}$ and $\tau_{\mathbb{Q}}$-comeagre set $A \subseteq 2^{\mathscr{L}_\alpha}$, $[p] \cap A \neq \varnothing$.*

Remark 2.16. The topology $\tau_{\mathbb{Q}}$ is Polish. To get a complete metric for this topology, fix an ω-ordering (σ_n) of T_α; for distinct $x, y \in 2^{\mathscr{L}_\alpha}$, the distance between x and y is 2^{-n}, where n is least such that $T^x(\sigma_n) \neq T^y(\sigma_n)$.

Definition 2.17. The quantifier "for all sufficiently generic G, …" means: there is a countable collection $\mathscr{D}$ of dense subsets of $\mathbb{Q}$ such that for every $\mathscr{D}$-generic filter G, …

By Lemma 2.14, for any $A \subseteq 2^{\mathscr{L}_\alpha}$, the following are equivalent: (1) For all sufficiently generic G, $x_G \in A$. (2) A is $\tau_{\mathbb{Q}}$-comeagre.

2.3. **The strong forcing relation.** The (very restricted) *forcing language* that we will use consists of codes of Borel subsets of our space.

Definition 2.18. We define, by recursion, the collection of *Borel codes* of subsets of $2^{\mathscr{L}_\alpha}$:

- Any finite partial function from $\mathscr{L}_\alpha$ to $\{0, 1\}$ is a Borel code;
- If φ is a Borel code, then $\neg\varphi$ is a Borel code;
- If $(\varphi_n)_{n<\omega}$ is an ω-sequence of Borel codes, then $\bigvee_n \varphi_n$ is a Borel code.

Thus, Borel codes can be identified as formulas in an infinitary propositional logic, or alternatively, as labelled well-founded trees (where the leaves are labelled by finite functions as described, and each non-leaf is labelled by either $\neg$ or $\vee$; a node labelled by $\neg$ has one child only). We omit conjunction to make some following definitions and arguments shorter.

For a Borel code φ, we let $[\varphi]$ denote the subset of $2^{\mathscr{L}_\alpha}$ that is coded by φ, namely:

- If $r \colon \mathscr{L}_\alpha \to \{0, 1\}$ is finite, then $[r] = \left\{x \in 2^{\mathscr{L}_\alpha} : r \subset x\right\}$ (this is a special case of Definition 2.6);
- $[\neg\varphi] = [\varphi]^{\complement} = 2^{\mathscr{L}_\alpha} \setminus [\varphi]$;
- $[\bigvee \varphi_n] = \bigcup_n [\varphi_n]$.

Note that at the basic level we take finite partial functions on $\mathscr{L}_\alpha$, not all elements of $\mathbb{Q}$. Thus, we are considering Borel sets as they are generated from the standard topology on $2^{\mathscr{L}_\alpha}$, rather than the $\tau_{\mathbb{Q}}$-topology.

Certainly, every Borel subset of $2^{\mathscr{L}_\alpha}$ has a code, indeed many different codes. The connection between generic points and Borel sets is given by the *forcing relation*, which is the heart of the theory. The standard forcing relation $p \Vdash \varphi$ (between conditions and Borel codes) is defined combinatorially, but is equivalent to $[\varphi]$ being $\tau_{\mathbb{Q}}$-comeagre in $[p]$. For our purposes, we will need a variation $\Vdash^*$, a stronger version of the usual forcing relation.

Definition 2.19. We define the relation $\Vdash^*$ between $\mathbb{Q}$ and Borel codes, by recursion on the complexity of the Borel code. Let $p \in \mathbb{Q}$.

- If $r \colon \mathscr{L}_\alpha \to 2$ is a finite partial function, then $p \Vdash^* r$ if p extends r.
- $p \Vdash^* \bigvee_n \varphi_n$ if there is some n such that $p \Vdash^* \varphi_n$.
- $p \Vdash^* \neg\varphi$ if there is no q extending p such that $q \Vdash^* \varphi$.

If $p \Vdash^* \varphi$, we say that p *strongly forces* φ.

We remark that the standard forcing relation can be defined as follows: $p \Vdash \varphi$ if the collection of $q \leqslant p$ that strongly force φ is dense below p (every extension of p has an extension that strongly forces φ). The difference between the two notions lies in disjunctions: suppose that densely below some p, we have conditions that strongly force some φ_n. Then p forces φ, but p itself may not force any particular φ_n, so does not strongly force φ. In contrast, $p \Vdash^* \neg\varphi$ if and only if p forces $\neg\varphi$.

Lemma 2.20. *Let φ be a Borel code.*

(a) *There is no $p \in \mathbb{Q}$ which strongly forces both φ and $\neg\varphi$.*
(b) *$\{p \in \mathbb{Q} : p \Vdash^* \varphi \vee p \Vdash^* \neg\varphi\}$ is dense.*
(c) *If q extends p and $p \Vdash^* \varphi$ then $q \Vdash^* \varphi$.*

Proof. (a) and (b) are immediate, from the definition of strongly forcing $\neg\varphi$. (c) is immediate except for disjunctions, and so is proved by induction on the complexity of the Borel code φ. □

The *forcing theorem* shows how Borel sets have the property of Baire for the topology $\tau_{\mathbb{Q}}$.

Proposition 2.21. *For each Borel code φ, for all sufficiently generic G,*

$$x_G \in [\varphi] \iff (\exists p \in G)\ (p \Vdash^* \varphi).$$

Proof. This is proved by induction on the complexity of φ. For each φ, we define a countable collection $\mathscr{D}_\varphi \supseteq \mathscr{D}_{\text{point}}$ of dense subsets of $\mathbb{Q}$ such that the equivalence above holds for every $\mathscr{D}_\varphi$-generic filter G.

If $\varphi = r$ is a finite partial function from $\mathscr{L}_\alpha$ to $\{0, 1\}$, then we can take $\mathscr{D}_r = \mathscr{D}_{\text{point}}$. The desired equivalence follows from Lemma 2.11.

If φ is $\bigvee_n \varphi_n$, then we let

$$\mathscr{D}_\varphi = \bigcup_n \mathscr{D}_{\varphi_n}.$$

Suppose that G is $\mathscr{D}_\varphi$-generic. Then $x_G \in [\varphi]$ if and only if $x_G \in [\varphi_n]$ for some n. By induction, this holds if and only if there is some n and some $p \in G$ such that $p \Vdash^* \varphi_n$. By definition of strong forcing, this holds if and only if there is some $p \in G$ such that $p \Vdash^* \varphi$.

If φ is $\neg\psi$, then we let $\mathscr{D}_\varphi$ be $\mathscr{D}_\psi$, together with the set

$$\{p \in \mathbb{Q} : p \Vdash^* \psi \vee p \Vdash^* \neg\psi\},$$

which is dense by Lemma 2.20(b).

Suppose that G is $\mathscr{D}_\varphi$-generic. In one direction, suppose that $p \in G$ and $p \Vdash^* \varphi$. That is, there is no $q \leqslant p$ with $q \Vdash^* \psi$. If $x_G \in [\psi]$ then by induction, there is some $r \in G$ such that $r \Vdash^* \psi$. Since G is a filter, there is some $q \in G$ extending both p and r. Since q extends r, by Lemma 2.20, $q \Vdash^* \psi$, contradicting the assumption on p. Hence, $x_G \notin [\psi]$, i.e., $x_G \in [\varphi]$.

In the other direction, suppose that $x_G \in [\varphi]$. Since G is $\mathscr{D}_\varphi$-generic, there is some $p \in G$ such that $p \Vdash^* \psi$ or $p \Vdash^* \varphi$. However, since $x_G \notin [\psi]$, by induction, $p \Vdash^* \psi$ is impossible, so $p \Vdash^* \varphi$. □

For each Borel code φ we let

$$U_\varphi = \bigcup \{[p] : p \Vdash^* \varphi\},$$

which is $\tau_{\mathbb{Q}}$-open. Translated, Proposition 2.21 says:

Proposition 2.22. *For any Borel code φ, $[\varphi]$ and U_φ are equivalent modulo a $\tau_{\mathbb{Q}}$-meagre set.*

2.4. **Untagging lemma.** The development so far is quite general, and can be applied to many notions of forcing other than our particular $\mathbb{Q}$. What comes next, however, is special to $\mathbb{Q}$ and the similar notions of forcing that we will use in this paper. We consider a fine-grained connection between strong forcing of statements in the Borel hierarchy, and intermediate topologies between the standard one and $\tau_{\mathbb{Q}}$.

First, we recall that codes can be placed in a syntactic hierarchy that mirrors the Borel hierarchy:

- The Π_0 codes are the codes r for clopen sets (finite functions from $\mathscr{L}_\alpha$ to $\{0,1\}$).
- For $\beta > 0$, a code φ is Σ_β if it is of the form $\bigvee_n \varphi_n$, where each φ_n is Π_γ for some $\gamma < \beta$.
- For $\beta > 0$, a code φ is Π_β if it is of the form $\neg\psi$, where ψ is Σ_β.

We observe that if φ is a Σ_β code then $[\varphi]$ is a $\mathbf{\Sigma}^0_\beta$ subset of $2^{\mathscr{L}_\alpha}$ (according to the standard topology), and similarly for Π_β.

Definition 2.23. For $p \in \mathbb{Q}$ and $\beta \leqslant \alpha + 1$, we define a condition $p\restriction\beta \subseteq p$ as follows: for all σ, $(p\restriction\beta)(\sigma)\downarrow$ if and only if $p(\sigma)\downarrow$,[1] and further, either

- $\mathrm{rk}(\sigma) < \beta$; or
- there is some k such that $p(\sigma\hat{\ }k) = 1$ and $\mathrm{rk}(\sigma\hat{\ }k) < \beta$.

That is, $p\restriction\beta$ records what p says about nodes of rank $< \beta$, and the consequences of that information to parents of nodes.

Lemma 2.24. *Let $p \in \mathbb{Q}$ and $\beta \leqslant \alpha + 1$.*

(a) *$p\restriction\beta \in \mathbb{Q}$ and p extends $p\restriction\beta$.*
(b) *If $0 < \gamma < \beta \leqslant \alpha + 1$ then $p\restriction\gamma = (p\restriction\beta)\restriction\gamma$, so $p\restriction\beta$ extends $p\restriction\gamma$.*

Definition 2.25. Let $\beta \leqslant \alpha + 1$. We say that $p \in \mathbb{Q}$ is *β-complete* if for all $\sigma \in \operatorname{dom} p$, if $\mathrm{rk}(\sigma) > \beta$ and $p(\sigma) = 1$, then $p(\sigma\hat{\ }k)\downarrow$ for all k such that $\mathrm{rk}(\sigma\hat{\ }k) < \beta$.

Note that for all such k, we must have $p(\sigma\hat{\ }k) = 0$. That is, for all k, p "knows" that $T^x(\sigma\hat{\ }k) = 0$ for all $x \in [p]$, but since $\operatorname{dom} p$ is required to be finite, we cannot actually have $p(\sigma\hat{\ }k) = 0$ for all k. The requirement of β-completeness is that at least below rank β, this knowledge of p is recorded in p itself (meaning that p determines witnesses for $p(\sigma\hat{\ }k) = 0$). Note again that the definition only cares about σ such that $\mathrm{rk}(\sigma)$ is a limit; otherwise, if $\mathrm{rk}(\sigma) > \beta$, then there are no k such that $\mathrm{rk}(\sigma\hat{\ }k) < \beta$. The key point, for the next lemma, is our observation early on that even in the limit case there will be only finitely many k such that $\mathrm{rk}(\sigma\hat{\ }k) < \beta < \mathrm{rk}(\sigma)$.

Lemma 2.26. *Let $\beta \leqslant \alpha + 1$. For all $p \in \mathbb{Q}$ there is a β-complete p' extending p.*

Proof. Do the obvious: define p' extending p by letting $p'(\sigma\hat{\ }k) = 0$ whenever $p(\sigma) = 1$ and $\mathrm{rk}(\sigma\hat{\ }k) < \beta < \mathrm{rk}(\sigma)$. For each such k, also define $p'(\sigma\hat{\ }k\hat{\ }l) = 1$ for some large l. As just discussed, $\operatorname{dom} p'$ is finite. By design, $p' \in \mathbb{Q}$. If $\tau \in \operatorname{dom} p' \setminus \operatorname{dom} p$ and $p'(\tau) = 1$ then $\mathrm{rk}(\tau) < \beta$; this implies that p' is β-complete. □

[1] Recall that $p(\sigma)\downarrow$ means $\sigma \in \operatorname{dom} p$, and $p(\sigma)\uparrow$ means $\sigma \notin \operatorname{dom} p$.

The following is the key technical lemma.

Lemma 2.27. *Let $0 < \beta \leqslant \alpha$. If $p \in \mathbb{Q}$ is β-complete and r extends $p\restriction(\beta+1)$ then p and $(r\restriction\beta)$ are compatible in $\mathbb{Q}$.*

Proof. By Lemma 2.5, we need to show that $p \cup (r\restriction\beta)$ is a function. Let $\sigma \in T_\alpha$, and suppose that $p(\sigma)\downarrow$ and $(r\restriction\beta)(\sigma)\downarrow$; we need to show that $p(\sigma) = (r\restriction\beta)(\sigma)$.

First, suppose that $\text{rk}(\sigma) \leqslant \beta$. Then $(p\restriction(\beta+1))(\sigma) = p(\sigma)$, and since $r \leqslant p\restriction(\beta+1)$, we have $r(\sigma) = p(\sigma)$; and $(r\restriction\beta)(\sigma) = r(\sigma)$.

Suppose that $\text{rk}(\sigma) > \beta$. Since $(r\restriction\beta)(\sigma)\downarrow$, we must have $r(\sigma) = 0$ and $r(\sigma\hat{\ }k) = 1$ for some k such that $\text{rk}(\sigma\hat{\ }k) < \beta$. We need to show that $p(\sigma) = 0$ as well. But if $p(\sigma) = 1$, then as p is β-complete, we would have $p(\sigma\hat{\ }k) = 0$ and as $r \leqslant p\restriction\beta$, we would have $r(\sigma\hat{\ }k) = 0$, which is not the case. ☐

The following is the “untagging lemma”.

Proposition 2.28. *Let $\gamma \leqslant \alpha$, and let φ be a Π_γ Borel code. If $p \in \mathbb{Q}$ is γ-complete and $p \Vdash^* \varphi$ then $p\restriction(\gamma+1) \Vdash^* \varphi$.*

Proof. The proposition is proved by induction on γ.

The base case $\gamma = 0$ follows from the definition of strong forcing; $p \Vdash^* r$ means $r \subseteq p$, and since r is only defined on $\sigma \in \mathscr{L}_\alpha$, this implies that $r \subseteq p\restriction 1$, so $p\restriction 1 \Vdash^* r$. (Note that every condition is 0-complete.)

Suppose that $\gamma > 0$, and that the proposition has been verified for all $\gamma' < \gamma$. Let φ be a Π_γ Borel code, and let $p \in \mathbb{Q}$ be γ-complete. We prove the contrapositive: if $p\restriction(\gamma+1) \nVdash^* \varphi$ then $p \nVdash^* \varphi$.

Suppose that $p\restriction(\gamma+1) \nVdash^* \varphi$. Since φ is $\neg\psi$ (where ψ is Σ_γ), by definition, this means that there is some r extending $p\restriction(\gamma+1)$ such that $r \Vdash^* \psi$. Now $\psi = \bigvee_n \psi_n$, and by definition, there is some n such that $r \Vdash^* \psi_n$; and ψ_n is $\Pi_{\gamma'}$ for some $\gamma' < \gamma$. By Lemma 2.20(c) and Lemma 2.26, we may assume that r is γ'-complete. Hence, by induction, $r\restriction(\gamma'+1) \Vdash^* \psi_n$, so $r\restriction(\gamma'+1) \Vdash^* \psi$. By Lemma 2.24, $r\restriction\gamma$ extends $r\restriction(\gamma'+1)$, so by Lemma 2.20(c) again, $r\restriction\gamma \Vdash^* \psi$.

By Lemma 2.27, $q = p \cup (r\restriction\gamma)$ extends both p and $r\restriction\gamma$. Since q extends $r\restriction\gamma$, $q \Vdash^* \psi$. Since q extends p, by definition, p does not strongly force φ, as required. ☐

2.5. **Interpretations of untagging.** We obtain a refinement of the forcing theorem (Proposition 2.21).

Definition 2.29. Let $\beta \leqslant \alpha + 1$. We let

$$\mathbb{Q}_\beta = \{p\restriction\beta \,:\, p \in \mathbb{Q}\}\,.$$

In other words, $\mathbb{Q}_\beta$ is the collection of all $p \in \mathbb{Q}$ such that for all $\sigma \in \operatorname{dom} p$, either $\text{rk}(\sigma) < \beta$, or $p(\sigma) = 0$ and $p(\sigma\hat{\ }k) = 1$ for some k with $\text{rk}(\sigma\hat{\ }k) < \beta$.

Proposition 2.30. *Suppose that $0 < \beta \leqslant \alpha + 1$, and that φ is a Σ_β code. For every sufficiently generic G,*

$$x_G \in [\varphi] \iff (\exists p \in G \cap \mathbb{Q}_\beta)\ p \Vdash^* \varphi.$$

Proof. For each $p \in \mathbb{Q}$ and $\gamma \leqslant \alpha + 1$, let $E_{p,\gamma}$ be the collection of $q \in \mathbb{Q}$ such that either

- q extends p and q is γ-complete; or
- q and p are incompatible.

By Lemma 2.26, each $E_{p,\gamma}$ is dense. Let $\mathscr{E} = \{E_{p,\gamma} : p \in \mathbb{Q} \;\&\; \gamma \leqslant \alpha+1\}$. By Proposition 2.21, let $\mathscr{D}$ be a countable collection of dense subsets of $\mathbb{Q}$ such that for all $\mathscr{D}$-generic G, $x_G \in [\varphi]$ if and only if $p \Vdash^* \varphi$ for some $p \in G$. We claim that if G is $\mathscr{E} \cup \mathscr{D}$-generic then the equivalence above holds.

In the direction which is not immediate, suppose that $x_G \in [\varphi]$. Let $p \in G$ be such that $p \Vdash^* \varphi$. Write $\varphi = \bigvee \varphi_n$. By definition of strong forcing, $p \Vdash^* \varphi_n$ for some n. Let $\gamma < \beta$ such that φ_n is Π_γ. Since G is $\mathscr{E}$-generic, find some $q \leqslant p$ in G that is γ-complete. By Proposition 2.28, $q\restriction(\gamma+1)$ strongly forces φ_n, and so strongly forces φ. Since $\gamma < \beta$, $q\restriction\beta$ strongly forces φ. Since q extends $q\restriction\beta$, $q\restriction\beta \in G$, and so is the desired condition. □

We interpret these results in the language of topology.

Definition 2.31. We let $\tau_\beta = \tau_{\mathbb{Q},\beta}$ denote the topology generated by $[p]$ for $p \in \mathbb{Q}_\beta$.

Hence, τ_1 is the standard topology on $2^{\mathscr{L}_\alpha}$, and $\tau_{\alpha+1}$ is $\tau_{\mathbb{Q}}$. Note that by Lemma 2.3, the generating sets of the topology τ_β are all finite Boolean combinations of $\Sigma^0_{<\beta}$ and $\Pi^0_{<\beta}$ sets, and so are all Δ^0_β sets; so each τ_β-open set is $\boldsymbol{\Sigma}^0_\beta$.

Recall that we let $U_\varphi = \bigcup\{[p] : p \Vdash^* \varphi\}$.

Proposition 2.32. *Let $0 < \beta \leqslant \alpha+1$. For any Σ_β Borel code φ, the set U_φ is τ_β-open, and is equivalent to $[\varphi]$ modulo a τ_β-meagre set.*

Proof. Fix nonzero $\beta \leqslant \alpha+1$. We can repeat the development of the beginning of this section, with $\mathbb{Q}_\beta$ replacing $\mathbb{Q}$:

- We define the notion of a filter $G \subset \mathbb{Q}_\beta$, the notion of a dense subset of $\mathbb{Q}_\beta$, and of a $\mathscr{D}$-generic filter, where $\mathscr{D}$ is a countable family of dense subsets of $\mathbb{Q}_\beta$ (Definition 2.7).
- For a sufficiently generic filter $G \subset \mathbb{Q}_\beta$, we define x_G as in Definition 2.10. The analogue of Lemma 2.11 holds for $\mathbb{Q}_\beta$, with the same proof, except that $\bigcup G$ is the restriction of T^{x_G} to σ of rank $< \beta$, and parents of 1-labelled nodes of rank $< \beta$.
- The Rasiowa-Sikorski lemma (Lemma 2.13) holds for any notion of forcing, including $\mathbb{Q}_\beta$. This gives the analogues of Lemma 2.14 and Proposition 2.15 for the topology τ_β.
- We define the strong forcing relation, where for negation, we only search for extensions in $\mathbb{Q}_\beta$. Denote this notion by $\Vdash^*_\beta$. The forcing theorem, Proposition 2.21, holds for $\mathbb{Q}_\beta$ as well, with the same proof.

Hence, $[\varphi]$ is equivalent to the τ_β-open set $\bigcup\{[p] : p \Vdash^*_\beta \varphi\}$ modulo a τ_β-meagre set. The real content of Proposition 2.32 is that this open set is the same as the one that we get when we consider $\tau_{\mathbb{Q}}$. That is, if φ is Σ_β then

(∗): $U_\varphi = \bigcup\big\{[p] : p \in \mathbb{Q}_\beta \;\&\; p \Vdash^*_\beta \varphi\big\}$.

To see this, we observe that the argument proving Proposition 2.28 gives:

- If φ is Σ_β, then for all $p \in \mathbb{Q}_\beta$, $p \Vdash^* \varphi$ if and only if $p \Vdash^*_\beta \varphi$.

Now (∗) follows by an argument similar to that giving Proposition 2.30. Suppose that $x \in U_\varphi$; we need to show that there is some $r \in \mathbb{Q}_\beta$ such that $r \Vdash^* \varphi$ and $x \in [r]$. Let $p \in \mathbb{Q}$ such that $p \Vdash^* \varphi$ and $x \in [p]$. Then $p \Vdash^* \varphi_n$ for some n, and φ_n is Π_γ for some $\gamma < \beta$. There is some $q \leqslant p$ such that $x \in [q]$ and q is γ-complete (define q as in the proof of Lemma 2.26, adding witnesses according to T^x). Now $r = q\restriction\beta$ (which extends $q\restriction(\gamma+1)$) is as required. □

2.6. **Mátrai's result.** The work above is closely related to a result of Mátrai [MÓ4]:

Theorem 2.33 (Mátrai)**.** *Let $\alpha \geqslant 1$ be a countable ordinal. There is a $\mathbf{\Pi}^0_\alpha$ set $P \subseteq 2^\omega$ and a Polish topology τ_α on 2^ω such that:*

(i) *τ_α is finer than the standard topology on 2^ω;*
(ii) *P is τ_α-closed and τ_α-nowhere dense;*
(iii) *If B is a basic τ_α-open set meeting P, $D \subseteq 2^\omega$ is $\mathbf{\Pi}^0_{<\alpha}$ (in the standard topology), and $P \cap B \cap D$ is comeagre in $(P \cap B, \tau_\alpha)$, then there is a τ_α-open set B' such that $P \cap B = P \cap B'$ and $D \cap B'$ is comeagre in (B', τ_α).*

Proof. We work in $2^{\mathscr{L}_\alpha}$, which, as discussed above, is computably isomorphic to Cantor space. We let $\tau_\alpha = \tau_{\mathbb{Q},\alpha}$ defined above (rather than using $\tau_\mathbb{Q} = \tau_{\alpha+1}$). Note that $\mathbb{Q}_\alpha$ is the collection of $p \in \mathbb{Q}$ such that it is not the case that $p(\langle\rangle) = 1$. We let

$$P = \left\{x \in 2^{\mathscr{L}_\alpha} \,:\, T^x(\langle\rangle) = 1\right\},$$

which is Π^0_α by Lemma 2.3. The set P is τ_α-closed since it equals the intersection of the sets

$$\left\{x \in 2^{\mathscr{L}_\alpha} \,:\, T^x(\langle k\rangle) = 0\right\},$$

each of which is τ_α-clopen. The set P is τ_α-nowhere dense since any $p \in \mathbb{Q}_\alpha$ can be extended to some $p' \in \mathbb{Q}$ with $p'(\langle\rangle) = 0$; so every τ_α-open set intersects the complement of P. Note that for $q \in \mathbb{Q}_\alpha$, $[q]$ meets P if and only if $q(\langle k\rangle) = 0$ whenever $q(\langle k\rangle)\downarrow$.

Let $B = [p]$ for $p \in \mathbb{Q}_\alpha$ be a basic τ_α-open set that meets P. Let D be $\mathbf{\Pi}^0_{<\alpha}$; let φ be a $\Pi_{<\alpha}$ Borel code of D (that is, $D = [\varphi]$). We suppose that $P \cap B \cap D$ is comeagre in $(P \cap B, \tau_\alpha)$. This means that for any $q \in \mathbb{Q}_\alpha$ extending p, if $[q]$ intersects P then there is some r extending q in $\mathbb{Q}_\alpha$ which also meets P and strongly forces φ. Then

$$B' = [p] \cup \bigcup \{[q] \,:\, q \in \mathbb{Q}_\alpha, q \leqslant p, [q] \cap P = \varnothing, \text{ and } q \Vdash^* \varphi\}$$

is as required. □

3. A weak dichotomy

Let $\alpha \geqslant 1$ be a computable ordinal. We let

$$\mathbb{X}_\alpha = 2^{\mathscr{L}_\alpha} \times \mathscr{P}(\mathbb{N})$$

(both factors can be identified with Cantor space).

Definition 3.1. The directed graph K_α on $\mathbb{X}_\alpha$ is defined as follows: a pair (x, y) is connected to a pair (x', y') if $x' = x$, $T^x(\langle\rangle) = 0$, and for the least n such that $T^x(\langle n\rangle) = 1$, we have $y \triangle y' = \{n\}$, and $n \notin y$.

The directed graph K_α is Borel, indeed the collection of edges is Σ^0_α. Note that K_α is the graph of a bijection between two disjoint Σ^0_α sets.

Theorem 3.2. *Let G be a Σ^1_1 directed graph on a computably presented Polish space Y. The following are equivalent:*

(1) *There is a homomorphism $f\colon (\mathbb{X}_\alpha, K_\alpha) \to (Y, G)$ which is continuous when both spaces are equipped with the topology generated by the $\Sigma^0_\alpha(\Delta^1_1)$ sets.*[2]
(2) *There is a homomorphism $f\colon (\mathbb{X}_\alpha, K_\alpha) \to (Y, G)$ such that for every $A \in \Sigma^0_\alpha(\Delta^1_1)$, $f^{-1}(A)$ is $\mathbf{\Sigma}^0_\alpha$.*

[2]That is, sets that are Σ^0_α relative to some Δ^1_1 parameter.

(3) *There is no countable $\mathbf{\Sigma}^0_\alpha$ colouring of G.*

For $\alpha = 1$, we need to require that Y be 0-dimensional.

We remark that in conditions (1) and (2), we can replace the topology generated by the $\Sigma^0_\alpha(\Delta^1_1)$ sets by the topology generated by sets that are both Σ^1_1 and $\mathbf{\Pi}^0_{<\alpha}$.

Most of the work is in the following:

Theorem 3.3. *There is no countable $\mathbf{\Sigma}^0_\alpha$ colouring of K_α.*

Given this result, we can prove the weak dichotomy theorem.

Proof of Theorem 3.2. (1) $\Longrightarrow$ (2) is immediate.

(2) $\Longrightarrow$ (3) follows from Theorem 3.3, and [LZ14a, Theorem 2.1], which shows that if (3) fails, then there is a $\Sigma^0_\alpha(\Delta^1_1)$ colouring of G.

For the rest, suppose that (3) holds. By Proposition 2.4 of [LZ14a], there is some point $p \in Y$ which is an accumulation point of edges of G in the $\Sigma^0_\alpha(\Delta^1_1)$ topology: say $a_n, b_n \to p$ and $(a_n, b_n) \in G$. Define the following map F from $\mathbb{X}_\alpha$ to Y: for $(x, y) \in \mathbb{X}_\alpha$

- If $T^x(\langle\rangle) = 1$ then $F(x, y) = p$.
- If $T^x(\langle\rangle) = 0$, let n be least such that $T^x(\langle n\rangle) = 1$.
  - If $n \notin y$, let $F(x, y) = a_n$.
  - If $n \in y$, let $F(x, y) = b_n$.

The range of F is a countable set, and for any $\Sigma^0_\alpha(\Delta^1_1)$-open set A, $F^{-1}[A]$ has one of two forms. For $n < \omega$ and $i < 2$ let

$$B_{n,i} = \{(x, y) : n \text{ least s.t. } T^x(\langle n\rangle) = 1 \ \& \ y(n) = i\}.$$

Each set $B_{n,i}$ is $\Pi^0_{<\alpha}(\Delta^1_1)$.

- If $p \notin A$, then $F^{-1}[A]$ is the union of sets among $B_{n,i}$.
- Otherwise, $F^{-1}[A]$ is the complement of the union of finitely many of the $B_{n,i}$.

Thus, $F^{-1}[A]$ is $\Sigma^0_\alpha(\Delta^1_1)$ whenever A is $\Sigma^0_\alpha(\Delta^1_1)$. □

It remains to prove Theorem 3.3. The proof is an elaboration on the forcing argument of the previous section. We force with $\mathbb{Q}\times$Cohen.

Definition 3.4. We let $\mathbb{P} = \mathbb{Q} \times 2^{<\omega}$. For $p = (u, \zeta)$ and $q = (v, \xi)$ in $\mathbb{P}$, we write $q \leqslant p$ (q extends p) if $u \subseteq v$ and $\zeta \preccurlyeq \xi$.

When $p \in \mathbb{P}$ we often write $p = (u^p, \zeta^p)$. For $\zeta \in 2^{<\omega}$ we let $[\zeta] = \{y \in 2^\omega : \zeta \prec y\}$, and for $p \in \mathbb{P}$ we let

$$[p] = [u^p] \times [\zeta^p] \subseteq \mathbb{X}_\alpha.$$

The notions of a filter of $\mathbb{P}$ and a dense subset of $\mathbb{P}$ are defined as before. For a filter G of $\mathbb{P}$ we define

$$x_G = \bigcup \{u^p \restriction \mathscr{L}_\alpha : p \in G\}$$

and

$$y_G = \bigcup \{\zeta^p : p \in G\}.$$

The proof of Lemma 2.11 gives its analogue for $\mathbb{P}$: there is a countable collection $\mathscr{D}_{\text{point}}(\mathbb{P})$ of dense subsets of $\mathbb{P}$, such that for any $\mathscr{D}_{\text{point}}(\mathbb{P})$-generic filter G,

$(x_G, y_G) \in \mathbb{X}_\alpha$, and for all $p \in \mathbb{P}$,

$$(x_G, y_G) \in [p] \iff p \in G.$$ [3]

We obtain the same characterisation of dense open and comeagre subsets of $\mathbb{X}_\alpha$, equipped with the product topology $\tau_\mathbb{P}$ (the product of $\tau_\mathbb{Q}$ and the usual topology on $\mathscr{P}(\mathbb{N})$).

We define Borel codes for subsets of $\mathbb{X}_\alpha$ analogously to the definition above; the only difference is in the clopen level, where a code is a pair $p = (u, \zeta) \in \mathbb{P}$ such that $\operatorname{dom} u \subset \mathscr{L}_\alpha$, and its interpretation is $[p]$ as defined above. The strong forcing relation $p \Vdash^* \varphi$ is defined exactly as in Definition 2.19. Lemma 2.20 holds. The forcing theorem, Proposition 2.21, holds, with the same proof.

Fix some β with $0 < \beta \leqslant \alpha + 1$. We define

$$\mathbb{P}_\beta = \mathbb{Q}_\beta \times 2^{<\omega},$$

and for $p \in \mathbb{P}$, we let

$$p{\upharpoonright}\beta = (u^p{\upharpoonright}\beta, \zeta^p).$$

Immediately from Lemma 2.24 we get its analogue for $\mathbb{P}$: $p{\upharpoonright}\beta \in \mathbb{P}_\beta$ and $p \leqslant p{\upharpoonright}\beta$. We say that p is *β-complete* if u^p is β-complete (Definition 2.25). Lemma 2.26 implies its analogue for $\mathbb{P}$. We get an analogue of Lemma 2.27:

Lemma 3.5. *Suppose that $p \in \mathbb{P}$ is β-complete and r extends $p{\upharpoonright}(\beta+1)$. Then p and $(r{\upharpoonright}\beta)$ are compatible in $\mathbb{P}$.*

Proof. By Lemma 2.27, there is some $v \in \mathbb{Q}$ that extends both u^p and $u^r{\upharpoonright}\beta = u^{r{\upharpoonright}\beta}$ (namely $v = u^p \cup (u^r{\upharpoonright}\beta)$). Then (v, ζ^r) extends both p and $r{\upharpoonright}\beta$ (note that $\zeta^r \succcurlyeq \zeta^p$ follows from $r \leqslant p{\upharpoonright}(\beta+1)$). □

Finally, we obtain the untagging lemma for $\mathbb{P}$. The proofs are identical, and so we obtain:

Lemma 3.6. *Suppose that $0 < \beta \leqslant \alpha + 1$, and that φ is a Σ_β code for a subset of $\mathbb{X}_\alpha$. For every sufficiently generic $G \subset \mathbb{P}$,*

$$(x_G, y_G) \in [\varphi] \iff (\exists p \in G \cap \mathbb{P}_\beta)\ p \Vdash^* \varphi.$$

Proof of Theorem 3.3. Let $\mathscr{C}$ be a countable collection of $\mathbf{\Sigma}^0_\alpha$ sets. Let $\mathscr{D}$ be a countable collection of dense subsets of $\mathbb{P}$, so that for every set $C \in \mathscr{C}$ there is a Σ_α code φ of C such that for any $\mathscr{D}$-generic filter $G \subset \mathbb{P}$,

$$(x_G, y_G) \in C \iff (\exists p \in G \cap \mathbb{P}_\alpha)\ p \Vdash^* \varphi.$$

Let p^* be the condition defined by ζ^{p^*} being the empty string, $\operatorname{dom} u^{p^*} = \{\langle\rangle\}$, and $u^{p^*}(\langle\rangle) = 1$. By the Rasiowa-Sikorski lemma, let G be a $\mathscr{D}$-generic filter containing p^*.

We claim that for every set $C \in \mathscr{C}$, if $(x_G, y_G) \in C$ then there are two points in C connected by an edge of K_α. Thus, $\mathscr{C}$ cannot be a partition of $\mathbb{X}_\alpha$ into K_α-independent sets. Note though that (x_G, y_G) itself is not part of an edge of K_α.

Let $C \in \mathscr{C}$ and suppose that $(x_G, y_G) \in C$. Let φ be a Σ_α code for C such that there is some $p \in G \cap \mathbb{P}_\alpha$ that strongly forces φ. Since p is compatible with p^*, $u^p(\langle k\rangle) = 0$ whenever defined. Since $p \in \mathbb{P}_\alpha$, it is not the case that $u^p(\langle\rangle) = 1$.

[3] To be specific, $\mathscr{D}_{\text{point}}(\mathbb{P})$ consists of the sets $\{p : \sigma \in \operatorname{dom} u^p\}$ for $\sigma \in T_\alpha$; the sets $\{p : |\zeta^p| \geqslant m\}$ for all $m \in \mathbb{N}$; and the sets $\{q : q \leqslant p,$ or q and p are incompatible$\}$ for $p \in \mathbb{P}$.

Let n be large, so $n > |\zeta^p|$ and every $\tau \in \operatorname{dom} u^p$ extends $\langle k\rangle$ for some $k < n$. Define p' extending p by setting $u^{p'}(\langle\rangle) = 0$, $u^{p'}(\langle k\rangle) = 0$ for all $k < n$, and $u^{p'}(\langle n\rangle) = 1$. Now define two conditions q_0 and r_0, both extending p', by setting $|\zeta^{q_0}| = |\zeta^{r_0}| = n+1$, $\zeta^{q_0}\restriction n = \zeta^{r_0}\restriction n$, and $\zeta^{q_0}(n) = 0$ while $\zeta^{r_0}(n) = 1$; $u^{q_0} = u^{r_0} = u^{p'}$.

We now define two filters Q and R starting with $q_0 \in Q$ and $r_0 \in R$. They will both be $\mathscr{D}$-generic, but not mutually so. Indeed, we will let Q be the upward closure of a decreasing sequence (q_ℓ) of conditions, and R be the upward closure of a sequence (r_ℓ) of conditions, and for each ℓ we will ensure:

- $u^{q_\ell} = u^{r_\ell}$; and
- $|\zeta^{q_\ell}| = |\zeta^{r_\ell}|$, and for all $m < |\zeta^{r_\ell}|$ other than n, $\zeta^{q_\ell}(m) = \zeta^{r_\ell}(m)$.

This is done by an interleaving construction: suppose that q_ℓ and r_ℓ have been determined. We first extend q_ℓ to some q'_ℓ meeting the ℓ^{th} dense set in $\mathscr{D}$. We let r'_ℓ extend r_ℓ by copying over the new values of q'_ℓ. We extend r'_ℓ to $r_{\ell+1}$ in the same dense set, and then copy the new values of $r_{\ell+1}$ to define $q_{\ell+1}$. This construction ensures that (x_Q, y_Q) is connected by an edge to (x_R, y_R). Both points lie in C, as both filters contain p. □

Using topological language. As discussed, one of our aims is to help bridge the gap between practitioners more comfortable with forcing, and ones more comfortable with topology. We therefore give a translation of the proof above of Theorem 3.3, using topological notions.

The main idea of the proof is passing between the two topologies, τ_α and $\tau_{\mathbb{P}} = \tau_{\alpha+1}$. We abuse notation by allowing τ_α to refer both to the topology on $2^{\mathscr{L}_\alpha}$ and on $\mathbb{X}_\alpha$ (using the product with the usual topology on $\mathscr{P}(\mathbb{N})$). As above, let

$$P = \left\{x \in 2^{\mathscr{L}_\alpha} \,:\, T^x(\langle\rangle) = 1\right\}.$$

As discussed in the proof of Theorem 2.33, according to τ_α, P is closed and nowhere dense, whereas it is clopen according to $\tau_{\alpha+1}$.

Suppose that $\mathscr{C}$ is a partition of $\mathbb{X}_\alpha$ into $\mathbf{\Sigma}^0_\alpha$ sets. Since P is $\tau_{\alpha+1}$-clopen, there is some $C \in \mathscr{C}$ such that $C \cap (P \times \mathscr{P}(\mathbb{N}))$ is $\tau_{\alpha+1}$-nonmeagre. Let $\tilde{p} \in \mathbb{P}$ such that $[u^{\tilde{p}}] \subseteq P$ and C is $\tau_{\alpha+1}$-comeagre in $[\tilde{p}]$.

The untagging lemma for $\mathbb{P}$ ensures that there is some $p \in \mathbb{P}_\alpha$ such that $[u^p] \cap P \neq \varnothing$, and such that C is τ_α-comeagre in $[p]$. Since $[u^p] \cap P \neq \varnothing$, we can extend p to q_0 and r_0 as in the proof of Theorem 3.3. Letting n be the same as in that proof, the map $(x, y) \mapsto (x, y \cup \{n\})$ is a homeomorphism between $[q_0]$ and $[r_0]$. Since C is τ_α-comeagre in both $[q_0]$ and $[r_0]$, there is some $(x, y) \in [q_0] \cap C$ such that $(x, y \cup \{n\}) \in C$ as well, showing that C contains an edge of K_α.

4. "Smaller" non-colourable graphs

The graph K_3 is not a least graph with no $\mathbf{\Sigma}^0_3$ colouring, with respect to continuous homomorphisms. To explain why, we define a family of graphs L_α for $\alpha \geqslant 3$ that are "smaller" in some sense than the graphs K_α. We will show that L_α has no countable $\mathbf{\Sigma}^0_\alpha$ colouring, and that there is no continuous homomorphism from K_3 to L_3.

4.1. **The graphs L_α.** Fix $\alpha \geqslant 3$. This implies that we may assume that no $\sigma \in T_\alpha$ of height $\leqslant 2$ is a leaf of T_α.[4]

Below, to avoid excess notation, for $x \in 2^{\mathscr{L}_\alpha}$, we write $T^x(m)$ instead of $T^x(\langle m\rangle)$, $T^x(m,k)$ instead of $T^x(\langle m,k\rangle)$, etc. We similarly write $\mathrm{rk}(m)$, $\mathrm{rk}(m,k)$, and so on.

Definition 4.1. Let $x \in 2^{\mathscr{L}_\alpha}$.

(a) We let $n^x = \sup\{n : (\forall m < n)\ T^x(m) = 0\}$.
(b) For $m < n^x$ we let $k^x(m)$ be the least k such that $T^x(m,k) = 1$.

Thus, if $T^x(\langle\rangle) = 0$, then n^x is the least n such that $T^x(n) = 1$, so $(x,y) \sim (x,y')$ in K_α exactly when $y \triangle y' = \{n^x\}$. If $T^x(\langle\rangle) = 1$ then $n^x = \omega$.

Definition 4.2. We let (M_k) be a uniformly computable and dense list of elements of Cantor space.

We think of each number k as a "code" of every finite initial segment of M_k.

Definition 4.3. We let $\mathbb{W}_\alpha$ be the collection of $(x,y) \in \mathbb{X}_\alpha$ such that for all $m < n^x$, $k^x(m)$ codes $y\restriction(m+1)$ (meaning that $y\restriction(m+1) \prec M_{k^x(m)}$).

We let L_α be the restriction of K_α to $\mathbb{W}_\alpha$.

Definition 4.4. We let $\alpha^* = \alpha - 2$. More precisely,

$$\alpha^* = \begin{cases} \alpha - 2, & \text{if } \alpha \text{ is the successor of a successor;} \\ \alpha - 1, & \text{if } \alpha \text{ is the successor of a limit ordinal;} \\ \alpha, & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Proposition 4.5. *$\mathbb{W}_\alpha$ is $\Pi^0_2(\tau_{\alpha^*})$.*

Proof. For all m and k, the collection of $x \in 2^{\mathscr{L}}$ such that $T^x(m,k) = 0$ is $\Sigma^0_1(\tau_{\alpha^*})$. For each finite tuple $\bar{k} = (k_0, k_1, \dots, k_m)$ of natural numbers, let $A_{\bar{k}}$ be the collection of $(x,y) \in \mathbb{X}_\alpha$ such that:

- there is some $\ell \leqslant m$ such that $T^x(\ell, k_\ell) = 0$; or
- there is some $\ell \leqslant m$ and some $k < k_\ell$ such that $T^x(\ell,k) = 1$; or
- for all $\ell \leqslant m$, k_ℓ is a code of $y\restriction(\ell+1)$.

then $A_{\bar{k}}$ is $\Delta^0_2(\tau_{\alpha^*})$, and $\mathbb{W}_\alpha$ is the intersection of all the sets $A_{\bar{k}}$. □

In particular, $\mathbb{W}_3$ is a Π^0_2 set, and so equipped with the subspace topology is Polish.

4.2. **Non-colourability of L_α.** We cannot use the notion of forcing $\mathbb{P}$ above to always obtain points in $\mathbb{W}_\alpha$; indeed, $\mathbb{W}_\alpha \cap \{(x,y) : T^x(\langle\rangle) = 1\}$ is meagre in the $\tau_{\alpha+1}$ topology.

Definition 4.6. For $u \in \mathbb{Q}$ we let

$$n^u = \max\{n : (\forall m < n)\ u(m) = 0\}.$$

For each $m < n^u$ we let

$$k^u(m) = \min\{k : u(m,k) = 1\}.$$

[4] We may assume that for all limit $\delta \leqslant \alpha$, every element δ_k of the cofinal sequence given by the computable presentation of α, is a successor of a successor ordinal. Hence, for example, if α is a limit ordinal, then none of the children of the root are leaves, nor can nodes of height 2 be leaves.

Here we use the same notation as above: $u(m) = u(\langle m\rangle)$, $u(m,k) = u(\langle m,k\rangle)$, etc.

Definition 4.7. We let $\tilde{\mathbb{Q}}$ be the collection of $u \in \mathbb{Q}$ satisfying:

(i) if $u(n^u)\uparrow$ then for all $m > n^u$, $u(m)\uparrow$;
(ii) There is no $m \geqslant n^u$ such that $u(m) = 0$;
(iii) For all $m < n^u$, for all $k < k^u(m)$, $u(m,k)\downarrow$;
(iv) For all $m < m' < n^u$, $\mathrm{rk}(m, k^u(m)) \leqslant \mathrm{rk}(m', k^u(m'))$.

Note that requirement (iv) in the definition is only relevant when α is the successor of a limit ordinal; otherwise, it holds automatically. Note that for $u \in \tilde{\mathbb{Q}}$, $u(n^u)\downarrow$ if and only if $u(\langle\rangle) = 0$. If $u(n^u)\uparrow$ we can have either $u(\langle\rangle)\uparrow$ or $u(\langle\rangle) = 1$.

Lemma 4.8. *Suppose that $u, v \in \tilde{\mathbb{Q}}$ and $u \cup v$ is a function; then $u \cup v \in \tilde{\mathbb{Q}}$.*

Proof. Lemma 2.5 implies that $u \cup v \in \mathbb{Q}$; the conditions for being in $\tilde{\mathbb{Q}}$ are easily verified. ☐

Lemma 4.9. *Let $u \in \tilde{\mathbb{Q}}$. For all $\beta \leqslant \alpha + 1$, $u\restriction\beta \in \tilde{\mathbb{Q}}$; $n^{u\restriction\beta} \leqslant n^u$, and for all $m < n^{u\restriction\beta}$, $k^{u\restriction\beta}(m) = k^u(m)$.*

Proof. We use the fact that for all σ and $k < k'$, $\mathrm{rk}(\sigma\hat{\ }k) \leqslant \mathrm{rk}(\sigma\hat{\ }k')$. This implies that for all $m < n^u$, $(u\restriction\beta)(m)\downarrow$ if and only if $\mathrm{rk}(m, k^u(m)) < \beta$.[5] Requirement (iv) implies that this is an initial segment of $m < n^u$. If $n^{u\restriction\beta} < n^u$ then for all $m \geqslant n^u$, $(u\restriction\beta)(m)\uparrow$, since $\mathrm{rk}(m) \geqslant \beta$ and $u(m) \neq 0$. ☐

Definition 4.10. We let $\mathbb{S}$ be the collection of pairs $(u,\zeta) \in \tilde{\mathbb{Q}} \times 2^{<\omega}$ such that $|\zeta| \geqslant n^u$, and for all $m < n^u$, $k^u(m)$ is a code of $\zeta\restriction(m+1)$.

The collection of conditions $\mathbb{S}$ is partially ordered by co-ordinatewise extension, just like $\mathbb{P}$ (it is a sub-ordering of $\mathbb{P}$).

Lemma 4.11.

(a) *The condition p^* defined by $\operatorname{dom} u^{p^*} = \{\langle\rangle\}$, $u^{p^*}(\langle\rangle) = 1$, and $\zeta^{p^*} = \langle\rangle$ is in $\mathbb{S}$.*

Suppose that $(u,\zeta) \in \mathbb{S}$.

(b) *For all $\xi \succcurlyeq \zeta$, $(u,\xi) \in \mathbb{S}$.*
(c) *Suppose that $u(n^u) = 1$. Then for all $v \supseteq u$ in $\tilde{\mathbb{Q}}$, $(v,\zeta) \in \mathbb{S}$.*
(d) *Suppose that $u(\langle\rangle)\uparrow$. Let v extend u by defining $v(n^u) = 1$ (and $v(\langle\rangle) = 0$). Then $(v,\zeta) \in \mathbb{S}$.*
(e) *Suppose that $u(n^u)\uparrow$. Then there is some $v \supseteq u$ in $\tilde{\mathbb{Q}}$ such that $n^v = |\zeta|$, $v(n^v)\uparrow$, and $(v,\zeta) \in \mathbb{S}$.*

Proof. For (c), the point is that $n^v = n^u$, so there are no further coding requirements on (v,ξ). For (d), note that $u(\langle\rangle)\uparrow$ implies $u(n^u)\uparrow$; we again get $n^v = n^u$.

For (e): let $l = |\zeta|$. For each m with $n^u \leqslant m < l$, choose some k_m that codes ζ, and such that for $m < n^u \leqslant m' < m'' < l$ we have $\mathrm{rk}(m, k^u(m)) \leqslant \mathrm{rk}(m', k_{m'}) \leqslant \mathrm{rk}(m'', k_{m''})$. Extend u to v by setting, for all m with $n^u \leqslant m < l$:

- $v(m) = 0$, and $v(m, k_m) = 1$; and

[5] In detail: if $\mathrm{rk}(m, k^u(m)) < \beta$ then since $u(m, k^u(m)) = 1$, we have $(u\restriction\beta)(m)\downarrow$. If $(u\restriction\beta)(m)]\downarrow$, then either $\mathrm{rk}(m) < \beta$, in which case $\mathrm{rk}(m,k) < \beta$ for all k; or there is some k such that $\mathrm{rk}(m,k) < \beta$ and $u(m,k) = 1$. The minimality of $k^u(m)$ implies $k^u(m) \leqslant k$, so $\mathrm{rk}(m, k^u(m)) \leqslant \mathrm{rk}(m,k) < \beta$.

- for all $k < k_m$, $v(m,k) = 0$, and $v(m,k,\ell) = 1$ for some large ℓ.

Then v is as required. □

Lemma 4.12. *For all $\sigma \in T_\alpha$, the collection of $q \in \mathbb{S}$ such that $u^q(\sigma)\downarrow$ is dense in $\mathbb{S}$.*

Proof. Let $p = (u,\zeta) \in \mathbb{S}$ and let σ such that $u(\sigma)\uparrow$. If $\sigma = \langle\rangle$, then (d) of Lemma 4.11 allows us to extend p to $(v,\zeta) \in \mathbb{S}$ with $v(\langle\rangle)\downarrow$. Suppose that $\sigma \neq \langle\rangle$.

If $u(n^u) = 1$, then (c) allows us to easily extend p to $(v,\zeta) \in \mathbb{S}$ with $v(\sigma)\downarrow$. Suppose that $u(n^u)\uparrow$. Let m such that $\langle m\rangle \preccurlyeq \sigma$. By (b), we may assume that $|\zeta| > m$; by (e), we may assume that $n^u = |\zeta|$, so $m < n^u$. Hence, if $\sigma = \langle m\rangle$, we are done. If $\sigma = \langle m,k\rangle$ for some k, and $k \leqslant k^u(m)$, we are also done. In all other cases, we may extend u to v by defining $v(\sigma) = 0$ and $v(\sigma\hat{\ }l) = 1$ for some large l, and keep $(v,\zeta) \in \mathbb{S}$, since $n^v = n^u$. □

Lemma 4.13. *Conditions $p,q \in \mathbb{S}$ are compatible if and only if $u^p \cup u^q$ is a function and ζ^p, ζ^q are comparable.*

Proof. Suppose that $v = u^p \cup u^q$ is a function and ζ^p, ζ^q are comparable; without loss of generality, $\zeta^p \preccurlyeq \zeta^q$. By Lemma 4.8, $v \in \tilde{\mathbb{Q}}$. To see that $(v,\zeta^q) \in \mathbb{S}$, let $m < n^v$. Then either $m < n^{u^q}$ or $m < n^{u^p}$. If the former, then $k^v(m) = k^{u^q}(m)$ codes $\zeta^q\restriction(m+1)$, as $q \in \mathbb{S}$. If the latter, then $k^v(m) = k^{u^p}(m)$ codes $\zeta^p\restriction(m+1)$, as $p \in \mathbb{S}$, but $\zeta^p \preccurlyeq \zeta^q$. □

For sufficiently generic $G \subset \mathbb{S}$, we define (x_G, y_G) as above.

Lemma 4.14. *If $G \subset \mathbb{S}$ is sufficiently generic, then $(x_G,y_G) \in \mathbb{W}_\alpha$, and for all $p \in \mathbb{S}$, $(x_G,y_G) \in [p]$ if and only if $p \in G$.*

Proof. Lemma 4.12 implies that for a sufficiently generic G, $\bigcup\{u^p : p \in G\}$ is defined on all of T_α (so in particular, $x_G \in 2^{\mathscr{L}_\alpha}$), and $y_G \in 2^\omega$. Hence, $(x_G,y_G) \in \mathbb{X}_\alpha$; the proof that $p \in G$ iff $(x_G,y_G) \in [p]$ is as for Lemma 2.11, using Lemma 4.13.

To show that in fact $(x_G,y_G) \in \mathbb{W}_\alpha$, let $m < n^{x_G}$. There is some $p = (u,\zeta) \in G$ such that $m < n^u$; the fact that $(u,\zeta) \in \mathbb{S}$ and $\zeta \prec y_G$ implies that $k^{x_G}(m) = k^u(m)$ codes $y_G\restriction(m+1)$. □

The definition of a β-complete condition is the same as for $\mathbb{P}$.

Lemma 4.15. *For every $\beta \leqslant \alpha+1$, every $p \in \mathbb{S}$ can be extended to a β-complete condition in $\mathbb{S}$.*

Proof. Write $p = (u,\zeta)$. Suppose that $p(\sigma) = 1$, and $\mathrm{rk}(\sigma) > \beta$ is a limit. If $|\sigma| \geqslant 1$, then adding 0-labels to those $\sigma\hat{\ }k$ of rank $< \beta$ (as in the proof of Lemma 2.26) does not affect n^u, and hence being in $\mathbb{S}$. If σ is the root $\langle\rangle$, we apply Lemma 4.11 to add 0-labels to finitely many children of the root (first extend ζ by (b), then extend as in (e)). □

Lemma 4.9 implies that for all $p \in \mathbb{S}$ and $\beta \leqslant \alpha$, $p\restriction\beta \in \mathbb{S}$. We obtain the analogue of Lemma 3.5:

Lemma 4.16. *Suppose that $p \in \mathbb{S}$ is β-complete and $r \in \mathbb{S}$ extends $p\restriction(\beta+1)$. Then p and $(r\restriction\beta)$ are compatible in $\mathbb{S}$.*

Proof. As in the proof of Lemma 3.5, let $v = u^p \cup (u^r\restriction\beta)$. By Lemma 2.27, $v \in \mathbb{Q}$. By Lemma 4.8, $v \in \tilde{\mathbb{Q}}$. By Lemma 4.13, $(v,\zeta^r) \in \mathbb{S}$. □

Again, the proof of the untagging lemma is the same, so we obtain the analogue of Proposition 2.30 and Lemma 3.6: if φ is Σ_β and $G \subset \mathbb{S}$ is sufficiently generic, then $(x_G, y_G) \in [\varphi] \iff (\exists p \in G \cap \mathbb{S}_\beta)\ p \Vdash^* \varphi$, where $\mathbb{S}_\beta = \mathbb{P}_\beta \cap \mathbb{S}$.

Theorem 4.17. *Let $\alpha \geqslant 3$. There is no countable $\mathbf{\Sigma}^0_\alpha$-colouring of L_α.*

Proof. Follow the proof of Theorem 3.3, using Lemma 4.11. By (a) of the lemma, we may start with the condition p^* described. We extend this condition to a sufficiently generic $G \subset \mathbb{S}$. By the untagging lemma, we obtain $p \in G \cap \mathbb{S}_\alpha$ that strongly forces into one of the $\mathbf{\Sigma}^0_\alpha$ sets $C \in \mathscr{C}$. Writing $p = (u, \zeta)$, we have $u(\langle\rangle)\uparrow$, so $u(n^u)\uparrow$. By (e), we may assume that $|\zeta| = n^u$. Extend u to v by setting $v(n^u) = 1$ and $v(\langle\rangle) = 0$. By (d) and (b), $q_0 = (v, \zeta\hat{\ }0)$ and $r_0 = (v, \zeta\hat{\ }1)$ are both in $\mathbb{S}$. We then use (c) to extend q_0 and r_0 to filters Q and R as in the proof of Theorem 3.3, to show that the set C contains an edge of L_α. □

Remark 4.18. In our coding scheme, instead of using a number k to code all initial segments of a real M_k, we could instead have fixed a single finite binary string coded by k (as long as each finite string has infinitely many codes). We will require our more flexible coding scheme in the next section.

4.3. **Approximating values at heights 1 and 2.** Our next goal is Proposition 4.23 below: there is no continuous embedding of K_3 into L_3. To prove this, we will diagonalise against a given continuous function from $\mathbb{X}_3$ to $\mathbb{W}_3$. In this kind of argument, we construct approximations to various points in $\mathbb{X}_3$, and somehow ensure that there is a pair of points a, b that we construct that is connected by an edge in K_3, but such that $F(a)$ and $F(b)$ cannot be connected by an edge in L_3. To do that, we will need to observe, during the construction, approximations to points such as $F(a)$ and $F(b)$ built by our opponent, and make guesses about whether they are connected by an edge or not. We now describe this guessing procedure.

Recall that an isomorphism between $2^{\mathscr{L}_\alpha}$ and 2^ω can be determined by an ω-ordering of the leaves of T_α. Fix such a computable ordering $<^*$. This gives us a notion of "initial segments" of elements of $2^{\mathscr{L}_\alpha}$:

Definition 4.19. We let $2^{<\mathscr{L}_\alpha}$ be the collection of all finite $r\colon \mathscr{L}_\alpha \to \{0,1\}$ such that if $\sigma <^* \tau$ and $r(\tau)\downarrow$ then $r(\sigma)\downarrow$.

For the rest of this section, we fix $\alpha = 3$; we will later generalise the guessing machinery that we develop now for $\alpha = 3$. We now define, for each $t \in 2^{<\mathscr{L}_3}$,

- n^t; and
- For $m < n^t$, $k^t(m)$,

that will serve as an approximation for n^x and $k^x(m)$ for $x \in 2^{\mathscr{L}_3}$ extending t. The idea is similar to Dekker's "deficiency stages". Suppose that at the previous stage, we are guessing that $k^x(m)$ is some k, but we have just discovered that this is wrong: there is some a such that $t(m,k,a) = 1$, so we cannot have $T^x(m,k) = 1$ for any $x \succ t$. Then $k^t(m)$ is increased compared to the previous stage, telling us that there's a chance it will go to ∞. That is, at that stage, we are guessing that $T^x(m) = 1$, so $n^x \leqslant m$. We therefore arrange that $n^t \leqslant m$ as well.

Definition 4.20. For $t \in 2^{<\mathscr{L}_3}$ we define numbers n^t, and for $m < n^t$, numbers $k^t(m)$, by recursion on $|t|$. If $t = \langle\rangle$ then we set $n^t = 0$.

Suppose that $t \neq \langle\rangle$; let t^- be the initial segment of t of length $|t| - 1$.

- If there is some $m < n^{t^-}$ and some a such that $t(m, k, a) = 1$, where $k = k^{t^-}(m)$, then we let n^t be the least such m.
- Otherwise, we let $n^t = n^{t^-} + 1$.

Then, for each $m < n^t$, we let $k^t(m)$ be the least k such that there is no a with $t(m, k, a) = 1$.

Lemma 4.21. *For all $x \in 2^{\mathscr{L}_3}$,*

$$n^x = \liminf \left\{n^t \, : \, t \prec x\right\}.$$

Proof. Let $x \in 2^{\mathscr{L}_3}$. By induction on $m < n^x$ we show that for all but finitely many $t \prec x$ we have $n^t > m$ and $k^t(m) = k^x(m)$. Suppose that $t_0 \prec x$ is such that for all t with $t_0 \preccurlyeq t \prec x$, we have $n^t \geqslant m$ and for all $m' < m$, $k^t(m') = k^x(m')$. Let t_1 such that $t_0 \prec t_1 \prec x$ and for all $k < k^x(m)$ there is some a such that $t_1(m, k, a) = 1$. Then for all t with $t_1 \preccurlyeq t \prec x$, if $n^t > m$ then $k^t(m) = k^x(m)$. This implies that for all t with $t_1 \prec t \prec x$ we have $n^t > m$.

On the other hand, for any $s \prec x$ there is some t with $s \preccurlyeq t \prec x$ and $n^t \leqslant n^x$. For suppose that $n^s > n^x$; let $k = k^s(n^x)$. Since $T^x(n^x, k) = 0$, there is some $t \prec x$ with $t(n^x, k, a) = 1$ for some a. For the least such t we must have $s \prec t$. Suppose that for all r with $s \preccurlyeq r \prec t$ we have $n^r > n^x$. Then the minimality of t implies that $k^{t^-}(n^x) = k$; now the definition implies that $n^t \leqslant n^x$. ☐

Lemma 4.22. *Let $s \in 2^{<\mathscr{L}_3}$.*

(a) *There is some $t \succ s$ in $2^{<\mathscr{L}_3}$ such that $n^t = 0$.*
(b) *For all $m \geqslant n^s$, there is some $t \succ s$ in $2^{<\mathscr{L}_3}$ such that $n^t = m$, and for all r with $s \preccurlyeq r \preccurlyeq t$ we have $n^r \geqslant n^s$.*

Proof. The main point is that if r is obtained from s by only adding 0's, then $n^r \geqslant n^s$ and for all $m < n^s$, $k^r(m) = k^s(m)$. For (a), keep extending s by 0's, until we get to place a 1 at a location $(0, k^s(0), a)$ for some a. For (b), keep extending s by 0's, until we get to place a 1 at a location (m, k, a), where $k = k^r(m)$, where r is a sufficiently long extension of s by adding 0's. ☐

4.4. K_3 **is not minimal.** Together with Theorem 4.17, the following implies that K_3 is not a least graph with no $\mathbf{\Sigma}^0_3$ colouring (with respect to continuous homomorphisms).

Proposition 4.23. *There is no continuous embedding of K_3 into L_3.*

Proof. Before we prove this proposition, let us explain informally how it is done. As described above, we are given a continuous function $F\colon \mathbb{X}_3 \to \mathbb{W}_3$, and our goal is to build a pair of points in $\mathbb{X}_3$ that are connected by an edge of K_3, but whose images under F are not, thus, showing that F is not a graph homomorphism.

We start by building two points a and b and plan for them to be connected in K_3: the intention will be to have $a = (z, w^a)$ and $b = (z, w^b)$, where we currently plan that $n^z = 0$ and $w^a \triangle w^b = \{0\}$ (we can set in advance $w^a = 0^\omega$ and $w^b = 1\hat{}0^\omega$). The opponent, playing F, will have to show us how $F(a)$ and $F(b)$ are connected by an edge: it will show us finite pieces of $F(a) = (x^a, y^a)$ and $F(b) = (x^b, y^b)$. To counter our move, the opponent better ensure that $x^a = x^b = x$ and $y^a \triangle y^b = \{n^*\}$ for some n^*; and then ensure that $n^x = n^*$.

The main point of the construction, our advantage over the opponent, is that since he has to ensure that $F(a), F(b) \in \mathbb{W}_3$, if he showed us this much, he cannot have $n^x > n^*$; otherwise, at least one of $F(a)$ or $F(b)$ cannot be in $\mathbb{W}_3$. Thus, when we see this information, we can defeat the opponent by creating a new point c, very close to a, and connected to a by an edge, which requires us to reset n^z to be some large number ℓ^*. This is sufficiently large so that the symmetric difference of the second coordinates of $F(a)$ and $F(c)$ must be greater than n^*, ruling out the possibility that $F(a)$ and $F(c)$ are connected by an edge.

We can now give the details of the construction. Let $F\colon \mathbb{X}_3 \to \mathbb{W}_3$ be continuous.

At stages $\ell = 0, 1, \dots$ of the construction, we define $s_\ell \in 2^{<\mathscr{L}_3}$; we will ensure that

$$s_0 \prec s_1 \prec s_2 \prec \cdots$$

For each ℓ, we let t^a_ℓ be the longest $t \in 2^{<\mathscr{L}_3}$ such that $|t| \leqslant \ell$ and

$$F[s_\ell, 0^\ell] \subseteq [t] \times 2^\omega;$$

here $[s, \xi] = [s] \times [\xi] \subseteq \mathbb{X}_3$ and $F[s, \xi]$ is the pointwise image under F of this set. We similarly let t^b_ℓ be the longest t such that $|t| \leqslant \ell$ and

$$F[s_\ell, 1\hat{\ }0^{\ell-1}] \subseteq [t] \times 2^\omega.$$

We let ζ^a_ℓ be the longest $\zeta \in 2^{<\omega}$ such that $|\zeta| \leqslant \ell$

$$F[s_\ell, 0^\ell] \subseteq [t^a_\ell, \zeta],$$

and similarly we let ζ^b_ℓ be the longest $\zeta \in 2^{<\omega}$ such that $|\zeta| \leqslant \ell$ and

$$F[s_\ell, 1\hat{\ }0^{\ell-1}] \subseteq [t^b_\ell, \zeta].$$

Note that the fact that (s_ℓ) is increasing implies that (t^a_ℓ), (t^b_ℓ), (ζ^a_ℓ), and (ζ^b_ℓ) are all (weakly) increasing with ℓ.

We start with $s_0 = \langle\rangle$. Let $\ell \geqslant 0$, and suppose that we have defined s_ℓ. There are three "phases" that we consider.

Phase 1: t^a_ℓ and t^b_ℓ are comparable, and ζ^a_ℓ and ζ^b_ℓ are also comparable. We let $s_{\ell+1}$ be an extension of s_ℓ satisfying $n^{s_{\ell+1}} = 0$ (use Lemma 4.22).

Phase 2: t^a_ℓ and t^b_ℓ are comparable, and ζ^a_ℓ and ζ^b_ℓ are incomparable. In this case, let $\ell^* \leqslant \ell$ be the least stage at which this case applied. We will ensure that for all ℓ, $n^{s_\ell} \leqslant \ell^*$.[6] Extend s_ℓ to $s_{\ell+1} \in 2^{<\mathscr{L}_3}$ so that $n^{s_{\ell+1}} = \ell^*$, and $n^r \geqslant n^{s_\ell}$ for all r with $s_\ell \preccurlyeq r \preccurlyeq s_{\ell+1}$ (again, this is possible by Lemma 4.22).

Phase 3: t^a_ℓ and t^b_ℓ are incomparable. We let $s_{\ell+1}$ be an extension of s_ℓ satisfying $n^{s_{\ell+1}} = 0$.

This completes the construction of the sequence (s_ℓ). The "phases" are named so because we never "go back": when the construction starts, phase 1 applies. If we ever move to phase 2, we never later go back to phase 1. If we ever move to phase 3, we never later go back to phase 1 or phase 2.

We now argue that there are two points in $\mathbb{X}_3$ that are connected by an edge of K_3, but whose images under F are not connected by an edge.

Let $z = \bigcup_\ell s_\ell$, and let

- $a = (z, 0^\omega)$;
- $b = (z, 1\hat{\ }0^\omega)$; and

[6] If $\ell = \ell^*$ then $n^{s_\ell} = 0$.

- if the construction ever enters phase 2, at stage ℓ^*, then we let $c = (z, 0^{\ell^*}\hat{\ }1\hat{\ }0^\omega)$.

Write $F(a) = (x^a, y^a)$ and $F(b) = (x^b, y^b)$; if the construction enters phase 2, write $F(c) = (x^c, y^c)$. We note that $x_a = \bigcup_\ell t^a_\ell$, $y^a = \bigcup_\ell \zeta^a_\ell$, and similarly for b. We consider which is the last phase that the construciton reaches.

The construction is always in phase 1. In this case, the sequence (n^{s_ℓ}) shows that $n^z = \liminf \{n^r : r \prec z\} = 0$ (Lemma 4.21), so a and b are connected by an edge in K_3. We claim that $F(a)$ and $F(b)$ are not connected by an edge; indeed, in this case, $F(a) = F(b)$.

The construction eventually settles in phase 2. Let ℓ^* be the stage at which the construction enters stage 2. In this case, the sequence $(n^{s_\ell})_{\ell \geqslant \ell^*}$, as well as the instructions of how to pass from s_ℓ to $s_{\ell+1}$, show that $n^z = \ell^*$. This means that a and c are connected by an edge. We claim that $F(a)$ and $F(c)$ are not connected by an edge.

Suppose that they were. Then $x^a = x^c$ (call this common value x), and $y^a \triangle y^c$ is the singleton $\{n^x\}$. Also, since we never leave phase 2, $x^b = x$. Since we did enter phase 2, $y^a \neq y^b$; let $n_{a,b} = \min(y^a \triangle y^b)$.

We obtain a contradiction by showing that $n^x > n_{a,b}$ and $n^x \leqslant n_{a,b}$.

- Since $F[s_{\ell^*}, 0^{\ell^*}] \subseteq [t^a_{\ell^*}, \zeta^a_{\ell^*}]$, and the second coordinates of a and c both extend 0^{ℓ^*}, we must have that y^a and y^c both extend $\zeta^a_{\ell^*}$, so their point of difference, n^x, must be greater than $|\zeta^a_{\ell^*}|$, and so greater than $n_{a,b}$.
- On the other hand, if $n^x > n_{a,b}$, then since the strings $y^a \restriction (n_{a,b} + 1)$ and $y^b \restriction (n_{a,b} + 1)$ are incomparable, they cannot be both coded by the number $k^x(n_{a,b})$. This would imply that $F(a) \notin \mathbb{W}_3$ or $F(b) \notin \mathbb{W}_3$.

The construction eventually settles in phase 3. In this case, as in the first case, $n^z = 0$, so a and b are connected by an edge. But in this case, $x^a \neq x^b$, so $F(a)$ and $F(b)$ are not connected by an edge. □

5. Candidates for minimal graphs

Unfortunately, even the graph L_3 is not a least graph with no $\mathbf{\Sigma}^0_3$ colouring, with respect to continuous homomorphisms. We will present a graph H_3, which is in fact a least graph with no $\mathbf{\Sigma}^0_3$ colouring, and observe that there is no continuous homomorphism from L_3 into H_3. The graph H_3 is a variant of the minimal graph constructed by Lecomte and Zeleny in [LZ14a]. We show how this graph can potentially be generalised to levels $\alpha > 3$, all using an elaboration of the machinery presented so far.

The difference between H_3 and L_3 is the location at which we connect two points. Instead of requiring that $y \triangle y' = \{n^x\}$, we will require $y \triangle y' = \{c^x(n^x)\}$, where $c^x(n^x)$ is a number (likely larger than n^x) that is computed using the approximation to n^x introduced above. First, we explain how to similarly approximate n^x even when $\alpha > 3$. To do so, we need to approximate values of $T^x(\sigma)$ for various σ (eventually, we will be interested in $|\sigma| = 3$).

5.1. **True stages for T_α.** Fix a computable $\alpha \geqslant 1$. Recalling that α is actually a concrete computable ordinal (a nice computable well-ordering of a computable subset of $\mathbb{N}$), for $\beta < \alpha$ (considered as a sub-ordering of α) we let $a_\beta \in \mathbb{N}$ be the least upper bound of β in α.

For each $\beta < \alpha$, we (uniformly) fix a computable ω-ordering $<^*_\beta$ on the set of nodes of T_α of rank β. This is done in a reasonable way, in particular, if $\sigma\hat{}k$ and $\sigma\hat{}k'$ are two sibling nodes of the same rank β, and $k < k'$, then $\sigma\hat{}k <^*_\beta \sigma\hat{}k'$. Thus, if α is a successor ordinal (so all children of the root have the same rank $\alpha - 1$), then $<^*_{\alpha-1}$ agrees with the "natural" ordering on the children of the root.

As above, we use the ordering $<^*_0$ on the leaves to give us a notion of an initial segment of an element of $2^{\mathscr{L}_\alpha}$, that is, we define the collection $2^{<\mathscr{L}_\alpha}$ as in Definition 4.19.

We will define, for each $t \in 2^{<\mathscr{L}_\alpha}$, a partial labelling T^t of T_α, to serve as an approximation (or guess) for T^x for some $x \in 2^{\mathscr{L}_\alpha}$ extending t. This will be done by induction on $|t|$. We start with:

(i) If $t = \langle\rangle$ then T^t is empty (no nodes are labelled).

Now let $t \in 2^{<\mathscr{L}_\alpha}$ be nonempty. To define T^t, by induction on $\beta \leqslant \alpha$, we will define $T^t(\sigma)$ for nodes of rank β. If this is done for all $\gamma < \beta$, then we let T^t_β be the partial labelling defined so far (the restriction of T^t to nodes of rank $< \beta$).

(ii) We start by setting $T^t_1 = t$.

That is, for nodes σ of rank 0, $T^t(\sigma) = t(\sigma)$. Now, let $\beta \leqslant \alpha$ be nonzero, and suppose that T^t_β is already defined.

(iii) Suppose that there are at most a_β-many $s \prec t$ such that $T^s_\beta \subseteq T^t_\beta$. We then let $T^t(\sigma)\uparrow$ for all σ of rank β.

(iv) Suppose that this is not the case. Let s be the longest $s \prec t$ such that $T^s_\beta \subseteq T^t_\beta$. Let τ be the $<^*_\beta$-least node such that either:
- $T^s(\tau)\uparrow$; or
- $T^s(\tau) = 1$, and there is some k such that $T^t(\tau\hat{}k) = 1$.

We define $T^t(\sigma)$ for σ of rank β as follows:
- For $\sigma <^*_\beta \tau$, $T^t(\sigma) = T^s(\sigma)$.
- For $\sigma >^*_\beta \tau$, $T^t(\sigma)\uparrow$.
- We let $T^t(\tau) = 1$, unless there is some k with $T^t(\tau\hat{}k) = 1$, in which case we let $T^t(\tau) = 0$.

This completes the definition of T^t for all $t \in 2^{<\mathscr{L}_\alpha}$. For $x \in 2^{\mathscr{L}_\alpha}$, T^x is already defined (Definition 2.2); for $\beta \leqslant \alpha + 1$, we let T^x_β be the restriction of T^x to nodes of rank $< \beta$.

Definition 5.1. For $s, t \in 2^{\leqslant\mathscr{L}_\alpha}$, and nonzero $\beta \leqslant \alpha+1$, we write $s \preccurlyeq_\beta t$ if $T^s_\beta \subseteq T^t_\beta$.

We list some basic properties of these partial labellings and relations.

Lemma 5.2. *Let $\beta \leqslant \alpha + 1$ be nonzero, and let $r, s, t \in 2^{\leqslant\mathscr{L}_\alpha}$.*

(a) *$s \prec_1 t$ if and only if $s \prec t$.*
(b) *For all t, $\langle\rangle \preccurlyeq_\beta t$.*
(c) *If $\gamma \leqslant \beta$ and $s \prec_\beta t$ then $s \prec_\gamma t$.*
(d) *If β is a limit, then $s \prec_\beta t$ if and only if for all $\gamma < \beta$, $s \prec_\gamma t$.*
(e) *If $r \prec_\beta s \prec_\beta t$ then $r \prec_\beta t$.*
(f) *If $T^t(\tau)\downarrow$ then for all $\sigma <^*_{\mathrm{rk}(\tau)} \tau$, $T^t(\sigma)\downarrow$.*
(g) *For every non-leaf σ, if $T^t(\sigma)\downarrow$, then $T^t(\sigma) = 0$ if and only if there is some k such that $T^t(\sigma\hat{}k) = 1$.*
(h) *If $s \prec_\beta t$, $\mathrm{rk}(\sigma) = \beta$, and $T^s(\sigma) = 0$, then it is not the case that $T^t(\sigma) = 1$.*
(i) *If $t \in 2^{<\mathscr{L}_\alpha}$ (i.e., is finite), then T^t is finite, and computable from t.*

(j) *If* $r \prec s \prec t$ *and* $r, s \prec_\beta t$ *then* $r \prec_\beta s$.

(k) *Suppose that* $\beta \leqslant \alpha$, $s \prec_\beta t$, *and* $s \not\prec_{\beta+1} t$. *Let* τ *be the* $<^*_\beta$*-least node such that it is not the case that* $T^s(\tau)\downarrow = T^t(\tau)$. *Then:*

- $T^s(\tau) = 1$ *and* $T^t(\tau) = 0$; *and*
- *For all* r *with* $s \preccurlyeq_\beta r \preccurlyeq_\beta t$, *for all* $\sigma <^*_\beta \tau$, *we have* $T^r(\sigma)\downarrow = T^s(\sigma)$.

(l) *If* $\beta \leqslant \alpha$, $r \prec_\beta s \prec_\beta t$ *and* $r \prec_{\beta+1} t$, *then* $r \prec_{\beta+1} s$.

Proof. (a)—(e) are immediate from the definitions.

(f) and (g) hold by definition when t is infinite. For finite t, they are proved by a straightforward induction on $|t|$.

For (h): by (g), there is some k such that $T^s(\sigma\hat{\ }k) = 1$. Since $\mathrm{rk}(\sigma\hat{\ }k) < \beta$ and $s \prec_\beta t$, we have $T^t(\sigma\hat{\ }k) = 1$. By (g) again, we cannot have $T^t(\sigma) = 1$.

(i) follows from the fact that there are only finitely many β with $a_\beta < |t|$, and induction on $|t|$; the entire construction is computable.

(j), (k) and (l) are proved by simultaneous induction on β. Let $\beta \leqslant \alpha + 1$, and suppose that these have been verified for all $\gamma < \beta$.

If $\beta = 1$, then (j)$_\beta$ is immediate. If β is a limit, then (j)$_\beta$ follows by induction, and (d). If $\beta = \gamma + 1 > 1$ is a successor, then (j)$_\beta$ follows from (j)$_\gamma$ and (l)$_\gamma$ (and (c)).

For (k)$_\beta$, fix some $s \in 2^{<\mathscr{L}_\alpha}$. If $T^s(\sigma)\uparrow$ for all σ of rank β, then $s \prec_{\beta+1} t$ whenever $s \prec_\beta t$. Hence, suppose that $T^s(\sigma)\downarrow$ for some σ of rank β. By induction on the length of $t \succcurlyeq_\beta s$, we prove that if $s \not\prec_{\beta+1} t$ then (k)$_\beta$ holds between s and t. This is vacuous when $s = t$.

Fix some $t \succ_\beta s$, and suppose that $s \not\prec_{\beta+1} t$. Let r be the $\prec_\beta$-predecessor of t. By (j)$_\beta$, $s \preccurlyeq_\beta r \prec_\beta t$. Since $T^s(\sigma)\downarrow$ for some σ, s has more than a_β-many $\prec_\beta$-predecessors. By (e), each such is a $\prec_\beta$-predecessor of t, so case (iii) does not apply in the definition of $T^t_{\beta+1}$.

Let $\tau_{s,t}$ be the $<^*_\beta$-least τ such that $T^s(\tau)\downarrow$ and $\neg(T^t(\tau) = T^s(\tau))$. If $s \not\prec_{\beta+1} r$, let $\tau_{s,r}$ be the $<^*_\beta$-least τ such that $T^s(\tau)\downarrow$ and $\neg(T^r(\tau) = T^s(\tau))$. Similarly, if $r \not\prec_{\beta+1} t$, we let $\tau_{r,t}$ be the $<^*_\beta$-least τ such that $T^r(\tau)\downarrow$ and $\neg(T^t(\tau) = T^r(\tau))$.

If $s \not\prec_{\beta+1} r$, then $\tau_{s,t} \leqslant^*_\beta \tau_{s,r}$. For suppose otherwise. By induction, $T^s(\tau_{s,r}) = 1$ and $T^r(\tau_{s,r}) = 0$. By minimality of $\tau_{s,t}$, $T^t(\tau_{s,r}) = T^s(\tau_{s,r}) = 1$. This contradicts (h) (between r and t). Hence, for all $\sigma <^*_\beta \tau_{s,t}$, $T^r(\sigma)\downarrow = T^s(\sigma)$.

Similarly, if $r \not\prec_{\beta+1} t$ then $\tau_{s,t} \leqslant^* \tau_{r,t}$. By the definition of T^t, in this case, $T^r(\tau_{r,t}) = 1$ and $T^t(\tau_{r,t}) = 0$. If $\tau_{r,t} <^*_\beta \tau_{s,t}$ then by the minimality of $\tau_{s,t}$, $T^s(\tau_{r,t}) = 0$, contradicting (h) between s and r.

Hence, if $\sigma <^*_\beta \tau_{s,t}$, then $T^r(\sigma)\downarrow = T^t(\sigma)$. This confirms the second part of (k)$_\beta$ between s and t. Also, by the definition of T^t, this implies that $T^t(\tau_{s,t})\downarrow$. By (h), we indeed have $T^t(\tau_{s,t}) = 0$ and $T^s(\tau_{s,t}) = 1$.

This concludes the proof of (k)$_\beta$ when t is finite. Suppose that $x \in 2^{\mathscr{L}_\alpha}$, $s \prec_\beta x$ but $s \not\prec_{\beta+1} x$. Define τ as in (k). Since $T^x(\tau)\downarrow$, by (h), we must have $T^x(\tau) = 0$ and $T^s(\tau) = 1$. For the second part of (k)$_\beta$ between s and x, let r with $s \preccurlyeq_\beta r \prec_\beta x$, and suppose that there is some $\sigma <^*_\beta \tau$ such that it is not the case that $T^r(\sigma) = T^s(\sigma)$. Choosing the $<^*_\beta$-least such σ, we have $T^r(\sigma) = 0$ and $T^x(\sigma) = T^s(\sigma) = 1$, as usual, contradicting (h) between r and x.

Finally, (l)$_\beta$ follows from (k)$_\beta$: suppose that $r \prec_\beta s \prec_\beta t$, and $r \not\prec_{\beta+1} s$. Let τ be $<^*_\beta$-minimal with $T^s(\tau) \neq T^r(\tau)$. By (k)$_\beta$, $T^r(\tau) = 1$ and $T^s(\tau) = 0$. By (h), we cannot have $T^t(\tau) = 1$. So $r \not\prec_{\beta+1} t$. □

Corollary 5.3. *Let $\beta \leqslant \alpha + 1$. Suppose that $s, t \in 2^{<\mathscr{L}_\alpha}$ and that T^s_β and T^t_β are compatible ($T^s_\beta \cup T^t_\beta$ is a function). Then $s \preccurlyeq_\beta t$ or $t \preccurlyeq_\beta s$.*

Proof. We have $s \preccurlyeq t$ or $t \preccurlyeq s$; say $s \preccurlyeq t$. Let γ be the greatest with $s \prec_\gamma t$. Suppose that $\gamma < \beta$. By (k) of Lemma 5.2 , there is some τ with $\text{rk}(\tau) = \gamma$ and $T^s(\tau) = 1$, $T^t(\tau) = 0$. □

Lemma 5.4. *For $x \in 2^{\mathscr{L}_\alpha}$ and nonzero $\beta \leqslant \alpha+1$, there are infinitely many $s \prec_\beta x$, and*

$$T^x_\beta = \bigcup \{T^s_\beta \,:\, s \prec_\beta x\}\,.$$

Proof. We prove this by induction on β. It is immediate for $\beta = 1$.

Suppose that β is a limit, and that the lemma holds for all $\gamma < \beta$. It suffices to show that there are infinitely many $t \prec_\beta x$. Well, let $r \prec x$; let $\gamma = \max\{\delta < \beta : a_\delta \leqslant |r|\}$. Let t be the shortest with $t \succcurlyeq r$ and $t \prec_{\gamma+1} x$. Let $\delta \in [\gamma+1, \beta)$, and suppose that $s \prec_\delta t$. Since $\delta > \gamma$, $s \prec_{\gamma+1} t$. Since $t \prec_{\gamma+1} x$, we have $s \prec_{\gamma+1} x$. The minimality of t implies that $s \preccurlyeq r$. Hence, t has at most $|r| + 1$ many $\prec_\delta$-predecessors. Since $a_\delta > |r|$, t has at most a_δ-many $\prec_\delta$-predecessors. By (iii) of the definition of T^t_δ, we have $T^t(\sigma)\uparrow$ for all σ of rank δ. Hence, $T^t_\beta = T^t_{\gamma+1}$, so $t \prec_\beta x$.

For the successor case, suppose that $\beta \leqslant \alpha$ and that the lemma has been verified for β. We first show:

($*$)**:** For all σ of rank β, for all but finitely many $s \prec_\beta x$ we have $T^s(\sigma)\downarrow = T^x(\sigma)$.

First, note that it suffices to show that for all but finitely many $s \prec_\beta x$ we have $T^s(\sigma)\downarrow$. This is because by Lemma 5.2(h), if $s \prec_\beta x$ and $T^s(\sigma) = 0$ then $T^x(\sigma) = 0$; on the other hand, if $T^x(\sigma) = 0$ then there is some k with $T^x(\sigma\hat{\ }k) = 1$, and then, by induction, for all but finitely $s \prec_\beta x$ we have $T^s(\sigma\hat{\ }k) = 1$, whence $T^s(\sigma) = 0$ if $T^s(\sigma)\downarrow$.

We prove ($*$) by induction on $<^*_\beta$. Let σ be a node of rank β; by the induction on $<^*_\beta$, we fix a long $s \prec_\beta x$ so that for all t with $s \preccurlyeq_\beta t \prec_\beta x$, for all $\rho <^*_\beta \sigma$, we have $T^t(\rho) = T^x(\rho)$. Also assume that s has more than a_β-many $\prec_\beta$-predecessors (this is required when σ is the $<^*_\beta$-least node, otherwise, this is implied by $T^s(\rho)\downarrow$ for some ρ). Then our construction ensures that for all t with $s \prec_\beta t \prec_\beta x$, $T^t(\sigma)\downarrow$. This completes the proof of ($*$).

So to prove the lemma for $\beta + 1$, it suffices to show that there are infinitely many $t \prec_{\beta+1} x$. Let $s \prec_\beta x$ and suppose that $s \not\prec_{\beta+1} x$. Let τ be $<^*_\beta$-least with $T^s(\tau)\downarrow \neq T^x(\tau)$. By Lemma 5.2(k), $T^x(\tau) = 0$ and $T^s(x) = 1$, and for all $\sigma <^*_\beta \tau$, for all t with $s \preccurlyeq_\beta t \prec_\beta x$, we have $T^t(\sigma)\downarrow = T^x(\sigma)$. Let t be the shortest $t \prec_\beta x$ such that $T^t(\tau) = 0$; then $s \prec_\beta t$. Let r be the $\prec_\beta$-predecessor of t. Since it is not the case that $T^t(\tau) = T^r(\tau)$, by the definition of T^t (case (iv)), for all $\sigma >^*_\beta \tau$ we have $T^t(\sigma)\uparrow$. Hence, $t \prec_{\beta+1} x$. □

Remark 5.5. In fact, for each $x \in 2^{\mathscr{L}_\alpha}$,

$$\{s \,:\, s \prec_\beta x\}$$

is the unique infinite path in the tree $(\{s : s \prec x\}, \prec_\beta)$. This is proved by induction on β; for the successor step, use (l).

5.2. **Dynamic coding locations.** For the rest of this section, we assume that α is the successor of a successor ordinal. We let

$$\alpha^* = \alpha - 2.$$

The point is that every node of length 2 has rank α^*, and so for all $t \in 2^{\leqslant \mathscr{L}_\alpha}$,

$$T^t_{\alpha*} = T^t \upharpoonright \{\sigma \in T_\alpha : |\sigma| \geqslant 3\}.$$

Using $T^t_{\alpha*}$, we can generalise Definition 4.20 to $\alpha > 3$.

Definition 5.6. For $t \in 2^{<\mathscr{L}_\alpha}$ we define numbers n^t, and for $m < n^t$, numbers $k^t(m)$, by induction on $|t|$. We let $n^{\langle\rangle} = 0$.

Suppose that $t \neq \langle\rangle$; let s be the longest such that $s \prec_{\alpha*} t$.[7]

- If there is some $m < n^s$ and some a such that $T^t(m, k^s(m), a) = 1$, then we let n^t be the least such m.
- Otherwise, we let $n^t = n^s + 1$.

Then, for each $m < n^t$, we let $k^t(m)$ be the least k such that there is no a with $T^t(m, k, a) = 1$.

Lemma 5.7. *Let $s, t \in 2^{\leqslant \mathscr{L}_\alpha}$.*

(a) *If $t \in 2^{<\mathscr{L}_\alpha}$, then $n^t \leqslant |t|$.*
(b) *If $s \preccurlyeq_{\alpha*} t$ then for all $m < \min\{n^s, n^t\}$, $k^s(m) \leqslant k^t(m)$.*
(c) *Suppose that $s \preccurlyeq_{\alpha*} t$, $m < \min\{n^s, n^t\}$, and $k^s(m) \neq k^t(m)$. Then there is some r with $s \prec_{\alpha*} r \prec_{\alpha*} t$ such that $n^r \leqslant m$.*

Proof. (a) is proved by induction on $|t|$, since $n^t \leqslant n^s + 1$, where s is the longest with $s \prec_{\alpha*} t$.

(b) is immediate from the definition of $k^t(m)$; if $s \preccurlyeq_{\alpha*} t$ and there is no a with $T^t(m, k, a) = 1$, then there is no a with $T^s(m, k, a) = 1$.

For (c), suppose that $s \preccurlyeq_{\alpha*} t$, $m < \min\{n^s, n^t\}$, and $k^s(m) \neq k^t(m)$. By (b), $k^t(m) > k^s(m)$. Choose $r \preccurlyeq_{\alpha*} t$ shortest such that $n^r > m$ and $k^r(m) > k^s(m)$. By Lemma 5.2(j), r and s are $\preccurlyeq_{\alpha*}$-comparable; by (b) again, $s \prec_{\alpha*} r$. Let $\bar{r}$ be the longest with $\bar{r} \prec_{\alpha*} r$; by Lemma 5.2(j) again, $s \preccurlyeq_{\alpha*} \bar{r} \prec_{\alpha*} r$. By minimality of r, either $n^{\bar{r}} \leqslant m$, or $k^{\bar{r}}(m) = k^s(m)$. The latter case is impossible, since then we would have $k^{\bar{r}}(m) \neq k^r(m)$, whence by definition we would have $n^r \leqslant m$. □

Definition 5.8. Let $r, s \in 2^{\leqslant \mathscr{L}_\alpha}$. We write

$$r \sqsubseteq_m s$$

if $r \preccurlyeq_{\alpha*} s$, $m \leqslant \min\{n^r, n^s\}$, and for all $\ell < m$, $k^s(\ell) = k^r(\ell)$.

Lemma 5.9. *Let $s, t \in 2^{<\mathscr{L}_\alpha}$.*

(a) *$\sqsubseteq_m$ is transitive.*
(b) *If $s \sqsubseteq_m t$ and $m' < m$ then $s \sqsubseteq_{m'} t$.*
(c) *If $s \sqsubseteq_m t$ then for all r with $s \preccurlyeq_{\alpha*} r \preccurlyeq_{\alpha*} t$ we have $n^r \geqslant m$.*
(d) *If $r \preccurlyeq_{\alpha*} s \preccurlyeq_{\alpha*} t$ and $r \sqsubseteq_m t$ then $r \sqsubseteq_m s$.*
(e) *If $r \preccurlyeq s$ and $r, s \sqsubseteq_m t$ then $r \sqsubseteq_m s$.*
(f) *If s is the shortest with $s \sqsubseteq_m t$, then $n^s = m$.*

[7] Such an s exists; by Lemma 5.2(b), $\langle\rangle \prec_{\alpha*} t$.

Proof. (a) follows from the transitivity of $\preccurlyeq_{\alpha *}$. (b) is immediate from the definition.

For (c), suppose that $s \prec_{\alpha *} t$, that $n^s, n^t \geqslant m$, and that there is some r such that $s \prec_{\alpha *} r \prec_{\alpha *} t$ and $n^r < m$. Let r be shortest such; let $\bar{r}$ be the $\prec_{\alpha *}$-predecessor of r. Again, $s \preccurlyeq_{\alpha *} \bar{r}$. The fact that $n^r < n^{\bar{r}}$ implies that there is some a such that $T^r(n,k,a) = 1$, where $n = n^r$ and $k = k^{\bar{r}}(n)$. Since $r \preccurlyeq_{\alpha *} t$, we have $T^t(n,k,a) = 1$, so $k^t(n) \neq k$. By Lemma 5.7(b), $k^t(n) > k^{\bar{r}}(n) \geqslant k^s(n)$, so $k^t(n) > k^s(n)$. This shows that $s \not\sqsubseteq_m t$.

For (d), suppose that $r \prec_{\alpha *} s \prec_{\alpha *} t$ and $r \sqsubseteq_m t$. By (c), $n^s \geqslant m$. Let $\ell < m$. By Lemma 5.7(b), $k^t(\ell) \geqslant k^s(\ell) \geqslant k^r(\ell)$. By assumption, $k^t(\ell) = k^r(\ell)$; so $k^s(\ell) = k^r(\ell)$ as well. This shows that $r \sqsubseteq_m s$.

(e): By Lemma 5.2(j), $r \preccurlyeq_{\alpha *} s$. Now $r \sqsubseteq_m s$ follows from (d).

For (f), suppose that $s \sqsubseteq_m t$ and that $n^s > m$. Since $n^s > 0$, $s \neq \langle\rangle$, so let $\bar{s}$ be the $\prec_{\alpha *}$-predecessor of s. There are two possibilities: either $n^s = n^{\bar{s}} + 1$, in which case $\bar{s} \sqsubset_{n^{\bar{s}}} s$; or $n^s < n^{\bar{s}}$, in which case $\bar{s} \sqsubset_{n^s} s$. In either case, $\bar{s} \sqsubseteq_m s$, hence $\bar{s} \sqsubset_m t$, so s is not the shortest with $s \sqsubseteq_m t$. □

Lemma 5.10. *Let $x \in 2^{\leqslant \mathscr{L}_\alpha}$.*

(a) *For all finite $m \leqslant n^x$, for all but finitely many $s \prec_{\alpha *} x$, $s \sqsubset_m x$.*
(b) $n^x = \liminf \{n^t : t \prec_{\alpha *} x\}$.
(c) *There are infinitely many s such that $s \sqsubset_{n^s} x$.*

Proof. (a) and (b) are similar to the proof of Lemma 4.21.

We prove (a) by induction on m. It is immediate for $m = 0$. Suppose that it holds for m, and that $n^x > m$. Let $s_0 \prec_{\alpha *} x$ such that for all s with $s_0 \preccurlyeq_{\alpha *} s \prec_{\alpha *} x$, $s \sqsubset_m x$. By Lemma 5.4, there is some s_1 such that $s_0 \preccurlyeq_{\alpha *} s_1 \prec_{\alpha *} x$ and for all $k < k^x(m)$, there is some a such that $T^{s_1}(m,k,a) = 1$. On the other hand, for all $s \prec_{\alpha *} x$, there is no a with $T^s(m, k^x(m), a) = 1$. Hence, for all s with $s_1 \preccurlyeq_{\alpha *} s \prec_{\alpha *} x$, if $n^s > m$ then $k^s(m) = k^x(m)$, so $s \sqsubset_{m+1} x$. We argue that for all s with $s_1 \prec_{\alpha *} s \prec_{\alpha *} x$ we have $n^s > m$. Let s be such, and let $\bar{s}$ be the $\prec_{\alpha *}$-predecessor of s. So $s_1 \preccurlyeq_{\alpha *} \bar{s} \prec_{\alpha *} x$. Since $n^{\bar{s}} \geqslant m$ and $\bar{s} \sqsubseteq_m x$, we have $\bar{s} \sqsubseteq_m s$. If $n^{\bar{s}} = m$ then by definition, $n^s = m+1$. If not, then $k^{\bar{s}}(m) = k^x(m)$ implies that $n^s > m$ as well.

One direction of (b) follows from (a): if $m \leqslant n^x$, then for all but finitely many $s \prec_{\alpha *} x$ we have $n^s \geqslant m$. On the other hand, suppose that n^x is finite, $s \prec_{\alpha *} x$, and $n^s > n^x$; and that $r \sqsubset_{n^x} x$ for all r with $s \preccurlyeq_{\alpha *} r \prec_{\alpha *} x$. Let $t \prec_{\alpha *} x$ be shortest with $T^t(n^x, k^s(n^x), a) = 1$ for some a. By induction on the length of u with $s \preccurlyeq_{\alpha *} u \prec_{\alpha *} t$ we see that $s \sqsubseteq_{n^x+1} u$. Applying this to u the $\prec^*$-predecessor of t, we see that $n^t = n^x$.

If n^x is finite, then (c) follows from (a) and (b): there are infinitely many $s \prec_{\alpha *}$ with $n^s = n^x$, and for all but finitely many of these, $s \sqsubseteq_{n^x} x$.

Suppose that $n^x = \omega$. Let $m < \omega$; by (a), let s be the shortest such that $s \sqsubset_m x$. By Lemma 5.9(f), $n^s = m$, so again $s \sqsubset_{n^s} x$. □

Definition 5.11. Let $t \in 2^{\leqslant \mathscr{L}_\alpha}$. For finite $m \leqslant n^t$ we let

$$c^t(m) = \min\{|s| : s \sqsubseteq_m t\}.$$

Lemma 5.12. *Let $t \in 2^{\leqslant \mathscr{L}_\alpha}$, and let $m \leqslant n^t$ be finite.*

(a) $c^t(m) \leqslant |t|$.
(b) $c^t(m) \geqslant m$.

(c) $c^t(m) < \omega$.
(d) *If* $s \sqsubseteq_m t$ *then* $c^s(m) = c^t(m)$.
(e) *If* $m < m' \leqslant n^t$ *then* $c^t(m) < c^t(m')$.

Proof. Let r be shortest with $r \sqsubseteq_m t$, so $c^t(m) = |r|$.

(a) is immediate. For (b), by Lemma 5.9(f), $n^r = m$; by Lemma 5.7(a), $m \leqslant |r|$.
(c) follows from Lemma 5.10(a).
(d) follows from Lemma 5.9(e), since we have $r \sqsubseteq_m s$, and r is shortest such.
(e) follows from Lemma 5.9(f); if r' is shortest with $r' \sqsubseteq_{m'} t$ then $n^{r'} = m' \neq m = n^r$; so $r' \neq r$. Since $r' \sqsubseteq_m t$ (Lemma 5.9(b)), the minimality of r implies that $r \prec r'$. □

Lemma 5.13. *Let* $t \in 2^{<\mathscr{L}_\alpha}$, $t \neq \langle\rangle$. *Suppose that* $s \prec_{\alpha*} t$ *is the longest such that* $n^s \leqslant n^t$.

(a) $s \sqsubseteq_{n^s} t$.
(b) *The value* $k^r(n^s)$ *is constant for all* r *with* $s \prec_{\alpha*} r \prec_{\alpha*} t$.
(c) *If* $n^s < n^t$ *then* s *is the* $\prec_{\alpha*}$*-predecessor of* t, *and* $c^t(n^t) = |t|$.

Proof. (a) follows from Lemma 5.7(c).

For (b), let r, r' with $s \prec_{\alpha*} r \prec_{\alpha*} r' \prec_{\alpha*} t$. If $k^{r'}(n^s) \neq k^r(n^s)$ then by Lemma 5.7(c), there is some u with $r \prec_{\alpha*} u \prec_{\alpha*} r'$ and $n^u \leqslant n^s \leqslant n^t$, contradicting the maximality of s.

(c): suppsoe that $n^s < n^t$. If there is some r with $s \prec_{\alpha*} r \prec_{\alpha*} t$, let r be the shortest such, so that s is the $\prec_{\alpha*}$-predecessor of r. By maximality of s, $n^r > n^t > n^s$ so $n^r \geqslant n^s + 2$, which is impossible. Note that this means that $n^t = n^s + 1$.

To see that $c^t(n^t) = |t|$, suppose, for a contradiction, that there is some $r \sqsubset_{n^t} t$. Then $r \preccurlyeq_{\alpha*} s$, so by Lemma 5.9(d), $r \sqsubseteq_{n^t} s$, which is impossible since $n^s < n^t$. □

5.3. **The graphs H_α.** We continue to use the list (M_k) of reals (Definition 4.2) for coding finite strings. We modify Definition 4.3: instead of coding an edge at n^x, we code it at the much higher number $c^x(n^x)$, and we similarly increase the lengths of the initial segments of y coded by $k^x(m)$ for $m < n^x$.

Definition 5.14.

(a) We let $\mathbb{Y}_\alpha$ be the collection of $(x, y) \in \mathbb{X}_\alpha$ such that for all $m < n^x$, $k^x(m)$ is a code for $y{\restriction}(c^x(m) + 1)$.
(b) For $(x, y), (x', y') \in \mathbb{Y}_\alpha$, we let (x, y) be related to (x', y') in the directed graph H_α if $x = x'$, $n^x < \omega$, $y \triangle y' = \{c^x(n^x)\}$, and $c^x(n^x) \notin y$.

Lemma 5.15. *The set* $\mathbb{Y}_\alpha$ *is* $\Pi^0_2(\tau_{\alpha*})$.

Proof. Similar to the proof of Proposition 4.5. We define $A_{\bar{k}}$ as in that proof, except that the third bullet point is:

- for the shortest $r \prec_{\alpha*} x$ such that $n^r = m$ and for all $\ell < m$, $k^r(\ell) = k_\ell$, k_m is a code for $y{\restriction}(|r| + 1)$. □

In particular, $\mathbb{Y}_3$ is Π^0_2 in the standard topology.

5.4. **Non-colorability when α is the successor of a successor.** Our aim is to design a notion of forcing similar to $\mathbb{S}$ from the previous section (Definition 4.10), with the purpose of obtaining generic points of $\mathbb{Y}_\alpha$. The extra complication comes from needing to code at the correct lengths. Specifically, suppose that we are building a generic point $(x_G, y_G) \in \mathbb{Y}_\alpha$, and let $m < n^{x_G}$. Let $k = k^{x_G}(m)$ and $c = c^{x_G}(m)$. Then k must code $y_G\restriction(c+1)$. Let $p = (u,\zeta) \in G$ be "sufficiently large", so that $n^u > m$, $|\zeta| > c$, and $k^u(m) = k$. Then the condition p should in some way "know" what c is, so that it can ensure that k codes $\zeta\restriction(c+1)$. However, c is computed by considering strings $s \sqsubset_m x_G$, in particular, strings $s \prec_{\alpha*} x_G$. Thus, we will require that u has some witness that could play the role of such s. Luckily, any two such witnesses will agree with each other, in that they will compute the same value of c.

For the following definition, recall the collection of conditions $\tilde{\mathbb{Q}}$ (Definition 4.7), and the numbers n^u and $k^u(m)$ (Definition 4.6).

Definition 5.16. We say that $s \in 2^{<\mathscr{L}_\alpha}$ *supports* a condition $u \in \tilde{\mathbb{Q}}$ if:

(i) $T^s_{\alpha*} \subseteq u$; and
(ii) $n^s \geqslant n^u$, and for all $m < n^u$, $k^s(m) = k^u(m)$.

We say that $u \in \tilde{\mathbb{Q}}$ is *supported* if some $s \in 2^{<\mathscr{L}_\alpha}$ supports u.

Note that if $s \sqsubseteq_{n^u} t$ and t supports u then s supports u as well.

Lemma 5.17. *Let $u, v \in \tilde{\mathbb{Q}}$ be compatible, with $n^u \leqslant n^v$. If s supports u and t supports v then s and t are $\sqsubseteq_{n^u}$-comparable.*

Proof. Let $w = u \cup v$. We have $T^s_{\alpha*}, T^t_{\alpha*} \subseteq w$, and so $T^s_{\alpha*} \cup T^t_{\alpha*}$ is a function. By Corollary 5.3, s and t are $\preccurlyeq_{\alpha*}$-comparable. For all $m < n^u$, $k^s(m) = k^u(m) = k^v(m) = k^t(m)$, so s and t are $\sqsubseteq_{n^u}$-comparable. □

By Lemma 5.12(d), if s and t both support u, then for all $m \leqslant n^u$, $c^s(m) = c^t(m)$. We therefore define:

Definition 5.18. If $u \in \tilde{\mathbb{Q}}$ is supported, then for $m \leqslant n^u$, we let $c^u(m) = c^s(m)$ for any (all) s that support u.

By Lemma 5.12(d) we get:

Lemma 5.19. *If $u, v \in \tilde{\mathbb{Q}}$ are supported and compatible, then for all $m \leqslant \min\{n^u, n^v\}$, $c^u(m) = c^v(m)$. In particular, if $u \subseteq v$ then for all $m < n^u$, $c^u(m) = c^v(m)$.*

Remark 5.20. The *minimal support* of a supported u is the $\sqsubseteq_{n^u}$-least s that supports u, which exists by Lemma 5.17. By Lemma 5.9(f), for this s we have $n^s = n^u$.

Lemma 5.21.

(a) *If $u, v \in \tilde{\mathbb{Q}}$ are compatible, and are both supported, then $u \cup v$ is supported.*
(b) *If $u \in \tilde{\mathbb{Q}}$ is supported, then for all $\beta \leqslant \alpha$, $u\restriction\beta$ is supported.*

Proof. (a): By Lemma 4.8, $w = u \cup v \in \tilde{\mathbb{Q}}$. Suppose that $n^u \leqslant n^v$. The main point is that $n^w = n^v$; if $n^u < n^v$ then necessarily $u(n^u)\uparrow$. Also, for all $m < n^v$, $k^w(m) = k^v(m)$ (note that for all $m < n^u$, $k^u(m) = k^v(m)$). Hence, if t supports v then it also supports w.

(b): There are two cases. If $\beta \leqslant \alpha^*$, then $n^{u\restriction\beta} = 0$: if $(u\restriction\beta)(\sigma) = 1$ then $\mathrm{rk}(\sigma) < \alpha^*$, meaning that $|\sigma| \geqslant 3$; hence there is no m with $(u\restriction\beta)(m)\downarrow$. In this case the empty string supports $u\restriction\beta$.

Suppose that $\beta > \alpha^*$. Then $n^{u\restriction\beta} = n^u$: for all $m < n^u$, $\mathrm{rk}(m, k^u(m)) \leqslant \alpha^* < \beta$, so $(u\restriction\beta)(m, k^x(m)) = 1$, implying that $(u\restriction\beta)(m) = 0$. This also shows that for all $m < n^u$, $k^{u\restriction\beta}(m) = k^u(m)$. Further, if $T^s_{\alpha*} \subseteq u$ then $T^s_{\alpha*} \subseteq u\restriction\beta$. Hence, any string that supports u also supports $u\restriction\beta$. ☐

Lemma 5.22. *For every $u \in \tilde{\mathbb{Q}}$ there is some $v \supseteq u$ in $\tilde{\mathbb{Q}}$ such that:*

(i) *v is supported;*
(ii) *$n^v = n^u$;*
(iii) *If $u(n^u)\uparrow$ then $v(n^v)\uparrow$.*

Proof. Choose any $x \in [u]$. By Lemma 5.10(a) and (b), find some $t \prec_{\alpha*} x$ such that $n^t \geqslant n^u$ and for all $m < n^u$, $k^t(m) = k^x(m)$. Note that since $x \in [u]$, for $m < n^u$, $k^x(m) = k^u(m)$. Since $T^t_{\alpha*} \subseteq T^x$ and $u \subseteq T^x$, $w = u \cup T^t_{\alpha*}$ is a function and does not contain any contradictions (there is no σ with $w(\sigma) = 1 = w(\sigma)\hat{\ }k$ for some k). By Lemma 5.2(i), $\mathrm{dom}\, w$ is finite. Since $u \in \tilde{\mathbb{Q}}$ and every $\sigma \in \mathrm{dom}\, T^t_{\alpha*}$ has height $\geqslant 3$, we can extend w to a condition in $v \in \mathbb{Q}$ without adding any labels to nodes of height 0 or 1, or any 1-labels to any nodes of height 2. This ensures that $v \in \tilde{\mathbb{Q}}$ and has the desired properties; t supports v. ☐

Definition 5.23. Let $\mathbb{R}$ be the collection of pairs $(u, \zeta) \in \tilde{\mathbb{Q}} \times 2^{<\omega}$ such that u is supported, $|\zeta| \geqslant c^u(n^u)$, and for all $m < n^u$, $k^u(m)$ is a code of $\zeta\restriction(c^u(m) + 1)$.

As above, pairs are ordered by co-ordinatewise extension. By Lemma 5.12(e), the requirement $|\zeta| \geqslant c^u(n^u)$ implies $|\zeta| > c^u(m)$ for all $m < n^u$, so the string $\zeta\restriction(c^u(m) + 1)$ makes sense.

Lemma 5.24.

(a) *The condition p^* defined by $\mathrm{dom}\, u^{p^*} = \{\langle\rangle\}$, $u^{p^*}(\langle\rangle) = 1$, and $\zeta^{p^*} = \langle\rangle$ is in $\mathbb{R}$.*

Suppose that $(u, \zeta) \in \mathbb{R}$.

(b) *For all $\xi \succcurlyeq \zeta$, $(u, \xi) \in \mathbb{R}$.*
(c) *Suppose that $u(n^u) = 1$. Then for all $v \supseteq u$ in $\tilde{\mathbb{Q}}$, $(v, \zeta) \in \mathbb{R}$.*
(d) *Suppose that $u(\langle\rangle)\uparrow$. Let v extend u by defining $v(n^u) = 1$ (and $v(\langle\rangle) = 0$). Then $(v, \zeta) \in \mathbb{R}$.*
(e) *Suppose that $u(n^u)\uparrow$. Then there is some $(w, \xi) \in \mathbb{R}$ extending (u, ζ) such that $w(n^w)\uparrow$ and $|\xi| = c^w(n^w)$.*

Proof. Most are the same as in the proof of Lemma 4.11; for example, for (c), again the point is that $n^u = n^v$, and $k^v(m) = k^u(m)$ for all $m < n^u$, so any s that supports u also supports v.

The argument for (e) is a little more elaborate. Let $(u, \zeta) \in \mathbb{R}$. Let l be a large number. Note that since we are assuming that $\alpha = \alpha^* + 2$ is the successor of a successor, the rank of every node of height 2 is α^*, in particular, requirement (iv) of Definition 4.7 holds automatically. (This is not really crucial, but simplifis notation.) We choose a large number k such that $\zeta \prec M_k$.

We first define v to be the extension of u defined by letting, for every m with $n^u \leqslant m < l$,

- $v(m) = 0$, and $v(m, k) = 1$;
- For all $k' < k$, $v(m, k') = 0$ and $v(m, k', l) = 1$.

Since $u \in \tilde{\mathbb{Q}}$ and $u(l)\uparrow$, this is consistent with u, and $v \in \tilde{\mathbb{Q}}$. Note that $v(n^v)\uparrow$.

By Lemma 5.22, extend v to a supported $w \in \tilde{\mathbb{Q}}$ with $n^w = n^v = l$ and $w(n^w)\uparrow$. Let $\xi = M_k \upharpoonright c^w(l)$. Then (w, ξ) is as required: if $m < n^u$, then as $c^w(m) = c^u(m)$ (Lemma 5.19), and $k^w(m) = k^u(m)$ codes $\zeta\upharpoonright(c^u(m)+1) = \xi\upharpoonright(c^w(m)+1)$. If $n^u \leqslant m < l$, then $k^w(m) = k$ codes every initial segment of ξ, in particular, $\xi\upharpoonright(c^w(m)+1)$. ☐

Remark 5.25. The proof of Lemma 5.24(e) is where we need the flexible coding mechanism given by the sequence (M_k): we need to choose some k that will code $\xi\upharpoonright(c^w(k)+1)$, without knowing what $c^w(k)$ is going to be.

Lemma 5.26. *For all $\sigma \in T_\alpha$, the collection of $q \in \mathbb{R}$ such that $u^q(\sigma)\downarrow$ is dense in $\mathbb{R}$.*

Proof. Let $p = (u, \zeta) \in \mathbb{R}$, and let $\sigma \in T_\alpha$. As in the proof of Lemma 4.12, if $\sigma = \langle\rangle$, then Lemma 5.24(d) allows us to add σ to the domain of u. Also, if $u(n^u)\downarrow$ then (c) allows us to similarly add any σ. Suppose that $u(n^u)\uparrow$. If $|\sigma| \geqslant 2$ then we can extend u to v by setting $v(\sigma) = 0$ and $v(\sigma\hat{\ }l) = 1$ for some large l, so that $n^v = n^u$, and $q = (v, \zeta)$ is as required. So to complete the proof of this lemma, it suffices to show that there is some $q \in \mathbb{R}$ extending p with $n^{u^q} > n^u$. By (e), we may assume that $|\zeta| = c^s(n^u)$. By (b), extend ζ to any longer string, say $\zeta\hat{\ }0$. Now apply (e) to the condition $(u, \zeta\hat{\ }0)$ to obtain $q = (w, \xi) \in \mathbb{R}$ with $|\xi| = c^t(n^w)$. Since $c^w(n^u) = c^u(n^u) = |\zeta| < |\xi|$ (Lemma 5.19), we must have $n^w > n^u$, as required. ☐

Lemma 5.27. *Two conditions $p, q \in \mathbb{R}$ are compatible in $\mathbb{R}$ if and only if $u^p \cup u^q$ is a function and ζ^p and ζ^q are comparable.*

Proof. Similar to the proof of Lemma 4.13, using Lemmas 5.19 and 5.21. ☐

Similarly, we obtain the analogue of Lemma 4.14:

Lemma 5.28. *If $G \subset \mathbb{R}$ is sufficiently generic, then:*

(i) *For all finite $m \leqslant n^{x_G}$, $c^{x_G}(m) = c^{u^p}(m)$ for some (or any) $p \in G$ with $n^{u^p} \geqslant m$.*
(ii) $(x_G, y_G) \in \mathbb{Y}_\alpha$.
(iii) *For all $p \in \mathbb{R}$, $(x_G, y_G) \in [p]$ if and only if $p \in G$.*

Proof. As in the proof of Lemma 4.14, we have $(x_G, y_G) \in \mathbb{X}_\alpha$, and property (iii) above holds.

For (i) and (ii), let $m \leqslant n^{x_G}$ be finite; let $p = (u, \zeta) \in G$ with $n^u \geqslant m$. Let s support p. Since $T^s_{\alpha*} \subseteq u \subset T^{x_G}$, we have $s \prec_{\alpha*} x_G$. For each $\ell < n^u$, since $u \subset T^{x_G}$, we have $k^u(\ell) = k^{x_G}(\ell)$. Since s supports u, $k^u(\ell) = k^s(\ell)$ for such ℓ. This shows that $s \sqsubseteq_{n^u} x_G$. By Lemma 5.12(d), for all $m \leqslant n^u$, $c^u(m)$, which is defined to be $c^s(m)$, equals $c^{x_G}(m)$.

For all $\ell < n^u$, since $u \in \mathbb{R}$, $k^{x_G}(\ell) = k^u(\ell)$ codes $\zeta\upharpoonright(c^{x_G}(\ell)+1)$, and $\zeta \prec y_G$. ☐

We define β-complete conditions as usual.

Lemma 5.29. *For every $\beta \leqslant \alpha + 1$, every $p \in \mathbb{R}$ can be extended to a β-complete condition in $\mathbb{R}$.*

Proof. The proof of Lemma 4.15 applies; since we are assuming that $\alpha = \alpha^* + 2$, we do not need to consider the case $|\sigma| \leqslant 1$. ☐

We define $p\upharpoonright\beta$ as usual.

Lemma 5.30. *For all $\beta \leqslant \alpha+1$, for every $p \in \mathbb{R}$, $p\restriction\beta \in \mathbb{R}$.*

Proof. Let $p = (u, \zeta) \in \mathbb{R}$. By Lemma 5.21, $u\restriction\beta$ is supported. If $m \leqslant n^{u\restriction\beta}$ then by Lemma 5.19 (or the proof of Lemma 5.21), $c^{u\restriction\beta}(m) = c^u(m)$, and when $m < n^{u\restriction\beta}$, $k^{u\restriction\beta}(m) = k^u(m)$, so codes correctly; we conclude that $p\restriction\beta = (u\restriction\beta, \zeta) \in \mathbb{R}$. □

We can now continue with analogues of Lemma 4.16 and of Proposition 2.30 and Lemma 3.6. Using the tools above, we can mimic the proof of Theorem 4.17 to obtain:

Theorem 5.31. *If $\alpha \geqslant 3$ is the successor of a successor ordinal, then the graph H_α does not have a countable $\mathbf{\Sigma}^0_\alpha$ colouring.*

5.5. **Non-minimality of L_3.** We show that L_3 is not least among graphs with no $\mathbf{\Sigma}^0_3$ colourings:

Proposition 5.32. *There is no continuous homomorphism from L_3 into H_3.*

The argument will be an elaboration on the proof of Proposition 4.23. The difficulty is that we must construct elements of $\mathbb{W}_3$, rather than $\mathbb{X}_3$; that is, we need to make sure that we are coding correctly. Our advantage over the opponent is that we can keep changing our mind between, say, coding at $n = 0$ and coding at some large value ℓ^*, whereas if the opponent stops coding at some current version of $c(n^*)$, and later wants to go back to $c(n^*)$, the new version of $c(n^*)$ is forced to be large.

We need an elaboration on Lemma 4.22.

Lemma 5.33. *Let $s \in 2^{<\mathscr{L}_3}$; let $m \geqslant n^s$, and let $\zeta \in 2^{<\omega}$ with $|\zeta| = m$. There is some $t \in 2^{<\mathscr{L}_3}$ such that:*

- *$n^t = m$;*
- *$s \sqsubseteq_{n^s} t$;*
- *For all ℓ with $n_s \leqslant \ell < m$, $k^t(\ell)$ codes $\zeta\restriction(\ell+1)$.*

The proof is similar; we can always just extend by mostly 0's, and place 1's in the correct places to increase $k^t(\ell)$ to a desired value if necessary, and bring down n^t to the required value.

Proof of Proposition 5.32. Let $F\colon \mathbb{W}_3 \to \mathbb{Y}_3$ be continuous. As in the proof of Proposition 4.23, we define a sequence (s_ℓ), define t^a_ℓ, ζ^a_ℓ, and t^b_ℓ, ζ^b_ℓ so that $F[s_\ell, 0^\ell] \subseteq [t^a_\ell, \zeta^a_\ell]$ and $F[s_\ell, 1\hat{\ }0^{\ell-1}] \subseteq [t^b_\ell, \zeta^b_\ell]$. It will be convenient for us to have $|\zeta^a_\ell| \leqslant |t^a_\ell|$, so we can define t^a_ℓ as above, and then let ζ^a_ℓ be the longest ζ such that $|\zeta| \leqslant |t^a_\ell|$ and $F[s_\ell, 0^\ell] \subseteq [t^a_\ell, \zeta]$.

Further, if the construction ever leaves phase 1 below, say at stage ℓ^*, then we will analogously define (t^d_ℓ, ζ^d_ℓ) such that $F[s_\ell, 0^{\ell^*}\hat{\ }1\hat{\ }0^\omega] \subseteq [t^d_\ell, \zeta^d_\ell]$.

To avoid repetition, we define the following.

- *Working toward* 0 *at stage* ℓ means extending s_ℓ to $s_{\ell+1}$ satisfying $n^{s_{\ell+1}} = 0$.
- Suppose that $\ell^* \in \mathbb{N}$, and that w is a binary string of length ℓ^*. A string $t \in 2^{<\mathscr{L}_3}$ is (ℓ^*, w)-*admissible* if $n^t = \ell^*$ and for all $m < \ell^*$, $k^t(m)$ codes $w\restriction(m+1)$.

  Working toward (ℓ^*, w) (at stage ℓ of the construction) means extending s_ℓ to a string $s_{\ell+1}$ that is (ℓ^*, w)-admissible, and further, if s_ℓ itself is already (ℓ^*, w)-admissible, then $s_\ell \sqsubset_{\ell^*} s_{\ell+1}$.

Below we describe the phases of the construction. To avoid clutter, instead of specifying a separate phase analogous to phase 3 of the previous construction, we just declare that if we ever see that t^a_ℓ and t^b_ℓ are incomparable, then we move to a terminal phase at which at every stage we work toward 0. Hence, for the rest of the description of the construction, we assume that t^a_ℓ and t^b_ℓ are comparable.

Similarly, if the construction ever leaves phase 1, say at stage ℓ^*, and we later see that t^a_ℓ and t^d_ℓ are incomparable, then the construction enters a special, terminal phase at which we always work toward $(\ell^*, 0^{\ell^*})$. Hence, for the rest of the description of the description of the construction, after we leave phase 1, we assume that t^a_ℓ and t^d_ℓ are comparable.

Phase 1: ζ^a_ℓ and ζ^b_ℓ are comparable. Work toward 0.

We leave phase 1 when we see that ζ^a_ℓ and ζ^b_ℓ are incomparable. If this happens, we define:

- $c_0 = \min(\zeta^a_\ell \bigtriangleup \zeta^b_\ell)$;
- $r_0 = t^a_\ell \restriction c_0$; and
- $n_0 = n^{r_0}$.

If there is some $r \sqsubset_{n_0} r_0$ then we enter a terminal phase in which we work toward 0. Also, if we ever see that $r_0 \not\sqsubseteq_{n_0} t^a_\ell$, we enter a terminal phase in which we work toward 0. (Observe that by Lemma 5.9(d), once we see that $r_0 \not\sqsubseteq_{n_0} t^a_\ell$, the same holds for all $\ell' > \ell$.)

Phase 2: ζ^a_ℓ and ζ^d_ℓ are comparable. Let ℓ^* be the stage at which the construction leaves phase 1. Work toward $(\ell^*, 0^{\ell^*})$.

We leave phase 2 when we see that ζ^a_ℓ and ζ^d_ℓ are incomparable. If this happens, we define:

- $c_1 = \min(\zeta^a_\ell \bigtriangleup \zeta^d_\ell)$;
- $r_1 = t^a_\ell \restriction c_1$; and
- $n_1 = n^{r_1}$.

If there is some $r \sqsubset_{n_1} r_1$ then we enter a terminal phase in which we work toward $(\ell^*, 0^{\ell^*})$. Also similarly to above, if we ever see that $r_1 \not\sqsubseteq_{n_1} t^a_\ell$, we enter a terminal phase in which we work toward $(\ell^*, 0^{\ell^*})$. If this is not the case, we stay in phase 3:

Phase 3. Work toward 0.

For the verification, we let $z = \bigcup_\ell s_\ell$, $a = (z, 0^\omega)$, $b = (z, 1\hat{}0^\omega)$, and if the construction ever leaves phase 1, we let $d = (z, 0^{\ell^*}\hat{}1\hat{}0^\omega)$. We observe:

- If in the final phase of the construction we work toward $(\ell^*, 0^{\ell^*})$ then $n^z = \ell^*$ and for all $m < \ell^*$, $k^z(m)$ codes 0^{m+1} (this follows from Lemma 5.10(a),(b)). Hence, in this case, both a and d are in $\mathbb{W}_3$, and a is connected to d by an edge of L_3.
- If in the final phase of the construction we work toward 0, then $n^z = 0$, and in this case, a and b are in $\mathbb{W}_3$ and are connected by an edge of L_3.

We define $F(a) = (x^a, y^a)$, $F(b) = (x^b, y^b)$. We obtain "easy victory" if $x^a \neq x^b$, or if we never leave phase 1. Suppose that we do leave phase 1 at stage ℓ^*, and that $x^a = x^b = x$. We observe:

- If $F(a)$ and $F(b)$ are connected by an edge of H_3 then $n^x = n_0$ and $c^x(n_0) = c_0$, so that r_0 is the shortest r with $r \sqsubset_{n_0} x$. This follows from $c^x(n^x) = c_0$, and Lemma 5.9(f).

Hence, if there is some $r \sqsubset_{n_0} r_0$, then a and b are connected by an edge, and $F(a)$ and $F(b)$ are not. Similarly, if $r_0 \not\sqsubset_{n_0} x$, i.e., if we ever see that $r_0 \not\sqsubset_{n_0} t^a_\ell$, then we win by the same outcome. Suppose that this is not the case.

Since we are assuming that we do leave phase 1, let $F(d) = (x^d, y^d)$. If $x^d \neq x^a$, or if we never leave phase 2, then we again obtain easy victory. So we assume that $x^a = x^b = x^d = x$, and that we leave phase 2 at some stage. Similarly to above:

- If $F(a)$ and $F(d)$ are connected by an edge then $n^x = n_1$ and r_1 is shortest with $r_1 \sqsubset_{n_1} x$.

Similarly to above, we may assume that $r_1 \sqsubset_{n_1} x$, and is the shortest such.

Now, the thing to observe is that $c_1 > c_0$. This is because, as in the previous construction, y^a and y^d must both extend $\zeta^a_{\ell *}$, and $|\zeta^a_{\ell *}| > c_0$. Hence, $r_0 \prec r_1 \prec x$.

So we are assuming that the final phase of the construction is phase 3; $r_0 \sqsubset_{n_0} x$ and $r_1 \sqsubset_{n_1} x$. By Lemma 5.9(d), $r_0 \sqsubset_{n_0} r_1$. The minimality of r_1 implies that $n_0 < n_1$. The fact that $r_1 \sqsubset_{n_1} x$ implies $n^x \geqslant n_1$, in particular, $n^x \neq n_0$. Hence, $F(a)$ and $F(b)$ are not connected by an edge, whereas a and b are. □

6. An embedding result

In this section we show that H_3 is minimal for non-$\mathbf{\Sigma}^0_3$-colourability.

Theorem 6.1. *Let X be a computably presented Polish space, let G be a Σ^1_1 directed graph on X, and suppose that there is no countable $\mathbf{\Sigma}^0_3$ colouring of G. Then there is a continuous graph homomorphism from $(\mathbb{Y}_3, H_3)$ to (X, G).*

The argument is similar to the one given in [LZ14a]; we give it for completeness.

Preparation. To define the continuous map (and further witnesses for its success), we define a collection of initial segments of elements of $\mathbb{Y}_3$.

Definition 6.2. We let $\mathbb{T}$ be the collection of pairs $(t, \zeta) \in 2^{<\mathscr{L}_3} \times 2^{<\omega}$ satisfying:

(i) $|\zeta| = |t| + 1$;
(ii) For all $m < n^t$, $k^t(m)$ is a code of $\zeta \upharpoonright (c^t(m) + 1)$.

Observe that by Lemma 5.12(a), for all $m < n^t$, $c^t(m) \leqslant |t|$, so $|\zeta| > |t|$ ensures that the strings $\zeta \upharpoonright (c^t(m) + 1)$ make sense.

For brevity, for $p = (t^p, \zeta^p) \in \mathbb{T}$ we write:

- $|p| = |t^p|$;
- $n^p = n^{t^p}$;
- For $m < n^p$, $k^p(m) = k^{t^p}(m)$;
- For $m \leqslant n^p$, $c^p(m) = c^{t^p}(m)$.

We also define, for $p, q \in \mathbb{T}$,

- $p \preccurlyeq q$ if $t^p \preccurlyeq t^q$ and $\zeta^p \preccurlyeq \zeta^q$;
- $p \sqsubseteq q$ if $p \preccurlyeq q$ and further, $t^p \sqsubseteq_{n^p} t^q$.

Note that there are two conditions $p \in \mathbb{T}$ with $|p| = 0$, namely, $(\langle\rangle, \langle 0 \rangle)$ and $(\langle\rangle, \langle 1 \rangle)$; $n^{\langle\rangle} = 0$, so no coding is required. For any $q \in \mathbb{T}$, If $\zeta^q \succcurlyeq \langle i \rangle$, then $(\langle\rangle, \langle i \rangle) \sqsubseteq q$. By Lemma 5.12(d), if $p \sqsubseteq q$ then for all $m \leqslant n^p$, $c^p(m) = c^q(m)$.

Lemma 6.3. *Let $p, q, r \in \mathbb{T}$.*

(a) *$\sqsubseteq$ is transitive.*
(b) *If $p \preccurlyeq q \preccurlyeq r$ and $p \sqsubseteq r$ then $p \sqsubseteq q$.*

Proof. (a): Suppose that $p \sqsubseteq q \sqsubseteq r$, so $p \preccurlyeq q \preccurlyeq r$ and $t^p \sqsubseteq_{n^p} t^q \sqsubseteq_{n^q} t^r$. Then $p \preccurlyeq r$. Since $n^p \leqslant n^q$, we have $t^q \sqsubseteq_{n^p} t^r$ (Lemma 5.9(b)); now apply Lemma 5.9(a).

(b) follows from Lemma 5.9(d). ◻

For the following lemma, we extend the relations $p \preccurlyeq q$ and $p \sqsubseteq q$ to elements of $\mathbb{Y}_3$.

Lemma 6.4.

(a) *Suppose that $p \in \mathbb{T}$ or $p = (t^p, \zeta^p) \in \mathbb{Y}_3$. If $s \sqsubset_{n^s} t^p$ then $q = (s, \zeta^p \restriction (|s|+1))$ is in $\mathbb{T}$ (and $q \sqsubseteq p$).*

(b) *For every $a \in \mathbb{Y}_3$ there are infinitely many $p \sqsubset a$ in $\mathbb{T}$.*

Proof. For (a), suppose that $s \sqsubset_{n^s} t^p$. Let $m < n^s$. Then $k^s(m) = k^{t^p}(m)$, and by Lemma 5.12(d), $c^s(m) = c^{t^p}(m)$. Since $p \in \mathbb{T} \cup \mathbb{Y}_3$, $k^{t^p}(m)$ codes $\zeta^p \restriction (c^{t^p}(m)+1)$. This shows that $(s, \zeta^p \restriction (|s|+1)) \in \mathbb{T}$.

(b) follows from (a) and Lemma 5.10(c). ◻

Definition 6.5. Let $p \in \mathbb{T}$.

(a) We let $\hat{p}$ be the pair $(t^p, \hat{\zeta}^p)$, where $|\hat{\zeta}^p| = |\zeta^p|$ and $\hat{\zeta}^p \triangle \zeta^p = \{c^p(n^p)\}$.

(b) We call p a *lefty* if $\zeta(c^p(n^p)) = 0$, a *righty* otherwise.

(c) If $|p| > 0$, we let $v(p)$ be the longest $q \sqsubset p$ in $\mathbb{T}$.

Note that $\hat{\hat{p}} = p$. The definition of $v(p)$ makes sense, since if $|p| > 0$ then there is some $q \in \mathbb{T}$ such that $q \sqsubset p$, namely one of $(\langle\rangle, \langle 0\rangle)$ or $(\langle\rangle, \langle 1\rangle)$.

For the proof of the following lemma, and below, we recall that since we are working with $\alpha = 3$, we have $\alpha^* = 1$; so for $s, t \in 2^{<\mathscr{L}_3}$, $s \prec_{\alpha*} t$ means $s \prec t$ (Lemma 5.2(a)).

Lemma 6.6. *Let $p \in \mathbb{T}$ with $|p| > 0$.*

(a) *If $v(p) \prec q \prec p$ then $n^q > n^p$.*

(b) *If $q \sqsubseteq p$ and $n^q = n^p$ then p and q have the same orientation (they are both lefties or both righties).*

(c) *If $n^{v(p)} = n^p$ then $v(\hat{p}) = \widehat{v(p)}$, and either $v(p)$ is the $\prec$-predecessor of p, or $v(\hat{p})$ is the $\prec$-predecessor of $\hat{p}$.*

(d) *If $n^{v(p)} < n^p$ then $v(\hat{p}) = v(p)$, and $v(p)$ is the $\prec$-predecessor of both p and $\hat{p}$.*

Proof. By Lemma 6.4(a), $t^{v(p)}$ is the longest s with $s \sqsubset_{n^s} t^p$. Then (a) follows from Lemma 5.13(a). Observe also that $|v(p)|$ only depends on t^p, hence, $t^{v(p)} = t^{v(\hat{p})}$, in particular, $|v(p)| = |v(\hat{p})|$.

(b) follows from the fact that $c^q(n^q) = c^p(n^p)$ (Lemma 5.12(d)) and $\zeta^q \preccurlyeq \zeta^p$.

(c): suppose that $n^{v(p)} = n^p$. Let $n = n^p$ and $c = c^p(n)$. By (b), $c^{v(p)}(n) = c$. Since $|v(p)| = |v(\hat{p})|$, we get that $v(\hat{p}) = \widehat{v(p)}$.

For any s with $t^{v(p)} \prec s \prec t^p$ we have $n^s > n$ and $t^{v(p)} \sqsubset_n s$ (Lemma 5.13(a) and Lemma 5.9(d)), so $c^s(n) = c$ (Lemma 5.12(d) again). By Lemma 5.13(b), let k be the constant value $k^s(n)$ for all s with $t^{v(p)} \prec s \prec t^p$.

Suppose that $v(p)$ is not the $\prec$-predecessor of p in $\mathbb{T}$, i.e., that there is some $q \in \mathbb{T}$ with $v(p) \prec q \prec p$. Since $q \in \mathbb{T}$, $n^q > n$, $c^q(n) = c$ and $k^q(n) = k$, k codes $\zeta^p \restriction (c+1)$. For any s with $t^{v(p)} \prec s \prec t^p$, $k = k^s(n)$ cannot code $\zeta^{\hat{p}} \restriction (c+1)$, since $\zeta^p(c) \neq \zeta^{\hat{p}}(c)$. Hence, $(s, \zeta^{\hat{p}} \restriction (|s|+1))$ cannot be in $\mathbb{T}$. This shows that $v(\hat{p})$ is the $\prec$-predecessor of $\hat{p}$ in $\mathbb{T}$.

(d): suppose that $n^{v(p)} < n^p$. By Lemma 5.13(c), $c^p(n^p) = |p|$. Since $|v(p)| < |p|$, this implies that $\zeta^p \restriction (|v(p)|+1) = \zeta^{\hat{p}} \restriction (|v(p)|+1)$, so $v(\hat{p}) = v(p)$. Further, Lemma 5.13(c) says that $t^{v(p)}$ is the $<$-predecessor of t^p, i.e., $|p| = |v(p)|+1$, whence there cannot be any q with $v(p) \prec q \prec p$ (or $\prec \hat{p}$). □

Construction. Suppose that X is a computably presented Polish space, G is a Σ^1_1 directed graph on X, and that there is no countable $\boldsymbol{\Sigma}^0_3$ colouring of G. Following [LZ14a, Theorem 5.1], we obtain a nonempty $\boldsymbol{\Pi}^0_2$, Σ^1_1 set $Y \subseteq X$ with the property that for every closed, Σ^1_1 set $V \subseteq X$, if $V \cap Y \neq \varnothing$ then $V \cap Y$ is not G-independent.

We replace G by $G \cap Y^2$ (note that this keeps the edge relation Σ^1_1), so we assume that $G \subseteq Y^2$.

For each $p \in \mathbb{T}$ we will define:

- A point $x_p \in X$;
- A rational open ball $X_p \subseteq X$;
- An effectively closed set $D_p \subseteq X^2 \times \omega^\omega$.

We will ensure that:

(i) $x_p \in X_p$.
(ii) $D_p = D_{\hat{p}}$.
(iii) The projection U_p of D_p onto X^2 is a subset of G.
(iv) If p is a lefty, then $(x_p, x_{\hat{p}}) \in U_p$.
(v) If p is a lefty, then $U_p \subseteq X_p \times X_{\hat{p}}$.
(vi) If $|p| > 0$ then the diameters of X_p and of D_p are $\leqslant 1/|p|$.
(vii) If $p \prec q$ then $\overline{X}_q \subseteq X_p$.
(viii) If $q \sqsubseteq p$ and $n^p = n^q$ then $D_p \subseteq D_q$.
(ix) If $q \sqsubseteq p$ and $n^q < n^p$, and q is a lefty, then every (left or right) endpoint of an edge in U_p is a limit point of points which are left endpoints of edges in U_q; analogously if q is a righty.

Note that (ii) and (iv) imply that if p is a righty, then $(x_{\hat{p}}, x_p) \in U_p$, and similarly, (v) implies that when p is a righty, $U_p \subseteq X_{\hat{p}} \times X_p$. Note that (iv) implies that for all p, $x_p \in Y$, since we modified G so that $G \subseteq Y^2$. Also observe that (iv) and (ix) imply that if $q \sqsubseteq p$ and $n^q < n^p$, then both x_p and $x_{\hat{p}}$ are limits of points that are left endpoints of edges in U_q (or right, depending on the orientation of q).

Let $p \in \mathbb{T}$; we suppose that the construction has been performed for all $q \prec p$ in $\mathbb{T}$. We will consider both p and $\hat{p}$ at the same time. There are several cases.

First case: $|p| = 0$. We let $D_p = D_{\hat{p}}$ be an effectively closed set projecting to G; $X_p = X_{\hat{p}} = X$ and we choose $x_p, x_{\hat{p}}$ so that $(x_p, x_{\hat{p}}) \in G$, where $p = (\langle\rangle, \langle 0 \rangle)$ is the lefty condition with $|p| = 0$.

Suppose that $|p| > 0$, so $v(p)$ and $v(\hat{p})$ are defined.

Second case: $n^{v(p)} < n^p$. This situation will be symmetric between p and $\hat{p}$, so suppose that p is a lefty.

For each q with $q \sqsubseteq v(p)$,

- If q is a lefty, let K_q be the collection of left end-points of edges in U_q;
- If q is a righty, let K_q be the collection of right end-points of edges in U_q.

Let Z be a rational open ball centered at $x_{v(p)}$ whose closure is contained in $X_{v(p)}$. Let

$$R = Y \cap \overline{Z} \cap \bigcap_{q \sqsubseteq v(p)} \overline{K_q}.$$

Note that R is the intersection with Y of a closed Σ^1_1 set. We claim that $x_{v(p)} \in R$. Let $q \sqsubseteq v(p)$. To see that $x_{v(p)} \in \overline{K_q}$, there are two cases. If $n^q < n^{v(p)}$, then $x_{v(p)} \in K_q$ is guaranteed by requirement (ix) of the construction, which by induction, holds for $v(p)$. Suppose that $n^q = n^{v(p)}$. Then by requirement (viii), $U_{v(p)} \subseteq U_q$. By Lemma 6.6(b), $v(p)$ and q have the same orientation. By (iv), $x_{v(p)}$ is a left / right end-point of an edge in $U_{v(p)}$, hence of U_q, and so $x_{v(p)} \in K_q$.

By the main property of Y, we can choose x_p and $x_{\hat{p}}$ in R that are connected by an edge of G. We then choose sufficiently small neighbourhoods X_p and $X_{\hat{p}}$ of x_p and $x_{\hat{p}}$, subsets of Z, and sufficiently small $D_p = D_{\hat{p}} \subseteq D_{v(p)}$ that projects to a subset of $G \cap R^2 \cap (X_p \times X_{\hat{p}})$, and whose projection contains the edge $(x_p, x_{\hat{p}})$.

Let us verify that the requirements of the construction hold for p and $\hat{p}$. The requirements (i)–(vi) are immediate by our choices. To verify (vii), let $q \in \mathbb{T}$, and suppose that either $q \prec p$ or $q \prec \hat{p}$. By Lemma 6.6(d), $q \preccurlyeq v(p)$. By induction, $X_{v(p)} \subseteq X_q$, and by construction, $\overline{X}_p, \overline{X}_{\hat{p}} \subseteq X_{v(p)}$.

To verify (viii) and (ix), suppose that $q \sqsubset p$ or $q \sqsubset \hat{p}$. By Lemma 6.3(b), $q \sqsubseteq v(p)$. Since $n^q \leqslant n^{v(p)} < n^p$, in this case (viii) holds vacuously; (ix) holds by construction, since we ensured that $U_p \subseteq R^2$, so all endpoints of edges in U_p are in $R \subseteq \overline{K_q}$.

In the third and fourth cases, we assume that $n^{v(p)} = n^p$.

Third case: $v(p)$ is the $\prec$-predecessor of p, and $v(\hat{p})$ is the $\prec$-predecessor of $\hat{p}$. In this case we let $x_p = x_{v(p)}$, $x_{\hat{p}} = x_{v(\hat{p})}$, and choose $X_p \subseteq X_{v(p)}$, $X_{\hat{p}} \subseteq X_{v(\hat{p})}$, and $D_p \subseteq D_{v(p)}$, appropriately small so that requirements (iv), (v), and (vi) are satisfied. All the other requirements follow by induction, using the fact that $\widehat{v(p)} = v(\hat{p})$ (Lemma 6.6(c)).

Fourth case: The third case does not hold. Without loss of generality, suppose that $v(p)$ is not the $\prec$-predecessor of p in $\mathbb{T}$ (we can therefore not assume that p is a lefty). Let p^- be the $\prec$-predecessor of p. By Lemma 6.6(a), $n^{p^-} > n^p = n^{v(p)}$. By induction, x_{p^-} is a limit point of endpoints of edges of $U_{v(p)}$, left or right depending on the orientation of $v(p)$, which is the same as the orientation of p (Lemma 6.6(b)). We therefore can choose x_p and $x_{\hat{p}}$ such that:

- $x_p \in X_{p^-}$; and
- The edge $(x_p, x_{\hat{p}})$ (or the reverse, according to parity) is in $U_{v(p)}$.

We observe that $x_{\hat{p}} \in X_{v(\hat{p})}$, since (v) holds for $v(p)$. As above, we choose small neighbourhoods X_p and $X_{\hat{p}}$ of x_p and $x_{\hat{p}}$, and choose sufficiently small $D_p \subseteq D_{v(p)}$ so that $(x_p, x_{\hat{p}})$ belongs to U_p, and $U_p \subseteq X_p \times X_{\hat{p}}$ (or the reverse).

For verifying that the requirements hold, the main fact is Lemma 6.6(c), that says that $\widehat{v(p)} = v(\hat{p})$ is the $\prec$-predecessor of $\hat{p}$ in $\mathbb{T}$ (this is really the heart of the construction, the reason that H_3 is minimal and L_3 is not). This mainly impacts (vii). Suppose that $q \prec p$. Then $q \preccurlyeq p^-$; so $\overline{X}_p \subseteq X_{p^-} \subseteq X_q$. On the other hand, if $q \prec \hat{p}$, then $q \preccurlyeq v(\hat{p})$, and $\overline{X}_{\hat{p}} \subseteq X_{v(\hat{p})} \subseteq X_q$.

For (viii) and (ix), suppose that $q \sqsubset p$ or $q \sqsubset \hat{p}$; then $q \sqsubseteq v(p)$ or $q \sqsubseteq v(\hat{p})$. If $n^q = n^p = n^{v(p)}$ then by induction, $D_{v(p)} = D_{v(\hat{p})} \subseteq D_q$, and $D_p \subseteq D_{v(p)}$. If $n^q < n^p$ then by induction, every end point of an edge in $U_{v(p)}$ is a limit of left / right endpoints of edges in U_q, and $U_p \subseteq U_{v(p)}$. [Note that it is only here, to keep the induction going, that we use the full (ix), rather than just assuming that x_p is a limit points of endpoints of U_q, which is what was used up until now.]

Verification. Having performed the construction of x_p, X_p, D_p for all $p \in \mathbb{T}$, we define $F\colon \mathbb{Y}_3 \to X$ as follows:

- For $a \in \mathbb{Y}_3$, we let $F(a)$ be the limit of $\{x_p : p \in \mathbb{T} \ \& \ p \prec a\}$.

Here we use the properties of the construction, in particular (vii), as well as Lemma 6.4(b), that ensures that the diameters of X_p for $p \prec a$ indeed go to 0, to see that F is well-defined and continuous.

We show that F is a graph homomorphism. Suppose that (a, b) is an edge of H_3. Then $a = (z, w)$ and $b = (z, w')$, where $n^z < \omega$ and $w \triangle w' = \{c^z(n^z)\}$. For all but finitely many $p \sqsubset a$ in $\mathbb{T}$ we have $n^p = n^z$, and for these p we have $\hat{p} \sqsubset b$. If $p, q \sqsubset a$ and $p \preccurlyeq q$ then $p \sqsubseteq q$ (Lemma 6.3(b)); if $n^p = n^z = n^q$ then by requirement (viii), $D_q \subseteq D_p$. Since each D_p is closed and their diameters shrink to 0, $\bigcap \{D_p : p \sqsubseteq a \ \& \ n^p = n^z\}$ is nonempty; by (v), this intersection necessarily projects to $(F(a), F(b))$, so we get $(F(a), F(b)) \in G$, as required.

This completes the proof of Theorem 6.1.

Remark 6.7. Instead of defining shrinking, closed $D_p \subseteq X^2 \times \omega^\omega$ that project to U_p, we can (as is done in [LZ14a]) define $U_p \subseteq \{(x, x') \in X^2 : \omega_1^{(x,x')} = \omega_1^{\text{ck}}\}$. The latter set (call it Ω_{X^2}) is Σ^1_1, and the restriction of the Gandy-Harrington topology to Ω_{X^2} is Polish (whereas the Gandy-Harrington topology on all of X^2 is not). We can then simply require that the sets U_p are shrinking in a metric that gives the Gandy-Harrington topology on Ω_{X^2}. The construction is essentially the same.

Remark 6.8. The argument above shows the following. Let $\alpha \geqslant 3$. Let X be a computably presented Polish space, let G be a Σ^1_1 graph on X, and suppose that there is no countable $\Sigma^0_\alpha(\Delta^1_1)$ colouring of G. Then there is a graph homomorphism $g\colon (\mathbb{Y}_\alpha, H_\alpha) \to (X, G)$ such that the pullback by g of any $\Sigma^0_{\alpha*}(\Delta^1_1)$ set is $\tau_{\alpha*}$-open.

To see this, we can repeat the argument, except that we define $p \preccurlyeq q$ to mean $t^p \preccurlyeq_{\alpha*} t^q$ and $\zeta^p \preccurlyeq \zeta^q$. There are no new ideas needed, so for length considerations, we omit the details.

7. Separators of iterated Fréchet ideals

Here we give a new proof of a result of Debs and Saint Raymond [DSR09], using our forcing methods and untagging. Day and Marks [DM18] gave another proof using forcing, though theirs is a different forcing notion and does not make use of untagging. The theorem (Theorem 7.2 below) is not stated explicitly in [DSR09], but follows immediately from Theorem 3.2 and the proof of Theorem 6.5 in that paper.

Recall that the first Fréchet ideal is the ideal of finite sets; the second is the ideal of sets, all but finitely many of whose columns are finite, and in general, the α^{th} iterate of the Fréchet ideal are those sets such that for almost all n, their n^{th} column belongs to the $(\alpha_n)^{\text{th}}$ iterate. Thus the natural "playing ground" of the α^{th} ideal is $2^{\mathscr{L}_\alpha}$. We cast the definition in these terms.

Definition 7.1. For $x \in 2^{\mathscr{L}_\alpha}$, the *filter labelling* $F^x = F^x_\alpha$ of T_α is defined as follows:

- For each $\sigma \in \mathscr{L}_\alpha$, $F^x(\sigma) = x(\sigma)$;
- For each $\sigma \in T_\alpha \setminus \mathscr{L}_\alpha$,

$$F^x(\sigma) = 1 \iff \{k \in \omega : F^x(\sigma\hat{\ }k) = 1\} \text{ is cofinite.}$$

The *ideal labelling* $I^x = I^x_\alpha$ of T_α is the dual:

- For each $\sigma \in \mathscr{L}_\alpha$, $I^x(\sigma) = x(\sigma)$;
- For each $\sigma \in T_\alpha \setminus \mathscr{L}_\alpha$,

$$I^x(\sigma) = 0 \iff \{k \in \omega : I^x(\sigma\hat{\ }k) = 0\} \text{ is cofinite.}$$

The α^{th} *iterate of the Fréchet filter* is the set

$$\mathscr{F}_\alpha = \{x \in 2^{\mathscr{L}_\alpha} : F^x(\langle\rangle) = 1\}.$$

The α^{th} *iterate of the Fréchet ideal* is the dual:

$$\mathscr{I}_\alpha = \{x \in 2^{\mathscr{L}_\alpha} : I^x(\langle\rangle) = 0\}.$$

Our objective is the following theorem.

Theorem 7.2 (Debs & Saint Raymond, [DSR09])**.** *The α^{th} iterates of the Fréchet filter and ideal cannot be separated by a $\mathbf{\Delta}^0_{\alpha+1}$ set.*

We will make use of a modified notion of forcing.

Definition 7.3. We let $\mathbb{U}$ be the collection of all finite partial functions $p : T_\alpha \to \{0, 1, \texttt{both}\}$ satisfying: if $\sigma \in \mathscr{L}_\alpha \cap \operatorname{dom} p$, then $p(\sigma) \in \{0,1\}$.

The set $\mathbb{U}$ is partially ordered as follows: for $p, q \in \mathbb{U}$, $q \leqslant p$ if and only if:

- $p \subseteq q$; and
- If $p(\sigma) \in \{0,1\}$, then for all k with $\sigma\hat{\ }k \in \operatorname{dom} q \setminus \operatorname{dom} p$, $q(\sigma\hat{\ }k) = p(\sigma)$.

Note that the set $\mathbb{U}$ is simpler than the set $\mathbb{Q}$, but the extension relation is more complicated.

For a filter $G \subset \mathbb{U}$, as above we define $x_G = \bigcup G \restriction \mathscr{L}_\alpha$. The labelling $\bigcup G$ is neither F^x or I^x, but it does indicate which pieces of x_G lie in the appropriate filters and ideals. The following is the analogue of Lemma 2.11, proven by induction on the rank of σ. In analogy with Definition 2.8, we let $\mathscr{D}_{\text{point}}(\mathbb{U})$ be the collection of the following dense subsets of $\mathbb{U}$:

- The sets $\{p \in \mathbb{U} : p(\sigma)\downarrow\}$ for $\sigma \in T_\alpha$;
- The sets $\{p \in \mathbb{U} : p \leqslant q \vee p \perp q\}$ for $q \in \mathbb{U}$; and
- For non-leaf $\sigma \in T_\alpha$, $i \in \{0,1\}$, and $k \in \mathbb{N}$, the sets

$$\{p \in \mathbb{U} : p(\sigma) = \texttt{both} \ \to \ (\exists m > k)\ p(\sigma\hat{\ }m) = i\}.$$

Lemma 7.4. *Suppose that $G \subset \mathbb{U}$ is $\mathscr{D}_{\text{point}}(\mathbb{U})$-generic. Then $x_G \in 2^{\mathscr{L}_\alpha}$, and for all σ, $F^{x_G}(\sigma) = 1 \leftrightarrow (\bigcup G)(\sigma) = 1$, and $I^{x_G}(\sigma) = 0 \leftrightarrow (\bigcup G)(\sigma) = 0$.*

Define $p_0, p_1 \in \mathbb{U}$ by $p_0(\langle\rangle) = 0$, $p_1(\langle\rangle) = 1$, and both are undefined everywhere else. Then $\mathscr{F}_\alpha$ is equivalent to $[p_1]$ modulo a $\tau_{\mathbb{U}}$-meagre set, and the same for $\mathscr{I}_\alpha$ and $[p_0]$.

We have a modified version of restriction.

Definition 7.5. For $p \in \mathbb{U}$ and $\beta \leqslant \alpha$, we define a condition $p\restriction\beta \subseteq p$ as follows: for $\sigma \in \operatorname{dom} p$, $\sigma \in \operatorname{dom} p\restriction\beta$ if and only if either:

- $\mathrm{rk}(\sigma) < \beta$; or
- $\mathrm{rk}(\sigma) = \beta$ and $p(\sigma) \neq \texttt{both}$.

Observe that we “gain an ordinal” compared to the previous notion of restriction; for example, $p \upharpoonright \mathscr{L}_\alpha$ is $p \upharpoonright 0$, not $p \upharpoonright 1$. This will occur again in the modified untagging lemma.

We use the same definition of strong forcing for this new notion of forcing, and the analog of Proposition 2.21 is by the same proof.

We again define a notion of β-completeness. Again the idea is that if $\mathrm{rk}(\sigma)$ is a limit, $\beta < \mathrm{rk}(\sigma)$, and $p(\sigma)$ tells us what the values of all undefined $p(\sigma\hat{}k)$ should be, then p records these values where $\mathrm{rk}(\sigma\hat{}k) < \beta$.

Definition 7.6. For $\beta \leqslant \alpha$, we say that $p \in \mathbb{U}$ is *β-complete* if for all $\sigma \in \mathrm{dom}\, p$ with $\mathrm{rk}(\sigma) > \beta$ and $p(\sigma) \neq \texttt{both}$, $p(\sigma\hat{}k)\downarrow$ for all k such that $\mathrm{rk}(\sigma\hat{}k) < \beta$.

The following density of β-complete conditions is straightforward.

Lemma 7.7. *Let $\beta \leqslant \alpha$. For all $p \in \mathbb{U}$, there is a β-complete p' extending p.*

We have our version of the key technical lemma.

Lemma 7.8. *Let $\gamma' < \gamma \leqslant \alpha$. If $p \in \mathbb{U}$ is γ-complete and r extends $p \upharpoonright \gamma$, then $(r \upharpoonright \gamma') \cup p$ extends both p and $r \upharpoonright \gamma'$.*

Proof. Extending $r \upharpoonright \gamma'$ is immediate by definition.

To show extension of p, the only concern is that there might be $\sigma \in \mathrm{dom}\, p$ with $p(\sigma) \neq \texttt{both}$, and k with $\sigma\hat{}k \in \mathrm{dom}(r \upharpoonright \gamma') \setminus \mathrm{dom}\, p$. But this indicates that $\mathrm{rk}(\sigma\hat{}k) \leqslant \gamma' < \gamma$, so by the γ-completeness of p we know that $\mathrm{rk}(\sigma) \leqslant \gamma$, and so $\sigma \in \mathrm{dom}(p \upharpoonright \gamma)$. By definition of the extension relation $(r \upharpoonright \gamma')(\sigma\hat{}k) = r(\sigma\hat{}k) = p(\sigma)$, so there is no obstacle to extending p. ☐

Now we have our untagging lemma. Note that in contrast with Proposition 2.28, the somewhat different definition of $p \upharpoonright \gamma$ allows us to “gain a quantifier”; to force a Π_γ fact, $p \upharpoonright \gamma$ suffices, we don't need $p \upharpoonright (\gamma + 1)$.

Proposition 7.9. *Let $\gamma \leqslant \alpha$, and let φ be a Π_γ Borel code. For $p \in \mathbb{U}$, if p is γ-complete and $p \Vdash^* \varphi$, then $p \upharpoonright \gamma \Vdash^* \varphi$.*

Proof. The proposition is proved by induction on γ.

The base case $\gamma = 0$ follows from the definition of strong forcing, along with the fact that $p(\sigma) \neq \texttt{both}$ for all $\sigma \in \mathscr{L}_\alpha$.

Suppose that $\gamma > 0$, and that the proposition has been verified for all $\gamma' < \gamma$. Let φ be a Π_γ Borel code, and let $p \in \mathbb{U}$ be γ-complete. We prove the contrapositive.

Suppose $p \upharpoonright \gamma \nVdash^* \varphi$. Since φ is $\neg \bigvee_n \psi_n$ (where each ψ_n is $\Pi_{\gamma'}$ for some $\gamma' < \gamma$), by definition this means there is some n and some r extending $p \upharpoonright \gamma$ such that $r \Vdash^* \psi_n$. We may assume r is γ'-complete. By induction, $r \upharpoonright \gamma' \Vdash^* \psi_n$. By Lemma 7.8, $(r \upharpoonright \gamma') \cup p$ extends p and $r \upharpoonright \gamma'$, and thus witnesses that $p \nVdash^* \varphi$. ☐

As usual we obtain:

Corollary 7.10. *Let $\gamma \leqslant \alpha$, and let φ be a $\Sigma_{\gamma+1}$ Borel code. If $G \subset \mathbb{U}$ is sufficiently generic, then $x_G \in [\varphi]$ if and only if there is some $p \in (G \cap \mathbb{U}_\gamma)$such that $p \Vdash^* \varphi$.*

We are now ready to prove Theorem 7.2.

Proof. Towards a contradiction, suppose Z were a $\mathbf{\Delta}^0_{\alpha+1}$ set with $\mathscr{I}_\alpha \subseteq Z$ and $\mathscr{F}_\alpha \subseteq Z^\complement$. Fix φ_i and φ_f, $\Sigma_{\alpha+1}$ Borel codes for Z and its complement, respectively.

Let $p_{\texttt{both}} \in \mathbb{U}$ be given by $p_{\texttt{both}}(\langle\rangle) = \texttt{both}$, and $p_{\texttt{both}}$ is undefined everywhere else. Let G be a sufficiently generic filter containing $p_{\texttt{both}}$, and let $x = x_G$.

Suppose $x \in Z$. Fix $q \in G \cap \mathbb{U}_\alpha$ with $q \Vdash^* \varphi_i$. Since $q \in \mathbb{U}_\alpha$, q is undefined at the root. Define q' extending q by the definition $q'(\langle\rangle) = 1$. Note that $q' \leqslant q$, and thus $q' \Vdash^* \varphi_i$.

Let H be a sufficiently generic filter containing q'. Then $x_H \in Z$, since $q' \Vdash^* \varphi_i$. But $F^{x_H}(\langle\rangle) = 1$ by Lemma 7.4, and so $x_H \in \mathscr{F}_\alpha$, a contradiction. If $x \notin Z$, mutatis mutandis. ☐

8. Questions

We state some open questions.

Question 8.1. *Is there a way to define graphs H_α for α which are not successors of successors?*

One possibility would be to replace the relation $s \prec_{\alpha*} t$ in the definition of n^t and $k^t(m)$ by the relation "for all σ with $|\sigma| \geqslant 3$, if $T^s(\sigma)\downarrow$ then $T^t(\sigma) = T^s(\sigma)$". What quickly goes wrong is the property Lemma 5.2(j). It would be interesting to see if the partial labellings T^t can be modified so that this property is recovered. Even then, it is not clear how to prove that such a graph is not $\mathbf{\Sigma}^0_\alpha$-colourable.

For the following question, let $\mathbb{A}_3 = \{(a,a) : a = (x,y) \in \mathbb{Y}_3 \ \& \ T^x(\langle\rangle) = 1\}$, and let $\mathbb{B}_3$ be the directed graph H_3. Recall that for Polish spaces X and Y, a set $C \subseteq X \times Y$ is $(\mathbf{\Sigma}^0_\alpha \times \mathbf{\Sigma}^0_\alpha)_\sigma$ if it is of the form $\bigcup_n (D_n \times E_n)$, where each D_n and E_n are $\mathbf{\Sigma}^0_\alpha$. The results above show that there is no set $C \subseteq (\mathbb{Y}_3)^2$ which is $(\mathbf{\Sigma}^0_3 \times \mathbf{\Sigma}^0_3)_\sigma$ and separates $\mathbb{A}_3$ from $\mathbb{B}_3$. The question is whether this is a least example:

Question 8.2. *Suppose that X, Y are Polish spaces, that $A, B \subseteq X \times Y$ are $\mathbf{\Sigma}^1_1$ and disjoint, and further, that there is no set $C \subseteq X \times Y$ which is $(\mathbf{\Sigma}^0_3 \times \mathbf{\Sigma}^0_3)_\sigma$ such that $A \subseteq C$ and $B \subseteq C^\complement$.*

Must there be continuous functions $f\colon \mathbb{Y}_3 \to X$ and $g\colon \mathbb{Y}_3 \to Y$ such that for all $(a,a) \in \mathbb{A}_3$, $(f(a), g(a)) \in A$, and for all $(a,b) \in \mathbb{B}_3$, $(f(a), f(b)) \in B$?

The background here is as follows. Lecomte [Lec07] derived from the $\mathbb{G}_0$ dichotomy (Theorem 1.1) a dichotomy result, characterising when two disjoint analytic sets can be separated by a countable union of Borel rectangles. He also showed that Theorem 1.1 is an easy corollary of this other dichotomy. In [LZ14a], Lecomte and Zelený found least examples of sets that are not separable by $(\mathbf{\Sigma}^0_1 \times \mathbf{\Sigma}^0_1)_\sigma$ sets, and by $(\mathbf{\Sigma}^0_2 \times \mathbf{\Sigma}^0_2)_\sigma$ sets; the problem for $\alpha = 3$ and higher is still open.

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