Universal and complete sets in martingale theory

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Abstract. The Doob convergence theorem implies that the set of divergence of any martingale has measure zero. We prove that, conversely, any $G_{\delta\sigma}$ subset of the Cantor space with Lebesgue-measure zero can be represented as the set of divergence of some martingale. In fact, this is effective and uniform. A consequence of this is that the set of everywhere converging martingales is $\mathbf{\Pi}^1_1$-complete, in a uniform way. We derive from this some universal and complete sets for the whole projective hierarchy, via a general method. We provide some other complete sets for the classes $\mathbf{\Pi}^1_1$ and $\mathbf{\Sigma}^1_2$ in the theory of martingales.

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1 Introduction

The reader should see [K2] for the notation used in this paper.

Definition 1.1 *We say that a map* $f:2^{<\omega}\rightarrow [0,1]$ *is a* **martingale** *if* $f(s)=\frac{f(s0)+f(s1)}{2}$ *for each* $s\in 2^{<\omega}$*. The set of martingales is denoted by* $\mathcal{M}$ *and is a compact subset of* $[0,1]^{2^{<\omega}}$ *(equipped with the usual product topology).*

This terminology is not the standard one, but the set $\mathcal{M}$ can be interpreted as the set of all discrete martingales (in the classical sense) taking values in [0,1], as follows. If $s\in 2^{<\omega}$, then

$$N_s:=\{\beta\in 2^\omega \mid s\subseteq\beta\}$$

is the usual basic clopen set. Let $f\in\mathcal{M}$. If $n\in\omega$, then let $\mathcal{S}_n$ be the σ-algebra on 2^ω generated by $\{N_s \mid s\in 2^n\}$, and $f_n:2^\omega\rightarrow [0,1]$ be defined by $f_n(\beta):=f(\beta|n)$. Then the sequence $(f_n)_{n\in\omega}$ is a discrete martingale taking values in [0,1] with respect to the sequence of σ-algebras $(\mathcal{S}_n)_{n\in\omega}$ and the usual Lebesgue product measure λ on 2^ω. Conversely, if $(f_n)_{n\in\omega}$ is any such martingale, it can be viewed as an element of $\mathcal{M}$ by setting $f(s):=f_{|s|}(\alpha)$ if $\alpha\in N_s$. This definition is correct because $f_{|s|}$, as a function measurable with respect to $\mathcal{S}_{|s|}$, has a constant value on N_s.

Definition 1.2 *Let* f *be a martingale and* $\beta\in 2^\omega$*. The* **oscillation** *of* f *at* β *is the number*

$$osc(f,\beta):=inf_{N\in\omega}\ sup_{p,q\geq N}\ |f(\beta|p)-f(\beta|q)|.$$

The **set of divergence** *of* f *is* $D(f):=\{\beta\in 2^\omega \mid osc(f,\beta)>0\}$.

By definition, if f is a martingale, then

$$\beta\in D(f)\Leftrightarrow \exists r\in\omega\ \ \forall N\in\omega\ \ \exists p,q\geq N\ \ |f(\beta|p)-f(\beta|q)|>2^{-r}.$$

This shows that $D(f)\in\mathbf{\Sigma}^0_3$. Moreover, $D(f)$ has λ-measure zero, by Doob's convergence theorem (see Chapter XI, Section 14 in [D]). So it is natural to ask whether any $\mathbf{\Sigma}^0_3$ subset of 2^ω with λ-measure zero is the set of divergence of some martingale (this question was asked by Louveau). We answer positively:

Theorem 1.3 *Let* B *be a subset of* 2^ω*. Then the following are equivalent:*
(a) B *is* $\mathbf{\Sigma}^0_3$ *and has* λ*-measure zero,*
(b) there is a martingale f *with* $B=D(f)$*.*

Definition 1.4 *Let* $\mathbf{\Gamma}$ *be a class of subsets of Polish spaces,* X,Y *be Polish spaces, and* $\mathcal{U}\subseteq Y\times X$*.*
(a) We say that $\mathcal{U}$ *is* Y**-universal for the** $\mathbf{\Gamma}$ **subsets of** X *if* $\mathcal{U}\in\mathbf{\Gamma}(Y\times X)$ *and* $\mathbf{\Gamma}(X)=\{\mathcal{U}_y \mid y\in Y\}$*.*
(b) We say that $\mathcal{U}$ *is* **uniformly** Y**-universal for the** $\mathbf{\Gamma}$ **subsets of** X *if* $\mathcal{U}$ *is* Y*-universal for the* $\mathbf{\Gamma}$ *subsets of* X *and, for each* $S\in\mathbf{\Gamma}(\omega^\omega\times X)$*, there is a Borel map* $b:\omega^\omega\rightarrow Y$ *such that* $S_\alpha=\mathcal{U}_{b(\alpha)}$ *for each* $\alpha\in\omega^\omega$*.*

Corollary 1.5 *Let* $\mathcal{G}$ *be a* G_δ *subset of* 2^ω *with* $\lambda(\mathcal{G})=0$*. Then the set* $\{(f,\beta)\in\mathcal{M}\times\mathcal{G} \mid \beta\in D(f)\}$ *is* $\mathcal{M}$*-universal for the* $\mathbf{\Sigma}^0_3$ *subsets of* $\mathcal{G}$*.*

In fact, we prove an effective and uniform version of the implication (a) $\Rightarrow$ (b) in Theorem 1.3.

In particular, we can associate, via a Borel map F, a martingale to a code α of an arbitrary G_δ subset G of $\mathcal{G}$ (as in the previous corollary), in such a way that $G=D\big(F(\alpha)\big)$. A consequence of this is the following:

Theorem 1.6 *The set $\mathcal{P}$ of everywhere converging martingales is $\mathbf{\Pi}^1_1$-complete.*

These statements are in the spirit of some results concerning the differentiability of functions due to Zahorski and Mazurkiewicz (see Section 4 for details). In fact, $\mathcal{P}$ is $\mathbf{\Pi}^1_1$-complete in a uniform way, which allows us to derive some universal and complete sets for the whole projective hierarchy, in spaces of continous functions, starting from $\mathcal{P}$. More precisely, let $P_1:=[0,1]^{2^{<\omega}}$ and $C_1:=\mathcal{P}$. We define, for each natural number $n\geq 1$,

- the space $P_{n+1}:=\mathcal{C}(2^\omega,P_n)$ of continuous functions from 2^ω into P_n, equipped with the topology of uniform convergence (inductively),

- $C_{n+1}:=\{h\in P_{n+1}\mid\forall\beta\in 2^\omega\ \ h(\beta)\notin C_n\}$ (inductively),
- $U_n:=\{(h,\beta)\in P_{n+1}\times 2^\omega\mid h(\beta)\in C_n\}$.

We prove the following:

Theorem 1.7 *Let $n\geq 1$ be a natural number. Then*
(a) the set U_n is uniformly P_{n+1}-universal for the $\mathbf{\Pi}^1_n$ subsets of 2^ω,
(b) the set C_n is $\mathbf{\Pi}^1_n$-complete.

In fact, our method is more general and works if we start with a $\mathbf{\Pi}^1_1$ set which is complete in a uniform way. In order to prove this, we give an effective refinement of the fact, proved in [K3], that a Borel $\mathbf{\Pi}^1_n$-complete set is actually $\mathbf{\Pi}^1_n$-complete (see Theorem 5.3).

Let f be a martingale. As $D(f)$ has λ-measure zero, we can associate to f the partial function $\psi(f)$ defined λ-almost everywhere by $\psi(f)(\beta):=\lim_{l\rightarrow\infty}f(\beta|l)$. The partial function $\psi(f)$ will be called the **associated partial function**. The martingale f is in $\mathcal{P}$ if and only if $\psi(f)$ is total, in which case $\psi(f)$ is called the **associated limit function**. Using the work in [B-Ka-L] and [K2] about spaces of continuous functions, we prove the following:

Theorem 1.8 *(a) The set $\big\{(f_k)_{k\in\omega}\in\mathcal{P}^\omega\mid\big(\psi(f_k)\big)_{k\in\omega}$ pointwise converges$\big\}$ is $\mathbf{\Pi}^1_1$-complete.*
(b) The set $\big\{(f_k)_{k\in\omega}\in\mathcal{P}^\omega\mid\big(\psi(f_k)\big)_{k\in\omega}$ pointwise converges to zero$\big\}$ is $\mathbf{\Pi}^1_1$-complete.
(c) The set

$$\big\{(f_k)_{k\in\omega}\in\mathcal{P}^\omega\mid\exists\gamma\in\omega^\omega\text{ strictly increasing such that }\big(\psi(f_{\gamma(i)})\big)_{i\in\omega}\text{ pointwise converges to zero}\big\}$$

is $\mathbf{\Sigma}^1_2$-complete.

2 $\mathbf{\Sigma}^0_3$ sets of measure zero

Notation. In the sequel, B will be a Borel subset of 2^ω, and M will be a λ-measurable subset of 2^ω. If $\beta\in 2^\omega$, then the **density of** M **at** β is the number $d(M,\beta):=\lim_{l\rightarrow\infty}\frac{\lambda(M\cap N_{\beta|l})}{\lambda(N_{\beta|l})}$ when it is defined. Note that $d(B,\beta)=1$ if $\beta\in B$ and B is open. We first recall the Lebesgue density theorem (see 17.9 in [K2]).

Theorem 2.1 *(Lebesgue) The equality* $\lambda(M)=\lambda\big(\{\beta\in M\mid d(M,\beta)=1\}\big)$ *holds for any* λ*-measurable subset* M *of* 2^ω.

The reader should see [C] for the next lemma. We include a proof to be self-contained and also because we will prove an effective and uniform version of it later.

Lemma 2.2 *(Lusin-Menchoff) Let* F *be a closed subset of* 2^ω*, and* $M\supseteq F$ *be a* λ*-measurable subset of* 2^ω *such that* $d(M,\beta)=1$ *for each* $\beta\in F$*. Then there is a closed subset* C *of* 2^ω *such that*

(1) $F\subseteq C\subseteq M$,

(2) $d(M,\beta)=1$ *for each* $\beta\in C$,

(3) $d(C,\beta)=1$ *for each* $\beta\in F$.

Proof. If F is 2^ω, then we can take $C:=F$. So we may assume that F is not 2^ω. We set $s^-:=s|(|s|-1)$ if $\emptyset\neq s\in 2^{<\omega}$. Note that $\neg F$ is the disjoint union of the elements of a sequence $(N_{s_n})_{n\in\omega}$, where $N_{s_n^-}\cap F\neq\emptyset$ for each $n\in\omega$. Fix $n\in\omega$. By Theorem 2.1,

$$\lambda(M\cap N_{s_n})=\lambda\big(\{\beta\in M\cap N_{s_n}\mid d(M\cap N_{s_n},\beta)=1\}\big).$$

The regularity of λ gives a closed subset F_n of 2^ω contained in $\{\beta\in M\cap N_{s_n}\mid d(M\cap N_{s_n},\beta)=1\}$ such that $\lambda(F_n)\geq(1-2^{-n})\lambda(M\cap N_{s_n})$. We set $C:=F\cup\bigcup_{n\in\omega}F_n$, which is closed since $|s_n|\to\infty$.

As Conditions (1) and (2) are clearly satisfied, pick $\beta\in F$. Note that

$$\begin{aligned}\lambda(N_{\beta|l}\setminus C)&=\Sigma_{s_n\supseteq\beta|l}\ \lambda(N_{s_n}\setminus C)\\&\leq\Sigma_{s_n\supseteq\beta|l}\ \lambda(N_{s_n}\setminus F_n)\\&\leq\Sigma_{s_n\supseteq\beta|l}\ 2^{-n}\lambda(M\cap N_{s_n})+\Sigma_{s_n\supseteq\beta|l}\ \lambda(N_{s_n}\setminus M)\\&\leq\Sigma_{s_n\supseteq\beta|l}\ 2^{-n}\lambda(N_{s_n})+\lambda(N_{\beta|l}\setminus M).\end{aligned}$$

This implies that the limit of $\frac{\lambda(N_{\beta|l}\setminus C)}{\lambda(N_{\beta|l})}$ is zero since $d(M,\beta)=1$. □

The next topology is considered in [Lu-Ma-Z], see Chapter 6.

Definition 2.3 *The* τ-**topology** *on* 2^ω *is generated by*

$$\mathcal{F}:=\{M\subseteq 2^\omega\mid M\textit{ is }\lambda\textit{-measurable}\wedge\forall\beta\in M\ \ d(M,\beta)=1\}.$$

The next result is proved in [Lu-Ma-Z], but in a much more abstract way. This is the reason why we include a much more direct proof here, since it is not too long.

Lemma 2.4 *The family* $\mathcal{F}$ *is a topology. In particular, any* τ*-open set is* λ*-measurable.*

Proof. Note first that $\mathcal{F}$ is closed under finite intersections, so that it is a basis for the τ-topology. Indeed, let M,M' be in $\mathcal{F}$, and $\beta\in M\cap M'$. Then we use the facts that

$$\lambda(M\cap M'\cap N_{\beta|l})=\lambda(M\cap N_{\beta|l})-\lambda\big((M\cap N_{\beta|l})\setminus M'\big)$$

and $\lambda\big((M\cap N_{\beta|l})\setminus M'\big)\leq\lambda(N_{\beta|l}\setminus M')$.

Let $\mathcal{H}$ be a subfamily of $\mathcal{F}$, and $H:=\cup\mathcal{H}$. We claim that there is a countable subfamily $\mathcal{C}$ of $\mathcal{H}$ such that $m:=\sup\{\lambda(\cup\mathcal{D})\mid\mathcal{D}\subseteq\mathcal{H}\text{ countable}\}=\lambda(\cup\mathcal{C})$. Indeed, for each $n\in\omega$ there is $\mathcal{D}_n\subseteq\mathcal{H}$ countable such that $\lambda(\cup\mathcal{D}_n)>m-2^{-n}$, and $\mathcal{C}:=\bigcup_{n\in\omega}\mathcal{D}_n$ is suitable. Let $C:=\cup\mathcal{C}$.

Let $\beta\in H$, and M in $\mathcal{H}$ with $\beta\in M$. Note that $\lambda(M\cup C)=\lambda(C)$ (consider the family $\mathcal{C}\cup\{M\}$). Thus $\lambda(M\setminus C)=0$. As $d(M,\beta)=1$, the equality $d(M\cap C,\beta)=1$ holds, and $d(\neg C,\beta)=0$. This implies that $H\setminus C$ is contained in $\{\beta\notin C\mid d(\neg C,\beta)<1\}$, which has λ-measure zero by Theorem 2.1. Therefore $H\setminus C$ has λ-measure zero and $H=C\cup(H\setminus C)$ is λ-measurable.

Pick $\beta\in H$, and $M\in\mathcal{H}$ with $\beta\in M$. Then $d(M,\beta)=1$, and thus $d(H,\beta)=1$. Therefore $H\in\mathcal{F}$. This finishes the proof. □

The next lemma is in the style of Urysohn's theorem (see [Lu-Ma-Z] for its version on the real line). We include a proof to be self-contained and also because we will prove an effective and uniform version of it later.

Lemma 2.5 *Let C be a closed subset of 2^ω, and G be a G_δ subset of 2^ω disjoint from C such that $\lambda(G)=0$. Then there is a τ-continuous map $h:2^\omega\to[0,1]$ such that $h_{|C}\equiv 0$ and $h_{|G}\equiv 1$.*

Proof. Let $(F_n)_{n\in\omega}$ be an increasing sequence of closed subsets of 2^ω with union $\neg G$ and $F_0=C$. We first construct a sequence $(C_{\frac{1}{2^n}})_{n\in\omega}$ of closed subsets of 2^ω with $F_n\subseteq C_{\frac{1}{2^n}}\subseteq\neg G$, $C_{\frac{1}{2^n}}\subseteq C_{\frac{1}{2^{n+1}}}$, and $d(C_{\frac{1}{2^{n+1}}},\beta)=1$ for each $\beta\in C_{\frac{1}{2^n}}$. We first apply Lemma 2.2 to $F:=F_0$ and $M:=\neg G$, which gives $F_0\subseteq C_1\subseteq\neg G$. Then, inductively, we apply Lemma 2.2 to $F:=C_{\frac{1}{2^n}}\cup F_{n+1}$ and $M:=\neg G$, which gives $C_{\frac{1}{2^n}}\cup F_{n+1}\subseteq C_{\frac{1}{2^{n+1}}}\subseteq\neg G$ such that $d(C_{\frac{1}{2^{n+1}}},\beta)=1$ for each $\beta\in C_{\frac{1}{2^n}}$.

Then we construct $C_{\frac{2k+1}{2^n}}$, for $0<k<2^{n-1}$ and $n\geq 2$. This will give us a family $(C_{\frac{k}{2^n}})_{n\in\omega,0<k\leq 2^n}$ of closed subsets of 2^ω. We want to ensure that $C_\zeta\subseteq C_{\zeta'}$ and $d(C_{\zeta'},\beta)=1$ for each $\beta\in C_\zeta$ if $\zeta'<\zeta$. We proceed by induction on n. We apply Lemma 2.2 to $F:=C_{\frac{k+1}{2^{n-1}}}$ and $M:=C_{\frac{k}{2^{n-1}}}$, which gives $C_{\frac{2k+1}{2^n}}$ such that $C_{\frac{k+1}{2^{n-1}}}\subseteq C_{\frac{2k+1}{2^n}}\subseteq C_{\frac{k}{2^{n-1}}}$, $d(C_{\frac{k}{2^{n-1}}},\beta)=1$ for each $\beta\in C_{\frac{2k+1}{2^n}}$, and $d(C_{\frac{2k+1}{2^n}},\beta)=1$ for each $\beta\in C_{\frac{k+1}{2^{n-1}}}$. This allows us to define $\tilde{h}$ by

$$\tilde{h}(\beta):=\begin{cases}0\text{ if }\beta\in G,\\ \sup\{\zeta\mid\beta\in C_\zeta\}\text{ if }\beta\notin G.\end{cases}$$

It remains to see that $\tilde{h}$ is τ-continuous (and then we will set $h(\beta):=1-\tilde{h}(\beta)$). So let $b\in(0,1]$, and $\beta\in 2^\omega$ with $\tilde{h}(\beta)<b$. Note that there is $\zeta<b$ with $\tilde{h}(\beta)<\zeta$, so that $\beta\notin C_\zeta$. If $\gamma\notin C_\zeta$, then $\tilde{h}(\gamma)\leq\zeta<b$, so that $\neg C_\zeta$ is an open (and thus τ-open since the τ-topology is finer than the usual one) neighborhood of β on which $\tilde{h}<b$. In particular, $\tilde{h}$ is Borel.

Now let $a\in[0,1)$. It is enough to see that $B:=\{\gamma\in 2^\omega\mid\tilde{h}(\gamma)>a\}$ is τ-open. So assume that $\tilde{h}(\gamma)>a$. Note that there are $\zeta>\zeta'>a$ with $\tilde{h}(\gamma)>\zeta$, so that $\gamma\in C_\zeta\subseteq C_{\zeta'}\subseteq B$. Thus $d(C_{\zeta'},\gamma)=1$, by construction of the family. As $\tilde{h}$ is Borel, B is Borel, $d(B,\gamma)$ is defined and equal to 1. □

Remark. We in fact proved that h is lower semi-continuous (in the standard topology on 2^ω).

Notation. If $h:2^\omega\to[0,1]$ is a λ-measurable map and $s\in 2^{<\omega}$, then we set $⨍_{N_s} h\ d\lambda:=\frac{\int_{N_s}h\ d\lambda}{\lambda(N_s)}$.

Lemma 2.6 *Let $h : 2^\omega \to [0,1]$ be a τ-continuous map, and $\beta \in 2^\omega$. Then*

$$\lim_{l\to\infty} \fint_{N_{\beta|l}} h\ d\lambda = h(\beta).$$

Proof. Let $\varepsilon > 0$, and $\beta \in M := h^{-1}\Big(B\big(h(\beta), \varepsilon\big)\Big)$. Note that $d(M, \gamma) = 1$ for each $\gamma \in M$ since h is τ-continuous. As h is λ-measurable, we can write

$$\int_{N_{\beta|l}} h\ d\lambda = \int_{M\cap N_{\beta|l}} h\ d\lambda + \int_{N_{\beta|l}\setminus M} h\ d\lambda.$$

Note that $0 \leq \int_{N_{\beta|l}\setminus M} h\ d\lambda \leq \lambda(N_{\beta|l}\setminus M)$, so that $0 \leq \fint_{N_{\beta|l}\setminus M} h\ d\lambda \leq \frac{\lambda(N_{\beta|l}\setminus M)}{\lambda(N_{\beta|l})} \to 0$. Similarly,

$$\fint_{M\cap N_{\beta|l}} h\ d\lambda \in \big[\big(h(\beta) - \varepsilon\big)\frac{\lambda(M\cap N_{\beta|l})}{\lambda(N_{\beta|l})}, \big(h(\beta) + \varepsilon\big)\frac{\lambda(M\cap N_{\beta|l})}{\lambda(N_{\beta|l})}\big],$$

and we are done since $\frac{\lambda(M\cap N_{\beta|l})}{\lambda(N_{\beta|l})}$ tends to 1 as l tends to ∞. □

Now we come to our main lemma, inspired by Zahorski (see [Za]).

Lemma 2.7 *Let G be a G_δ subset of 2^ω with λ-measure zero. Then there is a martingale f with $G = D(f)$ and $\{osc(f, \beta) \mid \beta \in 2^\omega\} \subseteq \{0\} \cup [\frac{1}{2}, 1]$.*

Proof. Let $(G_n)_{n\in\omega}$ be a decreasing sequence of open subsets of 2^ω with intersection G and $G_0 = 2^\omega$.

We construct $g_n : 2^\omega \to [0,1]$, open subsets G_n^*, G_n^{**} of 2^ω, subsets I_n of ω (ω itself or an interval containing 0), and a sequence $(s_j^n)_{j\in I_n}$ of pairwise incompatible finite binary sequences, by induction on $n \in \omega$, such that, if $S_n := \Sigma_{j\leq n}\ (-1)^j g_j$,

(1) $G \subseteq G_{n+1}^* \subseteq G_n^{**} = \bigcup_{j\in I_n}\ N_{s_j^n} \subseteq G_n^* \subseteq G_n\ \wedge\ G_0^* = 2^\omega$,
(2) $g_{n|G} \equiv 1\ \wedge\ g_{n|\neg G_n^*} \equiv 0$,
(3) g_n is τ-continuous,
(4) $g_{n+1} \leq g_n$,
(5) $\lambda(G_{n+1}^* \cap N_{s_j^n}) < 2^{-n-3}\lambda(N_{s_j^n})$,
(6) $|\fint_{N_{s_j^n}} S_n\ d\lambda - S_n(\beta)| < 2^{-3}$ if $\beta \in G \cap N_{s_j^n}$.

We set $g_0 :\equiv 1$, $G_0^*, G_0^{**} := 2^\omega$, $I_0 := \{0\}$ and $s_0^0 := \emptyset$. Assume that our objects are constructed up to n. We first construct an open subset G_{n+1}^* of 2^ω with $G \subseteq G_{n+1}^* \subseteq G_n^{**} \cap G_{n+1}$ such that

$$\lambda(G_{n+1}^* \cap N_{s_j^n}) < 2^{-n-3}\lambda(N_{s_j^n})$$

if $j \in I_n$. For each $j \in I_n$, there is an open set O_j with $G \cap N_{s_j^n} \subseteq O_j \subseteq G_{n+1} \cap N_{s_j^n}$ such that $\lambda(O_j) < 2^{-n-3}\lambda(N_{s_j^n})$. We then set $G_{n+1}^* := \bigcup_{j\in I_n}\ O_j$.

We now apply Lemma 2.5 to $C := \neg G^*_{n+1}$ and G, which gives a τ-continuous map $h : 2^\omega \to [0,1]$ with $h_{|\neg G^*_{n+1}} \equiv 0$ and $h_{|G} \equiv 1$. We set $g_{n+1} := \min(g_n, h)$, so that g_{n+1} satisfies (2)-(4).

By Lemma 2.6, $\lim_{l\to\infty} ⨍_{N_{\beta|l}} S_{n+1}\ d\lambda = S_{n+1}(\beta)$ for each $\beta \in G$. This gives $l(\beta) \in \omega$ minimal with $|⨍_{N_{\beta|l(\beta)}} S_{n+1}\ d\lambda - S_{n+1}(\beta)| < 2^{-3}$ and $N_{\beta|l(\beta)} \subseteq G^*_{n+1}$. The set G^{**}_{n+1} is the union of the $N_{\beta|l(\beta)}$'s, which defines I_{n+1} and $(s^{n+1}_j)_{j\in I_{n+1}}$ ($S_{n+1}(\beta)$ is 0 if n is even and 1 otherwise when $\beta \in G$).

We then define a partial map $f_\infty : 2^\omega \to [0,1]$ by $f_\infty := \Sigma_{j\in\omega}\ (-1)^j g_j$. If $\beta \in G$, then $S_n(\beta)$ takes alternatively the values 1 and 0, depending on the parity of n, so that $f_\infty(\beta)$ is not defined. If $\beta \notin G$, then there is n such that $\beta \in \neg G^*_{n+1} \subseteq \neg G^*_{n+2} \subseteq \ldots$ This implies that $f_\infty(\beta)$ is defined and equal to $S_n(\beta)$. As $0 \le \Sigma_{p\le q}\ (g_{2p} - g_{2p+1}) = S_{2q+1} \le S_{2q} = g_0 + \Sigma_{1\le p\le q}\ (g_{2p} - g_{2p-1}) \le g_0$, f_∞ takes values in $[0,1]$. So f_∞ is a partial λ-measurable map defined λ-almost everywhere since $\lambda(G) = 0$ (we use Lemma 2.4).

This allows us to define $f : 2^{<\omega} \to [0,1]$ by $f(s) := ⨍_{N_s} f_\infty\ d\lambda$. As $\lambda(N_s) = 2\lambda(N_{s\varepsilon})$ for each $\varepsilon \in 2$, $f(s) = ⨍_{N_s} f_\infty\ d\lambda = \frac{\int_{N_{s0}} f_\infty\ d\lambda + \int_{N_{s1}} f_\infty\ d\lambda}{\lambda(N_s)} = \frac{f(s0)}{2} + \frac{f(s1)}{2}$ and f is a martingale.

If $\beta \notin G$, then there is n with $\beta \in G^*_n \setminus G^*_{n+1}$, so that $f_\infty(\beta) = S_n(\beta)$. By Lemma 2.6, $k \ge n$ implies that $\lim_{l\to\infty} ⨍_{N_{\beta|l}} S_{k+1}\ d\lambda = S_{k+1}(\beta) = S_n(\beta)$ since S_{k+1} is τ-continuous. Note that, for each $k \ge n$,

$$\begin{aligned}\left|\int_{N_{\beta|l}} (f_\infty - S_{k+1})\ d\lambda\right| &\le \lambda(G^*_{k+2} \cap N_{\beta|l})\\ &\le \Sigma_{\beta|l \subseteq s^{k+1}_j}\ \lambda(G^*_{k+2} \cap N_{s^{k+1}_j})\\ &\le \Sigma_{\beta|l \subseteq s^{k+1}_j}\ 2^{-k-4}\lambda(N_{s^{k+1}_j})\\ &\le \lambda(N_{\beta|l})2^{-k-4}.\end{aligned}$$

Moreover,

$$\begin{aligned}|f(\beta|l) - f_\infty(\beta)| = |⨍_{N_{\beta|l}} f_\infty\ d\lambda - f_\infty(\beta)| &= |⨍_{N_{\beta|l}} \left(f_\infty - S_{k+1}\right) d\lambda + ⨍_{N_{\beta|l}} S_{k+1}\ d\lambda - S_{k+1}(\beta)|\\ &\le 2^{-k-4} + |⨍_{N_{\beta|l}} S_{k+1}\ d\lambda - S_{k+1}(\beta)|,\end{aligned}$$

so that $\lim_{l\to\infty} f(\beta|l) = f_\infty(\beta)$, $\mathrm{osc}(f,\beta) = 0$ and $\beta \notin D(f)$.

If $\beta \in G$ and $n \in \omega$, then there is $j \in \omega$ with $\beta \in N_{s^n_j}$. Note that

$$f(s^n_j) = ⨍_{N_{s^n_j}} f_\infty\ d\lambda = ⨍_{N_{s^n_j}} S_n\ d\lambda + ⨍_{N_{s^n_j}} (f_\infty - S_n)\ d\lambda$$

and $|\int_{N_{s^n_j}} (f_\infty - S_n)\ d\lambda| \le \lambda(G^*_{n+1} \cap N_{s^n_j}) < \frac{1}{8}\lambda(N_{s^n_j})$, so that $|⨍_{N_{s^n_j}} (f_\infty - S_n)\ d\lambda| < \frac{1}{8}$. By (6), $|f(s^n_j) - S_n(\beta)| < \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$. As $S_n(\beta)$ takes infinitely often the values 1 and 0, $\mathrm{osc}(f,\beta) \ge \frac{1}{2}$ and $\beta \in D(f)$. □

The main result will be a consequence of the main lemma and the following.

Lemma 2.8 *Let $(f_n)_{n\in\omega}$ be a sequence of martingales such that*

$$\{osc(f_n,\beta)\mid(n,\beta)\in\omega\times 2^\omega\}\subseteq\{0\}\cup[\frac{1}{2},1].$$

Then there is a martingale f with $D(f)=\bigcup_{n\in\omega}\ D(f_n)$.

Proof. We first observe the following facts. Let $g,h:2^{<\omega}\rightarrow\mathbb{R}$ be bounded, $\beta\in 2^\omega$ and $a\in\mathbb{R}$.

(1) $\text{osc}(g+h,\beta)\leq\text{osc}(g,\beta)+\text{osc}(h,\beta)$.

This comes from the triangle inequality.

(2) $\text{osc}(ag,\beta)=|a|\cdot\text{osc}(g,\beta)$.

(3) $\text{osc}(g+h,\beta)=\text{osc}(h,\beta)$ if $\text{osc}(g,\beta)=0$.

By (1), $\text{osc}(h,\beta)\leq\text{osc}(g+h,\beta)+\text{osc}(-g,\beta)=\text{osc}(g+h,\beta)\leq\text{osc}(g,\beta)+\text{osc}(h,\beta)=\text{osc}(h,\beta)$, so that $\text{osc}(h,\beta)=\text{osc}(g+h,\beta)$.

(4) $\text{osc}(g,\beta)\leq a$ if $g(\beta|l)\in[0,a]$ for each $l\in\omega$.

We set $D_n:=D(f_n)$ for each $n\in\omega$, and $f:=\Sigma_{n\in\omega}\ 4^{-n}f_n$. Note that f is defined and a martingale.

If $\beta\notin\bigcup_{n\in\omega}\ D_n$, then $\text{osc}(f_n,\beta)=0$ for each $n\in\omega$. In particular, $\text{osc}(4^{-n}f_n,\beta)=0$ for each $n\in\omega$, by (2). Let $\varepsilon>0$, and $M\in\omega$ with $\Sigma_{n>M}\ 4^{-n}\leq\varepsilon$. By (1), $\text{osc}(\Sigma_{n\leq M}\ 4^{-n}f_n,\beta)=0$. By (3) and (4), $\text{osc}(f,\beta)=\text{osc}(\Sigma_{n>M}\ 4^{-n}f_n,\beta)\leq\Sigma_{n>M}\ 4^{-n}\leq\varepsilon$. As ε is arbitrary, $\text{osc}(f,\beta)=0$, $\beta\notin D(f)$, which shows that $D(f)\subseteq\bigcup_{n\in\omega}\ D_n$.

If $\beta\in\bigcup_{n\in\omega}\ D_n$, then let m be minimal such that $\beta\in D_m$. Note that

$$f=\Sigma_{n<m}\ 4^{-n}f_n+4^{-m}f_m+\Sigma_{n>m}\ 4^{-n}f_n.$$

By (2) and (3), $\text{osc}(f,\beta)=\text{osc}(4^{-m}f_m+\Sigma_{n>m}\ 4^{-n}f_n,\beta)$. By (1), (2) and (4),

$$\text{osc}(f,\beta)\geq\text{osc}(4^{-m}f_m,\beta)-\text{osc}(\Sigma_{n>m}\ 4^{-n}f_n,\beta)\geq 4^{-m}\frac{1}{2}-4^{-m}\frac{1}{3}>0.$$

Thus $\beta\in D(f)$. $\square$

3 Effectivity and uniformity

- We refer to [M] for the basic notions of effective descriptive set theory. We first recall some material present in it.

- Let $(p_n)_{n\in\omega}$ be the sequence of prime numbers $2,3,...$

- If $l\in\omega$ and $s\in\omega^l$, then $\overline{s}:=\ <s(0),...,s(l-1)>\ :=p_0^{s(0)+1}...p_{l-1}^{s(l-1)+1}\in\omega$ codes s (if $l=0$, then $<>\ :=1$).

- If $\alpha\in\omega^\omega$ and $l\in\omega$, then $\overline{\alpha}(l):=\ <\alpha(0),...,\alpha(l-1)>\ \in\omega$ codes $\alpha|l\in\omega^l$, and α^* is defined by removing the first coordinate: $\alpha^*:=\big(\alpha(1),\alpha(2),...\big)$.

- If $\kappa\in\{2,\omega\}$, then $<.,.> : (\kappa^\omega)^2\rightarrow\kappa^\omega$ is a recursive homeomorphism with inverse map $\alpha\mapsto\big((\alpha)_0,(\alpha)_1\big)$ defined for example by $(\alpha)_\varepsilon(n):=\alpha(2n+\varepsilon)$ if $(n,\varepsilon)\in\omega\times 2$ (we will also consider recursive homeomorphisms $<.,.,.> : (\kappa^\omega)^3\rightarrow\kappa^\omega$ and $<.,.,...> : (\kappa^\omega)^\omega\rightarrow\kappa^\omega$).

- If $u\in\omega$, then $\text{Seq}(u)$ means that there are $l\in\omega$ and $s\in\omega^l$ (denoted by $s(u)$) such that $u=<s(0),...,s(l-1)>$. The natural number $(u)_i$ is $s(i)$ if $i<l$, and 0 otherwise. The number l is the **length** of u and is denoted by $\text{lh}(u)$. If $k\leq l$, then $\underline{u}(k):=<s(0),...,s(k-1)>$, so that $\underline{u}(l)=u$. The standard basic clopen set is $N^u:=\{\beta\in 2^\omega\mid\forall i<\text{lh}(u)\ \ \beta(i)=(u)_i\}$. We set $u^-:=<(u)_0,...,(u)_{\text{lh}(u)-2}>$ ($u^-:=<>$ if $\text{lh}(u)\leq 1$).

- Let X be a recursively presented Polish space. Then we will consider the effective basic open set $N(X,u)=B_X(r_{((u)_1)_0},\frac{((u)_1)_1}{((u)_1)_2+1})$.

- Let $n\geq 1$ be a natural number. A subset T of ω^n is a **tree** if $\text{Seq}(u_i)$ and $\text{lh}(u_i)=\text{lh}(u_0)$ for each $(u_0,...,u_{n-1})\in T$ and each $i<n$, and $\big(\underline{u_0}(k),...,\underline{u_{n-1}}(k)\big)\in T$ if $(u_0,...,u_{n-1})\in T$ and $k\leq\text{lh}(u_0)$.

- The next result is a part of 4A.1 in [M].

Theorem 3.1 *Let $m\geq 1$ be a natural number, and $B\in\Sigma^0_1\big(\omega^\omega\times(\omega^\omega)^m\big)$. Then there is a recursive subset T of $\omega^\omega\times\omega^m$ such that $(\alpha,\alpha_1,...,\alpha_m)\in B\Leftrightarrow\exists l\in\omega\ \ \big(\alpha,\overline{\alpha_1}(l),...,\overline{\alpha_m}(l)\big)\notin T$, and $T_\alpha:=\{(u_0,...,u_{m-1})\in\omega^m\mid(\alpha,u_0,...,u_{m-1})\in T\}$ is a tree for each $\alpha\in\omega^\omega$.*

- The next result is a part of 4A.7 in [M].

Theorem 3.2 *Let X be a recursively presented Polish space and $B\in\Delta^1_1(X)$. Then we can find a recursive function $\pi:\omega^\omega\rightarrow X$ and $C\in\Pi^0_1(\omega^\omega)$ such that π is injective on C and $\pi[C]=B$.*

- We then recall some material from [L].

Notation. Let X be a recursively presented Polish space. Recall that there is a pair $(\mathcal{W}^X,\mathcal{C}^X)$ such that

(1) $\mathcal{W}^X\subseteq\omega$ is a Π^1_1 set of codes for the Δ^1_1 subsets of X,

(2) $\mathcal{C}^X\subseteq\omega\times X$ is Π^1_1 and $\Delta^1_1(X)=\{\mathcal{C}^X_n\mid n\in\mathcal{W}^X\}$, which means that $\mathcal{C}^X$ is "universal" for the Δ^1_1 subsets of X,

(3) the relation "$n\in\mathcal{W}^X\ \wedge\ (n,x)\notin\mathcal{C}^X$" is Π^1_1 in (n,x).

If $X=\omega^\omega\times 2^\omega$, then we simply write $(\mathcal{W},\mathcal{C}):=(\mathcal{W}^X,\mathcal{C}^X)$.

The next result will be extremely useful in the sequel.

The uniformization lemma. *Let X,Y be recursively presented Polish spaces, and $P\in\Pi^1_1(X\times Y)$. Then the set $P^+:=\{x\in X\mid\exists y\in\Delta^1_1(x)\ \ (x,y)\in P\}$ is Π^1_1, and there is a partial Π^1_1-recursive map $f:X\rightarrow Y$ such that $\big(x,f(x)\big)\in P$ for each $x\in P^+$. If moreover $S\subseteq P^+$ is a Σ^1_1 subset of X, then there is a total Δ^1_1-recursive map $g:X\rightarrow Y$ such that $\big(x,g(x)\big)\in P$ for each $x\in S$.*

- The following definition is inspired by 3H.1 in [M].

Definition 3.3 *(a) Let Γ be a class of subsets of recursively presented Polish spaces, and $\mathbf{\Gamma}$ be the associated boldface class. A system of sets $\mathcal{U}^X \in \Gamma(\omega^\omega \times X)$, where is X is a recursively presented Polish space, is a* **nice parametrization** *in Γ for $\mathbf{\Gamma}$ if the following hold:*

(1) $\mathbf{\Gamma}(X)=\{\mathcal{U}^X_\alpha \mid \alpha\in\omega^\omega\}$,

(2) $\Gamma(X)=\{\mathcal{U}^X_\alpha \mid \alpha\in\omega^\omega$ recursive$\}$,

(3) if X is a recursively presented Polish space, then there is $\mathcal{R}:\omega^\omega\times\omega^\omega\rightarrow\omega^\omega$ recursive such that $(\alpha,\gamma,x)\in\mathcal{U}^{\omega^\omega\times X} \Leftrightarrow \big(\mathcal{R}(\alpha,\gamma),x\big)\in\mathcal{U}^X$ if $(\alpha,\gamma,x)\in\omega^\omega\times\omega^\omega\times X$.

(b) If $\mathcal{U}$ belongs to a nice parametrization, then we will say that $\mathcal{U}$ is a **good universal set** .

(c) If $\mathcal{U}$ satisfies all these properties except maybe (3), then we will say that $\mathcal{U}$ is a **suitable universal set** .

By 3E.2, 3F.6 and 3H.1 in [M], there is a nice parametrization in Π^1_n for $\mathbf{\Pi}^1_n$, for each natural number $n\geq 1$.

- We now recall two results that can essentially be found in [K1]. The first one is Theorem 2.2.3.(a) (see also [T1]).

Theorem 3.4 *(Tanaka) Let $U\in\Sigma^1_1(\omega^\omega\times\omega^\omega)$ be ω^ω-universal for the analytic subsets of ω^ω. Then $L(U):=\big\{(\alpha,p)\in\omega^\omega\times\omega \mid \lambda(U_\alpha\cap 2^\omega)>\frac{(p)_0}{(p)_1+1}\big\}$ is Σ^1_1.*

Corollary 3.5 *Let $B\in\Delta^1_1(\omega^\omega\times 2^\omega)$.*

(a) The map $\lambda_B:\omega^\omega\rightarrow\mathbb{R}$ defined by $\lambda_B(\alpha):=\lambda(B_\alpha)$ is Δ^1_1-recursive, and the partial function $(n,\alpha)\mapsto\lambda(\mathcal{C}_{n,\alpha})$ is Π^1_1-recursive on its domain $\mathcal{W}\times\omega^\omega$.

(b) Let $D\subseteq\omega$, $O_0\in\Sigma^1_1(\omega\times\omega^\omega\times 2^\omega)$, and $O_1\in\Pi^1_1(\omega\times\omega^\omega\times 2^\omega)$ be such that $\lambda\big((O_0)_{n,\alpha}\big)=\lambda\big((O_1)_{n,\alpha}\big)$ if $n\in D$. Then the partial map $\lambda_O:D\times\omega^\omega\rightarrow\mathbb{R}$ defined by $\lambda_O(n,\alpha):=\lambda\big((O_0)_{n,\alpha}\big)$ is Σ^1_1-recursive and Π^1_1-recursive on its domain.

(c) The partial map $d_B:\omega^\omega\times 2^\omega\rightarrow\mathbb{R}$ defined by $d_B(\alpha,\beta):=d(B_\alpha,\beta)$ is Δ^1_1-recursive, and the partial map $(n,\alpha,\beta)\mapsto d(\mathcal{C}_{n,\alpha},\beta)$ is Π^1_1-recursive on its Π^1_1 domain

$$\{(n,\alpha,\beta)\in\mathcal{W}\times\omega^\omega\times 2^\omega \mid d(\mathcal{C}_{n,\alpha},\beta) \text{ exists}\}.$$

(d) Let $h:\omega^\omega\times 2^\omega\rightarrow\mathbb{R}$ be Δ^1_1-recursive taking values in $[0,1]$. Then the partial map $i_h:\omega^\omega\times\omega\rightarrow\mathbb{R}$ defined by $i_h(\alpha,u):=\int_{N^u} h(\alpha,.)\ d\lambda$ is Δ^1_1-recursive on its Δ^0_1 domain $\omega^\omega\times\{u\in\omega \mid Seq(u)\}$.

Proof. (a) It is enough to see that the relations $P_B(\alpha,p)\Leftrightarrow\lambda(B_\alpha)>r_p:=(-1)^{(p)_0}\cdot\frac{(p)_1}{(p)_2+1}$ and

$$Q_B(\alpha,p)\Leftrightarrow\lambda(B_\alpha)<r_p$$

are Δ^1_1 to see that λ_B is Δ^1_1-recursive. Note that there is $\phi:\omega^2\rightarrow\omega$ recursive with $r_{\phi(p,l)}=r_p-\frac{1}{l+1}$. Thus

$$\begin{aligned}Q_B(\alpha,p) &\Leftrightarrow \exists l\in\omega\ \ \lambda(B_\alpha)\leq r_p-\tfrac{1}{l+1}\\ &\Leftrightarrow \exists l\in\omega\ \ \neg\big(\lambda(B_\alpha)>r_p-\tfrac{1}{l+1}\big)\\ &\Leftrightarrow \exists l\in\omega\ \ \neg P_B\big(\alpha,\phi(p,l)\big),\end{aligned}$$

so that it is enough to see that P_B is Δ^1_1.

Now let $S\in\Sigma^1_1\big(\omega^\omega\times(\omega^\omega)^2\big)$ be a good ω^ω-universal for the analytic subsets of $(\omega^\omega)^2$. We set

$$U(\alpha,\gamma)\Leftrightarrow S\big((\alpha)_0,(\alpha)_1,\gamma\big),$$

so that $U\in\Sigma^1_1(\omega^\omega\times\omega^\omega)$ is ω^ω-universal for the analytic subsets of ω^ω. Let A be a Σ^1_1 subset of $\omega^\omega\times 2^\omega$. Then there is $\alpha_0\in\omega^\omega$ recursive with $A=S_{\alpha_0}$, so that

$$\gamma\in A_\alpha\Leftrightarrow(\alpha_0,\alpha,\gamma)\in S\Leftrightarrow(<\alpha_0,\alpha>,\gamma)\in U.$$

This implies that the relation $R_A(\alpha,p)\Leftrightarrow\lambda(A_\alpha)>r_p$, equivalent to

$$\big((p)_0\text{ is odd }\wedge\ (p)_1>0\big)\ \vee\ \big((p)_0\text{ is even }\ \wedge\ (<\alpha_0,\alpha>,<(p)_1,(p)_2>)\in L(U)\big),$$

is Σ^1_1, by Theorem 3.4.

In particular, this applies to $A:=B$, so that P_B is Σ^1_1. Now note that

$$P_B(\alpha,p)\Leftrightarrow\lambda\big((\neg B)_\alpha\big)<1-r_p\Leftrightarrow Q_{\neg B}\big(\alpha,\phi'(p)\big),$$

for some $\phi':\omega\to\omega$ is recursive, so that P_B is Π^1_1 by the previous computation.

We set $\mathcal{C}':=\big\{(\gamma,\beta)\in\omega^\omega\times 2^\omega\mid\gamma(0)\in\mathcal{W}\ \wedge\ \big(\gamma(0),\gamma^*,\beta\big)\in\mathcal{C}\big\}$. As $\mathcal{C}'$ is Π^1_1,

$$\mathcal{A}:=\big\{(\alpha,p)\in\omega^\omega\times\omega\mid\lambda\big((\neg\mathcal{C}')_\alpha\big)>r_p\big\}$$

is Σ^1_1, by the previous discussion. So let $n\in\mathcal{W}$. Note that

$$\begin{aligned}\lambda(\mathcal{C}_{n,\alpha})>r_p\ &\Leftrightarrow\lambda(\neg\mathcal{C}_{n,\alpha})<1-r_p\Leftrightarrow\lambda\big((\neg\mathcal{C}')_{n\alpha}\big)<1-r_p\\&\Leftrightarrow\exists l\in\omega\ \ \lambda\big((\neg\mathcal{C}')_{n\alpha}\big)\leq 1-r_p-\tfrac{1}{l+1}\Leftrightarrow\exists l\in\omega\ \ \big(n\alpha,\phi''(p,l)\big)\notin\mathcal{A},\end{aligned}$$

for some recursive $\phi'':\omega^2\to\omega$. Similarly, the relation "$\lambda(\mathcal{C}_{n,\alpha})<r_p$" is Π^1_1 in (n,α,p) since the relation "$n\in\mathcal{W}\ \wedge\ (n,\alpha,\beta)\notin\mathcal{C}$" is Π^1_1, so that $(n,\alpha)\mapsto\lambda(\mathcal{C}_{n,\alpha})$ is Π^1_1-recursive on $\mathcal{W}\times\omega^\omega$.

(b) Let $A:=\big\{(\alpha,\beta)\in\omega^\omega\times 2^\omega\mid\big(\alpha(0),\alpha^*,\beta\big)\in O_0\big\}$. Note that A is Σ^1_1. By (a), the relation $R_A(\alpha,p)\Leftrightarrow\lambda(A_\alpha)>r_p$ is Σ^1_1. Therefore the relation $R_{O_0}(n,\alpha,p)\Leftrightarrow R_A(n\alpha,p)$ is Σ^1_1 too. Moreover, $R_{O_0}(n,\alpha,p)\Leftrightarrow\lambda\big((O_0)_{n,\alpha}\big)>r_p\Leftrightarrow\lambda_O(n,\alpha)>r_p$.

Assume now that $n\in D$. Then as above there is $\phi'':\omega^2\to\omega$ recursive such that

$$\begin{aligned}\lambda_O(n,\alpha)>r_p\ &\Leftrightarrow\lambda\big((O_1)_{n,\alpha}\big)>r_p\Leftrightarrow\lambda\big((\neg O_1)_{n,\alpha}\big)<1-r_p\\&\Leftrightarrow\exists l\in\omega\ \ \lambda\big((\neg O_1)_{n,\alpha}\big)\leq 1-r_p-\tfrac{1}{l+1}\Leftrightarrow\exists l\in\omega\ \ \neg\Big(\lambda\big((\neg O_1)_{n,\alpha}\big)>r_{\phi''(p,l)}\Big)\\&\Leftrightarrow\exists l\in\omega\ \ \neg R_{\neg O_1}\big(n,\alpha,\phi''(p,l)\big),\end{aligned}$$

which shows the existence of $R'_{O_0}\in\Pi^1_1$ such that $\lambda_O(n,\alpha)>r_p\Leftrightarrow R'_{O_0}(n,\alpha,p)$ if $n\in D$.

Assume that $n\in D$. Then there is $\phi':\omega\to\omega$ recursive such that

$$\lambda_O(n,\alpha)<r_q\Leftrightarrow\lambda\big((O_1)_{n,\alpha}\big)<r_q\Leftrightarrow\lambda\big((\neg O_1)_{n,\alpha}\big)>1-r_q\Leftrightarrow R_{\neg O_1}\big(n,\alpha,\phi'(q)\big),$$

which shows the existence of $R''_{O_0}\in\Sigma^1_1$ such that $\lambda_O(n,\alpha)<r_q\Leftrightarrow R''_{O_0}(n,\alpha,q)$ if $n\in D$.

Assume that $n \in D$. Then there is $\phi'' : \omega^2 \to \omega$ recursive such that

$$\begin{aligned}\lambda_O(n,\alpha) < r_q \Leftrightarrow \lambda\big((O_0)_{n,\alpha}\big) < r_q \Leftrightarrow \exists l \in \omega \ \ \lambda\big((O_0)_{n,\alpha}\big) \leq 1 - r_q - \tfrac{1}{l+1} \\ \Leftrightarrow \exists l \in \omega \ \ \neg\Big(\lambda\big((O_0)_{n,\alpha}\big) > r_{\phi''(q,l)}\Big) \Leftrightarrow \exists l \in \omega \ \ \neg R_{O_0}\big(n,\alpha,\phi''(q,l)\big),\end{aligned}$$

which shows the existence of $R'''_{O_0} \in \Pi^1_1$ such that $\lambda_O(n,\alpha) < r_q \Leftrightarrow R'''_{O_0}(n,\alpha,q)$ if $n \in D$.

Finally, $r_p < \lambda_O(n,\alpha) < r_q \Leftrightarrow R_{O_0}(n,\alpha,p) \wedge R''_{O_0}(n,\alpha,q)$ and

$$r_p < \lambda_O(n,\alpha) < r_q \Leftrightarrow R'_{O_0}(n,\alpha,p) \wedge R'''_{O_0}(n,\alpha,q)$$

if $n \in D$, which shows that λ_O is Σ^1_1-recursive and Π^1_1-recursive on $D \times \omega$.

(c) We first prove the following. Let X, Y be a recursively presented Polish spaces and $g : X \times \omega \to Y$ be a Δ^1_1-recursive map. Then the partial map $h : X \to Y$ defined by

$$h(x) := \lim_{l \to \infty} g(x,l)$$

when this limit exists is Δ^1_1-recursive.

Indeed, the domain D of h is $\{x \in X \mid \forall r \in \omega \ \ \exists L \in \omega \ \ \forall k, l \geq L \ \ d_Y\big(g(x,k), g(x,l)\big) < 2^{-r}\}$, so that D is Δ^1_1. If $x \in D$, then $h(x) \in N(Y,u)$ is equivalent to

$$\exists p, q \in \omega \ \ \frac{p}{q+1} < \frac{\big((u)_1\big)_1}{\big((u)_1\big)_2 + 1} \ \wedge \ \exists L \in \omega \ \ \forall l \geq L \ \ g(x,l) \in N\big(Y, \langle 0, <\big((u)_1\big)_0, p, q> \rangle\big),$$

and we are done.

We set $B' := \big\{(\alpha,\gamma) \in \omega^\omega \times 2^\omega \mid \big((\alpha)_0, \gamma\big) \in B \ \wedge \ \gamma \in N_{(\alpha)^*_1|(\alpha)_1(0)}\big\}$, so that $B_\alpha \cap N_{\beta|l} = B'_{<\alpha, l\beta>}$ and B' is Δ^1_1. By (a), the map $g : \omega^\omega \times 2^\omega \times \omega \to [0,1]$ defined by $g(\alpha,\beta,l) := 2^{-l}\lambda(B_\alpha \cap N_{\beta|l})$ is Δ^1_1-recursive. By the previous point, the partial map $h : \omega^\omega \times 2^\omega \to [0,1]$ defined by

$$h(\alpha,\beta) := \lim_{l \to \infty} 2^{-l}\lambda(B_\alpha \cap N_{\beta|l})$$

when it exists is also Δ^1_1-recursive. But $h = d_B$.

Fix $n \in \mathcal{W}$. Then there is $q(n) \in \mathcal{W}$ such that

$$\mathcal{C}_{q(n)} = \big\{(\gamma,\delta) \in \omega^\omega \times 2^\omega \mid \big(n, (\gamma)_0, \delta\big) \in \mathcal{C} \ \wedge \ (\gamma)^*_1|(\gamma)_1(0) \subseteq \delta\big\}.$$

Moreover, we may assume that q is Π^1_1-recursive on $\mathcal{W}$, by the uniformization lemma. As Π^1_1 has the substitution property, the map $g' : (n,\alpha,\beta,l) \mapsto 2^{-l}\lambda(\mathcal{C}_{q(n),<\alpha,l\beta>}) = 2^{-l}\lambda(\mathcal{C}_{n,\alpha} \cap N_{\beta|l})$ is Π^1_1-recursive on $\mathcal{W} \times \omega^\omega \times 2^\omega \times \omega$. As above, the map

$$h' : (n,\alpha,\beta) \mapsto \lim_{l \to \infty} 2^{-l}\lambda(\mathcal{C}_{n,\alpha} \cap N_{\beta|l}) = d(\mathcal{C}_{n,\alpha}, \beta)$$

is Π^1_1-recursive on the Π^1_1 set $\{(n,\alpha,\beta) \in \mathcal{W} \times \omega^\omega \times 2^\omega \mid d(\mathcal{C}_{n,\alpha},\beta) \text{ exists}\}$.

(d) The argument here is partly similar to 11.6 and 17.25 in [K2]. We set, for $(k,l)\in\omega^2$,

$$A_{k,l}:=h^{-1}\big([\frac{k}{2^l},\frac{k+1}{2^l}))$$

and define $h_l:\omega^\omega\times 2^\omega\to[0,1]$ by $h_l=\Sigma_{k\leq 2^l}\ \frac{k}{2^l}\chi_{A_{k,l}}$. We also define $R\subseteq\omega^\omega\times 2^\omega\times\omega^3$ by

$$R(\alpha,\beta,u,k,l)\Leftrightarrow\frac{k}{2^l}\leq h(\alpha,\beta)<\frac{k+1}{2^l}\ \wedge\ \text{Seq}(u)\ \wedge\ \beta\in N^u,$$

so that R is Δ^1_1. Then we define $O\subseteq\omega^\omega\times 2^\omega$ by

$$O(\alpha,\beta)\Leftrightarrow\text{Seq}\big(\alpha(0)\big)\ \wedge\ \text{lh}\big(\alpha(0)\big)=3\ \wedge\ R\Big(\alpha^*,\beta,\big(\alpha(0)\big)_0,\big(\alpha(0)\big)_1,\big(\alpha(0)\big)_2\Big),$$

so that O is Δ^1_1.

Note that (h_l) is a sequence of Borel functions pointwise converging to h. By Lebesgue's dominated convergence theorem, $\int_{N^u}h(\alpha,.)\ d\lambda=\lim_{l\to\infty}\ \int_{N^u}h_l(\alpha,.)\ d\lambda$ if Seq(u). Note that

$$\int_{N^u}h_l(\alpha,.)\ d\lambda=\int_{N^u}\Sigma_{k\leq 2^l}\ \frac{k}{2^l}\chi_{A_{k,l}}(\alpha,.)\ d\lambda=\Sigma_{k\leq 2^l}\ \frac{k}{2^l}\lambda\big((A_{k,l})_\alpha\cap N^u\big)$$

$$=\Sigma_{k\leq 2^l}\ \frac{k}{2^l}\lambda(R_{\alpha,u,k,l})=\Sigma_{k\leq 2^l}\ \frac{k}{2^l}\lambda(O_{<u,k,l>\alpha}).$$

Using (a), this implies that the map $(\alpha,u,l)\mapsto\int_{N^u}h_l(\alpha,.)\ d\lambda$ is Δ^1_1-recursive on its Δ^0_1 domain $\omega^\omega\times\{u\in\omega\mid\text{Seq}(u)\}\times\omega$. As in the proof of (c), i_h is Δ^1_1-recursive on its domain. □

We now prove a uniform version of Theorem 4.3.2 in [K1] (due to Tanaka, see [T2]).

Theorem 3.6 *Let $B\in\Delta^1_1(\omega^\omega\times 2^\omega)$, and $\epsilon:\omega^\omega\to\mathbb{R}$ be Δ^1_1-recursive such that $\epsilon(\alpha)\in(0,1]$ for each $\alpha\in\omega^\omega$. Then there is $T\in\Delta^1_1(\omega^\omega\times\omega)$ such that*
(a) T_α is a tree for each $\alpha\in\omega^\omega$,
(b) if $K=\big\{(\alpha,\beta)\in\omega^\omega\times 2^\omega\mid\forall l\in\omega\ \ \big(\alpha,\overline{\beta}(l)\big)\in T\big\}$, then $K_\alpha\subseteq B_\alpha$ and $\lambda(K_\alpha)\geq\lambda(B_\alpha)-\epsilon(\alpha)$ for each $\alpha\in\omega^\omega$.

Proof. Theorem 3.2 gives $\pi:\omega^\omega\to\omega^\omega\times 2^\omega$ recursive and $C\in\Pi^0_1(\omega^\omega)$ such that π is injective on C and $\pi[C]=B$. We set $Q:=\{(\alpha,\beta,\gamma)\in(\omega^\omega)^3\mid\gamma\in C\wedge\pi(\gamma)=(\alpha,\beta)\}$. As $Q\in\Pi^0_1$, Theorem 3.1 gives a recursive subset $\overline{T}$ of $\omega^\omega\times\omega^2$ such that $(\alpha,\beta,\gamma)\in Q\Leftrightarrow\forall l\in\omega\ \ \big(\alpha,\overline{\beta}(l),\overline{\gamma}(l)\big)\in\overline{T}$ and $\overline{T}_\alpha$ is a tree for each $\alpha\in\omega^\omega$.

We set, for $u,v\in\omega$,

$$u\leq^a v\ \Leftrightarrow\ \text{Seq}(u),\text{Seq}(v)\ \wedge\ \text{lh}(u)=\text{lh}(v)\ \wedge\ \forall i<\text{lh}(u)\ \ (u)_i\leq(v)_i.$$

Then we set, for $u\in\omega$ with Seq(u) and $\alpha\in\omega^\omega$,

$$B^u_\alpha:=\big\{\beta\in 2^\omega\mid\exists\gamma\in\omega^\omega\ \ \overline{\gamma}\big(\text{lh}(u)\big)\leq^a u\ \wedge\ \forall l\in\omega\ \ \big(\alpha,\overline{\beta}(l),\overline{\gamma}(l)\big)\in\overline{T}\big\}$$

and $B':=\{(\alpha,\beta)\in\omega^\omega\times 2^\omega\mid\text{Seq}\big(\alpha(0)\big)\ \wedge\ \beta\in B^{\alpha(0)}_{\alpha^*}\}$. Note that B' is Σ^1_1. In fact, B' is Δ^1_1 by uniqueness of the witness γ.

We now define $\delta_\alpha \in \omega^\omega$ as follows. We define $\delta_\alpha(i)$ by induction on i. We first set

$$\delta_\alpha(0) := \min\{k \in \omega \mid \lambda(B_\alpha^{<k>}) > \lambda(B_\alpha) - \frac{\epsilon(\alpha)}{2}\}.$$

This number exists since B_α is the increasing union of the $B_\alpha^{<k>}$'s. Then

$$\delta_\alpha(i+1) := \min\{k \in \omega \mid \lambda(B_\alpha^{<\delta_\alpha(0),...,\delta_\alpha(i),k>}) > \lambda(B_\alpha) - \frac{\epsilon(\alpha)}{2} - ... - \frac{\epsilon(\alpha)}{2^{i+2}}\}.$$

Note that $\delta_\alpha \in \Delta^1_1(\alpha)$, by Corollary 3.5.(a).

We set $T := \{(\alpha, v) \in \omega^\omega \times \omega \mid \text{Seq}(v) \wedge \exists u \leq^a \overline{\delta_\alpha}(\text{lh}(v)) \ \ (\alpha, v, u) \in \overline{T}\}$, so that $T \in \Delta^1_1(\omega^\omega \times \omega)$ and T_α is a tree for each $\alpha \in \omega^\omega$.

We set $K := \{(\alpha, \beta) \in \omega^\omega \times 2^\omega \mid \forall l \in \omega \ \ \beta \in B_\alpha^{\overline{\delta_\alpha}(l)}\}$, so that $K_\alpha \subseteq B_\alpha$ and

$$\lambda(K_\alpha) = \lim_{l \to \infty} \lambda(B_\alpha^{\overline{\delta_\alpha}(l)}) \geq \lambda(B_\alpha) - \epsilon(\alpha)$$

for each $\alpha \in \omega^\omega$ since $(B_\alpha^{\overline{\delta_\alpha}(l)})_{l \in \omega}$ is decreasing. It remains to apply König's lemma to see that $K = \big\{(\alpha, \beta) \in \omega^\omega \times 2^\omega \mid \forall l \in \omega \ \ \big(\alpha, \overline{\beta}(l)\big) \in T\big\}$ since

$$\big\{s \in \omega^{<\omega} \mid <s(0), ..., s(|s|-1)> \leq^a \overline{\delta_\alpha}\big(|s|\big) \ \wedge \ \big(\alpha, \overline{\beta}(|s|), <s(0), ..., s(|s|-1)>\big) \in \overline{T}\big\}$$

is a finitely splitting tree. $\square$

- We want to prove an effective and uniform version of the Lusin-Menchoff lemma. We first need the following result, which slightly and uniformly refines Theorem A in [L] at the first level of the Borel hierarchy.

Lemma 3.7 *Let O be a Δ^1_1 subset of $\omega^\omega \times 2^\omega$ with open vertical sections. Then there is a Δ^1_1-recursive map $f : \omega^\omega \to \omega^\omega$ such that O_α is the disjoint union $\bigcup\big\{N^{f(\alpha)(u)} \mid u \in \omega \wedge Seq\big(f(\alpha)(u)\big)\big\}$, for each $\alpha \in \omega^\omega$.*

Proof. Let $P := \big\{(\alpha, u) \in \omega^\omega \times \omega \mid \text{Seq}(u) \ \wedge \ \big(\text{lh}(u) = 0 \ \vee \ (N^u \subseteq O_\alpha \ \wedge \ N^{u^-} \not\subseteq O_\alpha)\big)\big\}$. Note that P is Π^1_1, since a nonempty $\Delta^1_1(\alpha)$ closed subset of 2^ω contains a $\Delta^1_1(\alpha)$ point, by 4F.15 in [M]. We then define a relation R on $\omega^\omega \times 2^\omega \times \omega$ by $R(\alpha, \beta, u) \Leftrightarrow P(\alpha, u) \ \wedge \ \beta \in N^u$, so that R is Π^1_1. Note that, for each $(\alpha, \beta) \in O$ there is u with $R(\alpha, \beta, u)$. By 4B.5 in [M], there is a Δ^1_1-recursive map $g : \omega^\omega \times 2^\omega \to \omega$ such that $R\big(\alpha, \beta, g(\alpha, \beta)\big)$ for each $(\alpha, \beta) \in O$. Fix $\alpha \in \omega^\omega$. Note that $S^\alpha := \{g(\alpha, \beta) \mid \beta \in O_\alpha\}$ is a $\Sigma^1_1(\alpha)$ subset of ω contained in the $\Pi^1_1(\alpha)$ set P_α. By 4B.11 and 4C in [M], there is $D^\alpha \in \Delta^1_1(\alpha)$ with $S^\alpha \subseteq D^\alpha \subseteq P_\alpha$. Note that $O_\alpha \subseteq \bigcup_{u \in D^\alpha} \ N^u \subseteq O_\alpha$, so that O_α is the disjoint union of the sequence $(N^u)_{u \in D^\alpha}$. We define $\delta_\alpha \in \omega^\omega$ by

$$\delta_\alpha(u) := \begin{cases} u \text{ if } u \in D_\alpha, \\ 0 \text{ otherwise.} \end{cases}$$

Note that $\delta_\alpha \in \Delta^1_1(\alpha)$ and O_α is the disjoint union $\bigcup\big\{N^{\delta_\alpha(u)} \mid u \in \omega \wedge \text{Seq}\big(\delta_\alpha(u)\big)\big\}$. As the set $\Big\{(\alpha, \delta) \in \omega^\omega \times \omega^\omega \mid \delta \in \Delta^1_1(\alpha) \wedge O_\alpha \text{ is the disjoint union } \bigcup\big\{N^{\delta(u)} \mid u \in \omega \wedge \text{Seq}\big(\delta(u)\big)\big\}\Big\}$ is Π^1_1, it remains to apply the uniformization lemma to get the desired map f. $\square$

Notation. We set $\mathcal{W}_1 := \{n\in\mathcal{W} \mid \forall\alpha\in\omega^\omega\ \ \exists\gamma_n\in\Delta^1_1(\alpha)\ \ \mathcal{C}_{n,\alpha}=\bigcup\big\{N^{\gamma_n(u)} \mid u\in\omega\wedge\text{Seq}\big(\gamma_n(u)\big)\big\}$, so that, by Lemma 3.7, $\mathcal{W}_1$ is a Π^1_1 set of codes for the Δ^1_1 subsets of $\omega^\omega\times 2^\omega$ with open vertical sections.

Lemma 3.8 *Let F be a Δ^1_1 subset of $\omega^\omega\times 2^\omega$ with closed vertical sections, and B be a Δ^1_1 subset of $\omega^\omega\times 2^\omega$ such that $B\supseteq F$ and $d(B_\alpha,\beta)=1$ for each $(\alpha,\beta)\in F$. Then there is a Δ^1_1 subset C of $\omega^\omega\times 2^\omega$ with closed vertical sections such that*

(1) $F\subseteq C\subseteq B$,

(2) $d(B_\alpha,\beta)=1$ for each $(\alpha,\beta)\in C$,

(3) $d(C_\alpha,\beta)=1$ for each $(\alpha,\beta)\in F$.

Proof. Lemma 3.7 gives a Δ^1_1-recursive map $f:\omega^\omega\to\omega^\omega$ such that $(\neg F)_\alpha$ is the disjoint union $\bigcup\big\{N^{f(\alpha)(u)} \mid u\in\omega\wedge\text{Seq}\big(f(\alpha)(u)\big)\big\}$, for each $\alpha\in\omega^\omega$. We set

$$B':=\Big\{(\alpha,\gamma)\in\omega^\omega\times 2^\omega \mid \big((\alpha)_0,\gamma\big)\in B\ \wedge\ \text{Seq}\Big(f\big((\alpha)_0\big)\big((\alpha)_1(0)\big)\Big)\ \wedge\ \gamma\in N^{f((\alpha)_0)((\alpha)_1(0))}\Big\},$$

so that B' is Δ^1_1 and $B_\alpha\cap N^{f(\alpha)(u)}=B'_{<\alpha,u^\infty>}$ if $\text{Seq}\big(f(\alpha)(u)\big)$. By Corollary 3.5.(c), the partial map $(\alpha,\beta,u)\mapsto d(B_\alpha\cap N^{f(\alpha)(u)},\beta)$ is Δ^1_1-recursive. We then set

$$B'':=\{(\alpha,\gamma)\in B' \mid d(B_{(\alpha)_0}\cap N^{f((\alpha)_0)((\alpha)_1(0))},\gamma)=1\},$$

so that B'' is Δ^1_1 and $\{\beta\in B_\alpha\cap N^{f(\alpha)(u)} \mid d(B_\alpha\cap N^{f(\alpha)(u)},\beta)=1\}=B''_{<\alpha,u^\infty>}$ if $\text{Seq}\big(f(\alpha)(u)\big)$. We define $\epsilon:\omega^\omega\to\mathbb{R}$ by

$$\varepsilon(\alpha):=\begin{cases} 2^{-(\alpha)_1(0)}\lambda(B'_\alpha) \text{ if } \lambda(B'_\alpha)\neq 0,\\ 1 \text{ otherwise,}\end{cases}$$

so that ϵ is Δ^1_1-recursive by Corollary 3.5.(a), and $\epsilon(\alpha)\in(0,1]$ for each $\alpha\in\omega^\omega$. Theorem 3.6 gives $T\in\Delta^1_1(\omega^\omega\times\omega)$ such that

(a) T_α is a tree for each $\alpha\in\omega^\omega$,

(b) if $K=\big\{(\alpha,\beta)\in\omega^\omega\times 2^\omega \mid \forall l\in\omega\ \ \big(\alpha,\overline{\beta}(l)\big)\in T\big\}$, then $K_\alpha\subseteq B''_\alpha$ and $\lambda(K_\alpha)\geq\lambda(B''_\alpha)-\epsilon(\alpha)$ for each $\alpha\in\omega^\omega$.

We set, for $u\in\omega$,

$$F^u:=\big\{(\alpha,\beta)\in\omega^\omega\times 2^\omega \mid \text{Seq}\big(f(\alpha)(u)\big)\ \wedge\ (<\alpha,u^\infty>,\beta)\in K\ \wedge\ \lambda(B'_{<\alpha,u^\infty>})\neq 0\big\}.$$

As K is Δ^1_1 with closed vertical sections, so is F^u. If $\text{Seq}\big(f(\alpha)(u)\big)$ and $\lambda(B'_{<\alpha,u^\infty>})=0$, then $\lambda(B_\alpha\cap N^{f(\alpha)(u)})=0$ and $F^u_\alpha=\emptyset$, so that $F^u_\alpha\subseteq\{\beta\in B_\alpha\cap N^{f(\alpha)(u)} \mid d(B_\alpha\cap N^{f(\alpha)(u)},\beta)=1\}$ and $\lambda(F^u_\alpha)\geq(1-2^{-u})\lambda(B_\alpha\cap N^{f(\alpha)(u)})$. If $\text{Seq}\big(f(\alpha)(u)\big)$ and $\lambda(B'_{<\alpha,u^\infty>})\neq 0$, then

$$F^u_\alpha=K_{<\alpha,u^\infty>}\subseteq B''_{<\alpha,u^\infty>}=\{\beta\in B_\alpha\cap N^{f(\alpha)(u)} \mid d(B_\alpha\cap N^{f(\alpha)(u)},\beta)=1\}.$$

Moreover,

$$\lambda(F^u_\alpha)=\lambda(K_{<\alpha,u^\infty>})\geq\lambda(B''_{<\alpha,u^\infty>})-\epsilon(<\alpha,u^\infty>)=\lambda(B''_{<\alpha,u^\infty>})-2^{-u}\lambda(B'_{<\alpha,u^\infty>})$$
$$=(1-2^{-u})\lambda(B_\alpha\cap N^{f(\alpha)(u)})$$

since $\lambda(B_\alpha\cap N^{f(\alpha)(u)})=\lambda(\{\beta\in B_\alpha\cap N^{f(\alpha)(u)}\mid d(B_\alpha\cap N^{f(\alpha)(u)},\beta)=1\})$, by Theorem 2.1. It remains to set $C:=F\cup\bigcup_{u\in\omega}\ F^u$. We conclude as in the proof of Lemma 2.2. $\square$

- We now want to prove an effective and uniform version of Lemma 2.5.

Lemma 3.9 *Let C be a Δ^1_1 subset of $\omega^\omega\times 2^\omega$ with closed vertical sections, $\mathcal{G}$ be a Borel subset of 2^ω with $\lambda(\mathcal{G})=0$, and G be a Δ^1_1 subset of $\omega^\omega\times 2^\omega$ with G_δ vertical sections, contained in $\omega^\omega\times\mathcal{G}$ and disjoint from C. Then there is a Δ^1_1-recursive map $h:\omega^\omega\times 2^\omega\to\mathbb{R}$ such that $h(\alpha,\cdot):2^\omega\to[0,1]$ is τ-continuous for each $\alpha\in\omega^\omega$, $h_{|C}\equiv 0$ and $h_{|G}\equiv 1$.*

Proof. By Theorem 3.5 in [L], there is a Δ^1_1 subset F of $\omega\times\omega^\omega\times 2^\omega$ such that $F_{n,\alpha}$ is closed for each $(n,\alpha)\in\omega\times\omega^\omega$ and $\neg G=\bigcup_{n\in\omega}\ F_n$. Moreover, we may assume that $(F_n)_{n\in\omega}$ is increasing and $F_0=C$.

We will define, by primitive recursion, a partial map $f:\omega\to\omega$ which is Π^1_1-recursive on its domain such that $f(n)$ essentially codes the set $C_{\frac{1}{2^n}}$ constructed in the proof of Lemma 2.5. As this map will in fact be total, it will be Δ^1_1-recursive by the uniformization lemma.

We first apply Lemma 3.8 to $F:=F_0$ and $B:=\neg G$. This is possible because $G_\alpha\subseteq\mathcal{G}$, so that $(\neg G)_\alpha$ has λ-measure one and therefore density one at any point of 2^ω, for each $\alpha\in\omega^\omega$. Lemma 3.8 gives $C_1\in\Delta^1_1$ with closed vertical sections such that $\neg G\supseteq C_1\supseteq F_0$. Let $f(0)\in\mathcal{W}_1$ with $\mathcal{C}_{f(0)}=\neg C_1$.

More generally, we will have $\mathcal{C}_{f(n)}=\neg C_{\frac{1}{2^n}}$. As mentioned above, f will be defined by primitive recursion, which means that there will be a partial map $g:\omega^2\to\omega$ such that $f(n+1)=g\big(f(n),n\big)$. This partial map g will be Π^1_1-recursive on its Π^1_1 domain $\{m\in\mathcal{W}_1\mid\neg\mathcal{C}_m\subseteq\neg G\}\times\omega$, so that f will be Π^1_1-recursive on its domain by 7A.5 in [M]. The map g will take values in $\mathcal{W}_1$, and is constructed in such a way that, if $A:=\neg\mathcal{C}_m\subseteq\neg G$ and $A':=\neg\mathcal{C}_{g(m,n)}$, then

(1) $A\cup F_{n+1}\subseteq A'\subseteq\neg G$,
(2) $\forall(\alpha,\beta)\in A'\ \ d\big((\neg G)_\alpha,\beta\big)=1$,
(3) $\forall(\alpha,\beta)\in A\cup F_{n+1}\ \ d(A'_\alpha,\beta)=1$.

Lemma 3.8 ensures that such a $g(m,n)\in\omega$ exists if $(m,n)\in\{q\in\mathcal{W}_1\mid\neg\mathcal{C}_q\subseteq\neg G\}\times\omega$. As the properties (1)-(3) are Π^1_1 by Corollary 3.5, the uniformization lemma ensures the existence of g. So we constructed a Δ^1_1-recursive map $f:\omega\to\omega$, taking values in $\mathcal{W}_1$, such that $C_{\frac{1}{2^n}}:=\neg\mathcal{C}_{f(n)}$ is a Δ^1_1 subset of $\omega^\omega\times 2^\omega$ with closed vertical sections, $F_n\subseteq C_{\frac{1}{2^n}}\subseteq\neg G$, $C_{\frac{1}{2^n}}\subseteq C_{\frac{1}{2^{n+1}}}$, and

$$d\big((C_{\frac{1}{2^{n+1}}})_\alpha,\beta\big)=1$$

if $(\alpha,\beta)\in C_{\frac{1}{2^n}}$.

Similarly, we construct a Δ^1_1-recursive map $\tilde{F}:\omega\rightarrow\omega$ satisfying the following properties, if

$$D:=\{p\in\omega \mid \text{Seq}(p)\ \wedge\ \text{lh}(p)=2\ \wedge\ 0<(p)_1\leq 2^{(p)_0}\}.$$

(a) $\tilde{F}(p)\in\mathcal{W}_1$ if $p\in D$, in which case we set $C_p:=\neg\mathcal{C}_{\tilde{F}(p)}$,
(b) $C_p\subseteq C_{p'}$ if $p,p'\in D\ \wedge\ \frac{(p')_1}{2^{(p')_0}}\leq\frac{(p)_1}{2^{(p)_0}}$,
(c) $d\big((C_{p'})_\alpha,\beta\big)=1$ if $p,p'\in D\ \wedge\ \frac{(p')_1}{2^{(p')_0}}<\frac{(p)_1}{2^{(p)_0}}\ \wedge\ (\alpha,\beta)\in C_p$.

This allows us to define h by

$$1-h(\alpha,\beta):=\begin{cases} 0 \text{ if } (\alpha,\beta)\in G,\\ \sup\{\frac{(p)_1}{2^{(p)_0}}\mid p\in D\ \wedge\ (\alpha,\beta)\in C_p\} \text{ if } (\alpha,\beta)\notin G.\end{cases}$$

Note that h is Δ^1_1-recursive since $D\in\Delta^0_1$, so that the relation "$p\in D\wedge(\alpha,\beta)\in C_p$" is Δ^1_1 in (p,α,β). We conclude as in the proof of Lemma 2.5. $\square$

- We are now ready to prove the main lemma in this section. We equip the space $[0,1]^{2^{<\omega}}$ with the distance defined by $d(f,g):=\Sigma_{u\in\omega,\text{Seq}(u)}\ \frac{|f(s(u))-g(s(u))|}{2^{u+1}}$. We give a recursive presentation of $([0,1]^{2^{<\omega}},d)$. We set

$$f_n(s):=\begin{cases} \frac{((n)_{\overline{s}})_0}{((n)_{\overline{s}})_0+((n)_{\overline{s}})_1+1} \text{ if Seq}(n)\ \wedge\ \forall k<\text{lh}(n)\ \big(\text{Seq}\big((n)_k\big)\ \wedge\ \text{lh}\big((n)_k\big)=2\big)\ \wedge\ \overline{s}<\text{lh}(n),\\ 0 \text{ otherwise,}\end{cases}$$

so that (f_n) is dense in $[0,1]^{2^{<\omega}}$. It is now routine to check that the relations "$d(f_m,f_n)\leq\frac{p}{q+1}$" and "$d(f_m,f_n)<\frac{p}{q+1}$" are recursive in (m,n,p,q). It is also routine to check that $F:\omega^\omega\rightarrow[0,1]^{2^{<\omega}}$ is Δ^1_1-recursive if the map $F':\omega\times\omega^\omega\rightarrow\mathbb{R}$ defined by $F'(u,\alpha):=F(\alpha)\big(s(u)\big)$ if Seq(u), 0 otherwise, is Δ^1_1-recursive ($s(u)$ was defined at the beginning of Section 3).

Lemma 3.10 *Let $\mathcal{V}:=\{(f,\beta)\in\mathcal{M}\times 2^\omega\mid osc(f,\beta)>0\}$, $\mathcal{G}$ be a nonempty $G_\delta\cap\Delta^1_1$ subset of 2^ω with $\lambda(\mathcal{G})=0$, and G be a Δ^1_1 subset of $\omega^\omega\times 2^\omega$, contained in $\omega^\omega\times\mathcal{G}$, and with G_δ vertical sections. Then there is a Δ^1_1-recursive map $F:\omega^\omega\rightarrow[0,1]^{2^{<\omega}}$, taking values in $\mathcal{M}$, and such that $G_\alpha=\mathcal{V}_{F(\alpha)}$ for each $\alpha\in\omega^\omega$.*

Proof. We will define, by primitive recursion, $f:\omega\rightarrow\omega^4$ coding g_n, S_n, G^*_n, and $(s^n_j)_{j\in I_n}$ defining G^{**}_n considered in the proof of the Lemma 2.7. We must find $r:\omega^4{\times}\omega\rightarrow\omega^4$ with $f(n{+}1)=r\big(f(n),n\big)$. In practice,

(1) $f_0(n)\in\mathcal{W}_1$ codes $G^*_n\subseteq\omega^\omega\times 2^\omega$,
(2) $f_1(n)\in\mathcal{W}^{\omega^\omega\times 2^\omega\times\mathbb{R}}$ codes the graph of $g_n:\omega^\omega\times 2^\omega\rightarrow\mathbb{R}$,
(3) $f_2(n)\in\mathcal{W}^{\omega^\omega\times 2^\omega\times\mathbb{R}}$ codes the graph of $S_n:\omega^\omega\times 2^\omega\rightarrow\mathbb{R}$,
(4) $f_3(n)\in\mathcal{W}^{\omega^\omega\times\omega^\omega}$ codes the graph of the function $\alpha\mapsto(s^{n,\alpha}_j)_{j\in I_{n,\alpha}}$.

By Theorem 3.5 in [L], there is a Δ^1_1 subset O of $\omega\times\omega^\omega\times 2^\omega$ such that $O_{n,\alpha}$ is open for each $(n,\alpha)\in\omega\times\omega^\omega$ and $G=\bigcap_{n\in\omega}\ O_n$. Moreover, we may assume that $(O_n)_{n\in\omega}$ is decreasing and $O_0=\omega^\omega\times 2^\omega$.

Let $n_0\in\mathcal{W}_1$ with $\mathcal{C}_{n_0}=\omega^\omega\times 2^\omega$, $n_1\in\mathcal{W}^{\omega^\omega\times 2^\omega\times\mathbb{R}}$ with

$$\mathcal{C}_{n_1}^{\omega^\omega\times 2^\omega\times\mathbb{R}}=\{(\alpha,\beta,r)\in\omega^\omega\times 2^\omega\times\mathbb{R}\mid r=1\},$$

and $n_3\in\mathcal{W}^{\omega^\omega\times\omega^\omega}$ with $\mathcal{C}_{n_3}^{\omega^\omega\times\omega^\omega}=\{(\alpha,\gamma)\in\omega^\omega\times\omega^\omega\mid\gamma=10^\infty\}$. We set $f(0):=(n_0,n_1,n_1,n_3)$, so that $\mathcal{C}_{n_0}=G^*_0$, $\mathcal{C}_{n_1}^{\omega^\omega\times 2^\omega\times\mathbb{R}}=\text{Gr}(g_0)=\text{Gr}(S_0)$, $\mathcal{C}_{n_3}^{\omega^\omega\times\omega^\omega}=\text{Gr}(\alpha\mapsto 10^\infty)$,

$$\big\{u\in\omega\mid\text{Seq}\big((10^\infty)(u)\big)\big\}=\{0\}=I_0$$

and $(10^\infty)(0)=1=<>=s^0_0$. So $f(0)$ is as desired.

We now study the induction step. This means that we must define $r(n_0,n_1,n_2,n_3,n)\in\omega^4$.

(1) We first define $r_0(n_0,n_1,n_2,n_3,n)$ coding G^*_{n+1}. Fix $n_3\in\mathcal{W}^{\omega^\omega\times\omega^\omega}$ coding the graph of a Δ^1_1-recursive function $\phi:\omega^\omega\to\omega^\omega$ such that the sequences $s\big(\phi(\alpha)(u)\big)$ coded by the u's with $\text{Seq}\big(\phi(\alpha)(u)\big)$ are pairwise incompatible and $G_\alpha\subseteq\bigcup\big\{N^{\phi(\alpha)(u)}\mid u\in\omega\wedge\text{Seq}\big(\phi(\alpha)(u)\big)\big\}$ (we call P_3 the Π^1_1 set of such n_3's). Let $\alpha\in\omega^\omega$. Assume that $\text{Seq}\big(\phi(\alpha)(u)\big)$ (which intuitively means that $u\in I_{n,\alpha}$ and $s^{n,\alpha}_u$ is coded by $\phi(\alpha)(u)$). By continuity of λ,

$$0=\lambda(G_\alpha\cap N^{\phi(\alpha)(u)})=\lim_{j\to\infty}\lambda(O_{j,\alpha}\cap N^{\phi(\alpha)(u)}).$$

This gives $j(n,\alpha,u)>n$ minimal with $\lambda(O_{j(n,\alpha,u),\alpha}\cap N^{\phi(\alpha)(u)})<2^{-n-3-\text{lh}(\phi(\alpha)(u))}$ (note that $2^{-\text{lh}(\phi(\alpha)(u))}=\lambda(N^{\phi(\alpha)(u)})$). Moreover, $G_\alpha\cap N^{\phi(\alpha)(u)}\subseteq O_{j(n,\alpha,u),\alpha}\cap N^{\phi(\alpha)(u)}\subseteq O_{n+1,\alpha}\cap N^{\phi(\alpha)(u)}$, so that $O_{j(n,\alpha,u),\alpha}\cap N^{\phi(\alpha)(u)}$ satisfies the properties of the set O_j in the proof of Lemma 2.7. We will have $G^*_{n+1,\alpha}=\bigcup_{\text{Seq}(\phi(\alpha)(u))}\ O_{j(n,\alpha,u),\alpha}\cap N^{\phi(\alpha)(u)}$. By Corollary 3.5 and the uniformization lemma, we may assume that the map j is Δ^1_1-recursive on its Δ^1_1 domain

$$\big\{(n,\alpha,u)\in\omega\times\omega^\omega\times\omega\mid\ \text{Seq}\big(\phi(\alpha)(u)\big)\big\}.$$

Note that G^*_{n+1} is a Δ^1_1 subset of $\omega^\omega\times 2^\omega$ with open vertical sections, which gives $m\in\mathcal{W}_1$ such that $\mathcal{C}_m=G^*_{n+1}$. By incompatibility, $G^*_{n+1,\alpha}\cap N^{\phi(\alpha)(u)}=O_{j(n,\alpha,u),\alpha}\cap N^{\phi(\alpha)(u)}$. So we proved that, for each $(n_3,n)\in P_3\times\omega$, there is $m\in\mathcal{W}_1$ such that, for each $\alpha\in\omega^\omega$,

$$\begin{array}{l}(1)\ G_\alpha\subseteq\mathcal{C}_{m,\alpha}\subseteq O_{n+1,\alpha}\cap\bigcup\big\{N^{\phi(\alpha)(u)}\mid u\in\omega\wedge\text{Seq}\big(\phi(\alpha)(u)\big)\big\},\\ (5)\ \lambda(\mathcal{C}_{m,\alpha}\cap N^{\phi(\alpha)(u)})<2^{-n-3-\text{lh}(\phi(\alpha)(u))}\text{ if }u\in\omega\wedge\text{Seq}\big(\phi(\alpha)(u)\big).\end{array}$$

By Corollary 3.5 and the uniformization lemma, we may assume that the map $\tilde{r}_0:(n_3,n)\mapsto m$ is Π^1_1-recursive on $P_3\times\omega$. We set $r_0(n_0,n_1,n_2,n_3,n):=\tilde{r}_0(n_3,n)$, which defines a partial map r_0 which is Π^1_1-recursive on its Π^1_1 domain $\omega^3\times P_3\times\omega$.

(2) We now define $r_1(n_0,n_1,n_2,n_3,n)$ coding g_{n+1}. We use Lemma 3.9 and its proof. Note that $r_0(n_0,n_1,n_2,n_3,n)\in D_0:=\{m\in\mathcal{W}_1\mid G\subseteq\mathcal{C}_m\}$. The proof of Lemma 3.9 shows that for any $m\in D_0$ there is $\tilde{F}_m\in\omega^\omega\cap\Delta^1_1$ satisfying the conditions (a), (b), (c) and

$$(d)\ \forall p\in D\ \ \neg(0<(p)_1=2^{(p)_0})\ \vee\ \mathcal{C}_{\tilde{F}_m(p)}\subseteq\mathcal{C}_m.$$

The uniformization lemma shows that we may assume that the partial map $\tilde{F}:m\mapsto\tilde{F}_m$ is Π^1_1-recursive on D_0.

The definition of h in the proof of Lemma 3.9 and the uniformization lemma show the existence of a partial map $\tilde{H}:\omega\rightarrow\omega$, which is Π^1_1-recursive on D_0, and such that $\tilde{H}(m)$ is in $\mathcal{W}^{\omega^\omega\times 2^\omega\times\mathbb{R}}$ and codes the graph of a Δ^1_1-recurive map $h:\omega^\omega\times 2^\omega\rightarrow\mathbb{R}$ with

$$1-h(\alpha,\beta):=\left\{\begin{array}{l}0 \text{ if } (\alpha,\beta)\in G\\ \sup\{\frac{(p)_1}{2^{(p)_0}}\mid p\in D\ \wedge\ (\alpha,\beta)\notin\mathcal{C}_{\tilde{F}(m)(p)}\} \text{ if } (\alpha,\beta)\notin G\end{array}\right.$$

if $m\in D_0$. We set $P_1:=\{c\in\mathcal{W}^{\omega^\omega\times 2^\omega\times\mathbb{R}}\mid\mathcal{C}_c$ is the graph of a function $\zeta_c\}$. It is routine to check that there is a Π^1_1-recursive partial map $I:\omega^2\rightarrow\omega$ on its domain P_1^2 such that $I(c,c')\in\mathcal{W}^{\omega^\omega\times 2^\omega\times\mathbb{R}}$ is the graph of the function $\min(\zeta_c,\zeta_{c'})$ if $c,c'\in P_1$. We set

$$r_1(n_0,n_1,n_2,n_3,n):=I\Big(n_1,\tilde{H}\big(r_0(n_0,n_1,n_2,n_3,n)\big)\Big),$$

so that r_1 is Π^1_1-recursive on its Π^1_1 domain $\omega\times P_1\times\omega\times P_3\times\omega$.

(3) We now define $r_2(n_0,n_1,n_2,n_3,n)$ coding

$$S_{n+1}=\left\{\begin{array}{l}S_n+g_{n+1} \text{ if } n \text{ is odd,}\\ S_n-g_{n+1} \text{ if } n \text{ is even.}\end{array}\right.$$

It is routine to check that there is a Π^1_1-recursive partial map $S:\omega^3\rightarrow\omega$ on its domain $P_1^2\times\omega$ such that $S(c,c',n)\in\mathcal{W}^{\omega^\omega\times 2^\omega\times\mathbb{R}}$ codes the graph of the function

$$(\alpha,\beta)\mapsto\left\{\begin{array}{l}\zeta_c(\alpha,\beta)+\zeta_{c'}(\alpha,\beta) \text{ if } n \text{ is odd}\\ \zeta_c(\alpha,\beta)-\zeta_{c'}(\alpha,\beta) \text{ if } n \text{ is even}\end{array}\right.$$

if $(c,c',n)\in P_1^2\times\omega$. We set $r_2(n_0,n_1,n_2,n_3,n):=S\big(n_2,r_1(n_0,n_1,n_2,n_3,n),n\big)$, so that r_2 is Π^1_1-recursive on its Π^1_1 domain $\omega\times P_1^2\times P_3\times\omega$.

(4) We now define $r_3(n_0,n_1,n_2,n_3,n)$ coding the graph of the function $\alpha\mapsto(s_j^{n+1,\alpha})_{j\in I_{n+1,\alpha}}$. We want to ensure the two following conditions:

(1) $G_\alpha\subseteq\bigcup_{j\in I_{n+1,\alpha}}N_{s_j^{n+1,\alpha}}\subseteq G^*_{n+1,\alpha}$
(6) $|\fint_{N_{s_j^{n+1,\alpha}}}S_{n+1}(\alpha,.)\ d\lambda-S_{n+1}(\alpha,\beta)|<2^{-3}$ if $j\in I_{n+1,\alpha}\ \wedge\ \beta\in G_\alpha\cap N_{s_j^{n+1,\alpha}}$

Note first that in practice

$$S_{n+1}(\alpha,\beta)=\left\{\begin{array}{l}0 \text{ if } n \text{ is even}\\ 1 \text{ if } n \text{ is odd}\end{array}\right.$$

if $(\alpha,\beta)\in G$ since $g_p(\alpha,\beta)=1$ for each p in this case. So there is $\psi:\omega\rightarrow\mathbb{R}^2$ recursive with

$$|\fint_{N_{s_j^{n+1,\alpha}}}S_{n+1}(\alpha,.)\ d\lambda-S_{n+1}(\alpha,\beta)|<2^{-3}\Leftrightarrow\psi_0(n)<\fint_{N_{s_j^{n+1,\alpha}}}S_{n+1}(\alpha,.)\ d\lambda<\psi_1(n)$$

if $(\alpha,\beta)\in G$. We use Corollary 3.5 and its proof. Note that $r_2(n_0,n_1,n_2,n_3,n)\in P_1$.

We first consider $n'_0 \in \mathcal{W}_1$ and $n'_2 \in P_1$ (coding G^*_{n+1} and S_{n+1} respectively) as variables. We define $R_0, R_1 \subseteq \omega \times \omega^\omega \times 2^\omega \times \omega^3$ by

$$R_0(n'_2, \alpha, \beta, u, k, l) \Leftrightarrow \exists r \in \mathbb{R}\ \neg\big(n'_2 \in \mathcal{W}^{\omega^\omega \times 2^\omega \times \mathbb{R}} \ \wedge\ (n'_2, \alpha, \beta, r) \notin \mathcal{C}^{\omega^\omega \times 2^\omega \times \mathbb{R}}\big) \wedge \big(\tfrac{k}{2^l} \leq r < \tfrac{k+1}{2^l} \ \wedge\ \text{Seq}(u) \ \wedge\ \beta \in N^u\big)$$

$$R_1(n'_2, \alpha, \beta, u, k, l) \Leftrightarrow \forall r \in \mathbb{R}\ \big(n'_2 \in \mathcal{W}^{\omega^\omega \times 2^\omega \times \mathbb{R}} \ \wedge\ (n'_2, \alpha, \beta, r) \notin \mathcal{C}^{\omega^\omega \times 2^\omega \times \mathbb{R}}\big) \vee \big(\tfrac{k}{2^l} \leq r < \tfrac{k+1}{2^l} \ \wedge\ \text{Seq}(u) \ \wedge\ \beta \in N^u\big),$$

so that R_0 is Σ^1_1, R_1 is Π^1_1, and $R_0(n'_2, \alpha, \beta, u, k, l) \Leftrightarrow R_1(n'_2, \alpha, \beta, u, k, l)$ if $n'_2 \in P_1$. Then, as in the proof of Corollary 3.5.(d), we define $O_0, O_1 \subseteq \omega \times \omega^\omega \times 2^\omega$ by

$$O_\varepsilon(n'_2, \alpha, \beta) \Leftrightarrow \text{Seq}\big(\alpha(0)\big) \ \wedge\ \text{lh}\big(\alpha(0)\big) = 3 \ \wedge\ R_\varepsilon\Big(n'_2, \alpha^*, \beta, \big(\alpha(0)\big)_0, \big(\alpha(0)\big)_1, \big(\alpha(0)\big)_2\Big)$$

if $\varepsilon \in 2$, so that O_0 is Σ^1_1, O_1 is Π^1_1, and $O_0(n'_2, \alpha, \beta) \Leftrightarrow O_1(n'_2, \alpha, \beta)$ if $n'_2 \in P_1$. In particular, $n'_2 \in P_1$ and $\text{Seq}(u)$ imply that

$$\int_{N^u} S_{n+1}(\alpha, .)\ d\lambda = \lim_{l \to \infty} \Sigma_{k \leq 2^l}\ \frac{k}{2^l} \lambda\big((O_\varepsilon)_{n'_2, <u,k,l>\alpha}\big)$$

for each $\varepsilon \in 2$. Thus $a < \int_{N^u} S_{n+1}(\alpha, .)\ d\lambda < b$ is in this case equivalent to

$$\exists p_0, p_1, q_0, q_1, N \in \omega\ a < \frac{p_0}{p_1+1} \wedge \frac{q_0}{q_1+1} < b \wedge \forall l \geq N\ \frac{p_0}{p_1+1} \leq \Sigma_{k \leq 2^l} \frac{k}{2^l} \lambda\big((O_\varepsilon)_{n'_2, <u,k,l>\alpha}\big) \leq \frac{q_0}{q_1+1}.$$

By Corollary 3.5.(b) applied to $D := P_1$, the partial map $\lambda_O : P_1 \times \omega^\omega \to \mathbb{R}$ defined by

$$\lambda_O(n'_2, \alpha) := \lambda\big((O_0)_{n'_2, \alpha}\big)$$

is Σ^1_1-recursive and Π^1_1-recursive on its domain. By 3E.2, 3G.1 and 3G.2 in [M], these two classes of functions are closed under composition. In particular, the partial map

$$(n'_2, \alpha, u, l) \mapsto \Sigma_{k \leq 2^l} \frac{k}{2^l} \lambda\big((O_\varepsilon)_{n'_2, <u,k,l>\alpha}\big)$$

is Σ^1_1-recursive and Π^1_1-recursive on $P_1 \times \omega^\omega \times \omega^2$. This shows the existence of $Q_0 \in \Sigma^1_1(\omega^2 \times \omega^\omega \times \omega)$ and $Q_1 \in \Pi^1_1(\omega^2 \times \omega^\omega \times \omega)$ such that

$$Q_0(n'_2, n, \alpha, u) \Leftrightarrow Q_1(n'_2, n, \alpha, u) \Leftrightarrow \text{Seq}(u) \ \wedge\ \psi_0(n) < \int_{N^u} S_{n+1}(\alpha, .)\ d\lambda < \psi_1(n)$$

if $n'_2 \in P_1$. We now consider $n'_0 \in \mathcal{W}_1$ and $n'_2 \in P_1$ as parameters. We set

$$P_{n'_0, n'_2}(n, \alpha, u) \Leftrightarrow Q_1(n'_2, n, \alpha, u) \ \wedge\ N^u \subseteq \mathcal{C}_{n'_0, \alpha} \ \wedge\ \forall k < \text{lh}(u)\ \ \big(\neg Q_0\big(n'_2, n, \alpha, \underline{u}(k)\big) \ \vee\ N^{\underline{u}(k)} \not\subseteq \mathcal{C}_{n'_0, \alpha}\big).$$

Note that for each $(\alpha, \beta) \in G$ there is $l \in \omega$ minimal with the properties that $N_{\beta|l} \subseteq \mathcal{C}_{n'_0, \alpha}$ and $Q_1\big(n'_2, n, \alpha, <\beta(0), ..., \beta(l-1)>\big)$, so that $P_{n'_0, n'_2}\big(n, \alpha, <\beta(0), ..., \beta(l-1)>\big)$ since $n'_0 \in \mathcal{W}_1$ and $n'_2 \in P_1$. As $n'_0 \in \mathcal{W}_1$, $N^{\underline{u}(k)} \setminus \mathcal{C}_{n'_0, \alpha}$ is a $\Delta^1_1(\alpha)$ compact subset of 2^ω, so that it contains a $\Delta^1_1(\alpha)$ point if it is not empty (see 4F.15 in [M]). This shows that $P_{n'_0, n'_2}$ is Π^1_1.

The uniformization lemma provides a Δ^1_1-recursive map $L\!:\omega\times\omega^\omega\times 2^\omega\rightarrow\omega$ such that

$$P_{n'_0,n'_2}\Big(n,\alpha,<\beta(0),...,\beta\big(L(n,\alpha,\beta)-1\big)>\Big)$$

if $(\alpha,\beta)\in G$. Note that the Σ^1_1 set

$$\sigma:=\big\{(n,\alpha,u)\in\omega\times\omega^\omega\times\omega\mid\exists\beta\in G_\alpha\ \ u=<\beta(0),...,\beta\big(L(n,\alpha,\beta)-1\big)>\big\}$$

is contained in the Π^1_1 set $\pi:=\{(n,\alpha,u)\in\omega\times\omega^\omega\times\omega\mid P_{n'_0,n'_2}(n,\alpha,u)\}$. By 7B.3 in [M], there is a Δ^1_1 subset δ of $\omega\times\omega^\omega\times\omega$ such that $\sigma\subseteq\delta\subseteq\pi$. We now also consider n as a parameter and define $\varphi\!:\omega^\omega\rightarrow\omega^\omega$ by

$$\varphi(\alpha)(u):=\begin{cases}u\text{ if }(n,\alpha,u)\in\delta,\\ 0\text{ otherwise.}\end{cases}$$

Note that φ is Δ^1_1-recursive, and that $\text{Seq}\big(\varphi(\alpha)(u)\big)$ is equivalent to $(n,\alpha,u)\in\delta$. In particular,

$$\begin{array}{l}(1)\ G_\alpha\subseteq\bigcup\big\{N^{\varphi(\alpha)(u)}\mid u\in\omega\wedge\text{Seq}\big(\varphi(\alpha)(u)\big)\big\}\subseteq\mathcal{C}_{n'_0,\alpha}\\ (6)\ |\,⨍_{N^{\varphi(\alpha)(u)}}S_{n+1}(\alpha,.)\ d\lambda-S_{n+1}(\alpha,\beta)|<2^{-3}\text{ if Seq}\big(\varphi(\alpha)(u)\big)\ \wedge\ \beta\in G_\alpha\cap N^{\varphi(\alpha)(u)}\end{array}$$

for each $\alpha\in\omega^\omega$. Let $k\in\mathcal{W}^{\omega^\omega\times\omega^\omega}$ such that $\mathcal{C}^{\omega^\omega\times\omega^\omega}_k=\text{Gr}(\varphi)$. We now consider n'_0, n'_2 and n as variables again. Note that for each $(n'_0,n'_2,n)\in\mathcal{W}_1\times P_1\times\omega$ there is $k\in\omega$ such that

$$R(n'_0,n'_2,n,k)\Leftrightarrow\begin{cases}k\in\mathcal{W}^{\omega^\omega\times\omega^\omega}\ \wedge\\ \Big(\forall\alpha\in\omega^\omega\ \ \forall\gamma\in\omega^\omega\ \ \big(k\in\mathcal{W}^{\omega^\omega\times\omega^\omega}\wedge\neg\mathcal{C}^{\omega^\omega\times\omega^\omega}(k,\alpha,\gamma)\big)\vee\\ \big((1)\ G_\alpha\subseteq\bigcup\big\{N^{\gamma(u)}\mid u\in\omega\wedge\text{Seq}\big(\gamma(u)\big)\big\}\subseteq\mathcal{C}_{n'_0,\alpha}\\ \wedge\ (6)\ \forall u\in\omega\ \ \neg\text{Seq}\big(\gamma(u)\big)\ \vee\ Q_1(n'_2,n,\alpha,u)\big)\Big)\end{cases}$$

Note that $R\in\Pi^1_1(\omega^4)$. The uniformization lemma provides a partial map $K:\omega^3\mapsto\omega$ which is Π^1_1-recursive on its Π^1_1 domain $\mathcal{W}_1\times P_1\times\omega$, and $R\big(n'_0,n'_2,n,K(n'_0,n'_2,n)\big)$ if

$$(n'_0,n'_2,n)\in\mathcal{W}_1\times P_1\times\omega.$$

It remains to set $r_3(n_0,n_1,n_2,n_3,n):=K(n'_0,n'_2,n)$ if $n'_0=r_0(n_0,n_1,n_2,n_3,n)$ and

$$n'_2=r_2(n_0,n_1,n_2,n_3,n),$$

so that r_3 is Π^1_1-recursive on its Π^1_1 domain $\mathcal{W}_1\times P^2_1\times P_3\times\omega$.

Finally, r is Π^1_1-recursive on $\mathcal{W}_1\times P^2_1\times P_3\times\omega$, f is Π^1_1-recursive on ω, and thus f is Δ^1_1-recursive by the uniformization lemma since it is total.

We are now ready to define the dimension two versions of G^*_n, g_n, S_n, and $(s^n_j)_{j\in I_n}$:

$$\begin{array}{l}(1)\ G^*_n:=\mathcal{C}_{f_0(n)},\\ (2)\ g_n(\alpha,\beta)=\rho\Leftrightarrow\big(f_1(n),\alpha,\beta,\rho\big)\in\mathcal{C}^{\omega^\omega\times 2^\omega\times\mathbb{R}},\\ (3)\ S_n(\alpha,\beta)=\rho\Leftrightarrow\big(f_2(n),\alpha,\beta,\rho\big)\in\mathcal{C}^{\omega^\omega\times 2^\omega\times\mathbb{R}},\\ (4)\ \begin{cases}(i)\ j\in I_{n,\alpha}\Leftrightarrow\exists\delta\in\omega^\omega\ \ \big(f_3(n),\alpha,\delta\big)\in\mathcal{C}^{\omega^\omega\times\omega^\omega}\ \wedge\ \text{Seq}\big(\delta(j)\big),\\ (ii)\ s^{n,\alpha}_j=\delta(j)\text{ if }j\in I_{n,\alpha}.\end{cases}\end{array}$$

By construction of r, these objects satisfy the conditions (1)-(6) of the proof of Lemma 2.7.

Consequently, the martingale $F(\alpha)$ will be defined in such a way that if $u\in\omega$ codes $s\in 2^{<\omega}$, then $F(\alpha)(s)=⨍_{N^u} f_\infty(\alpha,.)\ d\lambda$. Note that $G=\bigcap_{n\in\omega}\ G^*_n$, so that $\neg G$ is the disjoint union of the $G^*_n\setminus G^*_{n+1}$'s. Thus

$$\begin{aligned}\int_{N^u} f_\infty(\alpha,.)\ d\lambda &= \int_{N^u\setminus G_\alpha} f_\infty(\alpha,.)\ d\lambda = \Sigma_{n\in\omega}\ \int_{N^u\cap(G^*_n)_\alpha\setminus(G^*_{n+1})_\alpha} f_\infty(\alpha,.)\ d\lambda\\ &= \Sigma_{n\in\omega}\ \Sigma_{j\leq n}\ (-1)^j \int_{N^u\cap(G^*_n)_\alpha\setminus(G^*_{n+1})_\alpha} g_j(\alpha,.)\ d\lambda\\ &= \lim_{l\to\infty}\ \Sigma_{n\leq l}\ \Sigma_{j\leq n}\ (-1)^j \int_{N^u\cap(G^*_n)_\alpha\setminus(G^*_{n+1})_\alpha} g_j(\alpha,.)\ d\lambda.\end{aligned}$$

Consequently, in order to prove that F is Δ^1_1-recursive, it is enough to check that the partial map $(u,\alpha,j,n)\mapsto\int_{N^u\cap(G^*_n)_\alpha\setminus(G^*_{n+1})_\alpha} g_j(\alpha,.)\ d\lambda$ is Δ^1_1-recursive from $\{u\in\omega\mid \text{Seq}(u)\}\times\omega^\omega\times\omega^2$ into $\mathbb{R}$. By Corollary 3.5, it is enough to check that the map $h:\omega^\omega\times 2^\omega\to\mathbb{R}$ defined by

$$h(\alpha,\beta):=\begin{cases} g_{(\alpha(0))_0}(\alpha^*,\beta) \text{ if } \text{Seq}\big(\alpha(0)\big)\ \wedge\ \text{lh}\big(\alpha(0)\big)=2\ \wedge\ (\alpha^*,\beta)\in G^*_{(\alpha(0))_1}\setminus G^*_{(\alpha(0))_1+1},\\ 0 \text{ otherwise,}\end{cases}$$

is Δ^1_1-recursive. This comes from the facts that

$$(\alpha,\beta)\in G^*_n\Leftrightarrow\big(f_0(n),\alpha,\beta\big)\in\mathcal{C}\Leftrightarrow\neg\Big(f_0(n)\in\mathcal{W}\ \wedge\ \big(f_0(n),\alpha,\beta\big)\notin\mathcal{C}\Big)$$

is Δ^1_1 in (α,β,n) and

$$\begin{aligned} g_n(\alpha,\beta)\in N(\mathbb{R},p)\ &\Leftrightarrow\exists\rho\in\mathbb{R}\ \neg\Big(f_1(n)\in\mathcal{W}^{\omega\times 2^\omega\times\mathbb{R}}\ \wedge\ \big(f_1(n),\alpha,\beta,\rho\big)\notin\mathcal{C}^{\omega\times 2^\omega\times\mathbb{R}}\Big)\ \wedge\\ &\qquad\qquad\rho\in N(\mathbb{R},p)\\ &\Leftrightarrow\forall\rho\in\mathbb{R}\ \Big(f_1(n)\in\mathcal{W}^{\omega\times 2^\omega\times\mathbb{R}}\ \wedge\ \big(f_1(n),\alpha,\beta,\rho\big)\notin\mathcal{C}^{\omega\times 2^\omega\times\mathbb{R}}\Big)\ \vee\\ &\qquad\qquad\rho\in N(\mathbb{R},p)\end{aligned}$$

is Δ^1_1 in (α,β,n,p).

Finally, the map F is Δ^1_1-recursive and is as required. □

4 First consequences

(A) Universal sets

- We first recall some material from [K2]. The first result can be found in Section 23.F (see also [Za]).

Theorem 4.1 *(Zahorski) Let B be a subset of $[0,1]$. The following are equivalent:*
(a) there are $S\in\mathbf{\Sigma}^0_2$ and $P\in\mathbf{\Pi}^0_3$ with $m(P)=1$, where m is the Lebesgue measure on $[0,1]$, such that $B=S\cap P$,
(b) there is $f\in C([0,1])$ with $B=\{x\in[0,1]\mid f'(x)\text{ exists}\}$ (we consider only one-sided derivatives at the endpoints).

The second result is 23.23.

Theorem 4.2 *Let $\mathcal{G}$ be a G_δ subset of $(0,1)$ with $m(\mathcal{G})=0$. Then*

$$\{(f,x)\in C([0,1])\times\mathcal{G} \mid f'(x) \text{ exists}\}$$

is $C([0,1])$-universal for $\mathbf{\Pi}^0_3(\mathcal{G})$.

- We prove results in that spirit here.

Theorem 4.3 *Let B be a subset of 2^ω. Then the following are equivalent:*
(a) B is $\mathbf{\Sigma}^0_3$ and has λ-measure zero,
(b) there is $f\in\mathcal{M}$ with $B=\{\beta\in 2^\omega \mid osc(f,\beta)>0\}$.

Proof. (a) $\Rightarrow$ (b) Write $B=\bigcup_{n\in\omega} G_n$, where the G_n's are G_δ. Lemma 2.7 gives, for each n, a martingale f_n with $G_n=D(f_n)$ and $\{\text{osc}(f_n,\beta) \mid \beta\in 2^\omega\}\subseteq\{0\}\cup[\frac{1}{2},1]$. Lemma 2.8 gives $f\in\mathcal{M}$ with $D(f)=B$.

(b) $\Rightarrow$ (a) We already noticed in the introduction that B is $\mathbf{\Sigma}^0_3$. By Doob's theorem, B has λ-measure zero (see [D]). $\square$

Corollary 4.4 *Let $\mathcal{G}$ be a G_δ subset of 2^ω with $\lambda(\mathcal{G})=0$. Then $\{(f,\beta)\in\mathcal{M}\times\mathcal{G} \mid osc(f,\beta)>0\}$ is $\mathcal{M}$-universal for $\mathbf{\Sigma}^0_3(\mathcal{G})$.*

(B) Complete sets

- By 33.G in [K2], there is a uniform version of Zahorski's theorem, which allows one to prove the following result

Theorem 4.5 *(Mazurkiewicz) The set of differentiable functions in $C([0,1])$ is $\mathbf{\Pi}^1_1$-complete.*

- Here again, there is a result in that spirit.

Theorem 4.6 *The set $\mathcal{P}:=\{f\in\mathcal{M} \mid \forall\beta\in 2^\omega \;\; osc(f,\beta)=0\}$ is $\mathbf{\Pi}^1_1$-complete.*

Notation. Let $\mathcal{K}:=\{\beta\in 2^\omega \mid \forall n\in\omega \;\; \beta(2n)=0\}$, which is a $\mathit{\Pi}^0_1$ copy of the Cantor space 2^ω with $\lambda(\mathcal{K})=0$. In particular, $\mathcal{K}$ is a nonempty $G_\delta\cap\mathit{\Delta}^1_1$ subset of 2^ω.

Proof. Let $U\in\mathit{\Pi}^1_1(\omega^\omega\times 2^\omega)$ be ω^ω-universal for the co-analytic subsets of 2^ω, and

$$\Pi:=\{\alpha\in\omega^\omega \mid \big((\alpha)_0,(\alpha)_1\big)\in U\}.$$

Note that $\Pi\in\mathit{\Pi}^1_1$. If $P\in\mathbf{\Pi}^1_1(2^\omega)$, then $P=U_\alpha$ for some $\alpha\in\omega^\omega$, so that the map $\beta\mapsto <\alpha,\beta>$ is a continuous reduction of P to Π and Π is $\mathbf{\Pi}^1_1$-complete. Let $H\in\mathit{\Pi}^0_2(\omega^\omega\times 2^\omega)$ with $\neg\Pi=\text{proj}_0[H]$. We set $G:=\big\{(\alpha,\beta)\in\omega^\omega\times 2^\omega \mid \big(\alpha,(\beta)_1\big)\in H \;\wedge\; \beta\in\mathcal{K}\big\}$, so that $G\in\mathit{\Delta}^1_1(\omega^\omega\times 2^\omega)$, has G_δ vertical sections and $G\subseteq\omega^\omega\times\mathcal{K}$. Lemma 3.10 gives $F:\omega^\omega\rightarrow\mathcal{M}$ Borel such that $G_\alpha=\mathcal{V}_{F(\alpha)}$ for each $\alpha\in\omega^\omega$.

Thus

$$\alpha\notin\Pi\Leftrightarrow\exists\beta\in 2^\omega \;\; (\alpha,\beta)\in H\Leftrightarrow\exists\beta\in 2^\omega \;\; (\alpha,\beta)\in G\Leftrightarrow\exists\beta\in 2^\omega \;\; \big(F(\alpha),\beta\big)\in\mathcal{V}\Leftrightarrow F(\alpha)\notin\mathcal{P}.$$

Thus $\Pi=F^{-1}(\mathcal{P})$ and $\mathcal{P}$ is Borel $\mathbf{\Pi}^1_1$-complete. By 26.C in [K2], $\mathcal{P}$ is $\mathbf{\Pi}^1_1$-complete. $\square$

- We now prove Theorem 1.8. Let X be a metrizable compact space and Y be a Polish space. We equip $\mathcal{C}(X,Y)$ with the topology of uniform convergence, so that it is a Polish space (see 4.19 in [K2]). We use the map ψ defined before Theorem 1.8.

Theorem 4.7 *(a) The set* $\mathcal{P}_1:=\big\{(f_k)_{k\in\omega}\in\mathcal{P}^\omega\mid\big(\psi(f_k)\big)_{k\in\omega}$ *pointwise converges*$\big\}$ *is* $\mathbf{\Pi}^1_1$*-complete.*
(b) The set $\mathcal{P}_2:=\big\{(f_k)_{k\in\omega}\in\mathcal{P}^\omega\mid\big(\psi(f_k)\big)_{k\in\omega}$ *pointwise converges to zero*$\big\}$ *is* $\mathbf{\Pi}^1_1$*-complete.*
(c) The set $\mathcal{S}$ *defined by the formula*

$$\big\{(f_k)_{k\in\omega}\in\mathcal{P}^\omega\mid\exists\gamma\in\omega^\omega \text{ strictly increasing such that } \big(\psi(f_{\gamma(i)})\big)_{i\in\omega} \text{ pointwise converges to zero}\big\}$$

is $\mathbf{\Sigma}^1_2$*-complete.*

Proof. We define $\varphi:\mathcal{C}(2^\omega,[0,1])\to\mathcal{M}$ by $\varphi(h)(s):=⨍_{N_s} h\ d\lambda$. As in the proof of Lemma 2.7, φ is well-defined. It is also continuous, and injective: if $h\neq h'$, then we can find $q\in\omega$ and $s\in 2^{<\omega}$ such that $h(\beta)-h'(\beta)>2^{-q}$ for each $\beta\in N_s$ or $h'(\beta)-h(\beta)>2^{-q}$ for each $\beta\in N_s$, so that

$$|\varphi(h)(s)-\varphi(h')(s)|=\frac{1}{\lambda(N_s)}|\int_{N_s} h\ d\lambda-\int_{N_s} h'\ d\lambda|\geq 2^{-q}.$$

This implies that the range $\mathcal{R}$ of φ is Borel and $\psi:=\varphi^{-1}:\mathcal{R}\to\mathcal{C}(2^\omega,[0,1])$ is Borel. As every continuous map $h:2^\omega\to[0,1]$ is τ-continuous,

$$\text{lim}_{l\to\infty}\ \varphi(h)(\beta|l)=\text{lim}_{l\to\infty}\ ⨍_{N_{\beta|l}} h\ d\lambda=h(\beta)$$

for each $\beta\in 2^\omega$, by Lemma 2.6. This implies that $f\in\mathcal{P}$ and $\psi(f)(\beta)=\text{lim}_{l\to\infty}\ f(\beta|l)$ for each $\beta\in 2^\omega$ if $f\in\mathcal{R}$.

(a) Note that the proof of 33.11 in [K2] shows that the set

$$P_1:=\big\{(h_k)_{k\in\omega}\in\big(\mathcal{C}(2^\omega,[0,1])\big)^\omega\mid(h_k)_{k\in\omega} \text{ pointwise converges}\big\}$$

is $\mathbf{\Pi}^1_1$-complete. As $\mathcal{E}:=\big\{(f_k)_{k\in\omega}\in\mathcal{R}^\omega\ \mid\ \big(\psi(f_k)\big)_{k\in\omega}$ pointwise converges$\big\}=(\psi^\omega)^{-1}(P_1)$, the equalities $P_1=(\varphi^\omega)^{-1}(\mathcal{E})=(\varphi^\omega)^{-1}(\mathcal{P}_1)$ hold and $\mathcal{P}_1$ is $\mathbf{\Pi}^1_1$-complete.

(b) We argue as in (a).

(c) As in [B-Ka-L], the set

$$S:=\big\{(h_k)_{k\in\omega}\in\big(\mathcal{C}(2^\omega,[0,1])\big)^\omega\mid\exists\gamma\in\omega^\omega\ \ \big(h_{\gamma(i)}\big)_{i\in\omega} \text{ pointwise converges to zero}\big\},$$

is $\mathbf{\Sigma}^1_2$-complete. Indeed, fix $Q\in\mathbf{\Sigma}^1_2(2^\omega)$.

Lemma 2.2 in [B-Ka-L] gives $(g_k)_{k\in\omega}\in\big(\mathcal{C}(2^\omega\times 2^\omega,2)\big)^\omega$ such that, for each $\delta\in 2^\omega$, the following are equivalent:

(i) $\delta\in Q$,
(ii) $\exists\gamma\in\omega^\omega\ \ \forall\beta\in 2^\omega\ \ \text{lim}_{i\to\infty}\ g_{\gamma(i)}(\delta,\beta)=0$.

We define, $g:2^\omega\rightarrow\big(\mathcal{C}(2^\omega,[0,1])\big)^\omega$ by $g(\delta)(k)(\beta):=g_k(\delta,\beta)$. Then g is continuous and reduces Q to S. As

$$\mathcal{E}':=\big\{(f_k)_{k\in\omega}\in\mathcal{R}^\omega\mid\exists\gamma\in\omega^\omega\ \ \big(\psi(f_{\gamma(i)})\big)_{i\in\omega}\text{ pointwise converges to zero}\big\}=(\psi^\omega)^{-1}(S),$$

$S=(\varphi^\omega)^{-1}(\mathcal{E}')=(\varphi^\omega)^{-1}(\mathcal{S})$ and $\mathcal{S}$ is $\mathbf{\Sigma}^1_2$-complete. □

5 Universal and complete sets in the spaces $\mathcal{C}(2^\omega,X)$

- It is known that if $\mathbf{\Gamma}$ is a self-dual Wadge class and X is a Polish space, then there is no set which is X-universal for the subsets of X in $\mathbf{\Gamma}$ (see 22.7 in [K2]). This is no longer the case if the space of codes is different from the space of coded sets.

Proposition 5.1 *Let X be a Polish space, $\mathbf{\Gamma}$ be a Wadge class with complete set $C\in\mathbf{\Gamma}(X)$, and $\mathcal{U}^{\mathbf{\Gamma}}:=\{(h,\beta)\in\mathcal{C}(2^\omega,X)\times 2^\omega\mid h(\beta)\in C\}$. Then $\mathcal{U}^{\mathbf{\Gamma}}$ is $\mathcal{C}(2^\omega,X)$-universal for the $\mathbf{\Gamma}$ subsets of 2^ω.*

Proof. As the evaluation map $(h,\beta)\mapsto h(\beta)$ is continuous, $\mathcal{U}^{\mathbf{\Gamma}}\in\mathbf{\Gamma}$. If $A\in\mathbf{\Gamma}(2^\omega)$, then $A=h^{-1}(C)$ for some $h\in\mathcal{C}(2^\omega,X)$, so that $A=\mathcal{U}^{\mathbf{\Gamma}}_h$. □

We will partially strengthen this result to get our uniform universal sets.

- Recall that it is proved in [K3] that a Borel $\mathbf{\Pi}^1_1$-complete set is actually $\mathbf{\Pi}^1_1$-complete. In fact, Kechris's proof shows the result for the classes $\mathbf{\Pi}^1_n$. Our main tool is a uniform version of this. Kechris's result has recently been strengthened in [P] as follows.

Theorem 5.2 *(Pawlikowski) Let $n\geq 1$ be a natural number, and $C\subseteq X\subseteq 2^\omega$. If Borel functions from 2^ω into X give as preimages of C all $\mathbf{\Pi}^1_n$ subsets of 2^ω, then so do continuous injections.*

The main tool mentioned above is the following:

Theorem 5.3 *Let $n\geq 1$ be a natural number, $\mathcal{U}^{\mathbf{\Pi}^1_n,2^\omega}$ be a suitable ω^ω-universal set for the $\mathbf{\Pi}^1_n$ subsets of 2^ω, X be a recursively presented Polish space, $C\in\Pi^1_n(X)$, $\mathcal{R}:\omega^\omega\times\omega^\omega\rightarrow\omega^\omega$ be a recursive map, and $b:\omega^\omega\rightarrow X$ be a Δ^1_1-recursive map such that*

$$(\alpha,\beta)\in\mathcal{U}^{\mathbf{\Pi}^1_n,2^\omega}\Leftrightarrow b\big(\mathcal{R}(\alpha,\beta)\big)\in C$$

for each $(\alpha,\beta)\in\omega^\omega\times 2^\omega$. Then there is a Δ^1_1-recursive map $f:\omega^\omega\rightarrow\mathcal{C}(2^\omega,X)$ such that

$$(\alpha,\beta)\in\mathcal{U}^{\mathbf{\Pi}^1_n,2^\omega}\Leftrightarrow f(\alpha)(\beta)\in C$$

for each $(\alpha,\beta)\in\omega^\omega\times 2^\omega$.

- We first recall some material from [K3].

Definition 5.4 *(a) A* **coding system** *for nonempty perfect binary trees is a pair $(\mathcal{D},\mathcal{O})$, where $\mathcal{D}\subseteq 2^\omega$ and $\mathcal{O}:\mathcal{D}\rightarrow\{T\in 2^{2^{<\omega}}\mid T\text{ is a nonempty perfect binary tree}\}$ is onto.*

(b) A coding system $(\mathcal{D},\mathcal{O})$ is **nice** *if*

(i) for any $\alpha\in\omega^\omega$ and any $\Delta^1_1(\alpha)$-recursive map $H:2^\omega\times 2^\omega\rightarrow\omega$, we can find $\beta\in\mathcal{D}\cap\Delta^1_1(\alpha)$ and $k\in\omega$ such that $H(\beta,\delta)=k$ for each δ in the body $[\mathcal{O}(\beta)]$ of $\mathcal{O}(\beta)$,

(ii) $\mathcal{D}$ is Π^1_1 and, for $\beta\in\mathcal{D}$, the relation

$$R(m,\beta)\Leftrightarrow \mathit{Seq}(m)\ \wedge\ \big((m)_0,...,(m)_{lh(m)-1}\big)\in\mathcal{O}(\beta)$$

is Δ^1_1, i.e., there are Π^1_1 relations Π_0,Π_1 such that $R(m,\beta)\Leftrightarrow\Pi_0(m,\beta)\Leftrightarrow\neg\Pi_1(m,\beta)$ if $\beta\in\mathcal{D}$.

Nice coding systems exist. If $\beta\in\mathcal{D}$, then there is a canonical homeomorphism β^* from $[\mathcal{O}(\beta)]$ onto 2^ω. We now check that the construction of β^* is effective.

Lemma 5.5 *(a) The partial function $e:(\beta,\delta)\mapsto\beta^*(\delta)$ is Π^1_1-recursive on its Π^1_1 domain*

$$\mathit{Domain}(e):=\{(\beta,\delta)\in\mathcal{D}\times 2^\omega\mid\delta\in[\mathcal{O}(\beta)]\}.$$

(b) The partial function $\iota:(\beta,\gamma)\mapsto$ the unique $\delta\in[\mathcal{O}(\beta)]$ with $\beta^*(\delta)=\gamma$ is Π^1_1-recursive on its Π^1_1 domain $\mathcal{D}\times 2^\omega$.

Proof. (a) We define a Π^1_1 relation $\mathcal{Q}$ on $\omega^2\times(2^\omega)^2$ by

$$\mathcal{Q}(p,p',\beta,\delta)\Leftrightarrow\Big(\big(\forall\varepsilon\in 2\ \ \Pi_0((\overline{(\delta|p')\varepsilon},\beta)\big)\ \wedge\ \big(\forall p\leq p''<p'\ \ \exists\varepsilon\in 2\ \ \Pi_1((\overline{(\delta|p'')\varepsilon},\beta))\big)\Big).$$

Note that

$$\beta^*(\delta)(n)=\varepsilon\Leftrightarrow\left\{\begin{array}{l}\exists l\in\omega\ \ \mathrm{Seq}(l)\ \wedge\ \mathrm{lh}(l)=n+1\ \wedge\ \delta\big((l)_n\big)=\varepsilon\ \wedge\ \mathcal{Q}\big(0,(l)_0,\beta,\delta\big)\ \wedge\\ \forall m<n\ \ (l)_m<(l)_{m+1}\ \wedge\ \mathcal{Q}\big((l)_m+1,(l)_{m+1},\beta,\delta\big)\end{array}\right.$$

if $\beta\in\mathcal{D}$. The proof of (b) is similar. □

- Let X be a recursively presented Polish space, and d_X and $(r^X_n)_{n\in\omega}$ be respectively a distance function and a recursive presentation of X. We now give a **recursive presentation of** $\mathcal{C}(2^\omega,X)$, equipped with the usual distance defined by

$$d(h,h'):=\mathrm{sup}_{\beta\in 2^\omega}\ d_X\big(h(\beta),h'(\beta)\big),$$

since this is not present in [M]. We define, by primitive recursion, a recursive map $\nu:\omega\rightarrow\omega$ such that $\nu(i)$ enumerates $\{s\in 2^{<\omega}\mid |s|=i\}$. We first set $\nu(0):=1=<>$. Then

$$\nu(i+1)=k\Leftrightarrow\mathrm{Seq}(k)\ \wedge\ \mathrm{lh}(k)=2^{i+1}\ \wedge\ \forall l<2^i\ \ \forall\varepsilon\in 2\ \ (k)_{\varepsilon 2^i+l}=\overline{s\Big(\big(\nu(i)\big)_l\Big)\varepsilon}.$$

If $\mathrm{Seq}(n)$ and $\mathrm{lh}(n)=2^i$ for some i $(<n)$, then we define $h_n:2^\omega\rightarrow X$ by $h_n(\beta):=r^X_{(n)_l}$ if

$$\beta|i=s^i_l:=s\Big(\big(\nu(i)\big)_l\Big).$$

If $\neg\mathrm{Seq}(n)$ or $\mathrm{lh}(n)\neq 2^i$ for each i, then we define $h_n:2^\omega\rightarrow X$ by $h_n(\beta):=r^X_0$ if $\beta\in 2^\omega$. In any case, $h_n\in\mathcal{C}(2^\omega,X)$ and takes finitely many values.

Lemma 5.6 *Let X be a recursively presented Polish space. Then the sequence $(h_n)_{n\in\omega}$ is a recursive presentation of $\mathcal{C}(2^\omega, X)$, equipped with d.*

Proof. We have to see that (h_n) is dense in $\mathcal{C}(2^\omega, X)$. So let $h\in\mathcal{C}(2^\omega, X)$, $\epsilon>0$ and $m\in\omega$ with $2^{-m}<\frac{\epsilon}{2}$. As h is uniformly continuous, there is $i\in\omega$ such that $d_X\big(h(\beta),h(\delta)\big)<2^{-m}$ if $\beta|i=\delta|i$. We choose, for each $l<2^i$, $n_l\in\omega$ such that $d_X\big(r^X_{n_l},h(s^i_l0^\infty)\big)<2^{-m}$. We set $n:=<n_0,...,n_{2^i-1}>$. If $\beta\in 2^\omega$ and $\beta|i=s^i_l$, then $d_X\big(h(\beta),h_n(\beta)\big)\leq d_X\big(h(\beta),h(s^i_l0^\infty)\big)+d_X\big(h(s^i_l0^\infty),r^X_{n_l}\big)\leq 2^{-m}+2^{-m}$, so that $d(h,h_n)<\epsilon$. It is routine to check that the relations "$d(h_m,h_n)\leq\frac{p}{q+1}$" and "$d(h_m,h_n)<\frac{p}{q+1}$" are recursive in (m,n,p,q). □

We saw in the proof of Proposition 5.1 that the evaluation map $(h,\beta)\mapsto h(\beta)$ is continuous from $\mathcal{C}(2^\omega,X)\times 2^\omega$ into X. We can say more if X is recursively presented.

Lemma 5.7 *Let X be a recursively presented Polish space. Then the evaluation map is recursive.*

Proof. Note that

$$\begin{array}{ll} h(\beta)\in N(X,n) & \Leftrightarrow d_X\big(h(\beta),r^X_{((n)_1)_0}\big)<\frac{((n)_1)_1}{((n)_1)_2+1}\\ & \Leftrightarrow \exists m,i,l\in\omega\ \ \text{Seq}(m)\ \wedge\ \text{lh}(m)=2^i\ \wedge\ \beta|i=s^i_l\ \wedge\ (m)_l=\big((n)_1\big)_0\ \wedge\\ & \hfill d(h,h_m)<\frac{((n)_1)_1}{((n)_1)_2+1}, \end{array}$$

which gives the result. □

- We then strengthen 7A.3 in [M] about **primitive recursion** as follows. If Z,Y are recursively presented Polish spaces, $g:Z\to Y$ and $h:Y\times\omega\times Z\to Y$ are Π^1_1-recursive and $f:\omega\times Z\to Y$ is defined by

$$\left\{\begin{array}{l} f(0,z):=g(z),\\ f(n+1,z):=h\big(f(n,z),n,z\big), \end{array}\right.$$

then f is also Π^1_1-recursive. If $m:Z\to Z$ is Π^1_1-recursive, then the proof of 7A.3 in [M] shows that the map $f':\omega\times Z\to Y$ defined by

$$\left\{\begin{array}{l} f'(0,z):=g(z),\\ f'(n+1,z):=h\Big(f'\big(n,m(z)\big),n,z\Big), \end{array}\right.$$

is also Π^1_1-recursive. As in 7A.5 in [M], this can be extended to partial functions which are Π^1_1-recursive on their domain.

- We are ready for the proof of our main tool.

Proof of Theorem 5.3. 3E.6 in [M] provides $\pi:\omega^\omega\to X$ recursive, $\mathcal{F}\in\Pi^0_1(\omega^\omega)$ and a Δ^1_1-recursive injection $\rho:X\to\omega^\omega$ such that $\pi_{|\mathcal{F}}$ is injective, $\pi[\mathcal{F}]=X$ and ρ is the inverse of $\pi_{|\mathcal{F}}$. Let us show that the map $\mu:h\mapsto\pi\circ h$ is Δ^1_1-recursive from $\mathcal{C}(2^\omega,\omega^\omega)$ into $\mathcal{C}(2^\omega,X)$. More generally, let Y be a recursively presented Polish space, and $\psi:Y\to\mathcal{C}(2^\omega,X)$. Note that

$$\begin{array}{ll} \psi(y)\in N\big(\mathcal{C}(2^\omega,X),n\big) & \Leftrightarrow d\big(\psi(y),h_{((n)_1)_0}\big)<\frac{((n)_1)_1}{((n)_1)_2+1}\\ & \Leftrightarrow \exists m\in\omega\ \ \sup_{\beta\in 2^\omega}\ d_X\big(\psi(y)(\beta),h_{((n)_1)_0}(\beta)\big)<\frac{((m)_1)_1}{((m)_1)_2+1}<\frac{((n)_1)_1}{((n)_1)_2+1}\\ & \Leftrightarrow \exists m\in\omega\ \ \forall\beta\in 2^\omega\ \ d_X\big(\psi(y)(\beta),h_{((n)_1)_0}(\beta)\big)<\frac{((m)_1)_1}{((m)_1)_2+1}<\frac{((n)_1)_1}{((n)_1)_2+1} \end{array}$$

and $h_{((n)_1)_0}(\beta)=r^X_{g(n,\beta)}$ for some recursive map $g:\omega\times 2^\omega\to\omega$.

In the present case, $Y = \mathcal{C}(2^\omega, \omega^\omega)$ and $\psi(y)(\beta) = \pi\big(y(\beta)\big)$. Thus

$$\begin{aligned} d_X\big(\psi(y)(\beta), h_{((n)_1)_0}(\beta)\big) < \tfrac{((m)_1)_1}{((m)_1)_2+1} &\Leftrightarrow d_X\Big(\pi\big(y(\beta)\big), r^X_{g(n,\beta)}\Big) < \tfrac{((m)_1)_1}{((m)_1)_2+1} \\ &\Leftrightarrow \pi\big(y(\beta)\big) \in N\big(X, \langle 0, <g(n,\beta), \big((m)_1\big)_1, \big((m)_1\big)_2>\rangle\big) \\ &\Leftrightarrow \big(y(\beta), \langle 0, <g(n,\beta), \big((m)_1\big)_1, \big((m)_1\big)_2>\rangle\big) \in G^\pi, \end{aligned}$$

where G^π is the Σ^0_1 neighborhood diagram of π. As the evaluation map is recursive, $h \mapsto \pi \circ h$ is Π^1_1-recursive and total, and thus Δ^1_1-recursive.

Let us show that there is a Δ^1_1-recursive map $f : \omega^\omega \to \mathcal{C}(2^\omega, X)$ such that $\mathcal{U}^{\mathbf{\Pi}^1_n, 2^\omega}_\alpha = \big(f(\alpha)\big)^{-1}(C)$ for each $\alpha \in \omega^\omega$. We adapt the proof of the main result in [K3]. We set $A := \pi^{-1}(C)$. As $C \in \Pi^1_n(X)$, $A \in \Pi^1_n(\omega^\omega)$. If $<\beta^0, \delta^0> \in 2^\omega$, then we inductively define, for $i \in \omega$, $m_i, \beta^{i+1}, \delta^{i+1}$ as follows. If (β^i, δ^i) is given and in Domain(e), then $(\beta^i)^*(\delta^i) = <x_i, \beta^{i+1}, \delta^{i+1}>$ and

$$m_i := \begin{cases} \text{the location of the first 0 in } x_i \text{ if it exists,} \\ 2 \text{ otherwise.} \end{cases}$$

We then set $Q := \big\{(\alpha, <\beta^0, \delta^0>) \in \omega^\omega \times 2^\omega \mid \forall i \in \omega \ \ (\beta^i, \delta^i) \in \text{Domain}(e) \ \wedge \ \big(\alpha, (m_i)\big) \in \mathcal{U}^{\mathbf{\Pi}^1_n, 2^\omega}\big\}$ and $B^* := Q_\alpha$, so that $Q \in \Pi^1_n(\omega^\omega \times 2^\omega)$ and $\beta \in B^* \Leftrightarrow (\alpha, \beta) \in Q$ for each $(\alpha, \beta) \in \omega^\omega \times 2^\omega$ (note that B^* depends on α, but we denote it like this to keep the notation of [K3]). We define $I : \omega^\omega \to 2^\omega$ by $I(\alpha) := 0^{\alpha(0)} 1 0^{\alpha(1)} 1 \ldots$ Note that I a Δ^1_1-recursive injection onto the Π^0_2 set

$$\mathbb{P}_\infty := \{\beta \in 2^\omega \mid \forall p \in \omega \ \ \exists q \geq p \ \ \beta(q) = 1\},$$

so that there is a Δ^1_1-recursive map $\phi : 2^\omega \to \omega^\omega$ which is the inverse of I on $\mathbb{P}_\infty$. We set

$$Q' := \Big\{\delta \in 2^\omega \mid (\delta)_0 \in \mathbb{P}_\infty \ \wedge \ \Big(\phi\big((\delta)_0\big), (\delta)_1\Big) \in Q\Big\},$$

so that $Q' \in \Pi^1_n(2^\omega)$. As $\mathcal{U}^{\mathbf{\Pi}^1_n, 2^\omega}$ is suitable, there is $\alpha_Q \in \omega^\omega$ recursive with $Q' = \mathcal{U}^{\mathbf{\Pi}^1_n, 2^\omega}_{\alpha_Q}$. Note that

$$\begin{aligned} \beta \in B^* \Leftrightarrow (\alpha, \beta) \in Q \Leftrightarrow <I(\alpha), \beta> \in Q' &\Leftrightarrow (\alpha_Q, <I(\alpha), \beta>) \in \mathcal{U}^{\mathbf{\Pi}^1_n, 2^\omega} \\ &\Leftrightarrow b\big(\mathcal{R}(\alpha_Q, <I(\alpha), \beta>)\big) \in C \Leftrightarrow \rho\Big(b\big(\mathcal{R}(\alpha_Q, <I(\alpha), \beta>)\big)\Big) \in A. \end{aligned}$$

We set $G := \rho\Big(b\big(\mathcal{R}(\alpha_Q, <I(\alpha), .>)\big)\Big)$, so that $G : 2^\omega \to \omega^\omega$ is $\Delta^1_1(\alpha)$-recursive and $<\beta^0, \delta^0>$ is in B^* if and only if $G(<\beta^0, \delta^0>) \in A$.

As in [K3], we can find $F : 2^{<\omega} \to (2^\omega \times \omega)^{<\omega}$ satisfying the following properties:

(1) $t \subseteq t' \Rightarrow F(t) \subseteq F(t')$
(2) $|F(t)| = |t| + 1$
(3) (i) if $F(\emptyset) = (\beta^0, k_0)$, then $\beta^0 \in \mathcal{D} \ \wedge \ \forall \delta^0 \in [\mathcal{O}(\beta^0)] \ \ G(<\beta^0, \delta^0>)(0) = k_0$
(ii) if $F(\varepsilon_0, ..., \varepsilon_n) = (\beta^0, k_0, \beta^1, k_1, ..., \beta^{n+1}, k_{n+1})$, then
(a) $\forall i \leq n+1 \ \ \beta^i \in \mathcal{D}$
(b) for all $\delta^{n+1} \in [\mathcal{O}(\beta^{n+1})]$, if $\delta^n, ..., \delta^0$ are the uniquely determined members of $[\mathcal{O}(\beta^n)], ..., [\mathcal{O}(\beta^0)]$ such that $\forall i \leq n \ \ (\beta^i)^*(\delta^i) = <\overline{\varepsilon_i}, \beta^{i+1}, \delta^{i+1}>$, where $\overline{\varepsilon_i} = 1^{\varepsilon_i} 0 1^\infty$, then $\forall i \leq n+1 \ \ G(<\beta^0, \delta^0>)(i) = k_i$.

We will need an effective version of this, so that we give the details of the construction of F. In fact, the β^i's involved in the definition of F can be $\Delta^1_1(\alpha)$. In order to see this, we first define

$$H_0 : 2^\omega \times 2^\omega \to \omega$$

by $H_0(\beta,\delta) := G(<\beta,\delta>)(0)$. As G is $\Delta^1_1(\alpha)$-recursive, H_0 too, and the niceness of the coding system gives $\beta^0 \in \mathcal{D} \cap \Delta^1_1(\alpha)$ and $k_0 \in \omega$ such that $G(<\beta^0,\delta^0>)(0) = k_0$ for each $\delta^0 \in [\mathcal{O}(\beta^0)]$. Now suppose that $n \in \omega$, $(\varepsilon_0, ..., \varepsilon_n)$ and $F(\varepsilon_0, ..., \varepsilon_{n-1}) = (\beta^0, k_0, ..., \beta^n, k_n)$ are given. We define

$$H_{n+1} : 2^\omega \times 2^\omega \to \omega$$

as follows.

Given $(\beta,\delta) \in 2^\omega \times 2^\omega$, let $\delta^n, ..., \delta^0$ be the uniquely determined members of $[\mathcal{O}(\beta^n)], ..., [\mathcal{O}(\beta^0)]$ resp., such that $(\beta^n)^*(\delta^n) = <\overline{\varepsilon_n}, \beta, \delta>$, and $(\beta^i)^*(\delta^i) = <\overline{\varepsilon_i}, \beta^{i+1}, \delta^{i+1}>$ if $i < n$. Put

$$H_{n+1}(\beta,\delta) := G(<\beta^0,\delta^0>)(n+1).$$

As H_{n+1} is $\Delta^1_1(\alpha)$ (it is total and $\Pi^1_1(\alpha)$-recursive since ι is Π^1_1-recursive), the niceness of the coding system gives $\beta^{n+1} \in \mathcal{D} \cap \Delta^1_1(\alpha)$ and $k_{n+1} \in \omega$ such that $G(<\beta^0,\delta^0>)(n+1) = k_{n+1}$ for each $\delta^{n+1} \in [\mathcal{O}(\beta^{n+1})]$. Then $F(\varepsilon_0, ..., \varepsilon_n) := (\beta^0, k_0, ..., \beta^{n+1}, k_{n+1})$, so that F is as desired. So we can assume that the β^i's are $\Delta^1_1(\alpha)$ in the conditions required for F.

By [K3] again, the map $h_\alpha : (\varepsilon_i) \mapsto (k_i)$ is continuous and $\mathcal{U}^{\mathbf{\Pi}^1_n, 2^\omega}_\alpha = h_\alpha^{-1}(A)$. As this is not too long to prove, we give the details for completeness. The map h_α is in fact more than continuous: it is Lipschitz, by definition. Fix (ε_i). We apply F to the initial segments of (ε_i), which gives (β^i). For each n, we define perfect sets $C^n_0, C^n_1, ..., C^n_n \subseteq 2^\omega$ with $C^n_i \subseteq [\mathcal{O}(\beta^i)]$ if $i \leq n$, as follows:

$$\begin{array}{l} C^n_n := \{\delta^n \in [\mathcal{O}(\beta^n)] \mid \exists \delta^{n+1} \in 2^\omega \ \ (\beta^n)^*(\delta^n) = <\overline{\varepsilon_n}, \beta^{n+1}, \delta^{n+1}>\}, \\ C^n_{n-1} := \{\delta^{n-1} \in [\mathcal{O}(\beta^{n-1})] \mid \exists \delta^n \in C^n_n \ \ (\beta^{n-1})^*(\delta^{n-1}) = <\overline{\varepsilon_{n-1}}, \beta^n, \delta^n>\}, \\ ... \\ C^n_0 := \{\delta^0 \in [\mathcal{O}(\beta^0)] \mid \exists \delta^1 \in C^n_1 \ \ (\beta^0)^*(\delta^0) = <\overline{\varepsilon_0}, \beta^1, \delta^1>\}. \end{array}$$

Note that

(4) $\delta^0 \in C^n_0 \Rightarrow \ <\beta^i, \delta^i> \in \text{Domain}(e)$ for each $i \leq n$, where $\delta^1, ..., \delta^n$ are computed according to the formula in $(3).(ii).(b)$,

(5) $n' \geq n \Rightarrow \forall i \leq n \ \ C^{n'}_i \subseteq C^n_i$.

This implies that $[\mathcal{O}(\beta^0)] \supseteq C^0_0 \supseteq C^1_0 \supseteq C^2_0 \supseteq ...$ and $\bigcap_{n\in\omega} \ C^n_0$ contains some δ^0. Note that $<\beta^i, \delta^i>$ is in Domain(e), and $(\beta^i)^*(\delta^i) = <\overline{\varepsilon_i}, \beta^{i+1}, \delta^{i+1}>$ for each $i \in \omega$. By $(3).(ii).(b)$,

$$G(<\beta^0,\delta^0>) = k_i$$

for each $i \in \omega$. As $<\beta^0,\delta^0> \in B^* \Leftrightarrow G(<\beta^0,\delta^0>) \in A$,

$$\big(\forall i \in \omega \ <\beta^i, \delta^i> \in \text{Domain}(e) \wedge (\varepsilon_i) \in \mathcal{U}^{\mathbf{\Pi}^1_n, 2^\omega}_\alpha\big) \Leftrightarrow (k_i) \in A.$$

As $<\beta^i, \delta^i>$ is in Domain(e) for each $i \in \omega$, $(\varepsilon_i) \in \mathcal{U}^{\mathbf{\Pi}^1_n, 2^\omega}_\alpha \Leftrightarrow h_\alpha\big((\varepsilon_i)\big) = (k_i) \in A$.

So we found, for each $\alpha\in\omega^\omega$, $h_\alpha\in\mathcal{C}(2^\omega,\omega^\omega)$ with $\mathcal{U}_\alpha^{\mathbf{\Pi}^1_n,2^\omega}=(\pi\circ h_\alpha)^{-1}(C)=\big(\mu(h_\alpha)\big)^{-1}(C)$. It remains to see that the map $\psi:\alpha\mapsto h_\alpha$, from ω^ω into $\mathcal{C}(2^\omega,\omega^\omega)$, can be Δ^1_1-recursive (then f will be $\mu\circ\psi$). By the previous discussion, it is enough to see that the relation "$k_i=k$" is Δ^1_1 in $\big(\alpha,(\varepsilon_i),i,k\big)\in\omega^\omega\times 2^\omega\times\omega^2$.

We will define, by primitive recursion, a Δ^1_1-recursive map $\tilde f:\omega\times\omega^\omega\times 2^\omega\rightarrow 2^\omega\times\omega$ such that $\tilde f\big(n,\alpha,(\varepsilon_i)\big)$ will be of the form $(<\tilde\beta^0,...,\tilde\beta^n,\tilde\beta^n,...>,<\tilde k^0,...,\tilde k^n>)$ and can play the role of $F(\varepsilon_0,...,\varepsilon_{n-1})$. We first set

$$P:=\Big\{\big(\alpha,(\varepsilon_i),\beta,k\big)\in\omega^\omega\times(2^\omega)^2\times\omega\mid$$
$$\forall i\in\omega\ \ (\beta)_i=(\beta)_0\in\mathcal{D}\cap\Delta^1_1(\alpha)\wedge\forall\delta\in\big[\mathcal{O}\big((\beta)_0\big)\big]\ \ G(<(\beta)_0,\delta>)(0)=k\Big\}.$$

Note that P is Π^1_1 and for any $\big(\alpha,(\varepsilon_i)\big)\in\omega^\omega\times 2^\omega$ there is $(\beta,k)\in 2^\omega\times\omega$ such that $\big(\alpha,(\varepsilon_i),\beta,k\big)\in P$. The uniformization lemma gives a Δ^1_1-recursive map $\tilde g:\omega^\omega\times 2^\omega\rightarrow 2^\omega\times\omega$ such that

$$\Big(\alpha,(\varepsilon_i),\tilde g\big(\alpha,(\varepsilon_i)\big)\Big)\in P$$

for each $\big(\alpha,(\varepsilon_i)\big)\in\omega^\omega\times 2^\omega$. Then we set

$$D:=\Big\{\big(\beta,p,n,\alpha,(\varepsilon_i)\big)\in 2^\omega\times\omega^2\times\omega^\omega\times 2^\omega\mid\text{Seq}(p)\wedge\text{lh}(p)=n+1\wedge\forall q\in\omega\ \ (\beta)_q\in\mathcal{D}\cap\Delta^1_1(\alpha)\Big\}.$$

Note that D is Π^1_1, as well as

$$R:=\Big\{\big(\beta,p,n,\alpha,(\varepsilon_i),\beta',k'\big)\in D\times 2^\omega\times\omega\ \mid\ \forall i>n\ \ (\beta')_i=(\beta')_{n+1}\in\mathcal{D}\cap\Delta^1_1(\alpha)\wedge$$
$$\text{Seq}(k')\wedge\text{lh}(k')=n+2\wedge\forall i\leq n\ \ (\beta')_i=(\beta)_i\wedge(k')_i=(p)_i\wedge$$
$$\forall\delta\in 2^\omega\ \ \Big(\exists i\leq n+1\ \ (\delta)_i\notin\big[\mathcal{O}\big((\beta')_i\big)\big]\ \vee\ \exists i\leq n\ \ (\beta')^*_i\big((\delta)_i\big)\neq<\overline{\varepsilon_i},(\beta')_{i+1},(\delta)_{i+1}>\vee$$
$$\forall i\leq n+1\ \ G\big(<(\beta')_0,(\delta)_0>\big)(i)=(k')_i\Big)\Big\}.$$

Moreover, for each $\big(\beta,p,n,\alpha,(\varepsilon_i)\big)\in D=\text{proj}_{2^\omega\times\omega^2\times\omega^\omega\times 2^\omega}[R]$ there is $(\beta',k')\in\big(2^\omega\cap\Delta^1_1(\alpha)\big)\times\omega$ such that $\big(\beta,p,n,\alpha,(\varepsilon_i),\beta',k'\big)\in R$. The uniformization lemma gives a partial map

$$\tilde h:2^\omega\times\omega^2\times\omega^\omega\times 2^\omega\rightarrow 2^\omega\times\omega$$

which is Π^1_1-recursive on its domain D, and such that $\Big(\beta,p,n,\alpha,(\varepsilon_i),\tilde h\big(\beta,p,n,\alpha,(\varepsilon_i)\big)\Big)\in R$ if $\big(\beta,p,n,\alpha,(\varepsilon_i)\big)\in D$. This implies that the partial map $\tilde f$ defined by

$$\left\{\begin{array}{l}\tilde f\big(0,\alpha,(\varepsilon_i)\big):=\tilde g\big(\alpha,(\varepsilon_i)\big),\\ \tilde f\big(n+1,\alpha,(\varepsilon_i)\big):=\tilde h\Big(\tilde f\big(n,\alpha,(\varepsilon_i)\big),n,\alpha,(\varepsilon_i)\Big),\end{array}\right.$$

is Π^1_1-recursive.

Moreover, an induction shows that $\Big(\tilde{f}\big(n,\alpha,(\varepsilon_i)\big),n,\alpha,(\varepsilon_i)\Big)\in D$ for each $\big(n,\alpha,(\varepsilon_i)\big)$, so that $\tilde{f}$ is in fact total, and thus Δ^1_1-recursive. More precisely, $\tilde{f}\big(n,\alpha,(\varepsilon_i)\big)$ is of the form

$$(<\beta^0,...,\beta^n,\beta^n,...>,<k_0,...,k_n>),$$

where $(\varepsilon_0,...,\varepsilon_{n-1})\mapsto(\beta^0,k_0,...,\beta^n,k_n)$ satisfies the properties (1)-(3) of F. It remains to note that $k_i=\tilde{f}\big(i,\alpha,(\varepsilon_i)\big)(1)(i)$. □

- We now prove the consequences of our main tool.

Definition 5.8 *Let Γ be a class of subsets of recursively presented Polish spaces, $\mathbf{\Gamma}$ be the corresponding boldface class, X,Y be recursively presented Polish spaces, and $\mathcal{U}\in\Gamma(Y\times X)$. We say that $\mathcal{U}$ is* **effectively uniformly Y-universal for the $\mathbf{\Gamma}$ subsets of** X *if the following hold:*

(1) $\mathbf{\Gamma}(X)=\{\mathcal{U}_y\mid y\in Y\}$,

(2) $\Gamma(X)=\{\mathcal{U}_y\mid y\in Y\ \Delta^1_1$-recursive$\}$,

(3) for each $S\in\mathbf{\Gamma}(\omega^\omega\times X)$, there is a Borel map $b:\omega^\omega\to Y$ such that $S_\alpha=\mathcal{U}_{b(\alpha)}$ for each $\alpha\in\omega^\omega$,

(4) for each $S\in\Gamma(\omega^\omega\times X)$, there is a Δ^1_1-recursive map $b:\omega^\omega\to Y$ such that $S_\alpha=\mathcal{U}_{b(\alpha)}$ for each $\alpha\in\omega^\omega$.

Notation. Let $\mathcal{U}^{\mathbf{\Pi}^1_1,2^\omega}\in\Pi^1_1$ be a good ω^ω-universal for the $\mathbf{\Pi}^1_1$ subsets of 2^ω, X_1 be a recursively presented Polish space, and $\mathcal{C}_1$ be a Π^1_1 subset of X_1 for which there is a Δ^1_1-recursive map $b:\omega^\omega\to X_1$ such that

$$(\alpha,\beta)\in\mathcal{U}^{\mathbf{\Pi}^1_1,2^\omega}\Leftrightarrow b(<\alpha,\beta>)\in\mathcal{C}_1$$

if $(\alpha,\beta)\in\omega^\omega\times 2^\omega$. We define, for each natural number $n\geq 1$,

- $X_{n+1}:=\mathcal{C}(2^\omega,X_n)$ (inductively),

- $\mathcal{C}_{n+1}:=\{h\in X_{n+1}\mid\forall\beta\in 2^\omega\ \ h(\beta)\notin\mathcal{C}_n\}$ (inductively),

- $\mathcal{U}_n:=\{(h,\beta)\in X_{n+1}\times 2^\omega\mid h(\beta)\in\mathcal{C}_n\}$.

Theorem 5.9 *Let $n\geq 1$ be a natural number. Then*
(a) the set $\mathcal{U}_n$ is effectively uniformly X_{n+1}-universal for the $\mathbf{\Pi}^1_n$ subsets of 2^ω,
(b) the set $\mathcal{C}_n$ is $\mathbf{\Pi}^1_n$-complete.

Proof. We argue by induction on n.

(a) Assume first that $n=1$, and fix $S\in\mathbf{\Pi}^1_1(\omega^\omega\times 2^\omega)$. Our assumption gives $b_1:\omega^\omega\to X_1$. As $\mathcal{U}^{\mathbf{\Pi}^1_1,2^\omega}\in\Pi^1_1$ is a good ω^ω-universal for the $\mathbf{\Pi}^1_1$ subsets of 2^ω, there is by Theorem 5.3 a Δ^1_1-recursive map $f_1:\omega^\omega\to\mathcal{C}(2^\omega,X_1)$ such that $(\alpha,\beta)\in\mathcal{U}^{\mathbf{\Pi}^1_1,2^\omega}\Leftrightarrow f_1(\alpha)(\beta)\in\mathcal{C}_1$ if $(\alpha,\beta)\in\omega^\omega\times 2^\omega$. Let $\alpha_S\in\omega^\omega$ with $S=\mathcal{U}^{\mathbf{\Pi}^1_1,\omega^\omega\times 2^\omega}_{\alpha_S}$. Note that

$$(\alpha,\beta)\in S\Leftrightarrow\big(\mathcal{R}(\alpha_S,\alpha),\beta\big)\in\mathcal{U}^{\mathbf{\Pi}^1_1,2^\omega}\Leftrightarrow f_1\big(\mathcal{R}(\alpha_S,\alpha)\big)(\beta)\in\mathcal{C}_1\Leftrightarrow\Big(f_1\big(\mathcal{R}(\alpha_S,\alpha)\big),\beta\Big)\in\mathcal{U}_1.$$

As $\mathcal{C}_1$ is $\varPi^1_1$, $\mathcal{U}_1$ too. If $A\in\mathbf{\Pi}^1_1(2^\omega)$, then $A=\mathcal{U}^{\mathbf{\Pi}^1_1,2^\omega}_\alpha$ for some $\alpha\in\omega^\omega$. Applying the previous discussion to $S:=\mathcal{U}^{\mathbf{\Pi}^1_1,2^\omega}$, we get $A=(\mathcal{U}_1)_{f_1(\mathcal{R}(\alpha_S,\alpha))}$, so that $\mathcal{U}_1$ is X_2-universal for the $\mathbf{\Pi}^1_1$ subsets of 2^ω, effectively and uniformly.

We now study $\mathcal{U}_{n+1}$. Fix $S\in\mathbf{\Pi}^1_{n+1}(\omega^\omega\times 2^\omega)$. Let $\mathcal{U}^{\mathbf{\Pi}^1_n,2^\omega}$ be a good ω^ω-universal set for the $\mathbf{\Pi}^1_n$ subsets of 2^ω. We set $\mathcal{V}^{\mathbf{\Pi}^1_{n+1},2^\omega}:=\big\{(\alpha,\beta)\in\omega^\omega\times 2^\omega\mid\forall\delta\in 2^\omega\ \ \big(\mathcal{R}(\alpha,\beta),\delta\big)\notin\mathcal{U}^{\mathbf{\Pi}^1_n,2^\omega}\big\}$, so that $\mathcal{V}^{\mathbf{\Pi}^1_{n+1},2^\omega}$ is a suitable ω^ω-universal for the $\mathbf{\Pi}^1_{n+1}$ subsets of 2^ω. Moreover, the induction assumption gives a $\varDelta^1_1$-recursive map $b_{n+1}:\omega^\omega\to X_{n+1}$ such that

$$\begin{aligned}(\alpha,\beta)\in\mathcal{V}^{\mathbf{\Pi}^1_{n+1},2^\omega}&\Leftrightarrow\forall\delta\in 2^\omega\ \ \big(\mathcal{R}(\alpha,\beta),\delta\big)\notin\mathcal{U}^{\mathbf{\Pi}^1_n,2^\omega}\Leftrightarrow\forall\delta\in 2^\omega\ \ \Big(b_{n+1}\big(\mathcal{R}(\alpha,\beta)\big),\delta\Big)\notin\mathcal{U}_n\\&\Leftrightarrow\forall\delta\in 2^\omega\ \ b_{n+1}\big(\mathcal{R}(\alpha,\beta)\big)(\delta)\notin\mathcal{C}_n\Leftrightarrow b_{n+1}\big(\mathcal{R}(\alpha,\beta)\big)\in\mathcal{C}_{n+1}\end{aligned}$$

Theorem 5.3 gives a $\varDelta^1_1$-recursive map f_{n+1} such that $(\alpha,\beta)\in\mathcal{V}^{\mathbf{\Pi}^1_{n+1},2^\omega}\Leftrightarrow f_{n+1}(\alpha)(\beta)\in\mathcal{C}_{n+1}$ if $(\alpha,\beta)\in\omega^\omega\times 2^\omega$. Let

$$Q\in\mathbf{\Pi}^1_n(\omega^\omega\times 2^\omega\times 2^\omega)\subseteq\mathbf{\Pi}^1_n(\omega^\omega\times\omega^\omega\times 2^\omega)$$

such that $(\alpha,\beta)\in S\Leftrightarrow\forall\delta\in 2^\omega\ \ (\alpha,\beta,\delta)\notin Q$, and $\alpha_Q\in\omega^\omega$ such that $Q=\mathcal{U}^{\mathbf{\Pi}^1_n,\omega^\omega\times\omega^\omega\times 2^\omega}_{\alpha_Q}$. Note that

$$\begin{aligned}(\alpha,\beta)\in S&\Leftrightarrow\forall\delta\in 2^\omega\ \ \Big(\mathcal{R}\big(\mathcal{R}'(\alpha_Q,\alpha),\beta\big),\delta\Big)\notin\mathcal{U}^{\mathbf{\Pi}^1_n,2^\omega}\Leftrightarrow\big(\mathcal{R}'(\alpha_Q,\alpha),\beta\big)\in\mathcal{V}^{\mathbf{\Pi}^1_{n+1},2^\omega}\\&\Leftrightarrow f_{n+1}\big(\mathcal{R}'(\alpha_Q,\alpha)\big)(\beta)\in\mathcal{C}_{n+1}\Leftrightarrow\Big(f_{n+1}\big(\mathcal{R}'(\alpha_Q,\alpha)\big),\beta\Big)\in\mathcal{U}_{n+1}.\end{aligned}$$

As $\mathcal{C}_n\in\varPi^1_n$, $\mathcal{C}_{n+1}\in\varPi^1_{n+1}$ and $\mathcal{U}_{n+1}\in\varPi^1_{n+1}$. If $A\in\mathbf{\Pi}^1_{n+1}(2^\omega)$, then $A=\mathcal{U}^{\mathbf{\Pi}^1_{n+1},2^\omega}_\alpha$ for some $\alpha\in\omega^\omega$. Applying the previous discussion to $S:=\mathcal{U}^{\mathbf{\Pi}^1_{n+1},2^\omega}$, we get $A=(\mathcal{U}_{n+1})_{f_{n+1}(\mathcal{R}'(\alpha_Q,\alpha))}$, so that $\mathcal{U}_{n+1}$ is X_{n+2}-universal for the analytic subsets of 2^ω, effectively and uniformly.

(b) By definition, $\mathcal{C}_1\in\varPi^1_1$, and $\mathcal{C}_{n+1}\in\varPi^1_{n+1}$ if $\mathcal{C}_n\in\varPi^1_n$. Assume first that $E\in\mathbf{\Pi}^1_n(2^\omega)$. Then $E=(\mathcal{U}_n)_h$ for some $h\in\mathcal{C}(2^\omega,X_n)$, by (a). Thus $E=h^{-1}(\mathcal{C}_n)$. If Z is a zero-dimensional Polish space and $D\in\mathbf{\Pi}^1_n(Z)$, then we may assume that Z is a G_δ subset of 2^ω by 7.8 in [K2], so that $D\in\mathbf{\Pi}^1_n(2^\omega)$. The previous discussion gives $g\in\mathcal{C}(2^\omega,X_n)$ with $D=g^{-1}(\mathcal{C}_n)$. Thus $D=(g_{|Z})^{-1}(\mathcal{C}_n)$ and $\mathcal{C}_n$ is $\mathbf{\Pi}^1_n$-complete. □

Proof of Theorem 1.7. By Theorem 5.9, it is enough to show that if $\mathcal{U}^{\mathbf{\Pi}^1_1,2^\omega}\in\varPi^1_1$ is a good ω^ω-universal set for the $\mathbf{\Pi}^1_1$ subsets of 2^ω, then there is a $\varDelta^1_1$-recursive map $b:\omega^\omega\to[0,1]^{2^{<\omega}}$ such that $(\alpha,\beta)\in\mathcal{U}^{\mathbf{\Pi}^1_1,2^\omega}\Leftrightarrow b(<\alpha,\beta>)\in\mathcal{P}$ if $(\alpha,\beta)\in\omega^\omega\times 2^\omega$. Let $H\in\varPi^0_2(\omega^\omega\times 2^\omega\times 2^\omega)$ such that $\neg\mathcal{U}^{\mathbf{\Pi}^1_1,2^\omega}=\text{proj}_{\omega^\omega\times 2^\omega}[H]$. We set $G:=\big\{(\alpha,\beta)\in\omega^\omega\times 2^\omega\mid\big((\alpha)_0,(\alpha)_1,(\beta)_1\big)\in H\ \wedge\ \beta\in\mathcal{K}\big\}$, so that $G\in\varDelta^1_1(\omega^\omega\times 2^\omega)$, has G_δ vertical sections and $G\subseteq\omega^\omega\times\mathcal{K}$. Lemma 3.10 gives a $\varDelta^1_1$-recursive map $F:\omega^\omega\to[0,1]^{2^{<\omega}}$, taking values in $\mathcal{M}$, and such that $G_\alpha=\mathcal{V}_{b(\alpha)}$ for each $\alpha\in\omega^\omega$. If $(\alpha,\beta)\in\omega^\omega\times 2^\omega$, then

$$\begin{aligned}(\alpha,\beta)\notin\mathcal{U}^{\mathbf{\Pi}^1_1,2^\omega}&\Leftrightarrow\exists\delta\in 2^\omega\ \ (\alpha,\beta,\delta)\in H\Leftrightarrow\exists\delta\in 2^\omega\ \ (<\alpha,\beta>,\delta)\in G\\&\Leftrightarrow\exists\delta\in 2^\omega\ \ \big(b(<\alpha,\beta>),\delta\big)\in\mathcal{V}\Leftrightarrow b(<\alpha,\beta>)\notin\mathcal{P}.\end{aligned}$$

This finishes the proof. □

Questions. Let U be a $\varPi^0_2$ subset of $\omega^\omega\times 2^\omega$ which is universal for $\mathbf{\Pi}^0_2(2^\omega)$. We set

$$G:=\big\{(\alpha,\beta)\in\omega^\omega\times\mathcal{K}\mid\big(\alpha,(\beta)_1\big)\in U\big\}.$$

Note that G is a Π^0_2 subset of $\omega^\omega\times 2^\omega$ contained in $\omega^\omega\times\mathcal{K}$ which is universal for $\mathbf{\Pi}^0_2(\mathcal{K})$. Indeed, fix $H\in\mathbf{\Pi}^0_2(\mathcal{K})$. Then $H':=\{\gamma\in 2^\omega\mid <0^\infty,\gamma>\in H\}$ is $\mathbf{\Pi}^0_2$, which gives $\alpha_0\in\omega^\omega$ with $H'=U_{\alpha_0}$. Then $H=G_{\alpha_0}$.

Let $\alpha\mapsto\big((\alpha)_k\big)_{k\in\omega}$ be a homeomorphism between ω^ω and $(\omega^\omega)^\omega$, with inverse map

$$(\alpha_k)_{k\in\omega}\mapsto <\alpha_0,\alpha_1,...>.$$

We set $S':=\{\alpha\in\omega^\omega\mid\exists\gamma\in\omega^\omega\ \ \forall i\in\omega\ \ \forall\beta\in 2^\omega\ \ \beta\notin G_{(\alpha)_{\gamma(i)}}\}$. Note that S' is Σ^1_2.

(1) *Is S' a Borel $\mathbf{\Sigma}^1_2$-complete set?*

Assume that this is the case. Then the set $\mathcal{S}_2:=\{(f_k)_{k\in\omega}\in\mathcal{M}^\omega\mid\exists\gamma\in\omega^\omega\ \ \forall i\in\omega\ \ f_{\gamma(i)}\in\mathcal{P}\}$ of sequences of martingales having a subsequence made of everywhere converging martingales is Borel $\mathbf{\Sigma}^1_2$-complete. Indeed, Lemma 3.10 gives a Borel map $F:\omega^\omega\rightarrow\mathcal{M}$ such that $G_\alpha=\mathcal{V}_{F(\alpha)}$ for each $\alpha\in\omega^\omega$. The map $\tilde{F}:\omega^\omega\rightarrow\mathcal{M}^\omega$ defined by $\tilde{F}(\alpha)(k):=F\big((\alpha)_k\big)$ is Borel. Moreover,

$$\begin{aligned}\tilde{F}(\alpha)\in\mathcal{S}_2 &\Leftrightarrow\exists\gamma\in\omega^\omega\ \ \forall i\in\omega\ \ \forall\beta\in 2^\omega\ \ \beta\notin D\Big(F\big((\alpha)_{\gamma(i)}\big)\Big)\\ &\Leftrightarrow\exists\gamma\in\omega^\omega\ \ \forall i\in\omega\ \ \forall\beta\in 2^\omega\ \ \beta\notin\mathcal{V}_{F((\alpha)_{\gamma(i)})}\\ &\Leftrightarrow\exists\gamma\in\omega^\omega\ \ \forall i\in\omega\ \ \forall\beta\in 2^\omega\ \ \beta\notin G_{(\alpha)_{\gamma(i)}}\\ &\Leftrightarrow\alpha\in S',\end{aligned}$$

so that $S'=\tilde{F}^{-1}(\mathcal{S}_2)$.

(2) *Is there a Borel map $f:\mathcal{C}(2^\omega,[0,1])\rightarrow\omega^\omega$ such that, for each $(h_k)_{k\in\omega}\in\big(\mathcal{C}(2^\omega,[0,1])\big)^\omega$ and each $\beta\in 2^\omega$, the following are equivalent:*

(a) lim$_{k\rightarrow\infty}$ $h_k(\beta)=0$,

(b) $\forall k\in\omega\ \ \beta\notin G_{f(h_k)}$?

Assume that this is the case. Then S' (and therefore $\mathcal{S}_2$) is Borel $\mathbf{\Sigma}^1_2$-complete, and thus $\mathbf{\Sigma}^1_2$-complete (see [P]). We define $F:\big(\mathcal{C}(2^\omega,[0,1])\big)^\omega\rightarrow\omega^\omega$ by $F\big((h_k)_{k\in\omega}\big):=<f(h_0),f(h_1),...>$, so that F is Borel. Note that

$$\begin{aligned}F\big((h_k)_{k\in\omega}\big)\in S' &\Leftrightarrow\exists\gamma\in\omega^\omega\ \ \forall i\in\omega\ \ \forall\beta\in 2^\omega\ \ \beta\notin G_{f(h_{\gamma(i)})}\\ &\Leftrightarrow\exists\gamma\in\omega^\omega\ \ \forall\beta\in 2^\omega\ \ \lim_{i\rightarrow\infty}h_{\gamma(i)}(\beta)=0\\ &\Leftrightarrow(h_k)_{k\in\omega}\in S,\end{aligned}$$

so that $S=F^{-1}(S')$.

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