Potential Wadge classes

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June 2011

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Abstract. Let $\mathbf{\Gamma}$ be a Borel class, or a Wadge class of Borel sets, and $2\leq d\leq\omega$ be a cardinal. A Borel subset B of $\mathbb{R}^d$ is $potentially\ in$ $\mathbf{\Gamma}$ if there is a finer Polish topology on $\mathbb{R}$ such that B is in $\mathbf{\Gamma}$ when $\mathbb{R}^d$ is equipped with the new product topology. We give a way to recognize the sets potentially in $\mathbf{\Gamma}$. We apply this to the classes of graphs (oriented or not), quasi-orders and partial orders.

2010 Mathematics Subject Classification. Primary: 03E15, Secondary: 54H05, 28A05, 26A21

Keywords and phrases. Borel classes, potentially, products, reduction, Wadge classes

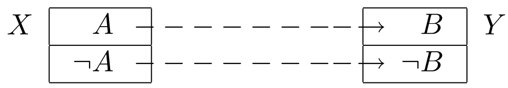
Acknowledgements. I would like to thank A. Louveau for making some useful remarks during the talks I gave at the Université Paris 6 Descriptive Set Theory Seminar. I am particularly grateful for his simplification of the proof of Lemma 2.4. I would also like to thank the referee for reading the first version of this paper, and for making some helpful suggestions to make it easier to read.

1 Introduction

The reader should see [K] for the descriptive set theoretic notation used in this paper. The standard way to compare the topological complexity of the subsets of the Baire space $\mathcal{N}:=\omega^\omega$ is to use the Wadge quasi-order $\leq_W$. Recall that if X (resp., Y) is a zero-dimensional Polish space and A (resp., B) is a subset of X (resp., Y), then

$$(X,A)\leq_W (Y,B)\ \Leftrightarrow\ \exists f\!:\!X\to Y \text{ continuous such that } A\!=\!f^{-1}(B).$$

This is a very natural definition since the continuous functions are the morphisms of topological spaces. So the diagram is as follows:



The "zero-dimensional" condition is here to ensure the existence of enough continuous functions (recall that the only continuous functions from $\mathbb{R}$ into $\mathcal{N}$ are the constant functions). In the sequel, $\mathbf{\Gamma}$ will be a subclass of the class of Borel sets in zero-dimensional Polish spaces. We denote by $\check{\mathbf{\Gamma}}:=\{\neg A\mid A\in\mathbf{\Gamma}\}$ the class of the complements of the elements of $\mathbf{\Gamma}$. We say that $\mathbf{\Gamma}$ is $self$-$dual$ if $\mathbf{\Gamma}=\check{\mathbf{\Gamma}}$. We also set $\Delta(\mathbf{\Gamma}):=\mathbf{\Gamma}\cap\check{\mathbf{\Gamma}}$. Following 4.1 in [Lo-SR2], we give the following definition:

Definition 1.1 *We say that $\mathbf{\Gamma}$ is a $Wadge\ class\ of\ Borel\ sets$ if there is a Borel subset $\mathbf{A}$ of $\mathcal{N}$ such that for any zero-dimensional Polish space X, and for any $A\subseteq X$, A is in $\mathbf{\Gamma}$ if and only if $(X,A)\leq_W(\mathcal{N},\mathbf{A})$. In this case, we also say that $\mathbf{A}$ is $\mathbf{\Gamma}$-$complete$.*

The Wadge hierarchy defined by $\leq_W$, i.e., the inclusion of Wadge classes, is the finest hierarchy of topological complexity in descriptive set theory. The goal of this paper is to study the descriptive complexity of the Borel sets in products of Polish spaces. More specifically, we are looking for a dichotomy of the following form, quite standard in descriptive set theory: either a set is simple, or it is more complicated than a well-known complicated set. Of course, we have to specify the notion of complexity and the notion of comparison that we consider. The two things are actually very much related. The usual notion of comparison between analytic equivalence relations is the Borel reducibility quasi-order $\leq_B$. Recall that if X (resp., Y) is a Polish space and E (resp., F) is an equivalence relation on X (resp., Y), then

$$(X,E)\leq_B (Y,F)\ \Leftrightarrow\ \exists f\!:\!X\to Y\ \text{ Borel such that } E\!=\!(f\times f)^{-1}(F).$$

Note that this makes sense even if E and F are not equivalence relations. The notion of complexity that we consider is a natural invariant for $\leq_B$ in dimension two. Its definition generalizes Definition 3.3 in [Lo3] to any dimension d making sense in the context of classical descriptive set theory, and also to any class $\mathbf{\Gamma}$. So in the sequel d will be a cardinal, and we will have $2\leq d\leq\omega$ since 2^{ω_1} is not metrizable.

Definition 1.2 *Let $(X_i)_{i\in d}$ be a sequence of Polish spaces, and B be a Borel subset of $\Pi_{i\in d}\ X_i$. We say that B is $potentially\ in\ \mathbf{\Gamma}$ $\big($or $B\in pot(\mathbf{\Gamma})\big)$ if, for each $i\in d$, there is a finer zero-dimensional Polish topology τ_i on X_i such that $B\in\mathbf{\Gamma}\big(\Pi_{i\in d}\ (X_i,\tau_i)\big)$.*

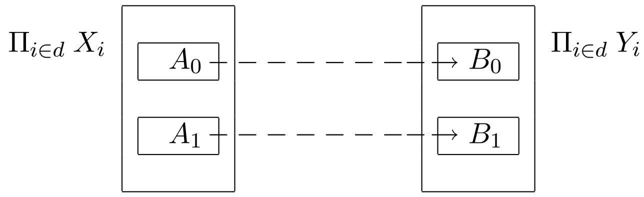
One should emphasize the fact that the point of this definition is to consider product topologies. Indeed, if B is a Borel subset of a Polish space X, then there is a finer Polish topology τ on X such that B is a clopen subset of (X,τ) (see 13.1 in [K]). This is not the case in products: if for example $\mathbf{\Gamma}$ is a non self-dual Wadge class of Borel sets, then there are sets in $\mathbf{\Gamma}(\mathcal{N}^2)$ that are not in pot$(\check{\mathbf{\Gamma}})$ (see Theorem 3.3 in [L1]). For example, the diagonal of $\mathcal{N}$ is not potentially open.

Note also that the "zero-dimensional" condition is not a restriction since we work up to finer Polish topologies. Indeed, if X is a Polish space, then there is a finer zero-dimensional Polish topology on X (see 13.5 in [K]). The notion of potential complexity is an invariant for $\leq_B$ in the sense that if $(X,E)\leq_B (Y,F)$ and $F\in$pot$(\mathbf{\Gamma})$, then $E\in$pot$(\mathbf{\Gamma})$ too.

The good notion of comparison is not the rectangular version of $\leq_B$. Instead of considering a Borel set E and its complement, we have to consider pairs of disjoint analytic sets. This leads to the following notation. Let $(X_i)_{i\in d}$, $(Y_i)_{i\in d}$ be sequences of Polish spaces, and A_0, A_1 (resp., B_0, B_1) be disjoint analytic subsets of $\Pi_{i\in d}\ X_i$ (resp., $\Pi_{i\in d}\ Y_i$). Then

$$\big((X_i)_{i\in d},A_0,A_1\big)\leq\big((Y_i)_{i\in d},B_0,B_1\big)\ \Leftrightarrow\ \forall i\in d\ \ \exists f_i\!:\!X_i\rightarrow Y_i \text{ continuous such that } \forall\varepsilon\in 2\ \ A_\varepsilon\subseteq(\Pi_{i\in d}\ f_i)^{-1}(B_\varepsilon).$$

So the good diagram of comparison is as follows:



The notion of potential complexity was studied in [L1]-[L7] when $d=2$ and $\mathbf{\Gamma}$ is a non self-dual Borel class. The main question of this long study was formulated by A. Louveau in 1990. He wanted to know whether Hurewicz's characterization of the G_δ sets can be generalized to the sets potentially in $\mathbf{\Gamma}$ when $\mathbf{\Gamma}$ is a Wadge class of Borel sets. The main result of this paper gives a complete and positive answer to this question:

Theorem 1.3 *Let $\mathbf{\Gamma}$ be a Wadge class of Borel sets, or the class $\mathbf{\Delta}^0_\xi$ for some $1\leq\xi<\omega_1$. Then there are Borel subsets $\mathbb{S}_0$, $\mathbb{S}_1$ of $(d^\omega)^d$ such that for any sequence of Polish spaces $(X_i)_{i\in d}$, and for any disjoint analytic subsets A_0, A_1 of $\Pi_{i\in d}\ X_i$, exactly one of the following holds:*
(a) The set A_0 is separable from A_1 by a pot$(\mathbf{\Gamma})$ set.
(b) The inequality $\big((d^\omega)_{i\in d},\mathbb{S}_0,\mathbb{S}_1\big)\leq\big((X_i)_{i\in d},A_0,A_1\big)$ holds.

Note that Theorem 1.3 is a result of continuous reduction. We already met the notion of continuous reduction when the Wadge quasi-order was defined. This is one of the motivations for trying to prove Theorem 1.3. This paper is the continuation of the article [L7], that was announced in [L6]. We generalize the main result of [L7], which was obtained by G. Debs and the author. The generalization goes in different directions: it works for

- any dimension d,
- the self-dual Borel classes $\mathbf{\Delta}^0_\xi$,
- any Wadge class of Borel sets (this is the hardest part).

We generalize the one-dimensional version of Theorem 1.3. This version was obtained by A. Louveau and J. Saint Raymond (see [Lo-SR1]), and is a generalization of the Hurewicz theorem. In fact, we give a new proof of this version. The games are not involved in the new proof. This proof gives a new approach for studying the Wadge classes.

Note that A. Louveau and J. Saint Raymond proved that if $\mathbf{\Gamma}$ is not self-dual, then the reduction map in (b) can be one-to-one (see Theorem 5.2 in [Lo-SR2]). We will see that there is no injectivity in general in Theorem 1.3. However, G. Debs proved that we can have the f_i's one-to-one when $d=2$, $\mathbf{\Gamma}\in\{\mathbf{\Pi}^0_\xi,\mathbf{\Sigma}^0_\xi\}$ and $\xi\geq 3$. Some details about the injectivity will be given in the last section.

We will prove a version of Theorem 1.3 for the following classes:

- graphs (i.e., irreflexive and symmetric relations),
- oriented graphs (i.e., irreflexive and antisymmetric relations),
- quasi-orders (i.e., reflexive and transitive relations),
- partial orders (i.e., reflexive, antisymmetric and transitive relations).

We will call $\mathfrak{C}$ the set of these four classes. Note that a reduction on the whole product is not possible in Theorem 1.3, for acyclicity reasons (see [L5]-[L7]). For example, the following result is proved in [L5]. Let X_0, X_1, Y_0, Y_1 be Polish spaces, and A (resp., B) be a subset of $X_0\times X_1$ (resp., $Y_0\times Y_1$). We set

$$(X_0,X_1,A)\leq^r_c(Y_0,Y_1,B)\ \Leftrightarrow\ \forall i\in 2\ \exists f_i:X_i\rightarrow Y_i\ \text{ continuous such that } A=(f_0\times f_1)^{-1}(B).$$

If $(X_0,X_1,A)\leq^r_c(Y_0,Y_1,B)$ with $X_0=X_1$, $Y_0=Y_1$ and $f_0=f_1$, then we write $(X_0,A)\leq_c(Y_0,B)$. In the sequel, we will denote by $\mathcal{C}$ the Cantor space 2^ω.

Theorem 1.4 *(a) There is a $\leq^r_c$-antichain $(\mathcal{C},\mathcal{C},A_\alpha)_{\alpha\in\mathcal{C}}$ such that $A_\alpha\in D_2(\mathbf{\Sigma}^0_1)$ is $\leq^r_c$-minimal among the $\mathbf{\Delta}^1_1\setminus$pot$(\mathbf{\Pi}^0_1)$ sets, for any $\alpha\in\mathcal{C}$.*
(b) There is a $\leq_c$-antichain $(\mathcal{C},R_\alpha)_{\alpha\in\mathcal{C}}$ such that R_α is $\leq_c$-minimal among the $\mathbf{\Delta}^1_1\setminus$pot$(\mathbf{\Pi}^0_1)$ sets, for any $\alpha\in\mathcal{C}$. Moreover, for any element C of $\mathfrak{C}$, we can ensure that $\{R_\alpha\mid\alpha\in\mathcal{C}\}\subseteq C$.

We prove the following corollary of Theorem 1.3:

Theorem 1.5 *Let $C\in\mathfrak{C}$, and $\mathbf{\Gamma}$ be a Wadge class of Borel sets, or the class $\mathbf{\Delta}^0_\xi$ for some $1\leq\xi<\omega_1$. Then there are Borel subsets $\mathbb{R}_0,\mathbb{R}_1$ of $\mathcal{C}\times\mathcal{C}$ with $\mathbb{R}_0,\mathbb{R}_0\cup\mathbb{R}_1\in C$ such that for any Polish space X, and for any Borel subset R of X^2 in C, exactly one of the following holds:*
(a) The set R is in pot$(\mathbf{\Gamma})$.
(b) There is $f:\mathcal{C}\rightarrow X$ continuous such that $\mathbb{R}_0\subseteq(f\times f)^{-1}(R)$ and $\mathbb{R}_1\subseteq(f\times f)^{-1}(\neg R)$.

We introduce the following notation and definition in order to dwell more deeply into Theorem 1.3. We define the notions of smallness that ensure the possibility of the reduction. We emphasize the fact that in this paper, there will be a constant identification between $(d^d)^l$ and $(d^l)^d$, for $l\leq\omega$, in order to simplify as much as possible the notations.

Notation. If $\mathcal{X}$ is a set, then $\vec{x}:=(x_i)_{i\in d}$ is an arbitrary element of $\mathcal{X}^d$. If $\mathcal{T}\subseteq\mathcal{X}^d$, then $G^{\mathcal{T}}$ is the graph whose set of vertices is $\mathcal{T}$, and whose set of edges is $\big\{\{\vec{x},\vec{y}\}\subseteq\mathcal{T}\mid\vec{x}\neq\vec{y}\text{ and }\exists i\in d\ \ x_i=y_i\big\}$ (see [B] for the basic notions about graphs). So $\vec{x}\neq\vec{y}\in\mathcal{T}$ are $G^{\mathcal{T}}$-related if they have at least a common coordinate.

Definition 1.6 *(a) We say that $\mathcal{T}$ is $one\text{-}sided$ if the following holds:*

$$\forall\vec{x}\neq\vec{y}\in\mathcal{T}\ \ \forall i\neq j\in d\ \ (x_i\neq y_i\vee x_j\neq y_j).$$

This means that if $\vec{x}\neq\vec{y}\in\mathcal{T}$, $\vec{x},\vec{y}$ have at most one common coordinate.
(b) We say that $\mathcal{T}$ is $almost\ acyclic$ if for every $G^{\mathcal{T}}$-cycle $(\overrightarrow{x^n})_{n\leq L}$ there are $i\in d$ and $k<m<n<L$ such that $x_i^k=x_i^m=x_i^n$. This means that every $G^{\mathcal{T}}$-cycle contains a "flat" subcycle, i.e., a subcycle in a fixed direction $i\in d$.
(c) We say that a tree T on d^d is a $tree\ with\ suitable\ levels$ if the set $\mathcal{T}^l:=T\cap(d^d)^l\subseteq(d^l)^d$ is finite, one-sided and almost acyclic for each natural number l.

We do not really need the finiteness of the levels, but it makes the proof of Theorem 1.3 much simpler. The following classical property will be crucial in the sequel:

Definition 1.7 *We say that $\mathbf{\Gamma}$ has the $separation\ property$ if for each $A,B\in\mathbf{\Gamma}(\mathcal{N})$ disjoint, there is $C\in\Delta(\mathbf{\Gamma})(\mathcal{N})$ separating A from B.*

The separation property is studied in [S] and [vW], which contain a proof of the following result:

Theorem 1.8 *(Steel-van Wesep) Let $\mathbf{\Gamma}$ be a non self-dual Wadge class of Borel sets. Then exactly one of the two classes $\mathbf{\Gamma}$, $\check{\mathbf{\Gamma}}$ has the separation property.*

We cut Theorem 1.3 into two parts.

Theorem 1.9 *There is a tree T_d with suitable levels such that, for each non self-dual Wadge class of Borel sets $\mathbf{\Gamma}$, the following statements hold.*
(1) There exists $\mathbb{S}\in\mathbf{\Gamma}(\lceil T_d\rceil)$ that is not separable from $\lceil T_d\rceil\setminus\mathbb{S}$ by a pot$(\check{\mathbf{\Gamma}})$ set.
(2) If $\mathbf{\Gamma}$ does not have the separation property, and $\mathbf{\Gamma}=\mathbf{\Sigma}^0_\xi$ or $\Delta(\mathbf{\Gamma})$ is a Wadge class, then we can find disjoint sets $\mathbb{S}_0,\mathbb{S}_1\in\mathbf{\Gamma}(\lceil T_d\rceil)$ which are not separable by a pot$\big(\Delta(\mathbf{\Gamma})\big)$ set.

Theorem 1.10 *Let T_d be a tree with suitable levels, $\mathbf{\Gamma}$ be a non self-dual Wadge class of Borel sets, $(X_i)_{i\in d}$ be a sequence of Polish spaces, and A_0, A_1 be disjoint analytic subsets of $\Pi_{i\in d}\ X_i$.*

(1) Assume that $S\in\mathbf{\Gamma}(\lceil T_d\rceil)$ is not separable from $\lceil T_d\rceil\setminus S$ by a pot$(\check{\mathbf{\Gamma}})$ set. Then exactly one of the following holds:
(a) The set A_0 is separable from A_1 by a pot$(\check{\mathbf{\Gamma}})$ set.
(b) The inequality $\big((d^\omega)_{i\in d},S,\lceil T_d\rceil\setminus S\big)\leq\big((X_i)_{i\in d},A_0,A_1\big)$ holds.

(2) Assume that $\mathbf{\Gamma}$ does not have the separation property, $\mathbf{\Gamma}=\mathbf{\Sigma}^0_\xi$ or $\Delta(\mathbf{\Gamma})$ is a Wadge class, and that $S_0,S_1\in\mathbf{\Gamma}(\lceil T_d\rceil)$ are disjoint and not separable by a pot$\big(\Delta(\mathbf{\Gamma})\big)$ set. Then exactly one of the following holds:
(a) The set A_0 is separable from A_1 by a pot$\big(\Delta(\mathbf{\Gamma})\big)$ set.
(b) The inequality $\big((d^\omega)_{i\in d},S_0,S_1\big)\leq\big((X_i)_{i\in d},A_0,A_1\big)$ holds.

We now come back to the new approach for studying the Wadge classes mentioned earlier. There are a lot of dichotomy results in descriptive set theory about the equivalence relations, the quasi-orders, the partial orders, or even the arbitrary analytic sets. So it is natural to look for common points between these dichotomies. B. Miller's recent work goes in this direction. He proved many known dichotomies without using the effective descriptive set theory, using some variants of the Kechris-Solecki-Todorčević dichotomy for analytic graphs (see [K-S-T]). Here we want to point out another common point, of effective nature. In these dichotomies, the first possibility of the dichotomy is equivalent to the emptyness of some Σ^1_1 set. For example, in the Kechris-Solecki-Todorčević dichotomy, the Σ^1_1 set is the complement of the union of the Δ^1_1 sets discrete with respect to the Σ^1_1 graph considered. We prove a strengthening of Theorem 1.10 in which such a Σ^1_1 set appears. We will state the first part of it, informally. Before that, we need the following notation.

Notation. Let X be a recursively presented Polish space. The topology on X generated by $\Delta^1_1(X)$ is denoted by Δ_X. This topology is Polish (see (iii) $\Rightarrow$ (i) in the proof of Theorem 3.4 in [Lo3]). The topology τ_1 on $\mathcal{N}^d$ is the product topology $\Delta^d_{\mathcal{N}}$.

Theorem 1.11 *Let T_d be a tree with Δ^1_1 suitable levels, $\mathbf{\Gamma}$ be a non self-dual Wadge class of Borel sets having a Δ^1_1 code, and A_0, A_1 be disjoint Σ^1_1 subsets of $\mathcal{N}^d$. Assume that $S \in \mathbf{\Gamma}(\lceil T_d \rceil)$ is not separable from $\lceil T_d \rceil \setminus S$ by a pot$(\check{\mathbf{\Gamma}})$ set. Then there is a Σ^1_1 subset R of $\mathcal{N}^d$ such that the following are equivalent:*

(a) The set A_0 is not separable from A_1 by a pot$(\check{\mathbf{\Gamma}})$ set.

(b) The set A_0 is not separable from A_1 by a $\Delta^1_1 \cap$ pot$(\check{\mathbf{\Gamma}})$ set.

(c) The set A_0 is not separable from A_1 by a $\check{\mathbf{\Gamma}}(\tau_1)$ set.

(d) $R \neq \emptyset$.

(e) The inequality $\big((d^\omega)_{i\in d}, S, \lceil T_d \rceil \setminus S\big) \leq \big((\mathcal{N})_{i\in d}, A_0, A_1\big)$ holds.

This Σ^1_1 set R is built with topologies based on τ_1. This use of these Σ^1_1 sets is the new approach for studying the Wadge classes.

We first prove Theorems 1.9 and 1.10 for the Borel classes, self-dual or not. Next, we consider the case of Wadge classes. In Section 2, we start to prove Theorem 1.9. We construct a concrete tree with suitable levels, and give a general condition ensuring the existence of complicated subsets of its body (see the statement of Theorem 1.9). We actually reduce the problem to a problem concerning the one-dimensional spaces. In Section 3, we prove Theorem 1.9 for the Borel classes. In Section 4, we prove Theorem 1.10 for the Borel classes, using some tools of effective descriptive set theory and the representation theorem for Borel sets proved in [D-SR]. In Section 5, we prove Theorem 1.9, using the description of the Wadge classes in [Lo-SR2]. In Section 6, we prove Theorems 1.3, 1.5, 1.10 and 1.11. Finally, in Section 7, we give some details about the injectivity.

2 A general condition ensuring the existence of complicated sets

We now build a tree with suitable levels. This tree has to be small enough since we cannot have a reduction on the whole product. But as the same time it has to be big enough to ensure the existence of complicated sets, as in Theorem 1.9.

Notation. Fix some standard bijection $< .,. >:\omega^2\rightarrow\omega$, for example

$$(n,p)\mapsto < n,p > := \frac{(n+p)(n+p+1)}{2}+p.$$

Let $b:\omega\rightarrow\omega^2$ be its inverse (b associates $\big((l)_0,(l)_1\big)$ with l).

In the introduction, we mentionned the idenfication between $(d^l)^d$ and $(d^d)^l$. More precisely, the bijection we use associates $\Big(\big(\alpha_i(j)\big)_{i\in d}\Big)_{j\in l}$ with $\vec{\alpha}\in(d^l)^d$.

Definition 2.1 *We say that* $E\subseteq\bigcup_{l\in\omega}\ (d^l)^d\equiv(d^d)^{<\omega}$ *is an* $effective\ frame$ *if*

(a) $\forall l\in\omega\ \exists!\ \overrightarrow{s^l}\in E\cap(d^l)^d$.

(b) $\forall p,q,r\in\omega\ \forall t\in d^{<\omega}\ \exists N\in\omega\ (s^q_i it0^N)_{i\in d}\in E$, $(|s^q_0 0t0^N|-1)_0=p$ *and* $\big((|s^q_0 0t0^N|-1)_1\big)_0=r$.

(c) $\forall l>0\ \exists q<l\ \exists t\in d^{<\omega}\ \forall i\in d\ \ s^l_i=s^q_i it$.

(d) The map $l\mapsto\overrightarrow{s^l}$ *can be coded by a recursive map from* ω *into* ω^d.

We will call T_d *the* $tree\ on\ d^d\ associated\ with\ an\ effective\ frame$ $E=\{\overrightarrow{s^l}\mid l\in\omega\}$*:*

$$T_d:=\big\{\vec{s}\in(d^d)^{<\omega}\mid(\forall i\in d\ \ s_i=\emptyset)\vee\big(\exists l\in\omega\ \exists t\in d^{<\omega}\ \forall i\in d\ \ s_i=s^l_i it\wedge\forall n<|s_0|\ s_0(n)\leq n\big)\big\}.$$

The uniqueness condition in (a) and Condition (c) ensure that T_d is small enough, and also the almost acyclicity. The definition of T_d ensures that T_d has finite levels. Note that $\mathcal{T}^l=T_d\cap(d^d)^l$ can be coded by a Π^0_1 subset of $\mathcal{N}^l$ when $d=\omega$. The existence condition in (a) and Condition (b) ensure that T_d is big enough. More precisely, if (X,τ) is a Polish space and σ is a finer Polish topology on X, then there is a dense G_δ subset of (X,τ) on which τ and σ coincide. The first part of Condition (b) ensures the possibility to get inside the products of dense G_δ sets. We use the examples in the articles [Lo-SR1] and [Lo-SR2] to build the examples in Theorem 1.9. Some conditions on the vertical sections are involved, and the second part of Condition (b) gives a control on the choice of the vertical sections. The very last part of Condition (b) is not necessary to get Theorem 1.9 for the Borel classes, but is useful to get Theorem 1.9 for the Wadge classes of Borel sets. Definition 2.1 is more restrictive than Definition 3.1 in [L7], with this very last part of Condition (b), with Condition (d) (ensuring the regularity of the levels of the tree), and also with the last part of the definition of the tree (ensuring the finiteness of the levels of the tree).

Proposition 2.2 *The tree* T_d *associated with an effective frame is a tree with* Δ^1_1 *suitable levels. In particular,* $\lceil T_d\rceil$ *is compact.*

Proof. Let $l\in\omega$. Let us prove that $\mathcal{T}^l$ is Δ^1_1 and finite. We argue by induction on l. The result is clear for $l\leq 1$ since $\mathcal{T}^0=\{\vec{\emptyset}\}$ and $\mathcal{T}^1=\{(i)_{i\in d}\}$. If $l\geq 1$ and $\vec{s}\in(d^d)^{<\omega}$, then

$$\vec{s}\in\mathcal{T}^l\ \Leftrightarrow\ |s_0|=l\wedge\exists q<l\ \ \exists t\in d^{<\omega}\ \ \forall i\in d\ \ s_i=s^q_i it\wedge\forall n<l\ \ s_0(n)\leq n.$$

But there are only finitely many possibilities for t since $s_0(n)\leq n$ for each $n<l$, which implies that $t(m)\leq q+1+m<l+1+l$ if $m<|t|$. This implies that $\mathcal{T}^l$ is Δ^1_1 and finite.

• Let ${}_dT$ be the tree generated by the effective frame:

$${}_dT := \big\{\vec{s}\in(d^d)^{<\omega} \mid (\forall i\in d\ \ s_i=\emptyset) \vee \big(\exists l\in\omega\ \exists t\in d^{<\omega}\ \forall i\in d\ \ s_i=s_i^l it\big)\big\}.$$

Note that $\mathcal{T}^l \subseteq {}^l\mathcal{T} := {}_dT \cap (d^d)^l$ for each natural number l since $T_d \subseteq {}_dT$. So it is enough to prove that ${}^l\mathcal{T}$ is one-sided and almost acyclic since these properties are hereditary.

• Let us prove that ${}^l\mathcal{T}$ is almost acyclic. We argue by induction on l. The result is clear for $l \leq 1$. So fix $l \geq 1$. We set, for $j \in d$, $C_j := \big\{(s_i^q it)_{i\in d} \in {}^{l+1}\mathcal{T} \mid t \neq \emptyset \wedge t(|t|-1) = j\big\}$. Note that ${}^{l+1}\mathcal{T} = \{(s_i^l i)_{i\in d}\} \cup \bigcup_{j\in d} C_j$, and this union is disjoint.

The restriction of $G^{{}^{l+1}\mathcal{T}}$ to each C_j is isomorphic to $G^{{}^l\mathcal{T}}$. The $G^{{}^{l+1}\mathcal{T}}$-edges are between two elements of the same C_j, or between $(s_i^l i)_{i\in d}$ and an element of one of the C_j's. If a $G^{{}^{l+1}\mathcal{T}}$-cycle exists, then we may assume that it involves only $(s_i^l i)_{i\in d}$ and some elements of a fixed C_j. But if $\vec{s} \in C_j$ is $G^{{}^{l+1}\mathcal{T}}$-related to $(s_i^l i)_{i\in d}$, then we must have $s_j^l j = s_j$. This implies the existence of $k<m<n$ showing that ${}^{l+1}\mathcal{T}$ is almost acyclic.

• Now assume that $\vec{x} \neq \vec{y} \in {}^l\mathcal{T}$, $i, j \in d$, $x_i = y_i$ and $x_j = y_j$. Then we can write $\vec{x} = (s_i^q it)_{i\in d}$ and $\vec{y} = (s_i^{q'} it')_{i\in d}$ since $\vec{x} \neq \vec{y}$. As $x_i = y_i$, the reverses t^{-1} and $(t')^{-1}$ of t and t' are compatible. If $t = t'$, then $q = |s_i^q| = l-1-|t| = l-1-|t'| = |s_i^{q'}| = q'$ and $\vec{x} = \vec{y}$, which is absurd. Thus $t \neq t'$, for example $|t'| < |t|$, and $t^{-1}(|t'|) = i$. This proves that $i = j$ and ${}^l\mathcal{T}$ is one-sided.

• We define $\pi_l : \mathcal{T}^{l+1} \to d^d$ by $\pi_l(\vec{s}\,) := \big(s_i(l)\big)_{i\in d}$. As $\mathcal{T}^{l+1}$ is finite, the range c_l of π_l is also finite. Thus $\lceil T_d \rceil$ is compact since $\lceil T_d \rceil \subseteq \Pi_{l\in\omega}\, c_l$. □

We now give a concrete effective frame.

Notation. Let $b_d : \omega \to d^{<\omega}$ be the following bijection.

• If $d < \omega$, then $b_d(0) := \emptyset$ is the sequence of length 0, $b_d(1) := 0$, ..., $b_d(d) := d-1$ are the sequences of length 1 in the lexicographical ordering, and so on.

• If $d = \omega$, then let $(p_n)_{n\in\omega}$ be the sequence of prime numbers, and $\mathcal{I} : \omega^{<\omega} \to \omega$ be defined by $\mathcal{I}(\emptyset) := 1$, and $\mathcal{I}(s) := p_0^{s(0)+1} ... p_{|s|-1}^{s(|s|-1)+1}$ if $s \neq \emptyset$. Note that $\mathcal{I}$ is one-to-one, so that there is an increasing bijection $\imath : \mathrm{Seq} := \mathcal{I}[\omega^{<\omega}] \to \omega$. We set $b_\omega := (\imath \circ \mathcal{I})^{-1} : \omega \to \omega^{<\omega}$.

Note that $|b_d(n)| \leq n$ if $n \in \omega$. Indeed, this is clear if $d < \omega$. If $d = \omega$, then

$$\mathcal{I}\big(b_\omega(n)|0\big) < \mathcal{I}\big(b_\omega(n)|1\big) < ... < \mathcal{I}\big(b_\omega(n)\big),$$

so that $(\imath \circ \mathcal{I})\big(b_\omega(n)|0\big) < (\imath \circ \mathcal{I})\big(b_\omega(n)|1\big) < ... < (\imath \circ \mathcal{I})\big(b_\omega(n)\big) = n$. This implies that $|b_\omega(n)| \leq n$.

Lemma 2.3 *There is a concrete effective frame.*

Proof. The idea is to code the properties that we want, using the bijection b. Fix $i\in d$. We set $s_i^0=\emptyset$, and $s_i^{l+1}:=s_i^{(((l)_1)_1)_0}\ i\ b_d((((l)_1)_1)_1)\ 0^{l-(((l)_1)_1)_0-|b_d((((l)_1)_1)_1)|}$. Note that

$$(l)_0+(l)_1=M(l):=\text{max}\{m\in\omega\mid\frac{m(m+1)}{2}\leq l\}\leq\frac{M(l)(M(l)+1)}{2}\leq l,$$

so that s_i^l is well defined and $|s_i^l|=l$, by induction on l. It remains to check that Condition (b) in the definition of an effective frame is fullfilled. We set $n:=b_d^{-1}(t)$, $s:=\langle\ r,<q,n>\rangle$ and $l:=<p,s>$. It remains to put $N:=l-q-|t|$: $(s_i^q it0^N)_{i\in d}=\overrightarrow{s^{l+1}}$. □

The previous lemma is essentially identical to Lemma 3.3 in [L7]. Now we come to the lemma crucial for proving Theorem 1.9. It strengthens Lemma 3.4 in [L7], even if the proof is essentially the same.

Notation. If $s\in\omega^{<\omega}$ and $q\leq|s|$, then $s-s|q$ is defined by $s=(s|q)(s-s|q)$. We extend this definition when $s\in\mathcal{N}$ and $q<\omega$. If $\emptyset\neq s\in\omega^{<\omega}$, then we define $s^-:=s|(|s|-1)$.

• We now define $p:\omega^{<\omega}\setminus\{\emptyset\}\rightarrow\omega$. The definition of $p(s)$ is by induction on $|s|$:

$$p(s):=\begin{cases} s(0)\text{ if }|s|=1,\\ \\ <p(s^-),s(|s|-1)>\ \text{ otherwise.}\end{cases}$$

Note that $p_{|\omega^n}:\omega^n\rightarrow\omega$ is a bijection, for each $n\geq 1$.

• Let $l\leq\omega$ be an ordinal. The map $\Delta:d^l\times d^l\rightarrow 2^l$ is the symmetric difference: for any $m\in l$,

$$(s\Delta t)(m):=\Delta(s,t)(m)=1\ \Leftrightarrow\ s(m)\neq t(m).$$

• By convention, $\omega-1:=\omega$.

Lemma 2.4 *Let T_d be the tree associated with an effective frame and, for any $i\in d$, G_i be a dense G_δ subset of $\Pi_i''\lceil T_d\rceil$. Then there are $\alpha_0\in G_0$ and $F:\mathcal{C}\rightarrow\Pi_{0<i<d}\ G_i$ continuous such that, for any $\alpha\in\mathcal{C}$,*
(a) $\big(\alpha_0,F(\alpha)\big)\in\lceil T_d\rceil$,
(b) for any $s\in\omega^{<\omega}$, and any $m\in\omega$,
(i) $\alpha\big(p(sm)\big)=1\ \Rightarrow\ \exists k\in\omega\ \ \big(\alpha_0\Delta F_0(\alpha)\big)\big(p(sk)+1\big)=1$,
(ii) $\big(\alpha_0\Delta F_0(\alpha)\big)\big(p(sm)+1\big)=1\ \Rightarrow\ \exists k\in\omega\ \ \alpha\big(p(sk)\big)=1$.

Moreover, there is an increasing bijection

$$B_\alpha:\{m\in\omega\mid\alpha(m)=1\}\rightarrow\{k\in\omega\mid\big(\alpha_0\Delta F_0(\alpha)\big)(k+1)=1\}$$

such that $(m)_0=\big(B_\alpha(m)\big)_0$ and $\big((m)_1\big)_0=\Big(\big(B_\alpha(m)\big)_1\Big)_0$ if $\alpha(m)=1$.

Proof. Let $(O^i_q)_{q\in\omega}$ be a decreasing sequence of dense open subsets of $\Pi''_i\lceil T_d\rceil$ whose intersection is G_i. We construct finite approximations of α_0 and F. The idea is to linearize the binary tree $2^{<\omega}$. This is the reason why we will use the bijection b_2 defined before Lemma 2.3. In order to construct $F(\alpha)$, we have to imagine, for each length l, the different possibilities for $\alpha|l$. More precisely, we construct a map $l\!:\!2^{<\omega}\!\rightarrow\!\omega\!\setminus\!\{0\}$. In order to simplify the notation, we set, for any $t\!\in\!2^{<\omega}$, ${}_it\!:=\!s_i^{l(t)}$. We want the map l to satisfy the following conditions:

$$(1)\ \forall t\!\in\!2^{<\omega}\ \ \forall i\!\in\!d\ \ (i\!\leq\!|t|\ \Rightarrow\ \emptyset\!\neq\!N_{{}_it}\cap\Pi''_i\lceil T_d\rceil\subseteq O^i_{|t|})$$

$$(2)\ \exists v_\emptyset\!\in\!d^{<\omega}\ \ \forall i\!\in\!d\ \ {}_i\emptyset\!=\!iv_\emptyset$$

$$(3)\ \forall t\!\in\!2^{<\omega}\ \ \forall\varepsilon\!\in\!2\ \ \exists v_{t\varepsilon}\!\in\!d^{<\omega}\ \ \forall i\!\in\!d\ \ {}_i(t\varepsilon)\!=\!({}_it)(i\!\cdot\!\varepsilon)v_{t\varepsilon}$$

$$(4)\ \forall r\!\in\!\omega\ \ \big({}_0b_2(r)\big)0\subseteq{}_0b_2(r\!+\!1)\wedge\forall t\!\in\!2^{<\omega}\ \ \forall n\!<\!l(t)\ \ ({}_0t)(n)\!\leq\!n$$

$$(5)\ \forall t\!\in\!2^{<\omega}\ \ \big(l(t)\!-\!1\big)_0\!=\!(|t|)_0\wedge\Big(\big(l(t)\!-\!1\big)_1\Big)_0\!=\!\big((|t|)_1\big)_0$$

• Assume that this construction is done. As ${}_0(0^q)\subsetneqq{}_0(0^{q+1})$ for each natural number q, we can define $\alpha_0\!:=\!\sup_{q\in\omega}{}_0(0^q)$. Similarly, as ${}_{i+1}\alpha|q\subsetneqq{}_{i+1}\alpha|(q+1)$, we can define, for any $\alpha\!\in\!\mathcal{C}$ and any $i\!<\!d\!-\!1$, $F_i(\alpha)\!:=\!\sup_{q\in\omega}{}_{i+1}\alpha|q$, and F is continuous.

(a) Fix $q\!\in\!\omega$. We have to see that $\big(\alpha_0,F(\alpha)\big)|q\!\in\!T_d$. Note first that $l(t)\geq|t|$ since $l(t\varepsilon)\!>\!l(t)$. Note also that ${}_0t\subseteq\alpha_0$ since ${}_0(0^{|t|})\subseteq{}_0t\subseteq{}_0(0^{|t|+1})$. Thus $\big(\alpha_0,F(\alpha)\big)|l(\alpha|q)\!=\!\overrightarrow{s^{l(\alpha|q)}}\!\in\!E$. This implies that $\big(\alpha_0,F(\alpha)\big)|l(\alpha|q)\!\in\!T_d$ since $({}_0\alpha|q)(n)\!\leq\!n$ if $n\!<\!l(\alpha|q)$. We are done since $l(\alpha|q)\!\geq\!q$.

Moreover, $\alpha_0\in\bigcap_{q\in\omega}\ N_{{}_0(0^q)}\cap\Pi''_0\lceil T_d\rceil\subseteq\bigcap_{q\in\omega}\ O^0_q\!=\!G_0$. Similarly,

$$F_i(\alpha)\in\bigcap_{q\in\omega}\ N_{{}_{i+1}\alpha|q}\cap\Pi''_{i+1}\lceil T_d\rceil\subseteq\bigcap_{q\geq i+1}\ O^{i+1}_q\!=\!G_{i+1}.$$

(b).(i) We set $t:=\alpha|p(sm)$, so that $({}_1t)1\subseteq{}_1(t1)={}_1\alpha|(p(sm)+1)\subseteq F_0(\alpha)$. As $\big(l(t)\!-\!1\big)_0=p(s)$ (or $(m)_0$ if $s=\emptyset$), there is k with $l(t)=p(sk)\!+\!1$ (or $l(t)=k\!+\!1$ and $(k)_0=(m)_0$ if $s=\emptyset$). But $({}_0t)0\subseteq{}_0(t1)\subseteq\alpha_0$, so that $\alpha_0\big(l(t)\big)\neq F_0(\alpha)\big(l(t)\big)$.

(ii) First notice that the only coordinates where α_0 and $F_0(\alpha)$ can differ are 0 and the $l(\alpha|q)$'s. Therefore there is a natural number q with $p(sm)\!+\!1=l(\alpha|q)$. In particular, $(q)_0=\big(l(\alpha|q)\!-\!1\big)_0=p(s)$ (or $(m)_0$ if $s=\emptyset$). Thus there is k with $q=p(sk)$ (or $q=k$ and $(k)_0=(m)_0$ if $s=\emptyset$). Note that $\alpha_0\big(l(\alpha|q)\big)=\big({}_0\alpha|(q+1)\big)\big(l(\alpha|q)\big)=0\neq F_0(\alpha)\big(l(\alpha|q)\big)=\big({}_1\alpha|(q+1)\big)\big(l(\alpha|q)\big)=\alpha(q)$. So $\alpha(q)\!=\!1$ and $\alpha\big(p(sk)\big)\!=\!1$.

Now it is clear that the formula $B_\alpha(m)\!:=\!l(\alpha|m)\!-\!1$ defines the bijection we are looking for.

• So let us prove that the construction is possible. We construct $l(t)$ by induction on $b_2^{-1}(t)$.

As $(i0^\infty)_{i\in d}\in\lceil T_d\rceil$, $0^\infty\in\Pi''_0\lceil T_d\rceil$ and O^0_0 is not empty. Thus there is $u\in d^{<\omega}\setminus\{\emptyset\}$ such that $\emptyset\neq N_u\cap\Pi''_0\lceil T_d\rceil\subseteq O^0_0$. Choose $\beta_0\in N_u\cap\Pi''_0\lceil T_d\rceil$, and $\vec{\alpha}\in\lceil T_d\rceil$ such that $\alpha_0=\beta_0$. Then $\vec{\alpha}||u|\in T_d$ and $u(n)\leq n$ for each $n<|u|$. Note that $u(0)=0$ and $(u-u|1)(n)=u(n+1)\leq 1+n$ for each $n<|u|-1$. We choose $L\in\omega$ with $\big(i\ (u-u|1)\ 0^L\big)_{i\in d}\in E$, $(|0\ (u-u|1)\ 0^L|-1)_0=(0)_0$ and $\big((|0\ (u-u|1)\ 0^L|-1)_1\big)_0=\big((0)_1\big)_0$. We put $v_\emptyset:=(u-u|1)\ 0^L$ and $l(\emptyset):=1+|v_\emptyset|$.

As $(iv_\emptyset 0^\infty)_{i\in d}\in\lceil T_d\rceil$, $N_{0v_\emptyset 0}\cap\Pi''_0\lceil T_d\rceil$ is a nonempty open subset of $\Pi''_0\lceil T_d\rceil$. Thus there is $u_0\in d^{<\omega}$ such that $\emptyset\neq N_{0v_\emptyset 0u_0}\cap\Pi''_0\lceil T_d\rceil\subseteq O^0_1$. As before we see that $u_0(n)\leq 1+|v_\emptyset|+1+n$ for each $n<|u_0|$. This implies that $(iv_\emptyset 0u_0 0^\infty)_{i\in d}\in\lceil T_d\rceil$. Thus $N_{1v_\emptyset 0u_0}\cap\Pi''_1\lceil T_d\rceil$ is a nonempty open subset of $\Pi''_1\lceil T_d\rceil$. So there is $u_1\in d^{<\omega}$ such that $\emptyset\neq N_{1v_\emptyset 0u_0u_1}\cap\Pi''_1\lceil T_d\rceil\subseteq O^1_1$. Choose $\beta_1\in N_{1v_\emptyset 0u_0u_1}\cap\Pi''_1\lceil T_d\rceil$, and $\vec{\gamma}\in\lceil T_d\rceil$ such that $\gamma_1=\beta_1$. Then $\vec{\gamma}||1v_\emptyset 0u_0u_1|\in T_d$ and $\gamma_0(n)\leq n$ for each $n<|1v_\emptyset 0u_0u_1|$. This implies that $\gamma_0(|1v_\emptyset 0u_0|+n)\leq|1v_\emptyset 0u_0|+n$ for each $n<|u_1|$. But $u_1(n)$ is either 1, or $\gamma_0(|1v_\emptyset 0u_0|+n)$. Thus $u_1(n)\leq|1v_\emptyset 0u_0|+n$ if $n<|u_1|$. We choose $M\in\omega$ such that $\big((_i\emptyset)\ 0u_0u_1\ 0^M\big)_{i\in d}\in E$, $\big(l(\emptyset)+|u_0u_1|+M\big)_0=(1)_0$ and $\big((l(\emptyset)+|u_0u_1|+M)_1\big)_0=\big((1)_1\big)_0$. We put $v_0:=u_0u_1\ 0^M$ and $l(0):=l(\emptyset)+1+|v_0|$.

Assume that $\big(l(t)\big)_{b_2^{-1}(t)\leq r}$ satisfying (1)-(5) have been constructed, which is the case for $r=1$. Fix $t\in 2^{<\omega}$ and $\varepsilon\in 2$ such that $b_2(r+1)=t\varepsilon$, with $r\geq 1$. Note that $b_2^{-1}(t)<r$, so that $l(t)<l\big(b_2(r)\big)$, by induction assumption.

As $N_{_0b_2(r)}\cap\Pi''_0\lceil T_d\rceil$ is nonempty, $N_{(_0b_2(r))0}\cap\Pi''_0\lceil T_d\rceil$ is nonempty too. Thus there is w_0 in $d^{<\omega}$ such that $\emptyset\neq N_{(_0b_2(r))0w_0}\cap\Pi''_0\lceil T_d\rceil\subseteq O^0_{|t|+1}$. As before we see that $w_0(n)\leq l\big(b_2(r)\big)+1+n$ for each $n<|w_0|$. Arguing as in the case $r=1$, we prove, for each $1\leq i\leq|t|+1$, the existence of $w_i\in d^{<\omega}$ such that $\emptyset\neq N_{(_it)(i\cdot\varepsilon)(_0b_2(r)-_0b_2(r)|(l(t)+1))0w_0\ldots w_i}\cap\Pi''_i\lceil T_d\rceil\subseteq O^i_{|t|+1}$ and

$$w_i(n)\leq l\big(b_2(r)\big)+1+|w_0\ldots w_{i-1}|+n$$

for each $n<|w_i|$ ($w_i(n)$ can be i, in which case we use the fact that $l(t)\geq|t|$). We choose $N\in\omega$ such that $\big((_it)\ (i\cdot\varepsilon)\ \big(_0b_2(r)-_0b_2(r)|(l(t)+1)\big)\ 0\ w_0\ldots w_{|t|+1}\ 0^N\big)_{i\in d}\in E$,

$$\big(l\big(b_2(r)\big)+|w_0\ldots w_{|t|+1}|+N\big)_0=(|t|+1)_0$$

and $\Big(\big(l\big(b_2(r)\big)+|w_0\ldots w_{|t|+1}|+N\big)_1\Big)_0=\big((|t|+1)_1\big)_0$. We put $l(t\varepsilon):=l(t)+1+|v_{t\varepsilon}|$, where by definition $v_{t\varepsilon}:=\big(_0b_2(r)-_0b_2(r)|(l(t)+1)\big)\ 0\ w_0\ldots w_{|t|+1}\ 0^N$. □

Now we come to the condition ensuring the existence of complicated sets announced in the introduction.

Notation. The map $\mathcal{S}:\mathcal{C}\to\mathcal{C}$ is the shift map: $\mathcal{S}(\alpha)(m):=\alpha(m+1)$.

Definition 2.5 *We say that $\mathbf{C}\subseteq\mathcal{C}$ is compatible with comeager sets (or ccs) if*

$$\alpha\in\mathbf{C}\ \Leftrightarrow\ \mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\in\mathbf{C},$$

for each $\alpha_0\in d^\omega$ and $F:\mathcal{C}\to(d^\omega)^{d-1}$ satisfying the conclusion of Lemma 2.4.(b).

Notation. Let T_d be the tree associated with an effective frame, and $\mathbf{C}\subseteq\mathcal{C}$. We put

$$S_{\mathbf{C}}:=\big\{\vec{\alpha}\in\lceil T_d\rceil \mid \mathcal{S}(\alpha_0\Delta\alpha_1)\in\mathbf{C}\big\}.$$

Lemma 2.6 *Let T_d be the tree associated with an effective frame, and $\mathbf{\Gamma}$ be a non self-dual Wadge class of Borel sets.*

(1) Assume that $\mathbf{C}$ is a $\mathbf{\Gamma}$-complete ccs set. Then $S_{\mathbf{C}}\in\mathbf{\Gamma}(\lceil T_d\rceil)$ is a Borel subset of $(d^\omega)^d$, and is not separable from $\lceil T_d\rceil\setminus S_{\mathbf{C}}$ by a pot($\check{\mathbf{\Gamma}}$) set.

(2) Assume that $\mathbf{C}_0$, $\mathbf{C}_1\in\mathbf{\Gamma}$ are disjoint, ccs, and not separable by a $\Delta(\mathbf{\Gamma})$ set. Then $S_{\mathbf{C}_0}, S_{\mathbf{C}_1}$ are in $\mathbf{\Gamma}(\lceil T_d\rceil)$, disjoint Borel subsets of $(d^\omega)^d$, and not separable by a pot$\big(\Delta(\mathbf{\Gamma})\big)$ set.

Proof. (1) It is clear that $S_{\mathbf{C}}\in\mathbf{\Gamma}(\lceil T_d\rceil)$ since $\mathcal{S}$ and Δ are continuous. So $S_{\mathbf{C}}$ is a Borel subset of $(d^\omega)^d$ since $\lceil T_d\rceil$ is a closed subset of $(d^\omega)^d$. Indeed, $\lceil T_\omega\rceil$ is closed:

$$\vec{\alpha}\in\lceil T_\omega\rceil \;\Leftrightarrow\; \forall n\in\omega\setminus\{0\}\;\exists l<n\;\forall i\in\omega\;\; s^l_i i\subseteq\alpha_i\wedge(\alpha_i|n-s^l_i i)=(\alpha_0|n-s^l_0 0)\wedge\alpha_0(n-1)\leq n-1.$$

We argue by contradiction to see that $S_{\mathbf{C}}$ is not separable from $\lceil T_d\rceil\setminus S_{\mathbf{C}}$ by a pot($\check{\mathbf{\Gamma}}$) set: this gives $P\in$ pot($\check{\mathbf{\Gamma}}$). For each $i\in d$ there is a dense G_δ subset G_i of the compact space $\Pi''_i\lceil T_d\rceil$ such that $P\cap(\Pi_{i\in d}\,G_i)\in\check{\mathbf{\Gamma}}(\Pi_{i\in d}\,G_i)$, and $S_{\mathbf{C}}\cap(\Pi_{i\in d}\,G_i)\subseteq P\cap(\Pi_{i\in d}\,G_i)\subseteq(\Pi_{i\in d}\,G_i)\setminus(\lceil T_d\rceil\setminus S_{\mathbf{C}})$.

Lemma 2.4 provides $\alpha_0\in G_0$ and $F:\mathcal{C}\rightarrow\Pi_{0<i<d}\,G_i$ continuous. Let

$$D:=\big\{\alpha\in\mathcal{C}\mid\big(\alpha_0,F(\alpha)\big)\in P\cap(\Pi_{i\in d}\,G_i)\big\}.$$

Then $D\in\check{\mathbf{\Gamma}}$. Let us prove that $\mathbf{C}=D$, which will contradict the fact that $\mathbf{C}\notin\check{\mathbf{\Gamma}}$. As $\mathbf{C}$ is ccs, $\alpha\in\mathbf{C}$ is equivalent to $\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\in\mathbf{C}$. Thus

$$\alpha\in\mathbf{C}\Rightarrow\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\in\mathbf{C}\Rightarrow\big(\alpha_0,F(\alpha)\big)\in S_{\mathbf{C}}\cap(\Pi_{i\in d}\,G_i)\subseteq P\cap(\Pi_{i\in d}\,G_i)\Rightarrow\alpha\in D.$$

Similarly, $\alpha\notin\mathbf{C}\Rightarrow\alpha\notin D$, and $\mathbf{C}=D$.

(2) We argue as in (1). □

This lemma reduces the problem of finding some complicated sets as in the statement of Theorem 1.9 to a problem concerning one-dimensional spaces.

3 The proof of Theorem 1.9 for the Borel classes

The full version of Theorem 1.9 for the Borel classes is as follows:

Theorem 3.1 *There are a concrete tree T_d with Δ^1_1 suitable levels, and, for any $1\leq\xi<\omega_1$,*

(1) a set $\mathbb{S}\in\mathbf{\Sigma}^0_\xi(\lceil T_d\rceil)$ not separable from $\lceil T_d\rceil\setminus\mathbb{S}$ by a pot($\mathbf{\Pi}^0_\xi$) set,

(2) disjoint sets $\mathbb{S}_0,\mathbb{S}_1\in\mathbf{\Sigma}^0_\xi(\lceil T_d\rceil)$ not separable by a pot($\mathbf{\Delta}^0_\xi$) set.

This is an application of Lemma 2.6. We now introduce the objects that will be used to define the $\mathbf{C}$'s in this lemma. These objects will also be useful in the general case. The following definition can be found in [Lo-SR2] (see Definition 2.2).

Definition 3.2 *A set* $\mathbf{H}$ *is* $\mathbf{\Gamma}$-*strategically complete if*

(a) $\mathbf{H}\in\mathbf{\Gamma}(\mathcal{C})$.

(b) If $A\in\mathbf{\Gamma}(\mathcal{N})$, *then Player 2 wins the Wadge game* $G(A,\mathbf{H})$ *(where Player 1 plays* $\alpha\in\mathcal{N}$, *Player 2 plays* $\beta\in\mathcal{C}$ *and Player 2 wins if* $\alpha\in A \Leftrightarrow \beta\in\mathbf{H}$*).*

The following definition can essentially be found in [Lo-SR1] (see Section 3) and [Lo-SR2] (see Definition 2.3).

Definition 3.3 *Let* $\eta<\omega_1$. *A function* $\zeta:\mathcal{C}\to\mathcal{C}$ *is an independent* η-*function if the following hold.*

(a) For some function $\pi:\omega\to\omega$, *the value* $\zeta(\alpha)(m)$ *depends only on the values of* α *on* $\pi^{-1}(\{m\})$.

(b) We set, for any natural number m, $\mathbf{Z}_m:=\{\alpha\in\mathcal{C}\mid\zeta(\alpha)(m)=1\}$.

(1) If $\eta=0$, *then* $\mathbf{Z}_m$ *is* $\mathbf{\Delta}^0_1$-*complete for any* m.

(2) If $\eta=\theta+1$ *is a successor ordinal, then* $\mathbf{Z}_m$ *is* $\mathbf{\Pi}^0_{1+\theta}$-*strategically complete for any* m.

(3) If η *is a limit ordinal, then there is a sequence* $(\theta_m)_{m\in\omega}$ *such that*

(i) $\theta_m<\eta$,

(ii) $\sup_{p\geq 1}\theta_{m_p}=\eta$, *for any one-to-one sequence* $(m_p)_{p\geq 1}$ *of natural numbers,*

(iii) the set $\mathbf{Z}_m$ *is* $\mathbf{\Pi}^0_{1+\theta_m}$-*strategically complete for any* m.

Note that we added a condition when $\eta=0$. Moreover, we do not ask the sequence $(\theta_m)_{m\in\omega}$ to be increasing, unlike in [Lo-SR2], Definition 2.3. Note also that an independent η-function has to be $\mathbf{\Sigma}^0_{1+\eta}$-measurable. Moreover, if ζ is an independent η-function, then π has to be onto.

Examples. In [Lo-SR1], Lemma 3.3, the map $\rho:\mathcal{C}\to\mathcal{C}$ defined as follows is introduced (it is in fact called ρ_0 in [Lo-SR1]):

$$\rho(\alpha)(m):=\begin{cases} 1 \text{ if } \alpha(<m,n>)=0 \text{ for any } n\in\omega,\\ \\ 0 \text{ otherwise.}\end{cases}$$

Note that ρ is an independent 1-function, with $\pi(k)=(k)_0$. In this paper, $\rho^\eta:\mathcal{C}\to\mathcal{C}$ is also defined for $\eta<\omega_1$ as follows, by induction on η (see the proof of Theorem 3.2). We put

- $\rho^0:=\mathrm{Id}_{\mathcal{C}}$.
- $\rho^{\theta+1}:=\rho\circ\rho^\theta$.
- If $\eta>0$ is a limit ordinal, then we fix a sequence $(\theta_m)_{m\in\omega}\subseteq\eta$ of successor ordinals satisfying the equality $\Sigma_{m\in\omega}\ \theta_m=\eta$. We define $\rho^{(m,m+1)}:\mathcal{C}\to\mathcal{C}$ by

$$\rho^{(m,m+1)}(\alpha)(i):=\begin{cases} \alpha(i) \text{ if } i<m,\\ \\ \rho^{\theta_m}\big(\mathcal{S}^m(\alpha)\big)(i-m) \text{ if } i\geq m.\end{cases}$$

We set $\rho^{(0,m+1)}:=\rho^{(m,m+1)}\circ\rho^{(m-1,m)}\circ\ldots\circ\rho^{(0,1)}$ and $\rho^\eta(\alpha)(m):=\rho^{(0,m+1)}(\alpha)(m)$. The authors prove that ρ^η is an independent η-function (see the proof of Theorem 3.2). In this paper, the set $\mathbf{H}_{1+\eta}:=(\rho^\eta)^{-1}(\{0^\infty\})$ is also introduced, and the authors prove that $\mathbf{H}_{1+\eta}$ is $\mathbf{\Pi}^0_{1+\eta}$-complete (see Theorem 3.2).

Notation. Let $1\leq\xi:=1+\eta<\omega_1$. We set $\mathbf{C}_\xi:=\neg\mathbf{H}_\xi$. If moreover $\varepsilon\in 2$, then we set

$$\mathbf{C}_\xi^\varepsilon:=\big\{\alpha\in\mathcal{C}\mid\exists m\in\omega\ \ \rho^\eta(\alpha)(m)=1\wedge\forall l<m\ \ \rho^\eta(\alpha)(l)=0\wedge(m)_0\equiv\varepsilon\ (\text{mod }2)\big\}.$$

Then we set $\mathbb{S}:=S_{\mathbf{C}_\xi}$ and $\mathbb{S}_\varepsilon:=S_{\mathbf{C}_\xi^\varepsilon}$.

Theorem 3.1 is a corollary of Proposition 2.2, Lemmas 2.3 and 2.6, and the following lemma.

Lemma 3.4 *Let* $1\leq\xi<\omega_1$.
(1) The set $\mathbf{C}_\xi$ *is a* $\mathbf{\Sigma}^0_\xi$*-complete ccs set.*
(2) The sets $\mathbf{C}_\xi^0$, $\mathbf{C}_\xi^1\in\mathbf{\Sigma}^0_\xi$*, are disjoint, ccs, and not separable by a* $\mathbf{\Delta}^0_\xi$ *set.*

Proof. (1) $\mathbf{C}_\xi$ is $\mathbf{\Sigma}^0_\xi$-complete since $\mathbf{H}_\xi$ is $\mathbf{\Pi}^0_\xi$-complete.

• Assume that α_0, F satisfy the conclusion of Lemma 2.4.(b). Let us prove that

$$\rho^\eta(\alpha)=\rho^\eta\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big),$$

for each $1\leq\eta<\omega_1$ and $\alpha\in\mathcal{C}$. For $\eta=1$ we apply the conclusion of Lemma 2.4.(b) to $s\in\omega$. Then note that $\rho^{\theta+1}(\alpha)=\rho\big(\rho^\theta(\alpha)\big)=\rho\left(\rho^\theta\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big)\right)=\rho^{\theta+1}\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big)$, by induction. From this we deduce that $\rho^{(0,1)}(\alpha)=\rho^{\theta_0}(\alpha)=\rho^{\theta_0}\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big)=\rho^{(0,1)}\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big)$ if $\lambda>0$ is a limit ordinal, by induction again. Thus $\rho^{(0,m+1)}(\alpha)=\rho^{(0,m+1)}\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big)$, and

$$\rho^\lambda(\alpha)(m)=\rho^{(0,m+1)}(\alpha)(m)=\rho^{(0,m+1)}\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big)(m)=\rho^\lambda\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big)(m).$$

• If we apply the previous point, or the conclusion of Lemma 2.4.(b) to $s:=\emptyset$, then we get

$$\alpha\in\mathbf{C}_\xi\Leftrightarrow\exists m\in\omega\ \ \rho^\eta(\alpha)(m)=1\Leftrightarrow\exists k\in\omega\ \ \rho^\eta\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big)(k)=1\Leftrightarrow\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\in\mathbf{C}_\xi.$$

Thus $\mathbf{C}_\xi$ is ccs.

(2) Note first that $\mathbf{C}_\xi^0$, $\mathbf{C}_\xi^1\in\mathbf{\Sigma}^0_\xi$ since ρ^η is $\mathbf{\Sigma}^0_{1+\eta}$-measurable, are clearly disjoint, and are ccs as in (1) since $(m)_0=\big(B_\alpha(m)\big)_0$ in Lemma 2.4.(b).

• We set, for $\varepsilon\in 2$, $\mathbf{V}_\varepsilon:=\big\{\alpha\in\mathcal{C}\mid\exists m\in\omega\ \ \rho^\eta(\alpha)(m)=1\wedge(m)_0\equiv\varepsilon\ (\text{mod }2)\big\}$. Then $\mathbf{V}_\varepsilon$ is a $\mathbf{\Sigma}^0_\xi$ set since ρ^η is $\mathbf{\Sigma}^0_{1+\eta}$-measurable. Let us prove that $\mathbf{V}_\varepsilon$ is $\mathbf{\Sigma}^0_\xi$-complete.

- If $\eta=0$, then $0^\infty\in\overline{\mathbf{V}_\varepsilon}\setminus\mathbf{V}_\varepsilon$, so that $\mathbf{V}_\varepsilon$ is $\mathbf{\Sigma}^0_1$-complete.

- If $\eta=\theta+1$, then ρ^η is an independent η-function. Let $(A_m)_{m\in\omega}$ be a sequence of $\mathbf{\Pi}^0_{1+\theta}(\mathcal{C})$ sets. Choose a continuous map $f_m:\mathcal{C}\to\mathcal{C}$ such that $A_m=f_m^{-1}(\mathbf{Z}_m)$. We define $f:\mathcal{C}\to\mathcal{C}$ by $f(\alpha)(k):=f_m(\alpha)(k)$ if $\pi_\eta(k)=m$, and f is continuous. Moreover,

$$\alpha\in A_m\Leftrightarrow f_m(\alpha)\in\mathbf{Z}_m\Leftrightarrow f(\alpha)\in\mathbf{Z}_m,$$

so that $\bigcup_{m\in\omega,(m)_0\equiv\varepsilon\ (\text{mod }2)}A_m=f^{-1}(\mathbf{V}_\varepsilon)$. Thus $\mathbf{V}_\varepsilon$ is $\mathbf{\Sigma}^0_\xi$-complete.

- If η is the limit of the θ_m's, then ρ^η is an independent η-function. We argue as in the successor case to see that $\mathbf{V}_\varepsilon$ is $\mathbf{\Sigma}^0_\xi$-complete.

• We argue by contradiction, which gives $D\in\mathbf{\Delta}^0_\xi$ separating C^0_ξ from C^1_ξ. Let v_0, v_1 be disjoint $\mathbf{\Sigma}^0_\xi$ subsets of $\mathcal{C}$. Then we can find a continuous map $f_\varepsilon:\mathcal{C}\rightarrow\mathcal{C}$ such that $v_\varepsilon=f_\varepsilon^{-1}(V_\varepsilon)$. As ρ^η_0 is an independent η-function, we get $\pi_\eta:\omega\rightarrow\omega$. We define a map $f:\mathcal{C}\rightarrow\mathcal{C}$ by $f(\alpha)(k):=f_\varepsilon(\alpha)(k)$ if $\big(\pi_\eta(k)\big)_0\equiv\varepsilon$ (mod 2), and f is continuous. Note that $\alpha\in v_\varepsilon\Leftrightarrow f_\varepsilon(\alpha)\in V_\varepsilon\Leftrightarrow f(\alpha)\in V_\varepsilon$, so that $v_\varepsilon=f^{-1}(V_\varepsilon)$. Thus $\alpha\in v_0\Leftrightarrow f(\alpha)\in V_0\Leftrightarrow f(\alpha)\in V_0\setminus V_1\subseteq C^0_\xi\subseteq D$ since v_0 is disjoint from v_1. Similarly, $\alpha\in v_1\Leftrightarrow f(\alpha)\in V_1\setminus V_0\subseteq C^1_\xi\subseteq\neg D$. Thus $f^{-1}(D)$ separates v_0 from v_1. As $f^{-1}(D)\in\mathbf{\Delta}^0_\xi$, this implies that $\mathbf{\Sigma}^0_\xi$ has the separation property, which contradicts 22.C in [K]. □

4 The proof of Theorem 1.10 for the Borel classes

The full versions of Theorems 1.10 and 1.3 for the Borel classes are as follows:

Theorem 4.1 *Let T_d be a tree with suitable levels, $1\leq\xi<\omega_1$, $(X_i)_{i\in d}$ be a sequence of Polish spaces, and A_0, A_1 be disjoint analytic subsets of $\Pi_{i\in d}\ X_i$.*

(1) Let $S\in\mathbf{\Sigma}^0_\xi(\lceil T_d\rceil)$. Then one of the following holds:

(a) The set A_0 is separable from A_1 by a pot($\mathbf{\Pi}^0_\xi$) set.

(b) The inequality $\big((d^\omega)_{i\in d},S,\lceil T_d\rceil\setminus S\big)\leq\big((X_i)_{i\in d},A_0,A_1\big)$ holds.

If moreover S is not separable from $\lceil T_d\rceil\setminus S$ by a pot($\mathbf{\Pi}^0_\xi$) set, then this is a dichotomy.

(2) Let $S_0,S_1\in\mathbf{\Sigma}^0_\xi(\lceil T_d\rceil)$ be disjoint. Then one of the following holds:

(a) The set A_0 is separable from A_1 by a pot($\mathbf{\Delta}^0_\xi$) set.

(b) The inequality $\big((d^\omega)_{i\in d},S_0,S_1\big)\leq\big((X_i)_{i\in d},A_0,A_1\big)$ holds.

If moreover S_0 is not separable from S_1 by a pot($\mathbf{\Delta}^0_\xi$) set, then this is a dichotomy.

Corollary 4.2 *Let $\mathbf{\Gamma}$ be Borel class. Then there are Borel subsets $\mathbb{S}_0$, $\mathbb{S}_1$ of $(d^\omega)^d$ such that for any sequence of Polish spaces $(X_i)_{i\in d}$, and for any disjoint analytic subsets A_0, A_1 of $\Pi_{i\in d}\ X_i$, exactly one of the following holds:*

(a) The set A_0 is separable from A_1 by a pot($\mathbf{\Gamma}$) set.

(b) The inequality $\big((d^\omega)_{i\in d},\mathbb{S}_0,\mathbb{S}_1\big)\leq\big((X_i)_{i\in d},A_0,A_1\big)$ holds.

4.1 Acyclicity

In this subsection we give a result that will be used later to prove Theorem 4.1. This is the place where the essence of the notion of a finite one-sided almost acyclic set is really used.

Lemma 4.1.1 *Assume that $\mathcal{T}\subseteq\mathcal{X}^d$ is finite. Then the following are equivalent:*

(a) The set $\mathcal{T}$ is one-sided and almost acyclic.

(b) For each $\vec{t}\in\mathcal{T}$, there is a natural number $0<l<d+2$ and a partition $(S_j)_{j\in l}$ of $\mathcal{T}\setminus\{\vec{t}\}$ with

(1) $\forall i\in d\ \ \forall j\neq k\in l\ \ \Pi_i[S_j]\cap\Pi_i[S_k]=\emptyset$.

(2) $\forall i\in d\ \ \forall j\in l\ \ \forall\vec{x}\in S_j\ \ x_i=t_i\ \Rightarrow\ i=j$.

Proof. (a) $\Rightarrow$ (b) If $\vec{y} \neq \vec{z} \in \mathcal{T}$ and $\left(\overrightarrow{y^j}\right)_{j \leq l}$ is a walk in $G^{\mathcal{T}}$ with $\overrightarrow{y^0} = \vec{y}$ and $\overrightarrow{y^l} = \vec{z}$, then we choose such a walk of minimal length, and we call it $w_{\vec{y},\vec{z}}$. We will define a partition of $\mathcal{T}$. We put, for $j \in d$,

$$N := \{ \vec{x} \in \mathcal{T} \mid \vec{x} \neq \vec{t} \wedge w_{\vec{x},\vec{t}} \text{ does not exist } \},$$
$$R_j := \{ \vec{x} \in \mathcal{T} \mid \vec{x} \neq \vec{t} \wedge \left(w_{\vec{x},\vec{t}}\,(|w_{\vec{x},\vec{t}}|-2)\right)_j = t_j \}.$$

So we defined a partition $\left(N, (R_j)_{j \in d}\right)$ of $\mathcal{T} \setminus \{\vec{t}\}$ since $\mathcal{T}$ is one-sided. As $\mathcal{T}$ is finite, there is $j_0 \in d$ minimal such that $R_j = \emptyset$ if $j > j_0$. We set $S_j := R_j$ if $j \leq j_0$, $S_{j_0+1} := N$ and $l := j_0 + 2$.

(1) Let us prove that $\Pi_i[R_j] \cap \Pi_i[N] = \emptyset$, for each $i, j \in d$. We argue by contradiction. This gives $x_i \in \Pi_i[R_j] \cap \Pi_i[N]$, $\vec{x} \in R_j$, and also $\vec{y} \in N$ such that $x_i = y_i$. As $\vec{x}, \vec{y} \in \mathcal{T}$ and $R_j \cap N = \emptyset$, $\vec{x} \neq \vec{y}$ and $\vec{x}, \vec{y}$ are $G^{\mathcal{T}}$-related. Note that $w_{\vec{y},\vec{t}}$ does not exist, and that $w_{\vec{x},\vec{t}}$ exists. Now the sequence $\left(\vec{y}, \vec{x}, ..., \vec{t}\,\right)$ shows the existence of $w_{\vec{y},\vec{t}}$, which is absurd.

It remains to see that $\Pi_i[R_j] \cap \Pi_i[R_k] = \emptyset$, for each $i, j, k \in d$ with $j \neq k$. We argue by contradiction. This gives $x_i \in \Pi_i[R_j] \cap \Pi_i[R_k]$, $\vec{x} \in R_j$, and also $\vec{y} \in R_k$ such that $x_i = y_i$. As $\vec{x}, \vec{y} \in \mathcal{T}$ and $j \neq k$, $\vec{x} \neq \vec{y}$ and $\vec{x}, \vec{y}$ are $G^{\mathcal{T}}$-related. We set $w_{\vec{x},\vec{t}} := \left(\overrightarrow{z^n}\right)_{n \leq I+1}$ and $w_{\vec{y},\vec{t}} := \left(\overrightarrow{y^n}\right)_{n \leq J+1}$. Note that $\overrightarrow{z^I} \neq \overrightarrow{y^J}$ since $z^I_j = t_j$ and $y^J_j \neq t_j$, since otherwise $\overrightarrow{y^J}$, $\vec{t} \in \mathcal{T}$, $\overrightarrow{y^J} \neq \vec{t}$ and $y^J_j = t_j$, $y^J_k = t_k$, which contradicts the fact that $\mathcal{T}$ is one-sided.

We denote by $W := \left(\overrightarrow{w^n}\right)_{n \leq K}$ the following $G^{\mathcal{T}}$-walk: $\left(\overrightarrow{z^I}, \overrightarrow{z^{I-1}}, ..., \overrightarrow{z^0}, \overrightarrow{y^0}, \overrightarrow{y^1}, ..., \overrightarrow{y^J}\right)$. If there are $k < n \leq K$ with $\overrightarrow{w^k} = \overrightarrow{w^n}$, then we put $W' := \left(\overrightarrow{w^0}, ..., \overrightarrow{w^k}, \overrightarrow{w^{n+1}}, ..., \overrightarrow{w^K}\right)$. If we iterate this construction, then we get a $G^{\mathcal{T}}$-walk without repetition $V := \left(\overrightarrow{v^n}\right)_{n \leq L}$ from $\overrightarrow{w^0}$ to $\overrightarrow{w^K}$.

If there are $i \in d$ and $k+1 < n \leq L$ with $v^k_i = v^n_i$, then we put $V' := \left(\overrightarrow{v^0}, ..., \overrightarrow{v^k}, \overrightarrow{v^n}, ..., \overrightarrow{v^L}\right)$. If we iterate this construction, then we get a $G^{\mathcal{T}}$-walk without repetition $U := \left(\overrightarrow{u^n}\right)_{n \leq M}$ from $\overrightarrow{w^0}$ to $\overrightarrow{w^K}$ for which it is not possible to find $i \in d$ and $k+1 < n \leq M$ with $u^k_i = u^n_i$.

Now $\vec{t}, \overrightarrow{u^0}, ..., \overrightarrow{u^M}, \vec{t}$ is a $G^{\mathcal{T}}$-cycle contradicting the almost acyclicity of $\mathcal{T}$.

(2) If $\vec{x} \in N$, then $w_{\vec{x},\vec{t}}$ does not exist. This implies that $x_i \neq t_i$ for each $i \in d$, since otherwise $\vec{x}$ and $\vec{t}$ would be $G^{\mathcal{T}}$-related, which contradicts the non-existence of $w_{\vec{x},\vec{t}}$.

If $\vec{x} \in R_j$, then i is the only coordinate for which the equality $x_i = t_i$ holds since $\mathcal{T}$ is one-sided. Note that $w_{\vec{x},\vec{t}} = \left(\vec{x}, \vec{t}\,\right)$. As $\vec{x} \in R_j$, we get $\left(w_{\vec{x},\vec{t}}\,(|w_{\vec{x},\vec{t}}|-2)\right)_j = t_j$. But $w_{\vec{x},\vec{t}}\,(|w_{\vec{x},\vec{t}}|-2) = \vec{x}$. Thus $x_j = t_j$ and $i = j$.

(b) $\Rightarrow$ (a) Let $\vec{t} \neq \vec{x} \in \mathcal{T}$, $i, j \in d$ such that $t_i = x_i$ and $t_j = x_j$, and $k \in l$ such that $\vec{x} \in S_k$. By (2) we get $i = k = j$ and $\mathcal{T}$ is one-sided. Now consider a $G^{\mathcal{T}}$-cycle $(\overrightarrow{x^n})_{n \leq L}$. By (1) there is $j \in l$ such that $\overrightarrow{x^n} \in S_j$ for each $0 < n < L$. Then by (2) we get $t_j = x^1_j = x^{L-1}_j$ and $\mathcal{T}$ is almost acyclic. $\square$

Definition 4.1.2 and Lemma 4.1.3 below are essentially due to G. Debs (see Subsection 2.1 in [L7]).

Definition 4.1.2 *(Debs) Let $\Theta : \mathcal{X}^d \to 2^{\mathcal{N}^d}$, $\mathcal{T} \subseteq \mathcal{X}^d$. We say that the map $\theta = \Pi_{i\in d}\, \theta_i \in (\mathcal{N}^{\mathcal{X}})^d$ is a $\pi\text{-}selector\ on\ \mathcal{T}\ for\ \Theta$ if*
(a) $\theta(\vec{x}) = \big(\theta_i(x_i)\big)_{i\in d}$ for each $\vec{x} \in \mathcal{X}^d$.
(b) $\theta(\vec{x}) \in \Theta(\vec{x})$ for each $\vec{x} \in \mathcal{T}$.

Lemma 4.1.3 *(Debs) Let l be a natural number, $\mathcal{X} := d^{l+1}$, $\mathcal{T} \subseteq \mathcal{X}^d$ be Δ^1_1, finite, one-sided, and almost acyclic, $\Theta : \mathcal{X}^d \to \Sigma^1_1(\mathcal{N}^d)$, and $\overline{\Theta} : \mathcal{X}^d \to \Sigma^1_1(\mathcal{N}^d)$ be defined by $\overline{\Theta}(\vec{x}) := \overline{\Theta(\vec{x})}^{\tau_1}$. Then Θ admits a π-selector on $\mathcal{T}$ if $\overline{\Theta}$ does.*

Proof. (a) Let $\vec{t} \in \mathcal{T}$, and $\Psi : \mathcal{X}^d \to \Sigma^1_1(\mathcal{N}^d)$. We assume that $\Psi(\vec{x}) = \Theta(\vec{x})$ if $\vec{x} \neq \vec{t}$, and that $\Psi\left(\vec{t}\right) \subseteq \overline{\Theta\left(\vec{t}\right)}^{\tau_1}$. We first prove that Θ admits a π-selector on $\mathcal{T}$ if Ψ does.

• Lemma 4.1.1 gives a finite partition $(S_j)_{j\in l}$ of $\mathcal{T} \setminus \{\vec{t}\}$. Fix a π-selector $\tilde{\psi}$ on $\mathcal{T}$ for Ψ, and let $M := \max\,(d \cap l)$. We define Σ^1_1 sets U_i, for $i \leq M$, by

$$U_i := \big\{\alpha \in \mathcal{N} \mid \exists \psi \in (\mathcal{N}^{\mathcal{X}})^d \ \ \alpha = \psi_i(t_i) \wedge \forall \vec{x} \in \mathcal{T} \ \ \psi(\vec{x}) \in \Psi(\vec{x})\big\}.$$

As $\tilde{\psi}\left(\vec{t}\right) = \big(\tilde{\psi}_i(t_i)\big)_{i\in d} \in \Psi\left(\vec{t}\right) \cap \big((\Pi_{i\leq M}\, U_i) \times \mathcal{N}^{d-M-1}\big)$ we get

$$\emptyset \neq \Psi\left(\vec{t}\right) \cap \big((\Pi_{i\leq M}\, U_i) \times \mathcal{N}^{d-M-1}\big) \subseteq \overline{\Theta\left(\vec{t}\right)}^{\tau_1} \cap \big((\Pi_{i\leq M}\, U_i) \times \mathcal{N}^{d-M-1}\big).$$

By the separation theorem this implies that $\Theta\left(\vec{t}\right) \cap \big((\Pi_{i\leq M}\, U_i) \times \mathcal{N}^{d-M-1}\big)$ is not empty and contains some point $\vec{\alpha}$. Fix $i \leq M$. As $\alpha_i \in U_i$ there is $\psi^i \in (\mathcal{N}^{\mathcal{X}})^d$ such that $\alpha_i = \psi^i_i(t_i)$ and $\psi^i(\vec{x}) \in \Psi(\vec{x})$ if $\vec{x} \in \mathcal{T}$.

• Now we can define $\theta_i : \mathcal{X} \to \mathcal{N}$, for each $i \in d$. We put

$$\theta_i(x_i) := \begin{cases} \alpha_i \text{ if } x_i = t_i, \\ \\ \psi^j_i(x_i) \text{ if } x_i \in \Pi_i[S_j] \setminus \{t_i\} \wedge j \leq M, \\ \\ \psi^0_i(x_i) \text{ otherwise.} \end{cases}$$

Then we set $\theta(\vec{x})(i) := \theta_i(x_i)$ if $i \in d$.

• It remains to see that $\theta(\vec{x}) \in \Theta(\vec{x})$ for each $\vec{x} \in \mathcal{T}$.

Note that $\theta\left(\vec{t}\right) = \vec{\alpha} \in \Theta\left(\vec{t}\right)$. So we may assume that $\vec{x} \neq \vec{t}$. So let $j \in l$ with $\vec{x} \in S_j$.

- If $x_i \neq t_i$ for each $i \in d$ and $j \leq M$, then $\theta(\vec{x}) = \big(\theta_i(x_i)\big)_{i\in d} = \psi^j(\vec{x}) \in \Psi(\vec{x}) = \Theta(\vec{x})$.

- Similarly, if $x_i \neq t_i$ for each $i \in d$ and $j > M$, then $\theta(\vec{x}) = \big(\theta_i(x_i)\big)_{i\in d} = \psi^0(\vec{x}) \in \Psi(\vec{x}) = \Theta(\vec{x})$.

- If $x_i = t_i$ for some $i \in d$, then $i = j \leq M$. This implies that $\theta_j(x_j) = \alpha_j = \psi^j_j(t_j) = \psi^j_j(x_j)$ and

$$\theta(\vec{x}) = \big(\theta_i(x_i)\big)_{i\in d} = \psi^j(\vec{x}) \in \Psi(\vec{x}) = \Theta(\vec{x}).$$

(b) Write $\mathcal{T} := \left\{\overrightarrow{x^1}, \ldots, \overrightarrow{x^n}\right\}$, and set $\Psi_0 := \overline{\Theta}$. We define $\Psi_{j+1} : \mathcal{X}^d \to \Sigma^1_1(\mathcal{N}^d)$ as follows. We put $\Psi_{j+1}(\vec{x}) := \Psi_j(\vec{x})$ if $\vec{x} \neq \overrightarrow{x^{j+1}}$, and $\Psi_{j+1}\left(\overrightarrow{x^{j+1}}\right) := \Theta\left(\overrightarrow{x^{j+1}}\right)$, for $j < n$. The result now follows from an iterative application of (a). $\square$

4.2 The topologies

In this subsection we give two other results that will be used to prove Theorem 4.1. We use some tools of effective descriptive set theory (the reader should see [M] for the basic notions). We first recall a classical result in the spirit of Theorem 3.3.1 in [H-K-Lo].

Notation. Let X be a recursively presented Polish space. Using the bijection between ω and ω^2 defined before Definition 2.1, we can build a bijection $(x_n)\mapsto < x_n >$ between $(X^\omega)^\omega$ and X^ω by the formula $< x_n > (l) := x_{(l)_0}\big((l)_1\big)$. The inverse map $x\mapsto \big((x)_n\big)$ is given by $(x)_n(p) := x(< n,p >)$. These bijections are recursive.

Lemma 4.2.1 *Let X be a recursively presented Polish space. Then there are Π^1_1 sets $W^X\subseteq\mathcal{N}$, $C^X\subseteq\mathcal{N}\times X$ with $\{(\alpha,x)\in\mathcal{N}\times X \mid \alpha\in W^X$ and $x\notin C^X_\alpha\}\in\Pi^1_1$, $\Delta^1_1(X)=\{C^X_\alpha \mid \alpha\in\Delta^1_1\cap W^X\}$, and $\mathbf{\Delta}^1_1(X)=\{C^X_\alpha \mid \alpha\in W^X\}$.*

Proof. By 3E.2, 3F.6 and 3H.1 in [M], there is $\mathcal{U}^X\in\Pi^1_1(\mathcal{N}\times X)$ which is universal for $\mathbf{\Pi}^1_1(X)$ and satisfies the two following properties.

- A subset P of X is Π^1_1 if and only if there is $\alpha\in\mathcal{N}$ recursive with $P=\mathcal{U}^X_\alpha$.
- There is $S:\mathcal{N}\times\mathcal{N}\to\mathcal{N}$ recursive such that $(\alpha,\beta,x)\in\mathcal{U}^{\mathcal{N}\times X}\Leftrightarrow\big(S(\alpha,\beta),x\big)\in\mathcal{U}^X$.

We set, for $\varepsilon\in 2$, $U_\varepsilon := \{(\alpha,x)\in\mathcal{N}\times X \mid \big((\alpha)_\varepsilon,x\big)\in\mathcal{U}^X\}$. Then $U_\varepsilon\in\Pi^1_1$. By 4B.10 in [M], Π^1_1 has the reduction property, which gives $V_0,V_1\in\Pi^1_1$ disjoint with $V_\varepsilon\subseteq U_\varepsilon$ and $V_0\cup V_1=U_0\cup U_1$. We set $W^X := \{\alpha\in\mathcal{N} \mid (V_0)_\alpha\cup(V_1)_\alpha=X\}$ and $C^X := V_0$, which defines Π^1_1 sets. Moreover,

$$\alpha\in W^X\wedge x\notin C^X_\alpha\Leftrightarrow\alpha\in W^X\wedge(\alpha,x)\in V_1$$

is Π^1_1 in (α,x). Assume that $A\in\Delta^1_1(X)$, which gives $\alpha_0,\alpha_1\in\mathcal{N}$ recursive with $A=\mathcal{U}^X_{\alpha_0}$ (resp., $\neg A=\mathcal{U}^X_{\alpha_1}$). We define $\alpha\in\mathcal{N}$ by $(\alpha)_\varepsilon := \alpha_\varepsilon$, so that α is recursive. We get

$$\begin{aligned} x\in A &\Leftrightarrow (\alpha_0,x)\in\mathcal{U}^X\Leftrightarrow(\alpha,x)\in U_0\Leftrightarrow(\alpha,x)\in U_0\setminus U_1\Leftrightarrow(\alpha,x)\in V_0,\\ x\notin A &\Leftrightarrow (\alpha_1,x)\in\mathcal{U}^X\Leftrightarrow(\alpha,x)\in U_1\Leftrightarrow(\alpha,x)\in U_1\setminus U_0\Leftrightarrow(\alpha,x)\in V_1,\end{aligned}$$

so that $\alpha\in W^X$ and $C^X_\alpha=A$. This also proves that $\mathbf{\Delta}^1_1(X)\subseteq\{C^X_\alpha \mid \alpha\in W^X\}$.

Conversely, let $\alpha\in\Delta^1_1\cap W^X$. Then $C^X_\alpha\in\Pi^1_1$, and $x\notin C^X_\alpha\Leftrightarrow\alpha\in W^X\wedge x\notin C^X_\alpha$, so that $\neg C^X_\alpha\in\Pi^1_1$ and $C^X_\alpha\in\Delta^1_1$. Note that this also proves that $\mathbf{\Delta}^1_1(X)\supseteq\{C^X_\alpha \mid \alpha\in W^X\}$. □

We now give some notation in order to state an effective version of Theorem 4.1.

Notation. Let X be a recursively presented Polish space.

• We will use the Gandy-Harrington topology Σ_X on X generated by $\Sigma^1_1(X)$. Recall that the set $\Omega_X := \{x\in X \mid \omega^x_1=\omega^{\text{CK}}_1\}$ is Borel and Σ^1_1, that (Ω_X,Σ_X) is a zero-dimensional Polish space (the intersection of Ω_X with any nonempty Σ^1_1 set is a nonempty clopen subset of (Ω_X,Σ_X)) (see [L8]).

• Recall the topology τ_1 defined before Theorem 1.11. We will also consider some topologies between τ_1 and $\Sigma_{\mathcal{N}^d}$. Let $2\leq\xi<\omega^{\text{CK}}_1$. The topology τ_ξ is generated by $\Sigma^1_1(\mathcal{N}^d)\cap\mathbf{\Pi}^0_{<\xi}(\tau_1)$. We have $\mathbf{\Sigma}^0_1(\tau_\xi)\subseteq\mathbf{\Sigma}^0_\xi(\tau_1)$, so that $\mathbf{\Pi}^0_1(\tau_\xi)\subseteq\mathbf{\Pi}^0_\xi(\tau_1)$. These topologies are similar to the ones considered in [Lo2] (see Definition 1.5). If $A\subseteq\mathcal{N}^d$ and $1\leq\xi<\omega^{\text{CK}}_1$, then we will write $\overline{A}^\xi$ instead of $\overline{A}^{\tau_\xi}$.

• We set $\text{pot}(\mathbf{\Pi}^0_0):=\{\Pi_{i\in d}\ A_i\ \mid\ A_i\in\mathbf{\Delta}^1_1(\mathcal{N})$, and $A_i=\mathcal{N}$ for almost every $i\in d\}$. We also set $W:=W^{\mathcal{N}^d}$ and $C:=C^{\mathcal{N}^d}$ (see Lemma 4.2.1). We will define precisely, for $\xi<\omega_1$,

$$\big\{(\beta,\gamma)\in\mathcal{N}\times W\mid \beta \text{ codes a pot}(\mathbf{\Pi}^0_\xi)\text{ set and } C_\gamma \text{ is the set coded by } \beta\big\}.$$

The way we will do it is not the simplest possible (we can in fact forget β, and work with $\gamma\in\omega$ instead of $\gamma\in\mathcal{N}$, see [L7]). We do it this way to start to give the flavor of what is going on with the Wadge classes.

• In order to do this, we set

$$V_0:=\left\{\ (m\beta,\gamma)\in\mathcal{N}\times W\mid \forall i<m\ \ (\beta)_i\in W^{\mathcal{N}}\wedge\gamma\in\Delta^1_1(m\beta)\wedge \left[\begin{array}{l} m=d\wedge C_\gamma=\Pi_{i<m}\ C^{\mathcal{N}}_{(\beta)_i}\text{ if } d<\omega\\ \\ C_\gamma=\left(\Pi_{i<m}\ C^{\mathcal{N}}_{(\beta)_i}\right)\times\mathcal{N}^\omega\text{ if } d=\omega\end{array}\right.\right\}.$$

We define an inductive operator $\mathfrak{F}$ over $\mathcal{N}\times\mathcal{N}$ (see [C]) as follows:

$$\mathfrak{F}(A):=A\cup V_0\ \cup\big\{(\beta,\gamma)\in\mathcal{N}\times W\mid\gamma\in\Delta^1_1(\beta)\wedge \exists\delta\in\Delta^1_1(\beta)\ \ \forall n\in\omega\ \ \big((\beta)_n,(\delta)_n\big)\in A\wedge\neg C_\gamma=\textstyle\bigcup_{n\in\omega}\ C_{(\delta)_n}\big\}.$$

Then $\mathfrak{F}$ is clearly a Π^1_1 monotone inductive operator. We set, for any ordinal ξ, $V_\xi:=\mathfrak{F}^\xi$ (which is coherent with the definition of V_0). We also set $V_{<\xi}:=\bigcup_{\eta<\xi}\ V_\eta$. The effective version of Theorem 4.1, which is the precise version of Theorem 1.11 for the Borel classes, is as follows:

Theorem 4.2.2 *Let T_d be a tree with Δ^1_1 suitable levels, $1\leq\xi<\omega_1^{CK}$, and A_0, A_1 be disjoint Σ^1_1 subsets of $\mathcal{N}^d$.*

(1) Assume that $S\in\mathbf{\Sigma}^0_\xi(\lceil T_d\rceil)$ is not separable from $\lceil T_d\rceil\setminus S$ by a pot$(\mathbf{\Pi}^0_\xi)$ set. Then the following are equivalent:

(a) The set A_0 is not separable from A_1 by a pot$(\mathbf{\Pi}^0_\xi)$ set.

(b) The set A_0 is not separable from A_1 by a $\Delta^1_1\cap$ pot$(\mathbf{\Pi}^0_\xi)$ set.

(c) $\neg\big(\exists(\beta,\gamma)\in(\Delta^1_1\times\Delta^1_1)\cap V_\xi\ \ A_0\subseteq C_\gamma\subseteq\neg A_1\big)$.

(d) The set A_0 is not separable from A_1 by a $\mathbf{\Pi}^0_\xi(\tau_1)$ set.

(e) $\overline{A_0}^\xi\cap A_1\neq\emptyset$.

(f) The inequality $\big((d^\omega)_{i\in d},S,\lceil T_d\rceil\setminus S\big)\leq\big((\mathcal{N})_{i\in d},A_0,A_1\big)$ holds.

(2) The sets V_ξ and $V_{<\xi}$ are Π^1_1.

(3) Assume that $S_0, S_1 \in \mathbf{\Sigma}^0_\xi(\lceil T_d \rceil)$ are disjoint and not separable by a pot($\mathbf{\Delta}^0_\xi$) set. Then the following are equivalent:

(a) The set A_0 is not separable from A_1 by a pot($\mathbf{\Delta}^0_\xi$) set.

(b) The set A_0 is not separable from A_1 by a $\Delta^1_1 \cap$ pot($\mathbf{\Delta}^0_\xi$) set.

(c) $\neg\big(\exists(\beta,\gamma),(\beta',\gamma') \in (\Delta^1_1 \times \Delta^1_1) \cap V_\xi \;\; C_{\gamma'} = \neg C_\gamma$ and $A_0 \subseteq C_\gamma \subseteq \neg A_1\big)$.

(d) The set A_0 is not separable from A_1 by a $\mathbf{\Delta}^0_\xi(\tau_1)$ set.

(e) $\overline{A_0}^\xi \cap \overline{A_1}^\xi \neq \emptyset$.

(f) The inequality $\big((d^\omega)_{i\in d}, S_0, S_1\big) \leq \big((\mathcal{N})_{i\in d}, A_0, A_1\big)$ holds.

The proofs of Theorems 4.1 and 4.2.2 will be by induction on ξ. This appears in the statement of the following lemma.

Lemma 4.2.3 *(1) The set V_0 is Π^1_1.*

(2) Let $1 \leq \xi < \omega_1^{CK}$. We assume that Theorem 4.2.2 is proved for $\eta < \xi$.

(a) The set $V_{<\xi}$ is Π^1_1.

(b) Fix $A \in \Sigma^1_1(\mathcal{N}^d)$. Then $\overline{A}^\xi \in \Sigma^1_1(\mathcal{N}^d)$.

(c) Let $n \geq 1$, $1 \leq \xi_1 < \xi_2 < \ldots < \xi_n \leq \xi$, and $S_1, \ldots, S_n$ be Σ^1_1 sets. Assume that $S_i \subseteq \overline{S_{i+1}}^{\xi_i+1}$ for $1 \leq i < n$. Then $S_n \cap \bigcap_{1\leq i<n} \overline{S_i}^{\xi_i}$ is τ_1-dense in $\overline{S_1}^1$.

Proof. (1) The set V_0 is clearly Π^1_1.

(2).(a) The proof is contained in the proof of Theorem 4.1 in [L7]. It is a consequence of Lemma 4.8 in [C].

(b) The proof is essentially the proof of Lemma 2.2.2.(a) in [L7].

(c) The proof is essentially the proof of Lemma 2.2.2.(b) in [L7]. □

Lemma 4.2.4 *Let $S, T \in \Sigma^1_1(\mathcal{N}^d)$ be such that S is τ_1-dense in T, $(X_i)_{i\in d}$ be a sequence of Σ^1_1 subsets of $\mathcal{N}$ such that $X_i = \mathcal{N}$ if $i \geq i_0$. Then $S \cap (\Pi_{i\in d}\, X_i)$ is τ_1-dense in $T \cap (\Pi_{i\in d}\, X_i)$.*

Proof. Let $(\Delta_i)_{i\in d}$ be a sequence of Δ^1_1 subsets of $\mathcal{N}$ such that $\Delta_i = \mathcal{N}$ if $i \geq j_0 \geq i_0$, and also $T \cap (\Pi_{i\in d}\, I_i) \neq \emptyset$, where $I_i := X_i \cap \Delta_i$. We have to see that $S \cap (\Pi_{i\in d}\, I_i) \neq \emptyset$. We argue by contradiction. This gives a sequence $(D_i)_{i\in d}$ of Δ^1_1 subsets of $\mathcal{N}$ such that $I_i \subseteq D_i$ if $i \in d$, and $S \cap (\Pi_{i\in d}\, D_i) = \emptyset$, by j_0 applications of the separation theorem. But $T \cap (\Pi_{i\in d}\, D_i) \neq \emptyset$, and $D_i = \mathcal{N}$ if $i \geq j_0$. So $S \cap (\Pi_{i\in d}\, D_i) \neq \emptyset$, by τ_1-density of S in T, which is absurd. □

4.3 Representation of Borel sets

Now we come to the representation theorem for Borel sets obtained by G. Debs and J. Saint Raymond (see [D-SR]). It is a refinement of the classical Lusin theorem asserting that any Borel set in a Polish space is the bijective continuous image of a closed subset of the Baire space. The material in this subsection can be found in Subsection 2.3 of [L7], but we recall most of it since it will be used iteratively in the case of the Wadge classes. The following definition can be found in [D-SR].

Definition 4.3.1 *(Debs-Saint Raymond) Let c be a countable set. A partial order relation R on $c^{<\omega}$ is a $tree\ relation$ if, for any $t\in c^{<\omega}$,*

(a) $\emptyset\ R\ t$,

(b) the set $P_R(t):=\{s\in c^{<\omega}\mid s\ R\ t\}$ is finite and linearly ordered by R.

For instance, the non strict extension relation $\subseteq$ is a tree relation.

• Let R be a tree relation. A R-$branch$ is a $\subseteq$-maximal subset of $c^{<\omega}$ linearly ordered by R. We denote by $[R]$ the set of all infinite R-branches.

We equip $(c^{<\omega})^\omega$ with the product of the discrete topology on $c^{<\omega}$. If R is a tree relation, then the space $[R]\subseteq(c^{<\omega})^\omega$ is equipped with the topology induced by that of $(c^{<\omega})^\omega$. The map $h:c^\omega\rightarrow[\subseteq]$ defined by $h(\gamma):=(\gamma|j)_{j\in\omega}$ is a homeomorphism.

• Let R, S be tree relations with $R\subseteq S$. The $canonical\ map$ $\Pi:[R]\rightarrow[S]$ is defined by

$$\Pi(\mathcal{B}):=\textit{the unique } S\textit{-branch containing } \mathcal{B}.$$

• Let S be a tree relation. We say that $R\subseteq S$ is $distinguished$ in S if

$$\forall s,t,u\in c^{<\omega}\qquad \left.\begin{array}{c} s\ S\ t\ S\ u\\ s\ R\ u\end{array}\right\}\Rightarrow s\ R\ t.$$

For example, let C be a closed subset of c^ω, and define

$$s\ R\ t\ \Leftrightarrow\ s\subseteq t\wedge N_s\cap C\neq\emptyset.$$

Then R is distinguished in $\subseteq$.

• Let $\eta<\omega_1$. A family $(R^\rho)_{\rho\leq\eta}$ of tree relations is a $resolution\ family$ if

(a) $R^{\rho+1}$ is a distinguished subtree of R^ρ, for all $\rho<\eta$.

(b) $R^\lambda=\bigcap_{\rho<\lambda}\ R^\rho$, for all limit $\lambda\leq\eta$.

We will use the following extension of the property of distinction:

Lemma 4.3.2 *Let $\eta<\omega_1$, $(R^\rho)_{\rho\leq\eta}$ be a resolution family, and $\rho<\eta$. Assume that $s\ R^0\ t\ R^\rho\ u$ and $s\ R^{\rho+1}\ u$. Then $s\ R^{\rho+1}\ t$.*

Notation. Let $\eta<\omega_1$, $(R^\rho)_{\rho\leq\eta}$ be a resolution family such that R^0 is a subrelation of $\subseteq$, $\rho\leq\eta$ and $v\in c^{<\omega}\setminus\{\emptyset\}$. We set $v^\rho:=v\mid\max\{r<|v|\mid v|r\ R^\rho\ v\}$. We enumerate $\{v^\rho\mid\rho\leq\eta\}$ by $\{v^{\xi_i}\mid 1\leq i\leq n\}$, where $1\leq n\in\omega$ and $\xi_1<\ldots<\xi_n=\eta$. We can write $v^{\xi_n}\subsetneqq v^{\xi_{n-1}}\subsetneqq\ldots\subsetneqq v^{\xi_2}\subsetneqq v^{\xi_1}\subsetneqq v$. By Lemma 4.3.2 we have $v^{\xi_{i+1}}\ R^{\xi_i+1}\ v^{\xi_i}$ for any $1\leq i<n$.

Lemma 4.3.3 *Let $\eta<\omega_1$, $(R^\rho)_{\rho\leq\eta}$ be a resolution family such that R^0 is a subrelation of $\subseteq$, v be in $c^{<\omega}\setminus\{\emptyset\}$ and $1\leq i<n$.*

(a) We set $\eta_i:=\{\rho\leq\eta\mid v^{\xi_i}\subseteq v^\rho\}$. Then η_i is a successor ordinal.

(b) We may assume that $v^{\xi_i+1}\subsetneqq v^{\xi_i}$.

The following result is a part of Theorem I-6.6 in [D-SR].

Theorem 4.3.4 *(Debs-Saint Raymond) Let $\eta<\omega_1$, R be a tree relation, and $(I_n)_{n\in\omega}$ be a sequence of $\mathbf{\Pi}^0_{\eta+1}$ subsets of $[R]$. Then there is a resolution family $(R^\rho)_{\rho\leq\eta}$ with*

(a) $R^0=R$,

(b) the canonical map $\Pi\colon[R^\eta]\rightarrow[R]$ is a continuous bijection,

(c) the set $\Pi^{-1}(I_n)$ is a closed subset of $[R^\eta]$ for each natural number n.

Now we come to the actual proof of Theorem 4.1.

4.4 Proof of Theorem 4.1

The next result is essentially Theorem 2.4.1 in [L7]. But we give its proof since it is the basis for further generalizations.

Theorem 4.4.1 *Let T_d be a tree with Δ^1_1 suitable levels, $\xi<\omega_1^{CK}$ be a successor ordinal, S be in $\mathbf{\Sigma}^0_\xi(\lceil T_d\rceil)$, and A_0, A_1 be disjoint Σ^1_1 subsets of $\mathcal{N}^d$. We assume that Theorem 4.2.2 is proved for $\eta<\xi$. Then one of the following holds:*

(a) $\overline{A_0}^\xi\cap A_1=\emptyset$.

(b) The inequality $\big((\Pi''_i\lceil T_d\rceil)_{i\in d},S,\lceil T_d\rceil\setminus S\big)\leq\big((\mathcal{N})_{i\in d},A_0,A_1\big)$ holds.

Proof. Fix $\eta<\omega_1^{\text{CK}}$ with $\xi=\eta+1$.

• Recall the finite sets c_l defined at the end of the proof of Proposition 2.2 (we only used the fact that T_d has finite levels to see that they are finite). Using the notation of Definition 4.3.1, we put $c:=\bigcup_{l\in\omega}\ c_l$, so that c is countable. The set $I:=h[\lceil T_d\rceil\setminus S]$ is a $\mathbf{\Pi}^0_{\eta+1}$ subset of $[\subseteq]$. Theorem 4.3.4 provides a resolution family. We put $D:=\big\{\vec{s}\in T_d\mid\vec{s}=\vec{\emptyset}\vee\exists\mathcal{B}\in\Pi^{-1}(I)\ \ \vec{s}\in\mathcal{B}\big\}$.

• Assume that $\overline{A_0}^\xi\cap A_1$ is not empty. Recall that (Ω_X,Σ_X) is a Polish space (see the notation at the beginning of Section 4.2). We fix a complete metric d_X on (Ω_X,Σ_X).

• We construct

- $(\alpha^i_s)_{i\in d,s\in\Pi''_iT_d}\subseteq\mathcal{N}$,
- $(O^i_s)_{i\leq|s|,i\in d,s\in\Pi''_iT_d}\subseteq\Sigma^1_1(\mathcal{N})$,
- $(U_{\vec{s}})_{\vec{s}\in T_d}\subseteq\Sigma^1_1(\mathcal{N}^d)$.

We want these objects to satisfy the following conditions.

$$(1)\ \alpha^i_s\in O^i_s\subseteq\Omega_\mathcal{N}\wedge(\alpha^i_{s_i})_{i\in d}\in U_{\vec{s}}\subseteq\Omega_{\mathcal{N}^d}$$

$$(2)\ O^i_{sq}\subseteq O^i_s$$

$$(3)\ \text{diam}_{d_\mathcal{N}}(O^i_s)\leq 2^{-|s|}\wedge\text{diam}_{d_{\mathcal{N}^d}}(U_{\vec{s}})\leq 2^{-|\vec{s}|}$$

$$(4)\ U_{\vec{s}}\subseteq\overline{A_0}^\xi\cap A_1 \text{ if } \vec{s}\in D$$

$$(5)\ U_{\vec{s}}\subseteq A_0 \text{ if } \vec{s}\notin D$$

$$(6)\ \big(1\leq\rho\leq\eta\wedge\vec{s}\ R^\rho\ \vec{t}\big)\Rightarrow U_{\vec{t}}\subseteq\overline{U_{\vec{s}}}^\rho$$

$$(7)\ \big((\vec{s},\vec{t}\in D\vee\vec{s},\vec{t}\notin D)\wedge\vec{s}\ R^\eta\ \vec{t}\big)\Rightarrow U_{\vec{t}}\subseteq U_{\vec{s}}$$

• Let us prove that this construction is sufficient to get the theorem.

- Fix $\vec{\beta}\in\lceil T_d\rceil$. Then we can define $(j_k)_{k\in\omega}:=(j^{\vec{\beta}}_k)_{k\in\omega}$ by $\Pi^{-1}\big((\vec{\beta}|j)_{j\in\omega}\big)=(\vec{\beta}|j_k)_{k\in\omega}$, with the inequalities $j_k<j_{k+1}$. In particular, $\vec{\beta}|j_k\ R^\eta\ \vec{\beta}|j_{k+1}$. Note that

$$\vec{\beta}\notin S\ \Leftrightarrow\ h(\vec{\beta})=(\vec{\beta}|j)_{j\in\omega}\in I\ \Leftrightarrow\ (\vec{\beta}|j_k)_{k\in\omega}\in\Pi^{-1}(I)\ \Leftrightarrow\ \forall k\geq k_0:=0\ \ \vec{\beta}|j_k\in D$$

since $\Pi^{-1}(I)$ is a closed subset of $[R^\eta]$. Similarly, $\vec{\beta}\in S$ is equivalent to the existence of $k_0\in\omega$ such that $\vec{\beta}|j_k\notin D$ for any $k\geq k_0$.

This implies that $(U_{\vec{\beta}|j_k})_{k\geq k_0}$ is a decreasing sequence of nonempty clopen subsets of the space $(\Omega_{\mathcal{N}^d},\Sigma_{\mathcal{N}^d})$ whose $d_{\mathcal{N}^d}$-diameters tend to zero, and we can define $\{F(\vec{\beta})\}:=\bigcap_{k\geq k_0}U_{\vec{\beta}|j_k}\subseteq\Omega_{\mathcal{N}^d}$. Note that $F(\vec{\beta})$ is the limit of $\big((\alpha^i_{\beta_i|j_k})_{i\in d}\big)_{k\in\omega}$.

- Now let $\gamma\in\Pi''_i\lceil T_d\rceil$, and $\vec{\beta}\in\lceil T_d\rceil$ such that $\beta_i=\gamma$. We set $f_i(\gamma):=F_i(\vec{\beta})$. This defines a map $f_i:\Pi''_i\lceil T_d\rceil\rightarrow\mathcal{N}$.

Note that $f_i(\gamma)$ is the limit of $(\alpha^i_{\gamma|j})_{j\in\omega}$. Indeed, $f_i(\gamma)$ is the limit of $(\alpha^i_{\gamma|j_k})_{k\in\omega}$. If $j\geq i$, then $\alpha^i_{\gamma|j}\in O^i_{\gamma|j}$, and the sequence $(O^i_{\gamma|j})_{j\geq i}$ is decreasing. Fix $\varepsilon>0$, $k\geq i$ such that $2^{-k}<\varepsilon$. Then we get, if $j\geq k$, $d_\mathcal{N}\big(f_i(\gamma),\alpha^i_{\gamma|j}\big)\leq\text{diam}_{d_\mathcal{N}}(O^i_{\gamma|j})\leq 2^{-j}\leq 2^{-k}<\varepsilon$. In particular, $f_i(\gamma)$ does not depend on the choice of $\vec{\beta}$. This also proves that f_i is continuous on $\Pi''_i\lceil T_d\rceil$.

- Note that $F_i(\vec{\beta})$ is the limit of some subsequence of $(\alpha^i_{\beta_i|j})_{j\in\omega}$, by continuity of the projections. Thus $F_i(\vec{\beta})=f_i(\beta_i)$, and $F(\vec{\beta})=(\Pi_{i\in d}\ f_i)(\vec{\beta})$. This implies that the inclusions $S\subseteq(\Pi_{i\in d}\ f_i)^{-1}(A_0)$ and $\lceil T_d\rceil\setminus S\subseteq(\Pi_{i\in d}\ f_i)^{-1}(A_1)$ hold.

• So let us prove that the construction is possible.

- Let $(\alpha^i_\emptyset)_{i\in d}\in\overline{A_0}^\xi\cap A_1\cap\Omega_{\mathcal{N}^d}$, which is nonempty since $\overline{A_0}^\xi\cap A_1\neq\emptyset$ is Σ^1_1, by Lemma 4.2.3.(2).(b). Then we choose a Σ^1_1 subset $U_{\vec{\emptyset}}$ of $\mathcal{N}^d$, with $d_{\mathcal{N}^d}$-diameter at most 1, such that

$$(\alpha^i_\emptyset)_{i\in d}\in U_{\vec{\emptyset}}\subseteq\overline{A_0}^\xi\cap A_1\cap\Omega_{\mathcal{N}^d}.$$

We choose a Σ^1_1 subset $O^0_\emptyset$ of $\mathcal{N}$, with $d_{\mathcal{N}}$-diameter at most 1, with $\alpha^0_\emptyset\in O^0_\emptyset\subseteq\Omega_{\mathcal{N}}$, which is possible since $\Omega_{\mathcal{N}^d}\subseteq\Omega^d_{\mathcal{N}}$. Assume that $(\alpha^i_s)_{|s|\leq l}$, $(O^i_s)_{|s|\leq l}$ and $(U_{\vec{s}})_{|\vec{s}|\leq l}$ satisfying conditions (1)-(7) have been constructed, which is the case for $l=0$.

- Let $v:=\overrightarrow{tm}\in T_d\cap(d^{l+1})^d$. Note that $v^\eta\in D$ if $v^\eta\in D$ is not equivalent to $v\in D$ (see the notation before Lemma 4.3.3).

- The conclusions in the assertions (a) and (b) of the following claim do not really depend on their respective assumptions, but we will use these assertions later in this form. We define $X_i:=O^i_{t_i}$ if $i\leq l$, and $\mathcal{N}$ if $i>l$.

Claim. Assume that $\eta>0$.

(a) The set $A_0\cap\bigcap_{1\leq\rho\leq\eta}\overline{U_{v^\rho}}^\rho\cap(\Pi_{i\in d}\ X_i)$ is τ_1-dense in $\overline{U_{v^1}}^1\cap(\Pi_{i\in d}\ X_i)$ if $v^\eta\in D$ and $v\notin D$.

(b) The set $U_{v^\eta}\cap\bigcap_{1\leq\rho<\eta}\overline{U_{v^\rho}}^\rho\cap(\Pi_{i\in d}\ X_i)$ is τ_1-dense in $\overline{U_{v^1}}^1\cap(\Pi_{i\in d}\ X_i)$ if $v^\eta,v\in D$ or $v^\eta,v\notin D$.

Indeed, let us forget $\Pi_{i\in d}\ X_i$ for the moment. We may assume that $v^{\xi_{i+1}}\subsetneqq v^{\xi_i}$ if $1\leq i<n$, by Lemma 4.3.3. We set $S_i:=U_{v^{\xi_i}}$, when $1\leq\xi_i\leq\eta$. As $v^{\xi_{i+1}}\ R^{\xi_i+1}\ v^{\xi_i}$, we can write $S_i\subseteq\overline{S_{i+1}}^{\xi_i+1}$, for $1\leq\xi_i<\eta$, by induction assumption. If $v^\eta\in D$ and $v\notin D$, then $S_n\subseteq\overline{A_0}^{\eta+1}$. Thus $A_0\cap\bigcap_{1\leq\xi_i\leq\eta}\overline{U_{v^{\xi_i}}}^{\xi_i}$ and $U_{v^\eta}\cap\bigcap_{1\leq\xi_i<\eta}\overline{U_{v^{\xi_i}}}^{\xi_i}$ are τ_1-dense in $\overline{U_{v^1}}^1$, by Lemma 4.2.3.(2).(c).

But if $1\leq\rho\leq\eta$, then there is $1\leq i\leq n$ with $v^\rho=v^{\xi_i}$. And $\rho\leq\xi_i$ since $v^{\xi_i+1}\subsetneqq v^{\xi_i}$ if $1\leq i<n$. We are done since $\bigcap_{1\leq\rho\leq\eta}\overline{U_{v^\rho}}^\rho=\bigcap_{1\leq\xi_i\leq\eta}\overline{U_{v^{\xi_i}}}^{\xi_i}$ and $U_{v^\eta}\cap\bigcap_{1\leq\rho<\eta}\overline{U_{v^\rho}}^\rho=U_{v^\eta}\cap\bigcap_{1\leq\xi_i<\eta}\overline{U_{v^{\xi_i}}}^{\xi_i}$. The claim now comes from Lemma 4.2.4. $\diamond$

- Let $\mathcal{X}:=d^{l+1}$. The map $\Theta:\mathcal{X}^d\to\Sigma^1_1(\mathcal{N}^d)$ is defined on $\mathcal{T}^{l+1}$ by

$$\Theta(v):=\begin{cases} A_0\cap\bigcap_{1\leq\rho\leq\eta}\overline{U_{v^\rho}}^\rho\cap(\Pi_{i\in d}\ X_i)\cap\Omega_{\mathcal{N}^d} \text{ if } v^\eta\in D\wedge v\notin D,\\ \\ U_{v^\eta}\cap\bigcap_{1\leq\rho<\eta}\overline{U_{v^\rho}}^\rho\cap(\Pi_{i\in d}\ X_i) \text{ if } v^\eta,v\in D\vee v^\eta,v\notin D.\end{cases}$$

By the claim, $\Theta(v)$ is τ_1-dense in $\overline{U_{v^1}}^1\cap(\Pi_{i\in d}\ X_i)$ if $\eta>0$. As $v^1\subseteq\vec{t}\subseteq v$ and R^1 is distinguished in $\subseteq$ we get $v^1\ R^1\ \vec{t}$ and $U_{\vec{t}}\subseteq\overline{U_{v^1}}^1$, by induction assumption. Therefore

$$\overline{U_{\vec{t}}}^1\cap(\Pi_{i\in d}\ X_i)\subseteq\overline{U_{v^1}}^1\cap(\Pi_{i\in d}\ X_i)\subseteq\overline{\Theta}(v),$$

and $(\alpha^i_{t_i})_{i\in d}\in U_{\vec{t}}\ \cap(\Pi_{i\in d}\ X_i)\subseteq\overline{\Theta}(v)$ (even if $\eta=0$). Therefore $\overline{\Theta}$ admits a π-selector on $\mathcal{T}^{l+1}$. Indeed, we define, for any $i\in d$, $\overline{\theta}_i:\mathcal{X}\to\mathcal{N}$ by $\overline{\theta}_i(t_im_i):=\alpha^i_{t_i}$ if $t_i\in\Pi''_iT_d$, 0^∞ otherwise.

- As T_d is a tree with Δ^1_1 suitable levels, we can apply Lemma 4.1.3. Thus Θ admits a π-selector θ on $\mathcal{T}^{l+1}$. We set, for $s\in\Pi_i[\mathcal{T}^{l+1}]$, $\alpha^i_s:=\theta_i(s)$.

- We choose Σ^1_1 sets U_v with $d_{\mathcal{N}^d}$-diameter at most 2^{-l-1} such that $\theta(v)\in U_v\subseteq\Theta(v)$ if $v\in\mathcal{T}^{l+1}$.

- Finally, we choose the O^i_{sq}'s. We first prove that $\alpha^i_{sq}\in O^i_s$ if $sq\in\Pi_i[\mathcal{T}^{l+1}]$, $i\in d$ and $i\leq l$.

Let $v:=\overrightarrow{tm}\in\mathcal{T}^{l+1}$ such that $sq=t_im_i$. Then $\alpha^i_{sq}=\theta_i(sq)=\theta_i(t_im_i)$. As $\theta(v)\in\Theta(v)$ and $i\leq l$, $\alpha^i_{sq}\in O^i_{t_i}=O^i_s$.

Now we can define the O^i_{sq}'s. If $sq\in\Pi_i[\mathcal{T}^{l+1}]$, then we choose a Σ^1_1 set O^i_{sq}, with $d_{\mathcal{N}}$-diameter at most 2^{-l-1}, such that

$$\alpha^i_{sq}\in O^i_{sq}\subseteq\begin{cases} O^i_s & \text{if } i\leq l,\\ \Omega_{\mathcal{N}} & \text{otherwise.}\end{cases}$$

- This finishes the proof since $\vec{u}\ R^\rho\ v$ and $\vec{u}\neq v\ \Rightarrow\ \vec{u}\ R^\rho\ v^\rho\ R^\rho\ v$, by Lemma 4.3.2. □

Now we come to the ambiguous classes.

Theorem 4.4.2 *Let T_d be a tree with Δ^1_1 suitable levels, $\xi<\omega_1^{CK}$ be a successor ordinal, S_0, S_1 be in $\mathbf{\Sigma}^0_\xi(\lceil T_d\rceil)$ disjoint, and A_0, A_1 be disjoint Σ^1_1 subsets of $\mathcal{N}^d$. We assume that Theorem 4.2.2 is proved for $\eta<\xi$. Then one of the following holds:*
(a) $\overline{A_0}^\xi\cap\overline{A_1}^\xi=\emptyset$.
(b) The inequality $\big((\Pi''_i\lceil T_d\rceil)_{i\in d}, S_0, S_1\big)\leq\big((\mathcal{N})_{i\in d}, A_0, A_1\big)$ holds.

Proof. Let us indicate the differences with the proof of Theorem 4.4.1. Assume that $\overline{A_0}^\xi\cap\overline{A_1}^\xi\neq\emptyset$. We set $I_\varepsilon:=h[\lceil T_d\rceil\setminus S_\varepsilon]$, so that I_ε is a $\mathbf{\Pi}^0_\xi$ subset of $[\subseteq]$. We also set, for $\varepsilon\in 2$,

$$D^1_\varepsilon:=\big\{\vec{s}\in T_d\mid\vec{s}=\vec{\emptyset}\vee\exists\mathcal{B}\in\Pi^{-1}(I_\varepsilon)\ \ \vec{s}\in\mathcal{B}\big\},$$

and $D^0_\varepsilon:=T_d\setminus D^1_\varepsilon$. We set, for $\theta_0,\theta_1\in 2$, $D_{\theta_0,\theta_1}:=D^{\theta_0}_0\cap D^{\theta_1}_1$. For example, $\vec{\emptyset}\in D_{1,1}$.

• Conditions (4), (5), and (7) become the following:

(4) $U_{\vec{s}}\subseteq\overline{A_0}^\xi\cap\overline{A_1}^\xi$ if $\vec{s}\in D_{1,1}$

(5) $U_{\vec{s}}\subseteq A_\varepsilon$ if $\vec{s}\in D_{\varepsilon,1-\varepsilon}$

(7) $(\vec{s},\vec{t}\in D_{\varepsilon,1-\varepsilon}\wedge\vec{s}\ R^\eta\ \vec{t}\,)\Rightarrow U_{\vec{t}}\subseteq U_{\vec{s}}$

• Fix $\vec{\alpha}\in\lceil T_d\rceil$. There are $(\theta_0,\theta_1)\in 2^2$ and $k_0\in\omega$ such that $\vec{\alpha}|j_k\in D_{\theta_0,\theta_1}$ if $k\geq k_0$. Thus $S_\varepsilon\subseteq(\Pi_{i\in d}\ f_i)^{-1}(A_\varepsilon)$.

• Let $(\alpha^i_{\emptyset})_{i\in d}\in\overline{A_0}^\xi\cap\overline{A_1}^\xi\cap\Omega_{\mathcal{N}^d}$, which is nonempty since $\overline{A_0}^\xi\cap\overline{A_1}^\xi\neq\emptyset$ is Σ^1_1. We choose $U_{\vec{\emptyset}}$ with $(\alpha^i_{\emptyset})_{i\in d}\in U_{\vec{\emptyset}}\subseteq\overline{A_0}^\xi\cap\overline{A_1}^\xi\cap\Omega_{\mathcal{N}^d}$.

• The statement of the claim is now as follows:

Claim. Assume that $\eta>0$.

(a) $A_\varepsilon\cap\bigcap_{1\leq\rho\leq\eta}\ \overline{U_{v^\rho}}^\rho\cap(\Pi_{i\in d}\ X_i)$ is τ_1-dense in $\overline{U_{v^1}}^1\cap(\Pi_{i\in d}\ X_i)$ if $v^\eta\notin D_{\varepsilon,1-\varepsilon}$ and $v\in D_{\varepsilon,1-\varepsilon}$.

(b) $U_{v^\eta}\cap\bigcap_{1\leq\rho<\eta}\ \overline{U_{v^\rho}}^\rho\cap(\Pi_{i\in d}\ X_i)$ is τ_1-dense in $\overline{U_{v^1}}^1\cap(\Pi_{i\in d}\ X_i)$ otherwise.

The point is that $v^\eta\in D_{1,1}$ if $v^\eta\notin D_{\varepsilon,1-\varepsilon}$ since $v^\eta\in D_{\theta_0,\theta_1}$ with $\varepsilon\leq\theta_0$ and $1-\varepsilon\leq\theta_1$.

• In the same fashion, $\Theta(v)$ is now defined as follows:

$$\Theta(v):=\begin{cases} A_\varepsilon\cap\bigcap_{1\leq\rho\leq\eta}\ \overline{U_{v^\rho}}^\rho\cap(\Pi_{i\in d}\ X_i)\cap\Omega_{\mathcal{N}^d} \text{ if } v^\eta\notin D_{\varepsilon,1-\varepsilon}\wedge v\in D_{\varepsilon,1-\varepsilon},\\ \\ U_{v^\eta}\cap\bigcap_{1\leq\rho<\eta}\ \overline{U_{v^\rho}}^\rho\cap(\Pi_{i\in d}\ X_i) \text{ otherwise.}\end{cases}$$

We conclude as in the proof of Theorem 4.4.1. □

Now we come to the limit case. We need some more definitions that can be found in [D-SR].

Definition 4.4.3 *(Debs-Saint Raymond) Let R be a tree relation on $c^{<\omega}$. If $t\in c^{<\omega}$, then $h_R(t)$ is the number of strict R-predecessors of t. Thus $h_R(t)=\text{Card}\big(P_R(t)\big)-1$.*

Let $\xi<\omega_1$ be an infinite limit ordinal. We say that a resolution family $(R^\rho)_{\rho\leq\xi}$ is $uniform$ if

$$\forall k\in\omega\ \exists\eta_k<\xi\ \forall s,t\in c^{<\omega}\ \ \Big(\mathit{min}\big(h_{R^\xi}(s),h_{R^\xi}(t)\big)\leq k\ \wedge s\ R^{\eta_k}\ t\Big)\Rightarrow s\ R^\xi\ t.$$

We may (and will) assume that $\eta_k\geq 2$.

The following result is a part of Theorem I-6.6 in [D-SR].

Theorem 4.4.4 *(Debs-Saint Raymond) Let $\xi<\omega_1$ be an infinite limit ordinal, R be a tree relation, and $(I_n)_{n\in\omega}$ be a sequence of $\mathbf{\Pi}^0_\xi$ subsets of $[R]$. Then there is a uniform resolution family $(R^\rho)_{\rho\leq\xi}$ with*

(a) $R^0=R$,

(b) the canonical map $\Pi:[R^\xi]\rightarrow[R]$ is a continuous bijection,

(c) the set $\Pi^{-1}(I_n)$ is a closed subset of $[R^\xi]$ for each natural number n.

Here again, the next result is essentially in [L7] (see Theorem 2.4.4).

Theorem 4.4.5 *Let T_d be a tree with Δ^1_1 suitable levels, $\xi<\omega_1^{CK}$ be an infinite limit ordinal, S be in $\mathbf{\Sigma}^0_\xi(\lceil T_d\rceil)$, and A_0, A_1 be disjoint Σ^1_1 subsets of $\mathcal{N}^d$. We assume that Theorem 4.2.2 is proved for $\eta<\xi$. Then one of the following holds:*

(a) $\overline{A_0}^\xi\cap A_1=\emptyset$.

(b) The inequality $\big((\Pi''_i\lceil T_d\rceil)_{i\in d},S,\lceil T_d\rceil\setminus S\big)\leq\big((\mathcal{N})_{i\in d},A_0,A_1\big)$ holds.

Proof. Let us indicate the differences with the proof of Theorem 4.4.1.

- The set $I:=h[\lceil T_d\rceil\setminus S]$ is in $\mathbf{\Pi}^0_\xi([\subseteq])$. Theorem 4.4.4 provides a uniform resolution family.

- If $\vec{t}\in c^{<\omega}$ then we set $\eta(\vec{t}\,):=\max\{\eta_{h_{R^\xi}(\vec{s})+1}\mid\vec{s}\subseteq\vec{t}\,\}$. Note that $\eta(\vec{s}\,)\leq\eta(\vec{t}\,)$ if $\vec{s}\subseteq\vec{t}$.

- Conditions (6) and (7) become

(6) $(1\leq\rho\leq\eta(\vec{s}\,)\wedge\vec{s}\ R^\rho\ \vec{t}\,)\Rightarrow U_{\vec{t}}\subseteq\overline{U_{\vec{s}}}^\rho$

(7) $\big((\vec{s},\vec{t}\in D\vee\vec{s},\vec{t}\notin D)\wedge\vec{s}\ R^\xi\ \vec{t}\,\big)\Rightarrow U_{\vec{t}}\subseteq U_{\vec{s}}$

Claim 1. Assume that $v^\rho\neq v^\xi$. Then $\rho+1\leq\eta(v^{\rho+1})$.

We argue by contradiction. Note that $\rho+1>\rho\geq\eta(v^{\rho+1})\geq\eta_{h_{R^\xi}(v^\xi)+1}=\eta_{h_{R^\xi}(v)}$. As $v^\rho\ R^\rho\ v$, we get $v^\rho\ R^\xi\ v$, and also $v^\rho=v^\xi$, which is absurd. $\diamond$

Note that $\xi_{n-1}<\xi_{n-1}+1\leq\eta(v^{\xi_{n-1}+1})\leq\eta(v)$. This implies that $v^{\eta(v)}=v^\xi$.

Claim 2. (a) The set $A_0\cap\bigcap_{1\leq\rho\leq\eta(v)}\ \overline{U_{v^\rho}}^\rho\cap(\Pi_{i\in d}\ X_i)$ is τ_1-dense in $\overline{U_{v^1}}^1\cap(\Pi_{i\in d}\ X_i)$ if $v^\eta\in D$ and $v\notin D$.

(b) The set $U_{v^\xi}\cap\bigcap_{1\leq\rho<\eta(v)}\ \overline{U_{v^\rho}}^\rho\cap(\Pi_{i\in d}\ X_i)$ is τ_1-dense in $\overline{U_{v^1}}^1\cap(\Pi_{i\in d}\ X_i)$ if $v^\xi,v\in D$ or $v^\xi,v\notin D$.

Indeed, we set $S_i:=U_{v^{\xi_i}}$, for $1\leq\xi_i\leq\xi$. By Claim 1 we can apply Lemma 4.2.3.(2).(c) and we are done. $\diamond$

- The map $\Theta:\mathcal{X}^d\rightarrow\Sigma^1_1(\mathcal{N}^d)$ is defined on $\mathcal{T}^{l+1}$ by

$$\Theta(v):=\begin{cases}A_0\cap\bigcap_{1\leq\rho\leq\eta(v)}\ \overline{U_{v^\rho}}^\rho\cap(\Pi_{i\in d}\ X_i)\cap\Omega_{\mathcal{N}^d}\text{ if }v^\eta\in D\wedge v\notin D,\\ U_{v^\xi}\cap\bigcap_{1\leq\rho<\eta(v)}\overline{U_{v^\rho}}^\rho\cap(\Pi_{i\in d}\ X_i)\text{ if }v^\xi,v\in D\vee v^\xi,v\notin D.\end{cases}$$

We conclude as in the proof of Theorem 4.4.1, using the facts that $\eta_k\geq 1$ and $\eta(.)$ is increasing. $\square$

Now we come to the ambiguous classes.

Theorem 4.4.6 *Let T be a tree with Δ^1_1 suitable levels, $\xi<\omega_1^{CK}$ be an infinite limit ordinal, S_0,S_1 be in $\mathbf{\Sigma}^0_\xi(\lceil T_d\rceil)$ disjoint, and A_0, A_1 be disjoint Σ^1_1 subsets of $\mathcal{N}^d$. We assume that Theorem 4.2.2 is proved for $\eta<\xi$. Then one of the following holds:*

(a) $\overline{A_0}^\xi\cap\overline{A_1}^\xi=\emptyset$.

(b) The inequality $\big((\Pi''_i\lceil T_d\rceil)_{i\in d},S_0,S_1\big)\leq\big((\mathcal{N})_{i\in d},A_0,A_1\big)$ holds.

Proof. Let us indicate the differences with the proofs of Theorems 4.4.1, 4.4.2 and 4.4.5.

• The set $I_\varepsilon := h[\lceil T_d\rceil\setminus S_\varepsilon]$ is in $\mathbf{\Pi}^0_\xi([\subseteq])$.

• The statement of Claim 2 is now as follows.

Claim 2. (a) $A_\varepsilon\cap\bigcap_{1\leq\rho\leq\eta(v)}\ \overline{U_{v^\rho}}^\rho\cap(\Pi_{i\in d}\ X_i)$ is τ_1-dense in $\overline{U_{v^1}}^1\cap(\Pi_{i\in d}\ X_i)$ if $v^\xi\notin D_{\varepsilon,1-\varepsilon}$ and $v\in D_{\varepsilon,1-\varepsilon}$.

(b) $U_{v^\xi}\cap\bigcap_{1\leq\rho<\eta(v)}\ \overline{U_{v^\rho}}^\rho\cap(\Pi_{i\in d}\ X_i)$ is τ_1-dense in $\overline{U_{v^1}}^1\cap(\Pi_{i\in d}\ X_i)$ otherwise.

• In the same fashion, $\Theta(v)$ is now defined as follows:

$$\Theta(v):=\begin{cases} A_\varepsilon\cap\bigcap_{1\leq\rho\leq\eta(v)}\ \overline{U_{v^\rho}}^\rho\cap(\Pi_{i\in d}\ X_i)\cap\Omega_{\mathcal{N}^d} \text{ if } v^\xi\notin D_{\varepsilon,1-\varepsilon}\wedge v\in D_{\varepsilon,1-\varepsilon},\\ \\ U_{v^\xi}\cap\bigcap_{1\leq\rho<\eta(v)}\ \overline{U_{v^\rho}}^\rho\cap(\Pi_{i\in d}\ X_i) \text{ otherwise.}\end{cases}$$

We conclude as in the proof of Theorem 4.4.5. □

Lemma 4.4.7 *Let $\mathbf{\Gamma}$ be a Wadge class of Borel sets. Then the class of pot$(\mathbf{\Gamma})$ sets is closed under pre-images by products of continuous maps.*

Proof. Assume that $A\in\text{pot}(\mathbf{\Gamma})$, $A\subseteq\Pi_{i\in d}\ Y_i$, and $f_i:X_i\rightarrow Y_i$ is continuous. Let τ_i be a finer zero-dimensional Polish topology on Y_i such that $A\in\mathbf{\Gamma}\big(\Pi_{i\in d}\ (Y_i,\tau_i)\big)$. As $f_i:X_i\rightarrow(Y_i,\tau_i)$ is Borel, there is a finer zero-dimensional Polish topology σ_i on X_i such that $f_i:(X_i,\sigma_i)\rightarrow(Y_i,\tau_i)$ is continuous. Thus $(\Pi_{i\in d}\ f_i)^{-1}(A)\in\mathbf{\Gamma}\big(\Pi_{i\in d}\ (X_i,\sigma_i)\big)$ and $(\Pi_{i\in d}\ f_i)^{-1}(A)\in\text{pot}(\mathbf{\Gamma})$. □

Proof of Theorem 4.1 for ξ, assuming that Theorem 4.2.2 is proved for $\eta<\xi$.

(1) We assume that (a) does not hold. This implies that the X_i's are not empty.

- We first prove that we may assume that $X_i=\mathcal{N}$ for each $i\in d$.

By 13.5 in [K], there is a finer zero-dimensional Polish topology τ_i on X_i, and, by 7.8 in [K], (X_i,τ_i) is homeomorphic to a closed subset K_i of $\mathcal{N}$, via a map φ_i. By 2.8 in [K], there is a continuous retraction $r_i:\mathcal{N}\rightarrow K_i$. Let A'_ε be the intersection of $\Pi_{i\in d}\ K_i$ with the pre-image of A_ε by the function $\Pi_{i\in d}\ (\varphi_i^{-1}\circ r_i)$. Then A'_0 and A'_1 are disjoint analytic subsets of $\mathcal{N}^d$. Moreover, A'_0 is not separable from A'_1 by a pot$(\mathbf{\Pi}^0_\xi)$ set, since otherwise (a) would hold.

This gives $g_i:d^\omega\rightarrow\mathcal{N}$ continuous with $S\subseteq(\Pi_{i\in d}\ g_i)^{-1}(A'_0)$ and $\lceil T_d\rceil\setminus S\subseteq(\Pi_{i\in d}\ g_i)^{-1}(A'_1)$. It remains to set $f_i(\alpha):=(\varphi_i^{-1}\circ r_i\circ g_i)(\alpha)$ if $\alpha\in d^\omega$.

- To simplify the notation, we may assume that T_d has Δ^1_1 levels, $\xi<\omega_1^{\text{CK}}$ and A_0, A_1 are in $\Sigma^1_1(\mathcal{N}^d)$. Notice that $\overline{A_0}^\xi\cap A_1$ is not empty, since otherwise A_0 would be separable from A_1 by a set in $\mathbf{\Pi}^0_1(\tau_\xi)\subseteq\mathbf{\Pi}^0_\xi(\tau_1)\subseteq\text{pot}(\mathbf{\Pi}^0_\xi)$ set, which is absurd. So (b) holds, by Theorems 4.4.1 and 4.4.5 (as $\Pi''_i\lceil T_d\rceil$ is compact, we just have to compose with continuous retractions to get functions defined on d^ω). So (a) or (b) holds.

If $P\in\text{pot}(\mathbf{\Pi}^0_\xi)$ separates A_0 from A_1 and (b) holds, then $S\subseteq(\Pi_{i\in d}\ f_i)^{-1}(P)\subseteq\neg(\lceil T_d\rceil\setminus S)$. This implies that S is separable from $\lceil T_d\rceil\setminus S$ by a pot($\mathbf{\Pi}^0_\xi$) set, by Lemma 4.4.7.

(2) We argue as in the proof of (1). Here we consider $\overline{A_0}^\xi\cap\overline{A_1}^\xi$, and we apply Theorems 4.4.2 and 4.4.6. This finishes the proof. □

Proof of Theorem 4.2.2. We assume that Theorem 4.1 is proved for ξ, and that Theorem 4.2.2 is proved for $\eta<\xi$.

(1) By Lemma 4.2.3, V_0 and $V_{<\xi}$ are $\mathit{\Pi}^1_1$.

(a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (d) are clear since $\mathit{\Delta}_\mathcal{N}$ is Polish.

(b) $\Rightarrow$ (c) We argue by contradiction. As $\gamma\in\mathit{\Delta}^1_1$ we get $C_\gamma\in\mathit{\Delta}^1_1$. If $(\beta,\gamma)\in V_{<\xi}$, then $C_\gamma\in\text{pot}(\mathbf{\Pi}^0_{<\xi})$, which is absurd. If $(\beta,\gamma)\in V_0$, then $C_\gamma\in\text{pot}(\mathbf{\Pi}^0_0)\subseteq\text{pot}(\mathbf{\Pi}^0_\xi)$, which is absurd. If $(\beta,\gamma)\notin V_{<\xi}\cup V_0$, then we get $\delta\in\mathit{\Delta}^1_1$ (see the definition of $\mathfrak{F}$ before Theorem 4.2.2). As $\big((\beta)_n,(\delta)_n\big)\in V_{<\xi}$, we get $C_{(\delta)_n}\in\text{pot}(\mathbf{\Pi}^0_{<\xi})$. Now the equality $\neg C_\gamma=\bigcup_{n\in\omega}\ C_{(\delta)_n}$ implies that $C_\gamma\in\text{pot}(\mathbf{\Pi}^0_\xi)$, which is absurd.

(d) $\Rightarrow$ (e) This comes from the proof of Theorem 4.1.(1).

(e) $\Rightarrow$ (f) This comes from Theorems 4.4.1 and 4.4.5.

(f) $\Rightarrow$ (a) This comes from Theorem 4.1.(1).

(c) $\Rightarrow$ (e) We argue by contradiction, so that $\overline{A_0}^\xi$ separates A_0 from A_1.

If $\xi=1$, then for each $\vec{\delta}\in A_1$ there is $(\tilde{\beta},\tilde{\gamma})\in(\mathit{\Delta}^1_1\times\mathit{\Delta}^1_1)\cap V_0$ such that $\vec{\delta}\in C_{\tilde{\gamma}}\subseteq\neg A_0$. The first reflection theorem gives $\beta,\delta\in\mathit{\Delta}^1_1$ such that $\big((\beta)_n,(\delta)_n\big)\in V_0$ for each natural number n and $A_1\subseteq U:=\bigcup_{n\in\omega}\ C_{(\delta)_n}\subseteq\neg A_0$. We choose $\gamma\in\mathit{\Delta}^1_1\cap W$ with $\neg C_\gamma=U$, and (β,γ) contradicts (c).

If $\xi\geq 2$, then by induction assumption and the first reflection theorem there are $\beta,\delta\in\mathit{\Delta}^1_1$ with $\big((\beta)_n,(\delta)_n\big)\in V_{<\xi}$ and $C_{(\delta)_n}\subseteq\neg A_0$, for each natural number n, and $A_1\subseteq U:=\bigcup_n\ C_{(\delta)_n}$. But U is $\mathit{\Delta}^1_1\cap\text{pot}(\mathbf{\Sigma}^0_\xi)$ and separates A_1 from A_0. So let $\gamma\in\mathit{\Delta}^1_1\cap W$ with $\neg C_\gamma=U$. Note that $(\beta,\gamma)\in V_\xi$ and C_γ separates A_0 from A_1, which is absurd.

(2) It is clear that V_ξ is $\mathit{\Pi}^1_1$.

(3) We argue as in the proof of (1), except for the implication (c) $\Rightarrow$ (e) (for the implication (e) $\Rightarrow$ (f) we use Theorems 4.4.2 and 4.4.6).

(c) $\Rightarrow$ (e) We argue by contradiction. By 4D.2 in [M], there are $W\in\mathit{\Pi}^1_1(\omega)$ and a partial function $\mathbf{d}:\omega\rightarrow\mathcal{N}$, $\mathit{\Pi}^1_1$-recursive on W, such that $\mathbf{d}''W$ is the set of $\mathit{\Delta}^1_1$ points of $\mathcal{N}$. We define

$$\Pi_{A_\varepsilon}:=\big\{n\in\omega\mid(n)_0,(n)_1\in W\wedge\big(\mathbf{d}\big((n)_0\big),\mathbf{d}\big((n)_1\big)\big)\in V_{<\xi}\wedge C_{\mathbf{d}((n)_1)}\cap A_\varepsilon=\emptyset\big\}.$$

Then $\Pi_{A_\varepsilon}\in\mathit{\Pi}^1_1$ and $\forall\vec{\beta}\in\mathcal{N}^d\ \ \exists n\in\Pi_{A_0}\cup\Pi_{A_1}\ \ \vec{\beta}\in C_{\mathbf{d}((n)_1)}$ since $\overline{A_0}^\xi\cap\overline{A_1}^\xi=\emptyset$ (we use the induction assumption). By the first reflection theorem there is $D\in\mathit{\Delta}^1_1(\omega)$ such that $D\subseteq\Pi_{A_0}\cup\Pi_{A_1}$ and $\forall\vec{\beta}\in\mathcal{N}^d\ \ \exists n\in D\ \ \vec{\beta}\in C_{\mathbf{d}((n)_1)}$.

As $\mathit{\Pi}^1_1$ has the reduction property, we can find $\Pi'_{A_\varepsilon}\in\mathit{\Pi}^1_1$ disjoint such that $\Pi'_{A_\varepsilon}\subseteq\Pi_{A_\varepsilon}$ and $\Pi'_{A_0}\cup\Pi'_{A_1}=\Pi_{A_0}\cup\Pi_{A_1}$. We set $\Delta:=\bigcup_{n\in D\cap\Pi'_{A_1}}\ C_{\mathbf{d}((n)_1)}\setminus(\bigcup_{q<n}\ C_{\mathbf{d}((q)_1)})$. Then

$$\neg\Delta=\bigcup_{n\in D\cap\Pi'_{A_0}} C^{\mathcal{N}^d}_{\mathbf{d}((n)_1)}\setminus(\bigcup_{q<n} C^{\mathcal{N}^d}_{\mathbf{d}((q)_1)}),$$

which proves that $\Delta\in\mathit{\Delta}^1_1\cap\text{pot}(\mathbf{\Delta}^0_\xi)$, and separates A_0 from A_1. Let $(\beta,\gamma),(\beta',\gamma')\in(\mathit{\Delta}^1_1\times\mathit{\Delta}^1_1)\cap V_\xi$ with $\Delta=C_\gamma$ and $\neg\Delta=C_{\gamma'}$. Then we get a contradiction with (c). $\square$

Remarks. The assertions 4.2.3.(2).(a) and 4.2.3.(2).(b) admit uniform versions in the following sense. By 3E.2, 3F.6 and 3H.1 in [M], there is $S:\mathcal{N}\times\mathcal{N}\rightarrow\mathcal{N}$ recursive such that for any recursively presented Polish space X there is a universal set $\mathcal{U}^X\in\mathit{\Pi}^1_1(\mathcal{N}^d)$ satisfying the following properties:
- $\mathbf{\Pi}^1_1(X)=\{\mathcal{U}^X_\alpha\mid\alpha\in\mathcal{N}\}$,
- $\mathit{\Pi}^1_1(X)=\{\mathcal{U}^X_\alpha\mid\alpha\in\mathcal{N}\text{ recursive}\}$,
- $(\alpha,\beta,x)\in\mathcal{U}^{\mathcal{N}\times X}\Leftrightarrow\big(S(\alpha,\beta),x\big)\in\mathcal{U}^X$.

We set $\mathcal{U}:=\mathcal{U}^{\mathcal{N}^d}$. The following relations are $\mathit{\Pi}^1_1$:

$$Q(\alpha,\beta,\gamma)\Leftrightarrow\alpha\in\text{WO}\wedge(\beta,\gamma)\in V_{|\alpha|},$$
$$R(\alpha,\beta,\vec{\delta}\,)\Leftrightarrow\alpha\in\mathit{\Delta}^1_1\cap\text{WO}\wedge|\alpha|\geq 1\wedge\vec{\delta}\notin\overline{\neg\mathcal{U}_\beta}^{|\alpha|}.$$

Indeed, this comes from the proof of Lemma 4.2.3.

• One can give simpler examples $\mathbb{S}_0,\mathbb{S}_1$ for which Corollary 4.2 holds when $\mathbf{\Gamma}=\mathbf{\Pi}^0_1$. Indeed, recall the map b_ω defined before Lemma 2.3. As $|b_\omega(n)|\leq n$ for each natural number n, we can define the sequence $s^\omega_n:=b_\omega(n)0^{n-|b_\omega(n)|}$. We set $\mathbb{S}_1:=\overline{\mathbb{S}_0}\setminus\mathbb{S}_0$, where

$$\mathbb{S}_0:=\Big\{\big(0s^\omega_n0\gamma,...,0s^\omega_nn\gamma,(n+1)s^\omega_n(n+1)\gamma,(n+1)s^\omega_n(n+2)\gamma,...\big)\mid(n,\gamma)\in\omega\times\mathcal{N}\Big\}$$

(we do not really need T_ω when $\mathbf{\Gamma}=\mathbf{\Pi}^0_1$). Note that $\mathbb{S}_0=(\Pi_{i\in d}\ f_i)^{-1}(A_0)\cap\overline{\mathbb{S}_0}$ if (b) holds. Let us denote this by $\mathbb{S}_0\leq A_0$ ($\leq$ is a quasi-order, by continuity of the f_i's).

• The fact that T_d has finite levels was used to give a proof of Corollary 4.2 as simple as possible. The tree T_d has finite levels when $d<\omega$, and not always when $d=\omega$. This is one of the main new points in the case of the infinite dimension. Let us dwell more deeply into this.

(a) We saw in the proof of Proposition 2.2 that the tree ${}_dT$ generated by an effective frame is a tree with one-sided almost acyclic levels. As before Lemma 2.6, we can define

$${}_{\mathbf{C}_1}S:=\{\vec{\alpha}\in\lceil_\omega T\rceil\mid\mathcal{S}(\alpha_0\Delta\alpha_1)\in\mathbf{C}_1\},$$

which is not separable from $\lceil_\omega T\rceil\setminus{}_{\mathbf{C}_1}S$ by a potentially closed set, since otherwise $S_{\mathbf{C}_1}$ would be separable from $\lceil T_\omega\rceil\setminus S_{\mathbf{C}_1}$ by a potentially closed set, which would contradict Lemmas 2.6 and 3.4.

But $\mathbb{A}_0:=\{0^{1+n}(1+n)^\infty\mid n\in\omega\}\subseteq\mathcal{N}$ is not potentially closed since $0^\infty\in\overline{\mathbb{A}_0}\setminus\mathbb{A}_0$ and the topology on ω is discrete. And one can prove, in a straightforward way, that ${}_{\mathbf{C}_1}S\not\leq\mathbb{A}_0$ and $\mathbb{A}_0\not\leq{}_{\mathbf{C}_1}S$. This proves that the finiteness of the levels of T_d is useful. But we will see that it is not necessary.

(b) We define $o\!:\!\{s\!\in\! 2^{<\omega}\mid 0\!\not\subseteq\! s\}\!\rightarrow\!\omega^{<\omega}$ such that $|o(s)|\!=\!|s|$ by

$$o(10^{n_0}10^{n_1}...10^{n_l}):=0^{1+n_0}(1+n_0)^{1+n_1}...\big((1+n_0)+...+(1+n_{l-1})\big)^{1+n_l}.$$

In other words, we can write $o(s)(i)=i$ if $s(i)=1$, and $o(s)(i)=o(s)(i-1)$ if $s(i)=0$. Note that o is an injective homomorphism, in the sense that $o(s)\subseteq o(t)$ if $s\subseteq t$. This implies that we can extend o to a continuous map from the basic clopen set N_1 into $\mathcal{N}$ by the formula $o(\alpha):=\text{sup}_{m\in\omega}\ o(\alpha|m)$.

We set $F_\omega := \big\{(m_i\alpha_i)_{i\in\omega}\in\mathcal{N}^\omega \mid \vec{\alpha}\in\lceil_\omega T\rceil \wedge \forall i\in\omega\ \ m_i=o(\alpha_0\Delta\alpha_1)(i)\big\}$, and we put $\underline{S}_{\mathbf{C}_\xi}:=\{(m_i\alpha_i)_{i\in\omega}\in F_\omega \mid \mathcal{S}(\alpha_0\Delta\alpha_1)\in\mathbf{C}_\xi\}$. One can take $\mathbb{S}_\xi=\underline{S}_{\mathbf{C}_\xi}$, and the proof is much more complicated than the one we gave. But the tree associated with $\overline{\underline{S}_{\mathbf{C}_\xi}}=F_\omega$ is

$$\{\vec{\emptyset}\}\cup\big\{(m_is_i)_{i\in\omega}\in\mathcal{N}^{<\omega}\mid (m_i)_{i\in\omega}\in o[N_1]\wedge \vec{s}\in{}_\omega T\wedge\forall i<|\vec{s}|\ \ m_i=o(s_0\Delta s_1)(i)\big\},$$

and has infinite levels. This proves that the finiteness of the levels of the tree associated with $\overline{\mathbb{S}_\xi}$ is not necessary.

(c) In [L8], an extension to any dimension of the Kechris-Solecki-Todorčević dichotomy for analytic graphs is proved. In [L5], it is proved that Corollary 4.2 is a consequence of the Kechris-Solecki-Todorčević dichotomy when $\mathbf{\Gamma}=\mathbf{\Pi}^0_1$. This works as well when $d<\omega$, but not when $d=\omega$. More specifically, let $\mathbb{G}:=\{\alpha\in\mathcal{N}\mid\forall m\in\omega\ \exists n\geq m\ \ s^\omega_n 0\subseteq\alpha\}$ and $\mathbb{A}_\omega:=\{(s^\omega_n i\gamma)_{i\in\omega}\mid n\in\omega\wedge\gamma\in\mathcal{N}\}$. Then the extension of the Kechris-Solecki-Todorčević dichotomy to the case $d\!=\!\omega$ works with the set $\mathbb{G}^\omega\cap\mathbb{A}_\omega$ (see [L8]). But one can prove the following result:

Theorem 4.4.8 *Let X be a recursively presented Polish space, σ_X be the topology on X^ω generated by $\{\Pi_{i\in\omega}\ C_i\mid C\in\Delta^1_1(\omega\times X)\}$, and A be a Δ^1_1 subset of X^ω. Then exactly one of the following holds:*
(a) $\overline{A}^{\sigma_X}\setminus A=\emptyset$.
(b) $\mathbb{G}^\omega\cap\mathbb{A}_\omega\leq A$.

In particular, $\mathbb{G}^\omega\cap\mathbb{A}_\omega\not\leq\mathbb{A}_0$ and we cannot take $\mathbb{S}_1=\mathbb{G}^\omega\cap\mathbb{A}_\omega$.

5 The proof of Theorem 1.9

5.1 Some one-dimensional material

The material in this subsection can be found in [Lo-SR1] or [Lo-SR2]. However, we need to make some changes for our purpose. Moreover some proofs are left to the reader in these papers. These are the reasons why we will give some proofs. The following definition can be found in [Lo-SR2] (see Definition 1.5).

Definition 5.1.1 *Let $1\leq\xi<\omega_1$, and $\mathbf{\Gamma}$, $\mathbf{\Gamma}'$ be two classes of sets. Then*

$$A\in S_\xi(\mathbf{\Gamma},\mathbf{\Gamma}')\ \Leftrightarrow\ A=\bigcup_{p\geq 1}(A_p\cap C_p)\cup\left(B\setminus\bigcup_{p\geq 1}C_p\right),$$

where $A_p\in\mathbf{\Gamma}$, $B\in\mathbf{\Gamma}'$, and $(C_p)_{p\geq 1}$ is a sequence of pairwise disjoint $\mathbf{\Sigma}^0_\xi$ sets.

Now we come to the definition of the $second\ type\ descriptions$ of the non self-dual Wadge classes of Borel sets, which are elements of ω_1^ω (we sometimes identify ω_1^ω with $(\omega_1^\omega)^\omega$). This definition can also be found in [Lo-SR2] (see Definition 1.6).

Definition 5.1.2 *The relations "u is a $second\ type\ description$" and "$u\ describes$ $\mathbf{\Gamma}$" (written $u\in\mathcal{D}$ and $\mathbf{\Gamma}_u=\mathbf{\Gamma}$ - ambiguously) are the least relations satisfying the following properties.*
(a) If $u=0^\infty$, then $u\in\mathcal{D}$ and $\mathbf{\Gamma}_u=\{\emptyset\}$.
(b) If $u=\xi^\frown 1^\frown v$, with $v\in\mathcal{D}$ and $v(0)=\xi$, then $u\in\mathcal{D}$ and $\mathbf{\Gamma}_u=\check{\mathbf{\Gamma}}_v$.
(c) If $u=\xi^\frown 2^\frown< u_p >$ satisfies $\xi\geq 1$, $u_p\in\mathcal{D}$, and $u_p(0)\geq\xi$ or $u_p(0)=0$, then $u\in\mathcal{D}$ and $\mathbf{\Gamma}_u=S_\xi(\bigcup_{p\geq 1}\ \mathbf{\Gamma}_{u_p},\mathbf{\Gamma}_{u_0})$.

Remark. If $A\in S_\xi(\bigcup_{p\geq 1}\ \mathbf{\Gamma}_{u_p},\mathbf{\Gamma}_{u_0})$, then A has a decomposition as in Definition 5.1.1, and A_p is in $\bigcup_{p\geq 1}\ \mathbf{\Gamma}_{u_p}$. But we may assume that $A_p\in\mathbf{\Gamma}_{u_{(p)_0+1}}$, using the fact that C_p may be empty if necessary. This remark will be useful in the sequel, since it specifies the class of A_p.

The following result can be found in [Lo-SR2] (see Section 3).

Theorem 5.1.3 *Let $\mathbf{\Gamma}$ be a non self-dual Wadge class of Borel sets. Then there is $u\in\mathcal{D}$ such that $\mathbf{\Gamma}(\mathcal{N})=\mathbf{\Gamma}_u(\mathcal{N})$. Conversely,*

$$\mathbf{\Gamma}_u:=\{f^{-1}(A)\mid f:X\rightarrow\mathcal{N}\text{ continuous }\wedge X\text{ zero-dimensional Polish }\wedge A\in\mathbf{\Gamma}_u(\mathcal{N})\}$$

is a non self-dual Wadge class of Borel sets if $u\in\mathcal{D}$.

If $\eta\leq\xi<\omega_1$, then $\xi-\eta$ is the unique ordinal θ with $\eta+\theta=\xi$. The following definition can be found in [Lo-SR2] (see Definition 1.9).

Definition 5.1.4 *Let $\eta<\omega_1$ and $u\in\mathcal{D}$. We define $u^\eta\in\mathcal{D}$ as follows.*
(a) If $u(0)=0$, then $u^\eta:=u$.
(b) If $u=\xi 1v$, with $\xi\geq 1$, then $u^\eta:=\big(1+\eta+(\xi-1)\big)1v^\eta$.
(c) If $u=\xi 2<u_p>$, with $\xi\geq 1$, then $u^\eta:=\big(1+\eta+(\xi-1)\big)2<(u_p)^\eta>$.

The following result can be found in [Lo-SR2] (see Proposition 1.10).

Proposition 5.1.5 *(a) If $f:\mathcal{N}\rightarrow\mathcal{N}$ is $\mathbf{\Sigma}^0_{1+\eta}$-measurable, and $A\in\mathbf{\Gamma}_u(\mathcal{N})$ for some $u\in\mathcal{D}$, then $f^{-1}(A)\in\mathbf{\Gamma}_{u^\eta}$.*
(b) The set $\mathcal{D}$ is the least subset $D\subseteq\mathcal{D}$ such that $0^\infty\in D$, $u(0)1u\in D$ if $u\in D$, $12<u_p>\in D$ if $u_p\in D$ for any $p\in\omega$, and $u^\eta\in D$ if $u\in D$ (for any $\eta<\omega_1$).

Recall the definition of an independent η-function (see Definition 3.3).

Example. Let $\tau:\omega\rightarrow\omega$ be one-to-one (in [Lo-SR2], just before Lemma 2.5, the authors consider increasing maps. In this paper, we work with this weaker property). We define $\tilde{\tau}:\mathcal{C}\rightarrow\mathcal{C}$ by the formula $\tilde{\tau}(\alpha):=\alpha\circ\tau$. The map $\tilde{\tau}$ is an independent 0-function (with witness π defined by the formula $\pi(k)=\tau^{-1}(k)$ if k is in the range of τ, 0 otherwise). We now describe an important example of this situation.

Example. Let n be a natural number, and $\mathcal{S}$ be the shift map (see the notation before Definition 2.5). Then $\mathcal{S}^n$ is an independent 0-function. Indeed, if we set $\tau^n(m) := m+n$, then $\mathcal{S}^n = \tilde{\tau^n}$, by induction on n. In particular, $\text{Id}_{\mathcal{C}} = \mathcal{S}^0$ is an independent 0-function.

The next result is essentially Lemma 2.5 in [Lo-SR2], which is given without proof. This is the reason why we give the details here.

Lemma 5.1.6 *Let $\tau : \omega \to \omega$ be one-to-one, and ζ be an independent η-function. Then $\tilde{\tau} \circ \zeta$ is an independent η-function.*

Proof. Let π be the map associated with ζ. We define $\pi' : \omega \to \omega$ by $\pi'(k) := \tau^{-1}\big(\pi(k)\big)$ if $\pi(k)$ is in the range of τ, 0 otherwise, so that $\pi'(k) = m$ if $\pi(k) = \tau(m)$. If m is a natural number, then $(\tilde{\tau} \circ \zeta)(\alpha)(m) = \zeta(\alpha)\big(\tau(m)\big)$ depends only of the values of α on $\pi^{-1}\big(\{\tau(m)\}\big) \subseteq (\pi')^{-1}(\{m\})$.

If $\xi = 0$ (resp., $\xi = \theta+1$, $\xi = \sup_{m\in\omega}\ \theta_m$), then $\mathbf{Z}_m = \{\alpha \in \mathcal{C} \mid \zeta(\alpha)\big(\tau(m)\big) = 1\}$ is $\mathbf{\Delta}^0_1$-complete (resp., $\mathbf{\Pi}^0_{1+\theta}$-strategically complete, $\mathbf{\Pi}^0_{1+\theta_{\tau(m)}}$-strategically complete). We are done since $\xi = \sup_{p\geq 1}\ \theta_{\tau(m_p)}$ if ξ is a limit ordinal (τ is one-to-one). □

After Definition 3.3, we saw that ρ^η is an independent η-function. We will actually prove more. In fact, we prove a result which is essentially Theorem 2.4.(b) in [Lo-SR2].

Theorem 5.1.7 *Let $\eta, \xi < \omega_1$, and ζ be an independent ξ-function. Then $\rho^\eta \circ \zeta$ is an independent $(\xi+\eta)$-function.*

Proof. Assume that $\varepsilon \in 2$ and $\zeta^\varepsilon : \mathcal{C} \to \mathcal{C}$ is equipped with π^ε such that $\zeta^\varepsilon(\alpha)(m)$ depends only on the values of α on $(\pi^\varepsilon)^{-1}(\{m\})$. Then $D := (\zeta^0 \circ \zeta^1)(\alpha)(m)$ depends only on the values of $\zeta^1(\alpha)$ on $(\pi^0)^{-1}(\{m\})$. Thus D depends only on the values of α on $(\pi^1)^{-1}\big((\pi^0)^{-1}(\{m\})\big)$. This implies that if we set $\pi := \pi^0 \circ \pi^1$, then D depends only on the values of α on $\pi^{-1}(\{m\})$.

• We argue by induction on η. The result is clear for $\eta = 0$. So assume that $\eta = \theta+1$, so that $\rho^\eta \circ \zeta = \rho \circ \rho^\theta \circ \zeta$. The induction assumption implies that $\rho^\theta \circ \zeta$ is an independent $(\xi+\theta)$-function. The fact that ρ is an independent 1-function and the previous point prove the existence of π_η such that $(\rho^\eta \circ \zeta)(\alpha)(m)$ depends only on the values of α on $\pi_\eta^{-1}(\{m\})$.

We set $\mathbf{A}_n := \{\alpha \in \mathcal{C} \mid (\rho^\theta \circ \zeta)(\alpha)(< m, n >) = 1\}$. Let us prove that $\bigcap_{n\in\omega}\ \neg\mathbf{A}_n$ is $\mathbf{\Pi}^0_{1+\xi+\theta}$-strategically complete.

Assume first that $\xi+\theta \neq 0$. As $\rho^\theta \circ \zeta$ is an independent $(\xi+\theta)$-function, $\mathbf{A}_n$ is $\mathbf{\Pi}^0_{1+\theta_n}$-strategically complete, for some $\theta_n < \xi+\theta$ satisfying $\theta_n+1 = \xi+\theta$ if $\xi+\theta$ is a successor ordinal, $\sup_{n\in\omega}\ \theta_n = \xi+\theta$ if $\xi+\theta$ is a limit ordinal. Note that $\xi+\theta = \sup_{n\in\omega}\ (\theta_n+1)$. As $\rho^\theta \circ \zeta$ is an independent $(\xi+\theta)$-function, there is π_θ such that $(\rho^\theta \circ \zeta)(\alpha)(q)$ depends only on the values of α on $\pi_\theta^{-1}(\{q\})$. We set $\pi(\alpha)(k) := \big(\pi_\theta(\alpha)\big)_1$, so that the fact that $\alpha \in \mathbf{A}_n$ depends only on the values of α on $\pi^{-1}(\{n\})$. By Lemma 3.7 in [Lo-SR1], $\bigcap_{n\in\omega}\ \neg\mathbf{A}_n$ is $\mathbf{\Pi}^0_{1+\xi+\theta}$-strategically complete.

Assume now that $\xi+\theta=0$. Then $\mathbf{A}_n := \{\alpha\in\mathcal{C} \mid \zeta(\alpha)(<m,n>)=1\}$ is $\mathbf{\Delta}^0_1$-complete since ζ is an independent 0-function. Let B be a closed subset of $\mathcal{N}$, $(B_n)_{n\in\omega}$ be a sequence of clopen subsets with $B=\bigcap_{n\in\omega}\ B_n$, and $g_n:\mathcal{N}\rightarrow\mathcal{C}$ be continuous with $B_n=g_n^{-1}(\neg\mathbf{A}_n)$. As ζ is an independent 0-function, there is π_ζ such that $\zeta(\alpha)(q)$ depends only on the values of α on $\pi_\zeta^{-1}(\{q\})$. We set $\pi(\alpha)(k):=\big(\pi_\rho(\alpha)\big)_1$, so that the fact that $\alpha\in\mathbf{A}_n$ depends only on the values of α on $\pi^{-1}(\{n\})$. We define $g:\mathcal{N}\rightarrow\mathcal{C}$ by $g(\beta)(k):=g_{\pi(k)}(\beta)(k)$, so that g is continuous. Moreover, $\beta\in B_n\Leftrightarrow g_n(\beta)\notin\mathbf{A}_n\Leftrightarrow g(\beta)\notin\mathbf{A}_n$ since the fact that $\alpha\in\mathbf{A}_n$ depends only on the values of α on $\pi^{-1}(\{n\})$. Thus $B=g^{-1}(\bigcap_{n\in\omega}\ \neg\mathbf{A}_n)$ and $\bigcap_{n\in\omega}\ \neg\mathbf{A}_n$ is $\mathbf{\Pi}^0_1$-complete. Therefore $\bigcap_{n\in\omega}\ \neg\mathbf{A}_n$ is $\mathbf{\Pi}^0_{1+\xi+\theta}$-strategically complete.

Now note that

$$\bigcap_{n\in\omega}\ \neg\mathbf{A}_n=\{\alpha\in\mathcal{C}\mid\forall n\in\omega\ \ (\rho^\theta\circ\zeta)(\alpha)(<m,n>)=0\}$$

$$=\{\alpha\in\mathcal{C}\mid(\rho\circ\rho^\theta\circ\zeta)(\alpha)(m)=1\}=\{\alpha\in\mathcal{C}\mid(\rho^\eta\circ\zeta)(\alpha)(m)=1\}.$$

Thus $\{\alpha\in\mathcal{C}\mid(\rho^\eta\circ\zeta)(\alpha)(m)=1\}$ is $\mathbf{\Pi}^0_{1+\xi+\theta}$-strategically complete for each m, and $\xi+\eta=\xi+\theta+1$, so that $\rho^\eta\circ\zeta$ is an independent $(\xi+\eta)$-function.

• Assume now that η is a limit ordinal. In the definition of ρ^η we fixed a sequence $(\eta_m)_{m\in\omega}\subseteq\eta$ of successor ordinals with $\Sigma_{m\in\omega}\ \eta_m=\eta$. As ρ^{η_m} is an independent η_m-function, we get $\pi^\eta_m:\omega\rightarrow\omega$. We define $\pi_{m,m+1}:\omega\rightarrow\omega$ by $\pi_{m,m+1}(k):=k$ if $k<m$, $\pi^\eta_m(k-m)+m$ if $k\geq m$. Let us check that $\rho^{(m,m+1)}(\alpha)(i)$ depends only on the values of α on $\pi^{-1}_{m,m+1}(\{i\})$. It is clearly the case if $i<m$. So assume that $i\geq m$. Note that $\pi_{m,m+1}(k)=i$ if $k\in(\pi^\eta_m)^{-1}(\{i-m\})+m$, and we are done. Now the first point of this proof gives $\pi_{0,m+1}:\omega\rightarrow\omega$ such that $\rho^{(0,m+1)}(\alpha)(i)$ depends only on the values of α on $\pi^{-1}_{0,m+1}(\{i\})$. We will check that $\rho^\eta(\alpha)(m):=\rho^{(0,m+1)}(\alpha)(m)$ depends only on the values of α on $E_m:=\pi^{-1}_{0,m+1}(\{m\})\cap\bigcap_{l<m}\ \pi^{-1}_{0,l+1}(\neg(l+1))$. We actually prove something stronger: for any natural number k, $\rho^{(0,m+1)}(\alpha)(k+m)$ depends only on the values of α on

$$\pi^{-1}_{0,m+1}(\{k+m\})\cap\bigcap_{l<m}\ \pi^{-1}_{0,l+1}(\neg(l+1)).$$

We argue by induction on m. For $m=0$, the result is clear. Assume that the result is true for m. Note that $\rho^{(0,m+2)}(\alpha)(k+m+1)$ depends only on the values of α on $\pi^{-1}_{0,m+2}(\{k+m+1\})$. But

$$\rho^{(0,m+2)}(\alpha)(k+m+1)=\rho^{(m+1,m+2)}\big(\rho^{(0,m+1)}(\alpha)\big)(k+m+1)=\rho^{\eta_{m+1}}\Big(\mathcal{S}^{m+1}\big(\rho^{(0,m+1)}(\alpha)\big)\Big)(k),$$

and we are done since $\rho^{(0,m+2)}(\alpha)(k+m+1)$ depends only on the values of $\mathcal{S}^{m+1}\big(\rho^{(0,m+1)}(\alpha)\big)$, which depends only on the values of α on $\pi^{-1}_{0,m+1}(\neg(m+1))\cap\bigcap_{l<m}\ \pi^{-1}_{0,l+1}(\neg(l+1))$.

As the E_m's are pairwise disjoint, we can define a map $\pi^\eta:\omega\rightarrow\omega$ by $\pi^\eta(k):=m$ if $k\in E_m$, and 0 if $k\notin\bigcup_{m\in\omega}\ E_m$. Now it is clear that $\rho^\eta(\alpha)(m)$ depends only on the values of α on $(\pi^\eta)^{-1}(\{m\})$. The first point of this proof gives $\pi_\eta:\omega\rightarrow\omega$ such that $(\rho^\eta\circ\zeta)(\alpha)(m)$ depends only on the values of α on $\pi^{-1}_\eta(\{m\})$.

Let ξ_m such that $\eta_m := \xi_m + 1$, and $\theta_m := \xi + \Sigma_{l<m}\ \eta_l + \xi_m$, so that $\theta_m < \xi + \eta$ and $\sup_{p\geq 1}\ \theta_{m_p} = \xi + \eta$ for any one-to-one sequence $(m_p)_{p\geq 1}$ of natural numbers. It remains to see that

$$\mathbf{Z}_m := \{\alpha\in\mathcal{C} \mid (\rho^\eta \circ \zeta)(\alpha)(m) = 1\}$$

is $\mathbf{\Pi}^0_{1+\theta_m}$-strategically complete for any natural number m.

Let us check that $\mathcal{S}^m \circ \rho^{(0,m+1)} = \rho^{\eta_m} \circ\ \circ_{l<m}\ (\mathcal{S}\circ\rho^{\eta_{m-l-1}})$ for any natural number m. We argue by induction on m. For $m=0$, the property is clear since $\rho^{(0,1)} = \rho^{\eta_0}$. Assume that the property is true for m. Then

$$\mathcal{S}^{m+1} \circ \rho^{(0,m+2)} = \rho^{\eta_{m+1}} \circ \mathcal{S}^{m+1} \circ \rho^{(0,m+1)} = \rho^{\eta_{m+1}} \circ \mathcal{S} \circ \mathcal{S}^m \circ \rho^{(0,m+1)}$$

$$= \rho^{\eta_{m+1}} \circ \mathcal{S} \circ \rho^{\eta_m} \circ\ \circ_{l<m}\ (\mathcal{S}\circ\rho^{\eta_{m-l-1}}) = \rho^{\eta_{m+1}} \circ\ \circ_{l\leq m}\ (\mathcal{S}\circ\rho^{\eta_{m-l}})$$

since in the last induction we proved that $\mathcal{S}^{m+1} \circ \rho^{(0,m+2)} = \rho^{\eta_{m+1}} \circ \mathcal{S}^{m+1} \circ \rho^{(0,m+1)}$. Thus

$$\mathbf{Z}_m = \{\alpha\in\mathcal{C} \mid \rho^{(0,m+1)}\big(\zeta(\alpha)\big)(m) = 1\} = \{\alpha\in\mathcal{C} \mid (\mathcal{S}^m \circ \rho^{(0,m+1)} \circ \zeta)(\alpha)(0) = 1\}$$

$$= \{\alpha\in\mathcal{C} \mid \big(\rho^{\eta_m} \circ\ \circ_{l<m}\ (\mathcal{S}\circ\rho^{\eta_{m-l-1}}) \circ \zeta\big)(\alpha)(0) = 1\}.$$

So it is enough to see that $\zeta_m := \rho^{\eta_m} \circ\ \circ_{l<m}\ (\mathcal{S}\circ\rho^{\eta_{m-l-1}}) \circ \zeta$ is an independent $(\theta_m + 1)$-function.

We argue by induction on m. For $m=0$, we are done since $\rho^{\eta_0}\circ\zeta$ is by induction assumption an independent $(\xi+\eta_0)$-function, and $\xi+\eta_0 = \xi+\xi_0+1 = \theta_0+1$. Assume that the property is true for m. Then $\zeta_{m+1} = \rho^{\eta_{m+1}} \circ \mathcal{S} \circ \zeta_m$. By induction assumption, ζ_m is an independent (θ_m+1)-function. By Lemma 5.1.6 and the example just before it, $\mathcal{S}\circ\zeta_m$ is also an independent (θ_m+1)-function. By induction assumption, ζ_{m+1} is an independent $(\theta_m+1+\eta_{m+1})$-function, and

$$\theta_m + 1 + \eta_{m+1} = \xi + \Sigma_{l<m}\ \eta_l + \xi_m + 1 + \eta_{m+1} = \xi + \Sigma_{l\leq m}\ \eta_l + \xi_{m+1} + 1 = \theta_{m+1} + 1.$$

This finishes the proof. □

5.2 Some complicated sets

Now we come to the existence of complicated sets, as in the statement of Theorem 1.9. Their construction is based on Theorem 2.7 in [Lo-SR2] that we now change. The main problem is that we want to ensure the ccs conditions in Lemma 2.6. In order to do this, we modify the definition of the maps τ_i in Lemma 2.11 in [Lo-SR2].

Notation. Let i be a natural number. We define $\tau_i : \omega \to \omega$ by

$$\tau_i(k) := \begin{cases} < 0, k > \text{ if } i=0,\\ \\ << i, (k)_0 >, (k)_1 > \text{ if } i\geq 1,\end{cases}$$

so that τ_i is one-to-one. This allows us to define, for any $\alpha\in\mathcal{C}$, $\alpha_i := \tilde{\tau}_i(\alpha)$. If $s\in\mathcal{F} := (\omega\setminus\{0\})^{<\omega}$, then we set $\tilde{\tau}_s := \tilde{\tau}_{s(0)} \circ ... \circ \tilde{\tau}_{s(|s|-1)}$.

Lemma 5.2.1 *Let $\mathbf{\Gamma}$ be a non self-dual Wadge class of Borel sets, and $\mathbf{H}$ be a $\mathbf{\Gamma}$-strategically complete set. Then the following hold.*

(a) The set $\tilde{\tau}_i^{-1}(\mathbf{H})$ is $\mathbf{\Gamma}$-strategically complete for any natural number i.

(b) Assume that $\tau:\omega\rightarrow\omega$ is one-to-one with the property that the fact that $\alpha\in\mathbf{H}$ depends only on $\alpha\circ\tau$. Then $\mathbf{M}:=\{\alpha\circ\tau\mid\alpha\in\mathbf{H}\}$ is $\mathbf{\Gamma}$-strategically complete.

Proof. (a) As $\tilde{\tau}_i$ is continuous, $\tilde{\tau}_i^{-1}(\mathbf{H})\in\mathbf{\Gamma}(\mathcal{C})$. We define a continuous map $f_{\tau_i}:\mathcal{C}\rightarrow\mathcal{C}$ by $f_{\tau_i}(\alpha)(m):=\alpha\big(\tau_i^{-1}(m)\big)$ if m is in the range of τ_i, 0 otherwise. Note that $\tilde{\tau}_i\big(f_{\tau_i}(\alpha)\big)=\alpha$, so that $\mathbf{H}=f_{\tau_i}^{-1}\big(\tilde{\tau}_i^{-1}(\mathbf{H})\big)$. This implies that $\tilde{\tau}_i^{-1}(\mathbf{H})$ is $\mathbf{\Gamma}$-strategically complete.

(b) As in (a), we consider the continuous map f_τ, so that $\tilde{\tau}\big(f_\tau(\beta)\big)=\beta$ for each $\beta\in\mathcal{C}$. Here again $f_\tau^{-1}(\mathbf{H})\in\mathbf{\Gamma}(\mathcal{C})$. Let $\beta\in\mathbf{M}$, which gives $\alpha\in\mathbf{H}$ with $\beta=\alpha\circ\tau$. As $f_\tau(\beta)\circ\tau=\tilde{\tau}\big(f_\tau(\beta)\big)=\beta$, we get $f_\tau(\beta)\circ\tau=\alpha\circ\tau$, and $f_\tau(\beta)\in\mathbf{H}$ by the assumption on $\mathbf{H}$. Conversely, if $f_\tau(\beta)\in\mathbf{H}$, then $\beta=\tilde{\tau}\big(f_\tau(\beta)\big)=f_\tau(\beta)\circ\tau\in\mathbf{M}$. Thus $\mathbf{M}=f_\tau^{-1}(\mathbf{H})$, and $\mathbf{M}\in\mathbf{\Gamma}(\mathcal{C})$.

If $\alpha\in\mathbf{H}$, then $\tilde{\tau}(\alpha)=\alpha\circ\tau\in\mathbf{M}$. Conversely, assume that $\tilde{\tau}(\alpha)\in\mathbf{M}$. Then there is $\beta\in\mathbf{H}$ with $\beta\circ\tau=\alpha\circ\tau$. The assumption on $\mathbf{H}$ implies that $\alpha\in\mathbf{H}$. Thus $\mathbf{H}=\tilde{\tau}^{-1}(\mathbf{M})$ and $\mathbf{M}$ is $\mathbf{\Gamma}$-strategically complete. □

Lemma 5.2.2 *Let $\mathbf{\Gamma}$ be a Wadge class of Borel sets, and $A\subseteq\mathcal{C}$. Then $A\in\mathbf{\Gamma}(\mathcal{C})$ if and only if there is $B\in\mathbf{\Gamma}(\mathcal{N})$ with $A=B\cap\mathcal{C}$.*

Proof. $\Rightarrow$ Let $r:\mathcal{N}\rightarrow\mathcal{C}$ be a continuous retraction. We just have to set $B:=r^{-1}(A)$.

$\Leftarrow$ Let $i:\mathcal{C}\rightarrow\mathcal{N}$ be the canonical injection. Then $A=i^{-1}(B)\in\mathbf{\Gamma}(\mathcal{C})$. □

This lemma shows that the notation $\mathbf{\Gamma}_u$ in Theorem 5.1.3 will not create any trouble, since it is equivalent to the one in Definition 5.1.2.

Notation. The following notation can essentially be found in [Lo-SR2] (after Lemma 2.5). Let $\mathcal{R}$ be the least set of functions from $\mathcal{C}$ into itself which contains the functions ρ^η, the functions $\tilde{\tau}_i$ for $i\geq 1$, and is closed under composition. By Lemma 5.1.6 and Theorem 5.1.7, each $\zeta\in\mathcal{R}$ is an independent η-function for some η called the $order\ o(\rho)$ of ρ.

Definition 5.2.3 *Let $u\in\mathcal{D}$. A set $\mathbf{H}\subseteq\mathcal{C}$ is $strongly\ u\text{-}strategically\ complete$ if, for each $\zeta\in\mathcal{R}$ of order η, $\zeta^{-1}(\mathbf{H})$ is $\mathbf{\Gamma}_{u^\eta}$-strategically complete and ccs.*

Theorem 5.2.4 *Let $u\in\mathcal{D}$. Then there exists a strongly u-strategically complete set $\mathbf{H}_u$. In particular, $\mathbf{H}_u$ is $\mathbf{\Gamma}_u$-complete and ccs.*

Proof. We will check that the sets $\mathbf{H}_u$ given by Theorem 2.7 in [Lo-SR2] essentially work, even if we change them.

The construction is by induction on $u\in\mathcal{D}$. Let us say that u is $nice$ if it satisfies the conclusion of the theorem. By Proposition 5.1.5, it is enough to prove that 0^∞ is nice, that $u(0)1u$ is nice if u is nice, that u^η is nice if u is nice and $\eta<\omega_1$, and that $12<u_p>$ is nice if the u_p's are nice.

- We set $\mathbf{H}_{0^\infty}:=\emptyset$, which is clearly strongly 0^∞-strategically complete.

• Assume that u is nice. We set $\mathbf{H}_{u(0)1u} := \neg \mathbf{H}_u$. Note that $\mathbf{H}_{u(0)1u}$ is strongly $u(0)1u$-strategically complete. Indeed, if $u(0)=0$, then $\mathbf{\Gamma}_{(u(0)1u)^\eta} = \mathbf{\Gamma}_{u(0)1u} = \check{\mathbf{\Gamma}}_u = \check{\mathbf{\Gamma}}_{u^\eta}$. If $u(0) \geq 1$, then

$$\mathbf{\Gamma}_{(u(0)1u)^\eta} = \mathbf{\Gamma}_{(1+\eta+(u(0)-1))1u^\eta} = \check{\mathbf{\Gamma}}_{u^\eta}$$

since $u^\eta(0) = 1+\eta+\big(u(0)-1\big)$.

• Assume that u is nice and $\eta < \omega_1$. We set $\mathbf{H}_{u^\eta} := (\rho^\eta)^{-1}(\mathbf{H}_u)$. Note that $\mathbf{H}_{u^\eta}$ is strongly u^η-strategically complete. Indeed, let $\zeta \in \mathcal{R}$ be of order ξ. Then $\zeta^{-1}(\mathbf{H}_{u^\eta}) = (\rho^\eta \circ \zeta)^{-1}(\mathbf{H}_u)$ is $\mathbf{\Gamma}_{u^{\xi+\eta}}$-strategically complete and compatible with comeager sets since u is nice and $\rho^\eta \circ \zeta$ is in $\mathcal{R}$ of order $\xi+\eta$. It remains to notice that $(u^\eta)^\xi = u^{\xi+\eta}$, which is clear by induction on u and by definition of the ordinal subtraction.

• Assume that the u_p's are nice. We set $v_n := u_{(n)_0+1}$ and

$$\alpha \in \mathbf{H}_{12<u_p>} \;\Leftrightarrow\; \begin{cases} \alpha_0 = 0^\infty \wedge \alpha_1 \in \mathbf{H}_{u_0} \\ \vee \\ \exists m \in \omega \;\; \alpha_0(m) = 1 \wedge \forall l < m \;\; \alpha_0(l) = 0 \wedge \alpha_{(m)_0+2} \in \mathbf{H}_{v_{(m)_0+2}}. \end{cases}$$

- Recall that $\mathbf{\Gamma}_{12<u_p>} = S_1(\bigcup_{p\geq 1} \mathbf{\Gamma}_{u_p}, \mathbf{\Gamma}_{u_0})$. We set $K_0 := \{\alpha \in \mathcal{C} \mid \alpha_1 \in \mathbf{H}_{u_0}\} = \tilde{\tau_1}^{-1}(\mathbf{H}_{u_0})$, and, for $n \geq 2$,

$$K_n := \{\alpha \in \mathcal{C} \mid \alpha_n \in \mathbf{H}_{v_n}\} = \tilde{\tau_n}^{-1}(\mathbf{H}_{v_n}),$$

$$C_n := \{\alpha \in \mathcal{C} \mid \exists m \in \omega \;\; \alpha_0(m) = 1 \wedge \forall l < m \;\; \alpha_0(l) = 0 \wedge (m)_0 + 2 = n\}.$$

Note that $(C_n)_{n\geq 2}$ is a sequence of pairwise disjoint open sets, and $K_0 \in \mathbf{\Gamma}_{u_0}$, $K_n \in \mathbf{\Gamma}_{v_n}$ if $n \geq 2$ by Lemma 5.2.1.(a). Moreover, $\mathbf{H}_{12<u_p>} = \bigcup_{n\geq 2} (K_n \cap C_n) \cup (K_0 \setminus \bigcup_{n\geq 2} C_n) \in \mathbf{\Gamma}_{12<u_p>}(\mathcal{C})$, by Lemma 5.2.2 and the reduction property for the class of the open sets (see 22.16 in [K]).

- Let $\zeta \in \mathcal{R}$ be of order η. Then $\zeta^{-1}(\mathbf{H}_{12<u_p>}) \in \mathbf{\Gamma}_{(12<u_p>)^\eta}(\mathcal{C})$, by Proposition 5.1.5.(a) and a retraction argument in the style of the proof of Lemma 5.2.2. Let π be associated with ζ, $e_0 : \omega \to \omega$ be a one-to-one enumeration of $\pi^{-1}\big(\text{Ran}(\tau_1)\big)$, and, for $n \geq 2$, $e_n : \omega \to \omega$ be a one-to-one enumeration of $\pi^{-1}\big(\text{Ran}(\tau_n)\big)$ and $e^n : \omega \to \omega$ be a one-to-one enumeration of

$$\pi^{-1}\big(\{j \in \text{Ran}(\tau_0) \mid \big(\tau_0^{-1}(j)\big)_0 + 2 = n\}\big).$$

As τ_i is one-to-one, Ran(τ_i) is infinite, and $\pi^{-1}\big(\text{Ran}(\tau_i)\big)$ is also infinite since π is onto. This proves the existence of the e_n's and of the e^n's. Note that the Ran(τ_i)'s are pairwise disjoint since $0 = <0, 0>$. This implies that the elements of $\{\text{Ran}(e_n) \mid n \neq 1\} \cup \{\text{Ran}(e^n) \mid n \geq 2\}$ are pairwise disjoint.

- Note that the fact that $\alpha \in \mathbf{L}_n := \zeta^{-1}(K_n)$ depends only on $\alpha \circ e_n$ if $n \neq 1$. We set, for $n \neq 1$,

$$\mathbf{M}_n := \{\alpha \circ e_n \mid \alpha \in \mathbf{L}_n\}.$$

Note that $\zeta^{-1}(K_0) = \zeta^{-1}\big(\tilde{\tau_1}^{-1}(\mathbf{H}_{u_0})\big) = (\tilde{\tau_1} \circ \zeta)^{-1}(\mathbf{H}_{u_0})$ is $\mathbf{\Gamma}_{u_0^\eta}$-strategically complete since u_0 is nice and $\tilde{\tau_1} \circ \zeta$ is in $\mathcal{R}$ of order η. Similarly, $\zeta^{-1}(K_n)$ is $\mathbf{\Gamma}_{v_n^\eta}$-strategically complete if $n \geq 2$. By Lemma 5.2.1.(b), $\mathbf{M}_0$ is $\mathbf{\Gamma}_{u_0^\eta}$-strategically complete, and $\mathbf{M}_n$ is $\mathbf{\Gamma}_{v_n^\eta}$-strategically complete if $n \geq 2$.

- We set, for $n\geq 2$, $\mathbf{D}_n:=\{\alpha\circ e^n\mid\exists m\in\omega\ \ \zeta(\alpha)_0(m)=1\ \text{ and }\ (m)_0+2=n\}$. Let us prove that $\mathbf{D}_n$ is $\mathbf{\Sigma}^0_{1+\eta}$-strategically complete.

Note first that $\{\alpha\in\mathcal{C}\mid f(\alpha)\neq 0^\infty\}$ is $\mathbf{\Sigma}^0_{1+\eta}$-strategically complete if f is an independent η-function. Indeed, using the notation of Definition 3.3, we can write

$$\{\alpha\in\mathcal{C}\mid f(\alpha)=0^\infty\}=\bigcap_{m\in\omega}\neg\,\mathbf{Z}_m.$$

Moreover, the fact that $\alpha\in\mathbf{Z}_m$ depends only of the values of α on $\pi_f^{-1}(\{m\})$.

Assume first that $\eta\geq 1$. As f is an independent η-function, $\mathbf{Z}_m$ is $\mathbf{\Pi}^0_{1+\theta_m}$-strategically complete, for some $\theta_m<\eta$ satisfying $\theta_m+1=\eta$ if η is a successor ordinal, and $\sup_{m\in\omega}\theta_m=\eta$ if η is a limit ordinal. Note that $\eta=\sup_{m\in\omega}(\theta_m+1)$. By Lemma 3.7 in [Lo-SR1], $\{\alpha\in\mathcal{C}\mid f(\alpha)=0^\infty\}$ is $\mathbf{\Pi}^0_{1+\eta}$-strategically complete.

Assume now that $\eta=0$. As in the proof of Theorem 5.1.7 we see that $\{\alpha\in\mathcal{C}\mid f(\alpha)=0^\infty\}$ is $\mathbf{\Pi}^0_{1+\eta}$-strategically complete.

Now we come back to the $\mathbf{D}_n$'s. We define $\tau:\omega\to\omega$ by $\tau(k):=<n-2,k>$, so that τ is one-to-one and $\text{Ran}(\tau)=\{m\in\omega\mid(m)_0=n-2\}$. As ζ is an independent η-function, $\tilde{\tau}_0\circ\zeta$ and $\tilde{\tau}\circ\tilde{\tau}_0\circ\zeta$ are also independent η-functions, by Lemma 5.1.6. The previous point shows that

$$\mathbf{P}:=\{\alpha\in\mathcal{C}\mid(\tilde{\tau}\circ\tilde{\tau}_0\circ\zeta)(\alpha)\neq 0^\infty\}$$

is $\mathbf{\Sigma}^0_{1+\eta}$-strategically complete. Note that

$$\mathbf{P}=\{\alpha\in\mathcal{C}\mid\exists k\in\omega\ \ \tilde{\tau}\big((\tilde{\tau}_0\circ\zeta)(\alpha)\big)(k)=1\}=\{\alpha\in\mathcal{C}\mid\exists k\in\omega\ \ (\tilde{\tau}_0\circ\zeta)(\alpha)\big(\tau(k)\big)=1\}$$

$$=\{\alpha\in\mathcal{C}\mid\exists m\in\omega\ \ (\tilde{\tau}_0\circ\zeta)(\alpha)(m)=1\ \text{ and }\ (m)_0+2=n\},$$

and the fact that $\alpha\in\mathbf{P}$ depends only on $\alpha\circ e^n$. By Lemma 5.2.1.(b), $\mathbf{D}_n$ is $\mathbf{\Sigma}^0_{1+\eta}$-strategically complete.

- Let $M\in\mathbf{\Gamma}_{(12<u_p>)^\eta}(\mathcal{N})$, say $M=\bigcup_{n\geq 2}(M_n\cap D_n)\cup(M_0\setminus\bigcup_{n\geq 2}D_n)$, with $D_n\in\mathbf{\Sigma}^0_{1+\eta}$ pairwise disjoint, $M_0\in\mathbf{\Gamma}_{u_0^\eta}$, and without loss of generality $M_n\in\mathbf{\Gamma}_{v_n^\eta}$. Then Player 2 has a winning strategy σ_n in $G(M_n,\mathbf{M}_n)$ (for any $n\neq 1$), and a winning strategy ρ_n in $G(D_n,\mathbf{D}_n)$ (for any $n\geq 2$). Then Player 2 plays in $G\big(M,\zeta^{-1}(\mathbf{H}_{u_{12<u_p>}})\big)$ against β by playing his strategies σ_n, ρ_n at the right places (the ranges of e_n and e^n respectively) against this same β, independently, and by playing 0 out of these ranges. The result is some α such that $\alpha\circ e_n$ wins against β in $G(M_n,\mathbf{M}_n)$ and $\alpha\circ e^n$ wins against β in $G(D_n,\mathbf{D}_n)$. This wins, since $\alpha\in\zeta^{-1}(K_n)$ exactly when $\beta\in M_n$, and $\zeta(\alpha)_0$ takes value 1 on some m with $(m)_0+2=n$ exactly when $\beta\in D_n$. But as the D_n's are pairwise disjoint, there is at most one n in $\{(m)_0+2\mid\zeta(\alpha)_0(m)=1\}$, and $\alpha\in\zeta^{-1}(C_n)$ exactly when $\beta\in D_n$. Thus $\zeta^{-1}(\mathbf{H}_{12<u_p>})$ is $\mathbf{\Gamma}_{(12<u_p>)^\eta}$-strategically complete.

- It remains to see that $\zeta^{-1}(\mathbf{H}_{12<u_p>})$ is ccs. So let $\alpha_0\in d^\omega$ and $F:\mathcal{C}\to(d^\omega)^{d-1}$ satisfying the conclusion of Lemma 2.4.(b).

$\circ$ Let $N\geq 1$ and $M\in\omega$. Then $\zeta(\alpha)_N\in\mathbf{H}_{u_M}\Leftrightarrow(\tilde{\tau_N}\circ\zeta)(\alpha)\in\mathbf{H}_{u_M}\Leftrightarrow\alpha\in(\tilde{\tau_N}\circ\zeta)^{-1}(\mathbf{H}_{u_M})$. As $N\geq 1$, $\tilde{\tau_N}\circ\zeta$ is in $\mathcal{R}$, and $(\tilde{\tau_N}\circ\zeta)^{-1}(\mathbf{H}_{u_M})$ is ccs since u_M is nice. Thus $\zeta(\alpha)_N\in\mathbf{H}_{u_M}$ if and only if $\zeta\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big)_N\in\mathbf{H}_{u_M}$.

$\circ$ Recall the notation before Lemma 2.4. We define $q:\omega^{<\omega}\setminus\{\emptyset\}\rightarrow\omega$ as follows:

$$q(t):=\begin{cases} t(0) \text{ if } |t|=1,\\ \\ <t(|t|-1),q(t^-)> \text{ if } |t|\geq 2.\end{cases}$$

$\circ$ Let us prove that $\tilde{\tau}_s(\alpha)(n)=\alpha(<q\big((n)_0s\big),(n)_1>)$ for any $s\in\mathcal{F}$.

We argue by induction on $|s|$. So assume that the result is proved for $|s|\leq l$, which is the case for $l=0$. Assume that $|s|=l+1$. We get

$$\tilde{\tau}_s(\alpha)(n)=\tilde{\tau}_{s|l}\big(\tilde{\tau_{s(l)}}(\alpha)\big)(n)=\tilde{\tau_{s(l)}}(\alpha)(<q\big((n)_0(s|l)\big),(n)_1>)=\alpha\big(\tau_{s(l)}(<q\big((n)_0(s|l)\big),(n)_1>)\big)$$

$$=\alpha\big(\big\langle<s(l),q\big((n)_0(s|l)\big)>,(n)_1\big\rangle\big)=\alpha(<q\big((n)_0s\big),(n)_1>).$$

$\circ$ Let us prove that $(\rho\circ\tilde{\tau}_s)(\alpha)=(\rho\circ\tilde{\tau}_s)\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big)$ for any $s\in\mathcal{F}$ and any $\alpha\in\mathcal{C}$. This comes from the following equivalences:

$$\begin{aligned}(\rho\circ\tilde{\tau}_s)(\alpha)(n)=0 &\Leftrightarrow\exists m\in\omega\ \ \tilde{\tau}_s(\alpha)(<n,m>)=1\Leftrightarrow\exists m\in\omega\ \ \alpha(<q(ns),m>)=1\\ &\Leftrightarrow\exists k\in\omega\ \ \mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)(<q(ns),k>)=1\\ &\Leftrightarrow(\rho\circ\tilde{\tau}_s)\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big)(n)=0.\end{aligned}$$

$\circ$ Let us prove that $(\rho^\eta\circ\tilde{\tau}_s)(\alpha)=(\rho^\eta\circ\tilde{\tau}_s)\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big)$ for any $1\leq\eta<\omega_1$, any $s\in\mathcal{F}$ and any $\alpha\in\mathcal{C}$.

We argue by induction on η. For $\eta=1$, this comes from the previous point. If $\theta\geq 1$ and $\eta=\theta+1$, then this comes from the fact that $\rho^\eta=\rho\circ\rho^\theta$. If η is a limit ordinal and m is a natural number, then

$$\begin{aligned}&(\rho^\eta\circ\tilde{\tau}_s)(\alpha)(m)\\ &=\rho^\eta\big(\tilde{\tau}_s(\alpha)\big)(m)=\rho^{(0,m+1)}\big(\tilde{\tau}_s(\alpha)\big)(m)\\ &=(\rho^{(m,m+1)}\circ ...\circ\rho^{(1,2)})\Big(\rho^{(0,1)}\big(\tilde{\tau}_s(\alpha)\big)\Big)(m)=(\rho^{(m,m+1)}\circ ...\circ\rho^{(1,2)})\Big(\rho^{\theta_0}\big(\tilde{\tau}_s(\alpha)\big)\Big)(m)\\ &=(\rho^{(m,m+1)}\circ ...\circ\rho^{(1,2)})\left(\rho^{\theta_0}\left(\tilde{\tau}_s\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big)\right)\right)(m)=(\rho^\eta\circ\tilde{\tau}_s)\Big(\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\Big)(m).\end{aligned}$$

$\circ$ Note that $\zeta(\alpha)_0 = 0^\infty \Leftrightarrow \alpha \in (\tilde{\tau}_0 \circ \zeta)^{-1}(\{0^\infty\})$. Let us prove that $(\tilde{\tau}_0 \circ \zeta)^{-1}(\{0^\infty\})$ is ccs.

We can write $\zeta = \circ_{j\leq l}\, \zeta^j$, where l is a natural number and each ζ^j is either of the form ρ^η, or one of the $\tilde{\tau}_i$'s for $i \geq 1$. By the previous point, we may assume that that each ζ^j is either $\rho^0 = \mathrm{Id}_{\mathcal{C}}$, or one of the $\tilde{\tau}_i$'s for $i \geq 1$. So there is $s \in \mathcal{F}$ such that $\zeta = \tilde{\tau}_s$. Note that

$$\begin{aligned}\alpha \notin (\tilde{\tau}_0 \circ \zeta)^{-1}(\{0^\infty\}) &\Leftrightarrow \exists m \in \omega \ \ (\tilde{\tau}_0 \circ \zeta)(\alpha)(m) = 1 \Leftrightarrow \exists m \in \omega \ \ \zeta(\alpha)\big(\tau_0(m)\big) = 1\\ &\Leftrightarrow \exists m \in \omega \ \ \tilde{\tau}_s(\alpha)(<0, m>) = 1 \Leftrightarrow \exists m \in \omega \ \ \alpha(<q(0s), m>) = 1\\ &\Leftrightarrow \exists m \in \omega \ \ \alpha\big(p(q(0s), m)\big) = 1\\ &\Leftrightarrow \exists k \in \omega \ \ \mathcal{S}\big(\alpha_0 \Delta F_0(\alpha)\big)\big(p(q(0s), k)\big) = 1\\ &\Leftrightarrow \mathcal{S}\big(\alpha_0 \Delta F_0(\alpha)\big) \notin (\tilde{\tau}_0 \circ \zeta)^{-1}(\{0^\infty\}).\end{aligned}$$

Thus $\zeta(\alpha)_0 = 0^\infty \Leftrightarrow \zeta\Big(\mathcal{S}\big(\alpha_0 \Delta F_0(\alpha)\big)\Big)_0 = 0^\infty$.

$\circ$ It remains to see that if $\zeta(\alpha)_0 \neq 0^\infty$ and m_α is minimal with $\zeta(\alpha)_0(m_\alpha) = 1$, then

$$(m_\alpha)_0 = (m_{\mathcal{S}(\alpha_0 \Delta F_0(\alpha))})_0.$$

As in the previous point we may assume that there is $s \in \mathcal{F}$ such that $\zeta = \tilde{\tau}_s$. The computations of the previous point show that $\zeta(\alpha)_0(m) = \alpha(<q(0s), m>)$ for each natural number m. Note that

$$n_\alpha := <q(0s), m_\alpha> = \min\{n \in \omega \mid \alpha(n) = 1 \wedge (n)_0 = q(0s)\}$$

since $<q(0s), .>$ is increasing, and, similarly,

$$<q(0s), m_{\mathcal{S}(\alpha_0 \Delta F_0(\alpha))}> = \min\{m \in \omega \mid \mathcal{S}\big(\alpha_0 \Delta F_0(\alpha)\big)(m) = 1 \wedge (m)_0 = q(0s)\}.$$

As B_α is a bijection satisfying $(n)_0 = \big(B_\alpha(n)\big)_0$,

$$B_\alpha[\{n \in \omega \mid \alpha(n) = 1 \wedge (n)_0 = q(0s)\}] = \{m \in \omega \mid \mathcal{S}\big(\alpha_0 \Delta F_0(\alpha)\big)(m) = 1 \wedge (m)_0 = q(0s)\}.$$

As B_α is increasing, $B_\alpha(n_\alpha) = <q(0s), m_{\mathcal{S}(\alpha_0 \Delta F_0(\alpha))}>$. Thus

$$(m_{\mathcal{S}(\alpha_0 \Delta F_0(\alpha))})_0 = \Big(\big(B_\alpha(n_\alpha)\big)_1\Big)_0 = \big((n_\alpha)_1\big)_0 = (m_\alpha)_0$$

and we are done. □

Corollary 5.2.5 *Let $\mathbf{\Gamma}$ be a non self-dual Wadge class of Borel sets. Then there is $\mathbf{C} \subseteq \mathcal{C}$ which is $\mathbf{\Gamma}$-complete and ccs.*

Proof. By Theorem 5.1.3 there is $u \in \mathcal{D}$ such that $\mathbf{\Gamma}(\mathcal{N}) = \mathbf{\Gamma}_u(\mathcal{N})$. By Theorem 5.2.4 there is $\mathbf{H}_u \subseteq \mathcal{C}$ which is strongly $\mathbf{\Gamma}_u$-strategically complete. It is clear that $\mathbf{C} := \mathbf{H}_u$ is suitable. □

Now we can prove Theorem 1.9.(1). But we need some more material to prove Theorem 1.9.(2).

Definition 5.2.6 *(a) A set* $\mathbf{U}\subseteq\mathcal{C}$ *is* $strongly\ ccs$ *if* $\tilde{\tau}_s^{-1}(\mathbf{U})$ *is ccs for any* $s\in\mathcal{F}$.

(b) Let $\mathbf{\Gamma}$ *be a Wadge class of Borel sets, and* $\mathbf{U}_0,\mathbf{U}_1\in\mathbf{\Gamma}(\mathcal{C})$ *be disjoint. We say that* $(\mathbf{U}_0,\mathbf{U}_1)$ $is\ complete\ for\ pairs\ of\ disjoint$ $\mathbf{\Gamma}$ $sets$ *if for any pair* (A_0,A_1) *of disjoint sets in* $\mathbf{\Gamma}(\mathcal{N})$ *there is* $f:\mathcal{N}\rightarrow\mathcal{C}$ *continuous such that* $A_\varepsilon=f^{-1}(\mathbf{U}_\varepsilon)$ *for any* $\varepsilon\in 2$. *Similarly, we can define the notion of a sequence* $(\mathbf{U}_p)_{p\geq 1}$ *complete for sequences of pairwise disjoint* $\mathbf{\Gamma}$ *sets.*

Lemma 5.2.7 *(a) There is* $(\mathbf{U}_0,\mathbf{U}_1)$ *complete for pairs of disjoint* $\mathbf{\Sigma}^0_1$ *sets with* $\mathbf{U}_\varepsilon$ *strongly ccs, and such that for any* $s\in\mathcal{F}$ *there is a pair* (O_0,O_1) *of ccs* $\mathbf{\Sigma}^0_1$ *sets reducing*

$$\big(\tilde{\tau}_{1s1}^{-1}(\mathbf{U}_0\cup\mathbf{U}_1),\tilde{\tau}_{1s2}^{-1}(\mathbf{U}_0\cup\mathbf{U}_1)\big).$$

(b) There is $(\mathbf{U}_p)_{p\geq 1}$ *complete for sequences of pairwise disjoint* $\mathbf{\Sigma}^0_1$ *sets with* $\mathbf{U}_p$ *strongly ccs, and such that for any* $s\in\mathcal{F}$ *there is a sequence* $(O^\varepsilon_p)_{\varepsilon\in 2,p\geq 1}$ *of ccs* $\mathbf{\Sigma}^0_1$ *sets reducing*

$$\big(\tilde{\tau}_{s(\varepsilon+1)}^{-1}(\mathbf{U}_p)\big)_{\varepsilon\in 2,p\geq 1}.$$

Proof. (a) Recall the definition of $\mathbf{H}_1$ after Definition 3.3: $\mathbf{H}_1:=\{0^\infty\}$. We saw that $\mathbf{H}_1\in\mathbf{\Pi}^0_1(\mathcal{C})$ and is $\mathbf{\Pi}^0_1$-complete. We set $\mathbf{U}:=\neg\mathbf{H}_1$, so that $\mathbf{U}$ is $\mathbf{\Sigma}^0_1$-complete. Let (A_0,A_1) be a pair of disjoint $\mathbf{\Sigma}^0_1$ subsets of $\mathcal{N}$. As $\mathbf{U}$ is complete there are $f_0,f_1:\mathcal{N}\rightarrow\mathcal{C}$ continuous such that $A_\varepsilon=f_\varepsilon^{-1}(\mathbf{U})$ for each $\varepsilon\in 2$. We define $f:\mathcal{N}\rightarrow\mathcal{C}$ by

$$f(\alpha)\big(\langle <\varepsilon+1,(k)_0>,(k)_1\rangle\big):=\begin{cases} f_\varepsilon(\alpha)(k) \text{ if } \varepsilon\in 2,\\ 0 \text{ otherwise,}\end{cases}$$

so that f is continuous and $f_\varepsilon=\tilde{\tau}_{\varepsilon+1}\circ f$. Now $A_\varepsilon=f^{-1}\big(\tilde{\tau}_{\varepsilon+1}^{-1}(\mathbf{U})\big)$ and $\big(\tilde{\tau}_1^{-1}(\mathbf{U}),\tilde{\tau}_2^{-1}(\mathbf{U})\big)$ is complete for pairs of $\mathbf{\Sigma}^0_1$ sets (not necessarily disjoint). Note that

$$\tilde{\tau}_{\varepsilon+1}^{-1}(\mathbf{U})=\big\{\alpha\in\mathcal{C}\mid\exists k\in\omega\ \ \alpha\big(\langle <\varepsilon+1,(k)_0>,(k)_1\rangle\big)=1\big\}$$

$$=\big\{\alpha\in\mathcal{C}\mid\exists N\in\omega\ \ \big((N)_0\big)_0=\varepsilon+1\wedge\alpha(N)=1\big\}.$$

We set $\mathbf{V}_\varepsilon:=\Big\{\alpha\in\mathcal{C}\mid\exists N\in\omega\ \ \big((N)_0\big)_0=\varepsilon+1\wedge\alpha(N)=1\wedge\forall l<N\ \ \Big(\big((l)_0\big)_0\notin\{1,2\}\vee\alpha(l)=0\Big)\Big\}$. Note that $\mathbf{V}_i\in\mathbf{\Sigma}^0_1$ and $(\mathbf{V}_0,\mathbf{V}_1)$ reduces $\big(\tilde{\tau}_1^{-1}(\mathbf{U}),\tilde{\tau}_2^{-1}(\mathbf{U})\big)$. Thus

$$\alpha\in A_\varepsilon\Leftrightarrow f(\alpha)\in\tilde{\tau}_{\varepsilon+1}^{-1}(\mathbf{U})\Leftrightarrow f(\alpha)\in\tilde{\tau}_{\varepsilon+1}^{-1}(\mathbf{U})\setminus\tilde{\tau}_{2-\varepsilon}^{-1}(\mathbf{U})\Leftrightarrow f(\alpha)\in\mathbf{V}_\varepsilon$$

and $(\mathbf{V}_0,\mathbf{V}_1)$ is complete for pairs of disjoint $\mathbf{\Sigma}^0_1$ sets. Recall the definition of τ_0 before Lemma 5.2.1. We set $\mathbf{U}_\varepsilon:=\tilde{\tau}_0^{-1}(\mathbf{V}_\varepsilon)$, which defines a pair of disjoint $\mathbf{\Sigma}^0_1$ sets. Now $g(\alpha):=<\alpha,\alpha,...>$ defines $g:\mathcal{C}\rightarrow\mathcal{C}$ continuous. Note that $\alpha\in A_\varepsilon\Leftrightarrow f(\alpha)\in\mathbf{V}_\varepsilon\Leftrightarrow\tilde{\tau}_0\Big(g\big(f(\alpha)\big)\Big)\in\mathbf{V}_\varepsilon\Leftrightarrow g\big(f(\alpha)\big)\in\mathbf{U}_\varepsilon$, which shows that $(\mathbf{U}_0,\mathbf{U}_1)$ is complete for pairs of disjoint $\mathbf{\Sigma}^0_1$ sets.

Fix $s\in\mathcal{F}$. The proof of Theorem 5.2.4 shows that $\tilde{\tau}_s(\alpha)(n)=\alpha\big(<q\big((n)_0s\big),(n)_1>\big)$. Thus

$$\tilde{\tau}_s^{-1}(\mathbf{U}_\varepsilon)=\Big\{\alpha\in\mathcal{C}\mid\exists N\in\omega\ \ \big((N)_0\big)_0=\varepsilon+1\wedge\alpha(<q(0s),N>)=1\wedge$$
$$\forall l<N\ \ \Big(\big((l)_0\big)_0\notin\{1,2\}\vee\alpha(<q(0s),l>)=0\Big)\Big\}.$$

Thus

$$\tilde{\tau}_s^{-1}(\mathbf{U}_\varepsilon)=\Big\{\alpha\in\mathcal{C}\mid\exists M\in\omega\ \ \Big(\big(((M)_1\big)_0\Big)_0=\varepsilon+1\wedge(M)_0=q(0s)\wedge\alpha(M)=1\wedge\\ \forall l<M\ \ \Big(\Big(\big(((l)_1\big)_0\Big)_0\notin\{1,2\}\vee(l)_0\neq q(0s)\vee\alpha(l)=0\Big)\Big\}.$$

Recall the conclusion of Lemma 2.4.(b). The bijection B_α induces an increasing bijection between $\Big\{M\in\omega\mid\Big(\big((M)_1\big)_0\Big)_0\in\{1,2\}\wedge(M)_0=q(0s)\wedge\alpha(M)=1\Big\}$ and

$$\Big\{K\in\omega\mid\Big(\big((K)_1\big)_0\Big)_0\in\{1,2\}\wedge(K)_0=q(0s)\wedge\mathcal{S}\big(\alpha_0\Delta F(\alpha)\big)(K)=1\Big\}$$

since $(M)_0=\big(B_\alpha(M)\big)_0$ and $\big((M)_1\big)_0=\Big(\big(B_\alpha(M)\big)_1\Big)_0$. A second application of this shows that $\tilde{\tau}_s^{-1}(\mathbf{U}_\varepsilon)$ is ccs. Thus $\mathbf{U}_\varepsilon$ is strongly ccs. Note that

$$\tilde{\tau}_{1s(\varepsilon+1)}^{-1}(\mathbf{U}_0\cup\mathbf{U}_1)=\Big\{\alpha\in\mathcal{C}\mid\exists M\in\omega\ \Big(\big((M)_1\big)_0\Big)_0\in\{1,2\}\wedge(M)_0=q\big(01s(\varepsilon+1)\big)\wedge\alpha(M)=1\Big\}.$$

We set

$$O_\varepsilon:=\Big\{\alpha\in\mathcal{C}\mid\exists M\in\omega\ \Big(\big((M)_1\big)_0\Big)_0\in\{1,2\}\wedge(M)_0=q\big(01s(\varepsilon+1)\big)\wedge\alpha(M)=1\wedge\\ \forall l<M\ \ \Big(\Big(\big((l)_1\big)_0\Big)_0\notin\{1,2\}\vee(l)_0\notin\{q(01s1),q(01s2)\}\vee\alpha(l)=0\Big)\Big\}.$$

This defines a pair of $\mathbf{\Sigma}^0_1$ sets reducing $\big(\tilde{\tau}_{1s1}^{-1}(\mathbf{U}_0\cup\mathbf{U}_1),\tilde{\tau}_{1s2}^{-1}(\mathbf{U}_0\cup\mathbf{U}_1)\big)$. We check that they are ccs as before.

(b) The proof is completely similar to that of (a). $\square$

The following result is a consequence of Theorem 1.9 and Lemmas 1.11, 1.23 in [Lo1], and also of Theorem 3 in [Lo-SR3]:

Theorem 5.2.8 *Let $\mathbf{\Gamma}$ be a self-dual Wadge class of Borel sets. Then there is a non self-dual Wadge class of Borel sets $\mathbf{\Gamma}'$ such that $\mathbf{\Gamma}(\mathcal{N})=\Delta(\mathbf{\Gamma}')(\mathcal{N})$, $\mathbf{\Gamma}'$ does not have the separation property, and one of the following holds:*
(1) There is $\overline{u}\in\mathcal{D}$ such that

$$\mathbf{\Gamma}'(\mathcal{N})=\Big\{(A_0\cap C_0)\cup(A_1\cap C_1)\mid A_0,\neg A_1\in\mathbf{\Gamma}_{\overline{u}}(\mathcal{N})\wedge C_0,C_1\in\mathbf{\Sigma}^0_1(\mathcal{N})\wedge C_0\cap C_1=\emptyset\Big\}.$$

(2) There is $\big((u')_p\big)_{p\geq 1}\in\mathcal{D}^\omega$ such that $\big(\mathbf{\Gamma}_{(u')_p}(\mathcal{N})\big)_{p\geq 1}$ is strictly increasing and

$$\mathbf{\Gamma}'(\mathcal{N})=\Big\{\bigcup_{p\geq 1}(A_p\cap C_p)\mid A_p\in\mathbf{\Gamma}_{(u')_p}(\mathcal{N})\wedge C_p\in\mathbf{\Sigma}^0_1(\mathcal{N})\wedge C_p\cap C_q=\emptyset\textit{ if }p\neq q\Big\}.$$

Lemma 5.2.9 *Let $\mathbf{\Gamma}'$ be as in the statement of Theorem 5.2.8. Then there are $\mathbf{C}_0,\mathbf{C}_1\in\mathbf{\Gamma}'(\mathcal{C})$ disjoint, ccs, and not separable by a $\Delta(\mathbf{\Gamma}')$ set.*

Proof. (1) Lemma 5.2.7.(a) gives $(\mathbf{U}_0,\mathbf{U}_1)$ complete for pairs of disjoint $\mathbf{\Sigma}^0_1$ sets with $\mathbf{U}_\varepsilon$ strongly ccs, and such that for any $s\in\mathcal{F}$ there is a pair (O_0,O_1) of ccs $\mathbf{\Sigma}^0_1$ sets reducing the pair

$$\big(\tilde\tau_{1s1}^{-1}(\mathbf{U}_0\cup\mathbf{U}_1),\tilde\tau_{1s2}^{-1}(\mathbf{U}_0\cup\mathbf{U}_1)\big).$$

Theorem 5.2.4 gives $\mathbf{H}_{\overline{u}}\subseteq\mathcal{C}$ which is $\mathbf{\Gamma}_{\overline{u}}$-complete and strongly ccs. We set

$$\mathbf{H}:=\big(\tilde\tau_2^{-1}(\mathbf{H}_{\overline{u}})\cap\tilde\tau_1^{-1}(\mathbf{U}_0)\big)\cup\big(\tilde\tau_3^{-1}(\neg\mathbf{H}_{\overline{u}})\cap\tilde\tau_1^{-1}(\mathbf{U}_1)\big)$$

and, for $\varepsilon\in 2$, $E_\varepsilon:=\tilde\tau_{\varepsilon+1}^{-1}(\mathbf{H})$. Finally, we set $\mathbf{C}_\varepsilon:=(O_\varepsilon\cap E_\varepsilon)\cup(O_{1-\varepsilon}\setminus E_{1-\varepsilon})$, where (O_0,O_1) is associated with $s:=\emptyset$.

• We set, for $\varepsilon,j\in 2$, $A^\varepsilon_0:=\tilde\tau_{2(\varepsilon+1)}^{-1}(\mathbf{H}_{\overline{u}})$, $A^\varepsilon_1:=\tilde\tau_{3(\varepsilon+1)}^{-1}(\neg\mathbf{H}_{\overline{u}})$, $F^\varepsilon_j:=\tilde\tau_{1(\varepsilon+1)}^{-1}(\mathbf{U}_j)$, so that

$$E_\varepsilon=(A^\varepsilon_0\cap F^\varepsilon_0)\cup(A^\varepsilon_1\cap F^\varepsilon_1).$$

Note that

$$\mathbf{C}_\varepsilon=(A^\varepsilon_0\cap F^\varepsilon_0\cap O_\varepsilon)\cup(A^\varepsilon_1\cap F^\varepsilon_1\cap O_\varepsilon)\cup(\neg A^{1-\varepsilon}_0\cap F^{1-\varepsilon}_0\cap O_{1-\varepsilon})\cup(\neg A^{1-\varepsilon}_1\cap F^{1-\varepsilon}_1\cap O_{1-\varepsilon})$$

$$=\Big(\big((A^\varepsilon_0\cap F^\varepsilon_0\cap O_\varepsilon)\cup(\neg A^{1-\varepsilon}_1\cap F^{1-\varepsilon}_1\cap O_{1-\varepsilon})\big)\cap\big((F^\varepsilon_0\cap O_\varepsilon)\cup(F^{1-\varepsilon}_1\cap O_{1-\varepsilon})\big)\Big)\cup$$

$$\Big(\big((A^\varepsilon_1\cap F^\varepsilon_1\cap O_\varepsilon)\cup(\neg A^{1-\varepsilon}_0\cap F^{1-\varepsilon}_0\cap O_{1-\varepsilon})\big)\cap\big((F^\varepsilon_1\cap O_\varepsilon)\cup(F^{1-\varepsilon}_0\cap O_{1-\varepsilon})\big)\Big),$$

and that $F^\varepsilon_0\cap O_\varepsilon$, $F^{1-\varepsilon}_1\cap O_{1-\varepsilon}$, $F^\varepsilon_1\cap O_\varepsilon$, $F^{1-\varepsilon}_0\cap O_{1-\varepsilon}$ are pairwise disjoint open subsets of $\mathcal{C}$. By Lemma 5.2.2 and the reduction property for $\mathbf{\Sigma}^0_1$ we can write $\mathbf{C}_\varepsilon$ as the intersection of $\mathcal{C}$ with

$$\Big(\big((\mathcal{A}^\varepsilon_0\cap\mathcal{O}^\varepsilon_0)\cup(\neg\mathcal{A}^{1-\varepsilon}_1\cap\mathcal{O}^{1-\varepsilon}_1)\big)\cap(\mathcal{O}^\varepsilon_0\cup\mathcal{O}^{1-\varepsilon}_1)\Big)\cup\Big(\big((\mathcal{A}^\varepsilon_1\cap\mathcal{O}^\varepsilon_1)\cup(\neg\mathcal{A}^{1-\varepsilon}_0\cap\mathcal{O}^{1-\varepsilon}_0)\big)\cap(\mathcal{O}^\varepsilon_1\cup\mathcal{O}^{1-\varepsilon}_0)\Big),$$

where $\mathcal{A}^\varepsilon_0,\neg\mathcal{A}^\varepsilon_1\in\mathbf{\Gamma}_{\overline{u}}(\mathcal{N})$ and $\mathcal{O}^\varepsilon_j$ are four pairwise disjoint open subsets of $\mathcal{N}$. By Lemma 1.4.(b) in [Lo1], $(\mathcal{A}^\varepsilon_0\cap\mathcal{O}^\varepsilon_0)\cup(\neg\mathcal{A}^{1-\varepsilon}_1\cap\mathcal{O}^{1-\varepsilon}_1),\neg\big((\mathcal{A}^\varepsilon_1\cap\mathcal{O}^\varepsilon_1)\cup(\neg\mathcal{A}^{1-\varepsilon}_0\cap\mathcal{O}^{1-\varepsilon}_0)\big)\in\mathbf{\Gamma}_{\overline{u}}(\mathcal{N})$, so that $\mathbf{C}_\varepsilon\in\mathbf{\Gamma}'(\mathcal{C})$, by Lemma 5.2.2 again.

• It is clear that $\mathbf{C}_0$ and $\mathbf{C}_1$ are disjoint and ccs.

• Assume, towards a contradiction, that $D\in\Delta(\mathbf{\Gamma}')$ separates $\mathbf{C}_0$ from $\mathbf{C}_1$. Let $D_0,D_1\in\mathbf{\Gamma}'(\mathcal{N})$ be disjoint. As $\mathbf{H}$ is complete there are $f_\varepsilon:\mathcal{N}\to\mathcal{C}$ continuous such that $D_\varepsilon=f_\varepsilon^{-1}(\mathbf{H})$. We define $f:\mathcal{N}\to\mathcal{C}$ by

$$f(\alpha)\big(\langle <\varepsilon+1,(k)_0>,(k)_1\rangle\big):=\begin{cases}f_\varepsilon(\alpha)(k)\text{ if }\varepsilon\in 2,\\ 0\text{ otherwise,}\end{cases}$$

so that $\big(f(\alpha)\big)_{\varepsilon+1}=f_\varepsilon(\alpha)$. Then f is continuous and $D_\varepsilon=f^{-1}(E_\varepsilon)$. Note that $E_\varepsilon\setminus E_{1-\varepsilon}\subseteq\mathbf{C}_\varepsilon$. This implies that $\alpha\in D_0\Leftrightarrow f(\alpha)\in E_0\Leftrightarrow f(\alpha)\in E_0\setminus E_1\Rightarrow f(\alpha)\in\mathbf{C}_0\subseteq D$. Similarly, $D_1\subseteq f^{-1}(\neg D)$, and $f^{-1}(D)\in\Delta(\mathbf{\Gamma}')(\mathcal{N})$ separates D_0 from D_1. Thus $\mathbf{\Gamma}'$ has the separation property, which is absurd.

(2) Lemma 5.2.7.(b) gives $(\mathbf{U}_p)_{p\geq 1}$ complete for sequences of pairwise disjoint $\mathbf{\Sigma}^0_1$ sets with $\mathbf{U}_p$ strongly ccs, and such that for any $s\in\mathcal{F}$ there is a sequence $(O^\varepsilon_p)_{\varepsilon\in 2,p\geq 1}$ of ccs $\mathbf{\Sigma}^0_1$ sets reducing $\big(\tilde\tau_{s(\varepsilon+1)}^{-1}(\mathbf{U}_p)\big)_{\varepsilon\in 2,p\geq 1}$. Theorem 5.2.4 gives $\mathbf{H}_{(u')_p}\subseteq\mathcal{C}$ which is $\mathbf{\Gamma}_{(u')_p}$-complete and strongly ccs. We set $\mathbf{H}:=\bigcup_{p\geq 1}\ \big(\tilde\tau_{2p}^{-1}(\mathbf{H}_{(u')_p})\cap\tilde\tau_1^{-1}(\mathbf{U}_p)\big)$ and, for $\varepsilon\in 2$, $E_\varepsilon:=\tilde\tau_{\varepsilon+1}^{-1}(\mathbf{H})$.

We also set $A_p^\varepsilon := \tilde{\tau}^{-1}_{(2p)(\varepsilon+1)}(\mathbf{H}_{(u')_p})$, $F_p^\varepsilon := \tilde{\tau}^{-1}_{1(\varepsilon+1)}(\mathbf{U}_p)$, so that $E_\varepsilon = \bigcup_{p\geq 1}\ (A_p^\varepsilon \cap F_p^\varepsilon)$. Finally, we set $\mathbf{C}_\varepsilon := (A_1^\varepsilon \cap O_1^\varepsilon) \cup \bigcup_{p\geq 1}\ \big((O_p^{1-\varepsilon}\setminus A_p^{1-\varepsilon}) \cup (A_{p+1}^\varepsilon \cap O_{p+1}^\varepsilon)\big)$, where $(O_p^\varepsilon)_{\varepsilon\in 2, p\geq 1}$ is associated with $s := 1$.

Note that $\mathbf{C}_\varepsilon \in \mathbf{\Gamma}'(\mathcal{C})$ since $\big(\mathbf{\Gamma}_{(u')_p}(\mathcal{N})\big)_{p\geq 1}$ is strictly increasing, using again Lemma 5.2.2, the generalized reduction property for $\mathbf{\Sigma}^0_1$ (see 22.16 in [K]), and Lemma 1.4.(b) in [Lo1]. Here again, $E_\varepsilon \setminus E_{1-\varepsilon} \subseteq \mathbf{C}_\varepsilon$ and we conclude as in (1). □

Proof of Theorem 1.9. It is clear that Proposition 2.2, Lemmas 2.3, 2.6, Corollary 5.2.5, Lemma 5.2.9 and Theorem 3.1 imply Theorem 1.9, if we set $\mathbb{S} := S_{\mathbf{C}}$ and $\mathbb{S}_\varepsilon := S_{\mathbf{C}_\varepsilon}$. □

6 The proof of Theorem 1.10

We first introduce an operator in the spirit of $\mathfrak{F}$ defined before Theorem 4.2.2, in dimension one. Another important difference to notice is the following. In Theorem 4.2.2, (f) for example, S is in a boldface class, while A_0 and A_1 are in a lightface class. The same phenomenon will hold in the case of Wadge classes, and in the new operator we introduce we have boldface conditions (for example, we do not ask δ to be $\Delta^1_1(\beta)$). We code the Borel classes, and define an operator $\mathfrak{G}$ on $\mathcal{N}\times\mathcal{N}$ to do it. Recall the definition of Seq before Lemma 2.3. We set

$$W_0 := \Big\{(n\beta,\gamma)\in\mathcal{N}\times W^{\mathcal{N}} \mid \Big(n\in\text{Seq} \wedge C^{\mathcal{N}}_\gamma = \big\{\alpha\in\mathcal{N} \mid \mathcal{I}^{-1}(n)\subseteq\alpha\big\}\Big) \vee \Big(n\notin\text{Seq} \wedge C^{\mathcal{N}}_\gamma = \emptyset\Big)\Big\},$$

$$\mathfrak{G}(A) := A\cup W_0\cup\Big\{(\beta,\gamma)\in\mathcal{N}\times W^{\mathcal{N}} \mid \exists\delta\in\mathcal{N}\ \ \forall n\in\omega\ \ \big((\beta)_n,(\delta)_n\big)\in A\ \wedge\neg C^{\mathcal{N}}_\gamma = \bigcup_{n\in\omega}\ C^{\mathcal{N}}_{(\delta)_n}\Big\}.$$

In the sequel, we will consider $\mathfrak{G}^{<\xi} := \bigcup_{\eta<\xi}\ \mathfrak{G}^\eta$.

Lemma 6.1 *Let $1\leq\xi<\omega_1$ and $B\subseteq\mathcal{N}$. Then $B\in\mathbf{\Pi}^0_\xi$ if and only if there is $(\beta,\gamma)\in\mathfrak{G}^\xi$ such that $C^{\mathcal{N}}_\gamma = B$.*

Proof. Note first that $B = N_s := \{\alpha\in\mathcal{N} \mid s\subseteq\alpha\}$ for some $s\in\omega^{<\omega}$ or $B=\emptyset$ if and only if there is $(\beta,\gamma)\in W_0 = \mathfrak{G}^0$ with $C^{\mathcal{N}}_\gamma = B$. Then

$$\begin{aligned} B\in\mathbf{\Pi}^0_1 &\Leftrightarrow \exists (s_n)_{n\in\omega}\in(\omega^{<\omega})^\omega\ \ \neg B = \textstyle\bigcup_{n\in\omega}\ N_{s_n}\vee\neg B=\emptyset\\ &\Leftrightarrow \exists\beta,\delta\in\mathcal{N}\ \ \forall n\in\omega\ \ \big((\beta)_n,(\delta)_n\big)\in\mathfrak{G}^0\wedge\neg B = \textstyle\bigcup_{n\in\omega}\ C^{\mathcal{N}}_{(\delta)_n}\\ &\Leftrightarrow \exists(\beta,\gamma)\in\mathfrak{G}^1\ \ C^{\mathcal{N}}_\gamma = B.\end{aligned}$$

Assume now that the result is proved for $1\leq\eta<\xi\geq 2$. Note that

$$\begin{aligned} B\in\mathbf{\Pi}^0_\xi &\Leftrightarrow \exists (B_n)_{n\in\omega}\in(\mathbf{\Pi}^0_{<\xi})^\omega\ \ \neg B = \textstyle\bigcup_{n\in\omega}\ B_n\\ &\Leftrightarrow \exists\beta,\delta\in\mathcal{N}\ \ \forall n\in\omega\ \ \big((\beta)_n,(\delta)_n\big)\in\mathfrak{G}^{<\xi}\wedge\neg B = \textstyle\bigcup_{n\in\omega}\ C^{\mathcal{N}}_{(\delta)_n}\\ &\Leftrightarrow \exists(\beta,\gamma)\in\mathfrak{G}^\xi\ \ C^{\mathcal{N}}_\gamma = B.\end{aligned}$$

This finishes the proof. □

We now define a Π^1_1 coding of $\mathcal{D}$ (recall Definition 5.1.2).

Notation. If $\alpha\in\mathcal{N}$ and $j,p,q\in\omega$, then we will denote $(\alpha)_{2+j}$ by ${}_j\alpha$, and $(\alpha)_{2+<p,q>}$ by ${}_{p,q}\alpha$. We define an inductive operator $\mathfrak{H}$ over $\mathcal{N}$ as follows:

$$\begin{aligned}\mathfrak{H}(D):=D\cup\big\{\alpha\in\mathcal{N}\mid\forall n\in\omega\ \ (\alpha)_n\in\mathrm{WO}\wedge|(\alpha)_n|=0\big\}\cup\\ \big\{\alpha\in\mathcal{N}\mid\forall n\in\omega\ \ (\alpha)_n\in\mathrm{WO}\wedge(\alpha)_0=(\alpha)_2\wedge|(\alpha)_1|=1\wedge<{}_j\alpha>\in D\big\}\cup\\ \big\{\alpha\in\mathcal{N}\mid\forall n\in\omega\ \ (\alpha)_n\in\mathrm{WO}\wedge|(\alpha)_0|\geq 1\wedge|(\alpha)_1|=2\wedge\\ \forall p\in\omega\ \ <{}_{p,q}\alpha>\in D\wedge\big(|{}_{p,0}\alpha|\geq|(\alpha)_0|\vee|{}_{p,0}\alpha|=0\big)\big\}.\end{aligned}$$

Then $\mathfrak{H}$ is a Π^1_1 monotone inductive operator, by 4A.2 in [M].

By 7C.1 in [M] we get $\mathfrak{H}^\infty:=\bigcup_\xi\ \mathfrak{H}^\xi=\mathfrak{H}(\mathfrak{H}^\infty)=\bigcap\{D\subseteq\mathcal{N}\mid\mathfrak{H}(D)\subseteq D\}$. An easy induction on ξ shows that $\mathfrak{H}^\infty\subseteq\big\{\alpha\in\mathcal{N}\mid\forall n\in\omega\ \ (\alpha)_n\in\mathrm{WO}\big\}$, so that the coding function c, partially defined by $c(\alpha):=\big(|(\alpha)_n|\big)_{n\in\omega}$, is defined on $\mathfrak{H}^\infty$.

Lemma 6.2 *The set $\mathfrak{H}^\infty$ is a Π^1_1 coding of $\mathcal{D}$, which means that $\mathfrak{H}^\infty\in\Pi^1_1(\mathcal{N})$ and $c[\mathfrak{H}^\infty]=\mathcal{D}$.*

Proof. We first prove that $\mathfrak{H}^\infty\in\Pi^1_1(\mathcal{N})$ (see 7C in [M] for that). We define a set relation $\varphi(\alpha,D)$ on $\mathcal{N}$ by $\varphi(\alpha,D)\Leftrightarrow\alpha\in\mathfrak{H}(D)$. As $\mathfrak{H}$ is monotone, φ is operative. If $Q\in\Pi^1_1(Z\times\mathcal{N})$, then the relation $\varphi(\alpha,\{\beta\in\mathcal{N}\mid(z,\beta)\in Q\})$ is in Π^1_1. Thus φ is Π^1_1 on Π^1_1. By 7C.8 in [M], $\varphi^\infty(\alpha)$ is in Π^1_1 and $\mathfrak{H}^\infty\in\Pi^1_1(\mathcal{N})$.

Let $\beta_\varepsilon\in\mathrm{WO}$ such that $|\beta_\varepsilon|=\varepsilon$, for $\varepsilon\in 3$. Then $<\beta_0\mid n\in\omega>\in\mathfrak{H}^0\subseteq\mathfrak{H}^\infty$, so that $0^\infty\in c[\mathfrak{H}^\infty]$. Let $v\in c[\mathfrak{H}^\infty]$, $\alpha\in\mathfrak{H}^\infty$ with $v=c(\alpha)$. Then $<(\alpha)_0,\beta_1,(\alpha)_0,(\alpha)_1,...>\in\mathfrak{H}(\mathfrak{H}^\infty)=\mathfrak{H}^\infty$, so that $v(0)1v=c\big(<(\alpha)_0,\beta_1,(\alpha)_0,(\alpha)_1,...>\big)\in c[\mathfrak{H}^\infty]$.

Now let $\xi\geq 1$, $u_p\in c[\mathfrak{H}^\infty]$ such that $u_p(0)\geq\xi$ or $u_p(0)=0$, for each $p\in\omega$. Choose $\alpha\in\mathrm{WO}$ with $|\alpha|=\xi$, and $\alpha^p\in\mathfrak{H}^\infty$ with $u_p=c(\alpha^p)$. Then $<\alpha,\beta_2,(\alpha^{(0)_0})_{(0)_1},(\alpha^{(1)_0})_{(1)_1},...>\in\mathfrak{H}(\mathfrak{H}^\infty)=\mathfrak{H}^\infty$, so that $\xi 2<u_p>=c\big(<\alpha,\beta_2,(\alpha^{(0)_0})_{(0)_1},(\alpha^{(1)_0})_{(1)_1},...>\big)\in c[\mathfrak{H}^\infty]$. Thus $\mathcal{D}\subseteq c[\mathfrak{H}^\infty]$.

Assume now that $\mathcal{E}\subseteq\omega_1^\omega$ satisfies the following properties:

(a) $0^\infty\in\mathcal{E}$.

(b) $v\in\mathcal{E}\Rightarrow v(0)1v\in\mathcal{E}$.

(c) $\Big(\xi\geq 1\wedge\forall p\in\omega\ \big(u_p\in\mathcal{E}\wedge(u_p(0)\geq\xi\vee u_p(0)=0)\big)\Big)\Rightarrow\xi 2<u_p>\in\mathcal{E}$.

We set $D:=\{\alpha\in\mathcal{N}\mid\forall n\in\omega\ \ (\alpha)_n\in\mathrm{WO}\wedge c(\alpha)\in\mathcal{E}\}$. It remains to see that $\mathfrak{H}(D)\subseteq D$. Indeed, this will imply that $\mathfrak{H}^\infty\subseteq D$, $c[\mathfrak{H}^\infty]\subseteq c[D]\subseteq\mathcal{E}$ and $c[\mathfrak{H}^\infty]\subseteq\mathcal{D}$.

As $0^\infty\in\mathcal{E}$, we get $\big\{\alpha\in\mathcal{N}\mid\forall n\in\omega\ \ (\alpha)_n\in\mathrm{WO}\wedge|(\alpha)_n|=0\big\}\subseteq D$. Assume that $(\alpha)_n\in\mathrm{WO}$ for each $n\in\omega$, that $(\alpha)_0=(\alpha)_2$, $|(\alpha)_1|=1$ and $<{}_j\alpha>\in D$. Then $v:=\big(|{}_j\alpha|\big)\in\mathcal{E}$, and $|(\alpha)_2|1v\in\mathcal{E}$. Thus $c(\alpha)\in\mathcal{E}$ and $\alpha\in D$.

Assume now that $(\alpha)_n\in\mathrm{WO}$ for any natural number n, $|(\alpha)_0|\geq 1$, $|(\alpha)_1|=2$, $<{}_{p,q}\alpha>\in D$, and $|{}_{p,0}\alpha|\geq|(\alpha)_0|$ or $|{}_{p,0}\alpha|=0$ for any $p\in\omega$. We set $\xi:=|(\alpha)_0|$. Then $u_p:=\big(|{}_{p,q}\alpha|\big)\in\mathcal{E}$, and $\xi 2<u_p>\in\mathcal{E}$. Thus $c(\alpha)\in\mathcal{E}$ and $\alpha\in D$. □

Note that just like Definition 5.1.2, the definition of $\mathfrak{H}$ is cut into three cases, that we will meet again later on: $|(\alpha)_1|=0$ (or, equivalently, $|(\alpha)_n|=0$ for each natural number n), $|(\alpha)_1|=1$ or $|(\alpha)_1|=2$.

Even if "$u\in\mathcal{D}$" is the least relation satisfying some conditions, some simplifications are possible. For example, $\mathbf{\Gamma}_{01010^\infty}=\mathbf{\Gamma}_{0^\infty}$. Some other simplifications are possible, and some of them will simplify the notation later on. This will lead to the notion of a normalized code of a description. To define it, we need to associate a tree with a code of a description. The idea is to describe the construction of a set in $\mathbf{\Gamma}_u$ using simpler and simpler sets, until we reach the simplest set, namely the empty set. More specifically, we define $\mathfrak{T}:\mathfrak{H}^\infty\rightarrow\{\text{trees on }\omega\times\mathfrak{H}^\infty\}$ as follows. Let $\alpha\in\mathfrak{H}^\xi\setminus\mathfrak{H}^{<\xi}$. We set

$$\mathfrak{T}(\alpha):=\begin{cases}\{\emptyset\}\cup\{<(0,\alpha)>\}\text{ if }|(\alpha)_1|=0,\\ \\ \{\emptyset\}\cup\big\{(0,\alpha)^\frown s\mid s\in\mathfrak{T}(<_j\alpha>)\big\}\text{ if }|(\alpha)_1|=1,\\ \\ \{\emptyset\}\cup\big\{(0,\alpha)^\frown s\mid s\in\mathfrak{T}(<_{0,q}\alpha>)\big\}\cup\\ \qquad\bigcup_{p\geq 1}\ \big\{(p,\alpha)^\frown s\mid s\in\mathfrak{T}(<_{(p)_0+1,q}\alpha>)\big\}\text{ if }|(\alpha)_1|=2.\end{cases}$$

An easy induction on η shows that $\mathfrak{T}(\alpha)$ is always a countable well-founded tree (the first coordinate of (p,α) ensures the well-foundedness). A sequence $s\in\mathfrak{T}(\alpha)$ is said to be $maximal$ if $s\subseteq t\in\mathfrak{T}(\alpha)$ implies that $s=t$. Note that $\big|\big(s_1(|s|-1)\big)_1\big|=0$ if s is maximal. We denote by $\mathcal{M}_\alpha$ the set of maximal sequences of $\mathfrak{T}(\alpha)$.

Definition 6.3 *We say that* $\alpha\in\mathfrak{H}^\infty$ *is* $normalized$ *if the following holds:*

$$\big(s\in\mathcal{M}_\alpha\wedge i<|s|\wedge\big|\big(s_1(i)\big)_1\big|=1\big)\Rightarrow i=|s|-2.$$

This means that in a maximal sequence s of $\mathfrak{T}(\alpha)$, $\big|\big(s_1(i)\big)_1\big|$ is 2, then possibly 1 once, and finally 0 once. The next lemma says that we can always assume that α is normalized. It is based on the fact that $\check{S}_\xi(\mathbf{\Gamma},\mathbf{\Gamma}')=S_\xi(\check{\mathbf{\Gamma}},\check{\mathbf{\Gamma}}')$.

Lemma 6.4 *Let* $\alpha\in\mathfrak{H}^\infty$*. Then there is* $\gamma\in\mathfrak{H}^\infty$ *normalized with* $(\gamma)_0=(\alpha)_0$ *and* $\mathbf{\Gamma}_{c(\gamma)}=\mathbf{\Gamma}_{c(\alpha)}$.

Proof. Assume that $\alpha\in\mathfrak{H}^\xi\setminus\mathfrak{H}^{<\xi}$. We argue by induction on ξ.

Case 1. $|(\alpha)_1|=0$.

We just set $\gamma:=\alpha$ since $\big|\big(s_1(i)\big)_1\big|=0$.

Case 2. $|(\alpha)_1|=1$.

• We first define $N:\mathfrak{H}^\infty\rightarrow\mathfrak{H}^\infty$ as follows. We ensure that $\big(N(\beta)\big)_0=(\beta)_0$ and $\mathbf{\Gamma}_{c(N(\beta))}=\check{\mathbf{\Gamma}}_{c(\beta)}$. Let $\beta_1\in\text{WO}$ with $|\beta_1|=1$. We set

$$N(\beta):=\begin{cases}<(\beta)_0,\beta_1,(\beta)_0,(\beta)_1,(\beta)_2,...>\ \text{ if }|(\beta)_1|=0,\\ <_j\beta>\ \text{ if }|(\beta)_1|=1,\\ <(\beta)_0,(\beta)_1,\Big(\big(N(<_{(i)_0,q}\beta>)\big)_{(i)_1}\Big)_{i\in\omega}>\ \text{ if }|(\beta)_1|=2,\end{cases}$$

and one easily checks that N is defined and suitable.

• As $<{}_j\alpha>\in\mathfrak{H}^{<\xi}$, the induction assumption gives $\delta\in\mathfrak{H}^\infty$ normalized satisfying the equalities $(\delta)_0=(\alpha)_2=(\alpha)_0$ and $\mathbf{\Gamma}_{c(\delta)}=\mathbf{\Gamma}_{c(<_j\alpha>)}$. In particular, $\mathbf{\Gamma}_{c(\alpha)}=\check{\mathbf{\Gamma}}_{c(<_j\alpha>)}=\check{\mathbf{\Gamma}}_{c(\delta)}=\mathbf{\Gamma}_{c(N(\delta))}$. So we have to find $\gamma\in\mathfrak{H}^\infty$ normalized with $(\gamma)_0=(\delta)_0$ and $\mathbf{\Gamma}_{c(\gamma)}=\mathbf{\Gamma}_{c(N(\delta))}$. Assume that δ is in $\mathfrak{H}^\eta\setminus\mathfrak{H}^{<\eta}$. We argue by induction on η.

Subcase 1. $|(\delta)_1|\leq 1$.

We just set $\gamma:=N(\delta)$.

Subcase 2. $|(\delta)_1|=2$.

Note that $<_{p,q}\delta>$ is normalized since $(0,\delta)^\frown s\in\mathcal{M}_\delta$ (resp., $(p,\delta)^\frown s\in\mathcal{M}_\delta$) if $s\in\mathcal{M}_{0,q}\delta$ (resp., $s\in\mathcal{M}_{(p)_0+1,q}\delta$ and $p\geq 1$). The induction assumption gives $<_{p,q}\gamma>\in\mathfrak{H}^\infty$ normalized with ${}_{p,0}\gamma={}_{p,0}\delta$ and $\mathbf{\Gamma}_{c(<_{p,q}\gamma>)}=\mathbf{\Gamma}_{c(N(<_{p,q}\delta>))}$. We set $(\gamma)_i:=(\delta)_i$ if $i\in 2$ and we are done.

Case 3. $|(\alpha)_1|=2$.

The induction assumption gives $<_{p,q}\gamma>\in\mathfrak{H}^\infty$ normalized satisfying ${}_{p,0}\gamma={}_{p,0}\alpha$ and

$$\mathbf{\Gamma}_{c(<_{p,q}\gamma>)}=\mathbf{\Gamma}_{c(<_{p,q}\alpha>)}.$$

We set $(\gamma)_i:=(\alpha)_i$ if $i\in 2$ and we are done. $\square$

Using $\mathfrak{G}$, we will now code the non self-dual Wadge classes of Borel sets, and define an operator $\mathfrak{I}$ on $\mathcal{N}^3$ to do it. We set

$$\begin{aligned}
\mathfrak{I}(A):=A\cup\Big\{&(\alpha,m\beta,\gamma)\in\mathcal{N}^2\times W^{\mathcal{N}}\mid\forall n\in\omega\ \ (\alpha)_n\in\text{WO}\wedge\\
&\Big(\forall n\in\omega\ \ |(\alpha)_n|=0\wedge m=0\wedge C^{\mathcal{N}}_\gamma=\emptyset\Big)\vee\\
&\Big(|(\alpha)_1|=1\wedge(\alpha)_0=(\alpha)_2\wedge m=1\wedge\exists\delta\in\mathcal{N}\ \ (<{}_j\alpha>,\beta,\delta)\in A\wedge C^{\mathcal{N}}_\gamma=\neg C^{\mathcal{N}}_\delta\Big)\vee\\
&\Big(|(\alpha)_1|=2\wedge|(\alpha)_0|\geq 1\wedge\forall p\in\omega\ \ \big(|{}_{p,0}\alpha|\geq|(\alpha)_0|\vee|{}_{p,0}\alpha|=0\big)\wedge\\
&m=2\wedge\exists\delta\in\mathcal{N}\ \ (<{}_{0,q}\alpha>,(\beta)_0,(\delta)_0)\in A\wedge\\
&\forall p\geq 1\ \ \big(<_{(p)_0+1,q}\alpha>,\big((\beta)_p\big)_0,\big((\delta)_p\big)_0\big)\in A\wedge\Big(\big(((\beta)_p\big)_1,\big((\delta)_p\big)_1\Big)\in\mathfrak{G}^{|(\alpha)_0|}\wedge\\
&\forall p\neq q\geq 1\ \ C^{\mathcal{N}}_{((\delta)_p)_1}\cup C^{\mathcal{N}}_{((\delta)_q)_1}=\mathcal{N}\wedge\\
&C^{\mathcal{N}}_\gamma=\textstyle\bigcup_{p\geq 1}\ (C^{\mathcal{N}}_{((\delta)_p)_0}\setminus C^{\mathcal{N}}_{((\delta)_p)_1})\cup(C^{\mathcal{N}}_{(\delta)_0}\cap\bigcap_{p\geq 1}\ C^{\mathcal{N}}_{((\delta)_p)_1})\Big)\Big\}.
\end{aligned}$$

Lemma 6.5 *Let ξ be an ordinal.*
(a) Assume that $(\alpha,m\beta,\gamma)\in\mathfrak{I}^\xi$. Then $\alpha\in\mathfrak{H}^\xi$.
(b) Let $\alpha\in\mathfrak{H}^\xi$ and $B\subseteq\mathcal{N}$. Then $B\in\mathbf{\Gamma}_{c(\alpha)}$ if and only if there are $m\in\omega$ and $\beta,\gamma\in\mathcal{N}$ such that $(\alpha,m\beta,\gamma)\in\mathfrak{I}^\xi$ and $C^{\mathcal{N}}_\gamma=B$.

Proof. (a) We argue by induction on ξ. So let $\alpha\in\mathfrak{I}^\xi\setminus\mathfrak{I}^{<\xi}$. We may assume that $|(\alpha)_1|\geq 1$. If $|(\alpha)_1|=1$, then $(<_j\alpha>,\beta,\delta)\in\mathfrak{I}^{<\xi}$ for some δ, and $<_j\alpha>\in\mathfrak{H}^{<\xi}$, by induction assumption, so we are done. If $|(\alpha)_1|=2$, then $(<_{0,q}\alpha>,(\beta)_0,(\delta)_0),\big(<_{(p)_0+1,q}\alpha>,\big((\beta)_p\big)_0,\big((\delta)_p\big)_0\big)\in\mathfrak{I}^{<\xi}$ for some δ, and $<_{p,q}\alpha>\in\mathfrak{H}^{<\xi}$, by induction assumption, for any natural number p.

(b) $\Rightarrow$ We argue by induction on ξ, and we may assume that $\alpha\notin\mathfrak{H}^{<\xi}$.

Case 1. $|(\alpha)_1|=0$.

Note that $c(\alpha)=0^\infty$ and $B=\emptyset$. We set $m:=0$, $\beta:=0^\infty$, and we choose $\gamma\in W^{\mathcal{N}}$ with $C^{\mathcal{N}}_\gamma=\emptyset$. Then $(\alpha,\beta,\gamma)\in\mathfrak{I}^0\subseteq\mathfrak{I}^\xi$.

Case 2. $|(\alpha)_1|=1$.

Note that $<_j\alpha>\in\mathfrak{H}^{<\xi}$, and $\neg B\in\mathbf{\Gamma}_{c(<_j\alpha>)}$. The induction assumption gives $\beta,\delta\in\mathcal{N}$ such that $(<_j\alpha>,\beta,\delta)\in\mathfrak{I}^{<\xi}$ and $C^{\mathcal{N}}_\delta=\neg B$. We set $m:=1$ and choose $\gamma\in W^{\mathcal{N}}$ with $C^{\mathcal{N}}_\gamma=\neg C^{\mathcal{N}}_\delta$.

Case 3. $|(\alpha)_1|=2$.

Note that $<_{p,q}\alpha>\in\mathfrak{H}^{<\xi}$ for any natural number p. We can write

$$B=\bigcup_{p\geq 1}(A_p\cap C_p)\cup(D\setminus\bigcup_{p\geq 1}C_p),$$

where $(C_p)_{p\geq 1}$ is a sequence of pairwise disjoint $\mathbf{\Sigma}^0_{|(\alpha)_0|}$ sets, $D\in\mathbf{\Gamma}_{c(<_{0,q}\alpha>)}$ and

$$A_p\in\mathbf{\Gamma}_{c(<_{(p)_0+1,q}\alpha>)}.$$

Lemma 6.1 gives $\big(\big((\beta)_p\big)_1,\big((\delta)_p\big)_1\big)\in\mathfrak{G}^{|(\alpha)_0|}$ such that $C^{\mathcal{N}}_{((\delta)_p)_1}=\neg C^{\mathcal{N}}_p$. The induction assumption gives $(\beta)_0,(\delta)_0\in\mathcal{N}$ such that $(<_{0,q}\alpha>,(\beta)_0,(\delta)_0)\in\mathfrak{I}^{<\xi}$ and $C^{\mathcal{N}}_{(\delta)_0}=D$, and $\big((\beta)_p\big)_0,\big((\delta)_p\big)_0\in\mathcal{N}$ such that $\big(<_{(p)_0+1,q}\alpha>,\big((\beta)_p\big)_0,\big((\delta)_p\big)_0\big)\in\mathfrak{I}^{<\xi}$ and $C^{\mathcal{N}}_{((\delta)_p)_0}=A_p$. We set $m:=2$ and choose $\gamma\in W^{\mathcal{N}}$ with $C^{\mathcal{N}}_\gamma=\bigcup_{p\geq 1}(C^{\mathcal{N}}_{((\delta)_p)_0}\setminus C^{\mathcal{N}}_{((\delta)_p)_1})\cup(C^{\mathcal{N}}_{(\delta)_0}\cap\bigcap_{p\geq 1}C^{\mathcal{N}}_{((\delta)_p)_1})$.

$\Leftarrow$ We argue by induction on ξ, and we may assume that $(\alpha,m\beta,\gamma)\notin\mathfrak{I}^{<\xi}$.

Case 1. $|(\alpha)_1|=0$.

Note that $B=C^{\mathcal{N}}_\gamma=\emptyset\in\mathbf{\Gamma}_{0^\infty}=\mathbf{\Gamma}_{c(\alpha)}$.

Case 2. $|(\alpha)_1|=1$.

Note that there is δ such that $(<_j\alpha>,\beta,\delta)\in\mathfrak{I}^{<\xi}$ and $C^{\mathcal{N}}_\gamma=\neg C^{\mathcal{N}}_\delta$, which implies that B is in $\check{\mathbf{\Gamma}}_{c(<_j\alpha>)}=\mathbf{\Gamma}_{c(\alpha)}$.

Case 3. $|(\alpha)_1|=2$.

Let δ be a witness for the fact that $(\alpha, m\beta, \gamma)\in\mathfrak{I}^\xi$. As

$$(<_{0,q}\alpha>, (\beta)_0, (\delta)_0), \big(<_{(p)_0+1,q}\alpha>, \big((\beta)_p\big)_0, \big((\delta)_p\big)_0\big)\in\mathfrak{I}^{<\xi},$$

the set $C^{\mathcal{N}}_{(\delta)_0}$ is in $\mathbf{\Gamma}_{c(<_{0,q}\alpha>)}$ and $C^{\mathcal{N}}_{((\delta)_p)_0}$ is in $\mathbf{\Gamma}_{c(<_{(p)_0+1,q}\alpha>)}$, by induction assumption. As

$$\big(((\beta)_p\big)_1, \big((\delta)_p\big)_1\big)\in\mathfrak{G}^{|(\alpha)_0|},$$

$C^{\mathcal{N}}_{((\delta)_p)_1}\in\mathbf{\Pi}^0_{|(\alpha)_0|}$, by Lemma 6.1. Thus $B\in S_{|(\alpha)_0|}(\bigcup_{p\geq 1}\ \mathbf{\Gamma}_{c(<_{p,q}\alpha>)}, \mathbf{\Gamma}_{c(<_{0,q}\alpha>)})=\mathbf{\Gamma}_{c(\alpha)}$. $\square$

Remark. We will also consider the operator $\mathfrak{J}$ defined just like $\mathfrak{I}$, except that
- we replace $(W^{\mathcal{N}}, C^{\mathcal{N}})$ with (W, C) (we work in $\mathcal{N}^d$ instead of $\mathcal{N}$),
- we replace the condition of the form $(\tilde\beta, \tilde\gamma)\in\mathfrak{G}^{|(\alpha)_0|}$ with $\big((\alpha)_0, \tilde\beta, \tilde\gamma\big)\in Q$ (see the remark at the end of Section 4 for the definition of Q),
- we ask β, γ, δ to be $\Delta^1_1(\alpha)$, so that $\mathfrak{J}$ is a Π^1_1 monotone inductive operator.

To prove Theorem 1.10, we will consider some tuples $\vec{v}:=(\alpha, a_0, a_1, b_0, b_1, r)$, where $\alpha\in\mathfrak{H}^\infty$ is a (normalized in practice) code for a description $u=c(\alpha)$. We will inductively define them through an inductive operator over $\mathcal{N}^6$ called $\mathfrak{K}$. The definition of $\mathfrak{K}$ is in the spirit of that of $\mathfrak{I}$. We will use the good universal set $\mathcal{U}$ for Π^1_1 defined after the proof of Theorem 4.2.2, at the end of Section 4, and the following lemma.

Lemma 6.6 *There is a recursive map $\mathcal{A}:\mathcal{N}^2\rightarrow\mathcal{N}$ such that $\mathcal{U}_{\mathcal{A}(\alpha,r)}=\mathcal{U}_{(r)_0}\cup\bigcup_{p\geq 1}\ \overline{\neg\mathcal{U}_{(r)_p}}^{|\alpha|}$ if $\alpha\in\Delta^1_1\cap$ WO and $|\alpha|\geq 1$.*

Proof. Note first that $P:=\{(\beta, \vec\delta)\in\mathcal{N}\times\mathcal{N}^d\mid (\beta)_0\in\Delta^1_1\cap\text{WO}\wedge|(\beta)_0|\geq 1\wedge$

$$\vec\delta\in\mathcal{U}_{((\beta)_1)_0}\cup\bigcup_{p\geq 1}\ \overline{\neg\mathcal{U}_{((\beta)_1)_p}}^{|(\beta)_0|}\}$$

is a Π^1_1 set, by the remark defining R at the end of Section 4. This gives $\gamma\in\mathcal{N}$ recursive with $P=\mathcal{U}^{\mathcal{N}\times\mathcal{N}^d}_\gamma$. Let $\alpha\in\Delta^1_1\cap$ WO with $|\alpha|\geq 1$, and $r\in\mathcal{N}$. Then

$$\begin{aligned}\vec\delta\in\mathcal{U}_{(r)_0}\cup\bigcup_{p\geq 1}\ \overline{\neg\mathcal{U}_{(r)_p}}^{|\alpha|} &\Leftrightarrow (<\alpha, r, r, ...>, \vec\delta\,)\in P\\ &\Leftrightarrow (\gamma, <\alpha, r, r, ...>, \vec\delta\,)\in\mathcal{U}^{\mathcal{N}\times\mathcal{N}^d}\\ &\Leftrightarrow \big(S(\gamma, <\alpha, r, r, ...>), \vec\delta\,\big)\in\mathcal{U}.\end{aligned}$$

We just have to set $\mathcal{A}(\alpha, r):=S(\gamma, <\alpha, r, r, ...>)$. $\square$

We are now ready to define $\mathfrak{K}$ (recall the remark defining Q at the end of Section 4).

The operator $\mathfrak{K}$ is defined as follows (recall the definition of $\mathfrak{H}$):

$$\mathfrak{K}(A):=A\cup\Bigg\{(\alpha,a_0,a_1,b_0,b_1,r)\in\big(\mathcal{N}\cap\Delta^1_1(\alpha)\big)^6\mid\forall n\in\omega\ \ (\alpha)_n\in\text{WO}\ \wedge$$
$$\Bigg(\forall n\in\omega\ \ |(\alpha)_n|=0\wedge\mathcal{U}_{a_0}\cup\mathcal{U}_{a_1}=\mathcal{N}^d\wedge(b_0,b_1)=(a_0,a_1)\wedge r=a_1\Bigg)\vee$$
$$\Bigg(|(\alpha)_1|=1\wedge(\alpha)_0=(\alpha)_2\wedge(<{}_j\alpha>,a_0,a_1,b_0,b_1,a_1)\in A\wedge r=a_0\Bigg)\vee$$
$$\Bigg(|(\alpha)_1|=2\wedge|(\alpha)_0|\geq 1\wedge\forall p\in\omega\ \ \big(|_{p,0}\alpha|\geq|(\alpha)_0|\vee|_{p,0}\alpha|=0\big)\wedge$$
$$\exists c_0,c_1,s\in\Delta^1_1(\alpha)\ \ \big(<{}_{0,q}\alpha>,a_0,a_1,(c_0)_0,(c_1)_0,(s)_0\big)\in A\wedge$$
$$\forall p\geq 1\ \ \big(<{}_{(p)_0+1,q}\alpha>,a_0,a_1,(c_0)_p,(c_1)_p,(s)_p\big)\in A\wedge$$
$$\forall i\in 2\ \ b_i=\mathcal{A}\big((\alpha)_0,<a_i,(s)_1,(s)_2,...>\big)\wedge$$
$$\exists d_0,d_1\in\Delta^1_1(\alpha)\ \ (<{}_{0,q}\alpha>,b_0,b_1,d_0,d_1,r)\in A\Bigg)\Bigg\}.$$

Then $\mathfrak{K}$ is a Π^1_1 monotone inductive operator.

Remark. Let ξ be an ordinal, and $\vec{v}:=(\alpha,a_0,a_1,b_0,b_1,r)\in\mathfrak{K}^\xi$. An induction on ξ shows the following properties.

- $\neg\mathcal{U}_{a_0}\cap\neg\mathcal{U}_{a_1}=\emptyset$.
- $B_i:=\neg\mathcal{U}_{b_i}\subseteq A_i:=\neg\mathcal{U}_{a_i}$ for any $i\in 2$. In particular, $B_0\cap B_1=\emptyset$.
- b_0,b_1,r are completely determined by (α,a_0,a_1). This is the reason why we will sometimes identify $b_i=b_i(\alpha,a_0,a_1)\simeq b_i(u,a_0,a_1)$ and $r=r(\alpha,a_0,a_1)\simeq r(u,a_0,a_1)$.
- If $\neg\mathcal{U}_{a_i}\subseteq\neg\mathcal{U}_{a'_i}$ for any $i\in 2$, then $\neg\mathcal{U}_{b_i}\subseteq\neg\mathcal{U}_{b'_i}$ for any $i\in 2$ and $\neg\mathcal{U}_{r(\alpha,a_0,a_1)}\subseteq\neg\mathcal{U}_{r(\alpha,a'_0,a'_1)}$.
- There is $i\in 2$ such that $\neg\mathcal{U}_r\subseteq\neg\mathcal{U}_{a_i}$.

Lemma 6.7 *(a) Let ξ be an ordinal, $\alpha\in\Delta^1_1$, and $(\alpha,m\beta,\gamma)\in\mathfrak{I}^\xi$. Then $\alpha\in\mathfrak{H}^\xi$ and the set C_γ is in $\Delta^1_1\cap\mathbf{\Gamma}_{c(\alpha)}(\tau_1)$.*
(b) Let $\alpha\in\Delta^1_1\cap\mathfrak{H}^\infty$ normalized, and $a_0,a_1\in\Delta^1_1$ with $A_0\cap A_1=\emptyset$. Then there are $b_0,b_1,r\in\mathcal{N}$ such that $(\alpha,a_0,a_1,b_0,b_1,r)\in\mathfrak{K}^\infty$.

Proof. (a) We argue as in the proof of Lemmas 6.5.(a) and 6.5.(b)$\Leftarrow$. The only thing to notice is that in the case $|(\alpha)_1|=2$, $\big((\alpha)_0,\big((\beta)_p\big)_1,\big((\delta)_p\big)_1\big)\in Q$. Proposition 2.2, Lemma 2.3 and Theorem 3.1 give a tree T_d with Δ^1_1 suitable levels and $S\in\mathbf{\Sigma}^0_{|(\alpha)_0|}(\lceil T_d\rceil)$ which is not separable from $\lceil T_d\rceil\setminus S$ by a pot($\mathbf{\Pi}^0_{|(\alpha)_0|}$) set. As $\alpha\in\Delta^1_1$, $|(\alpha)_0|<\omega_1^{\text{CK}}$ and Theorem 4.2.2 implies that $C_{((\delta)_p)_1}$ is in $\mathbf{\Pi}^0_{|(\alpha)_0|}(\tau_1)$. Thus $C_\gamma\in\mathbf{\Gamma}_{c(\alpha)}(\tau_1)$.

(b) Let ξ be an ordinal with $\alpha\in\mathfrak{H}^\xi$. Here again we argue by induction on ξ. So assume that $\alpha\notin\mathfrak{H}^{<\xi}$.

Case 1. $|(\alpha)_1|=0$.

Let $b_i:=a_i$ and $r:=a_1$. Then $(\alpha,a_0,a_1,b_0,b_1,r)\in\mathfrak{K}^0\subseteq\mathfrak{K}^\infty$.

Case 2. $|(\alpha)_1|=1$.

As $<{}_j\alpha>\in\mathfrak{H}^{<\xi}$, the induction assumption gives (b_0,b_1,s) with

$$(<{}_j\alpha>,a_0,a_1,b_0,b_1,s)\in\mathfrak{K}^\infty.$$

As α is normalized, $|_j\alpha|=0$ for any j, and $s=a_1$. We set $r:=a_0$. Then

$$(\alpha,a_0,a_1,b_0,b_1,r)\in\mathfrak{K}(\mathfrak{K}^\infty)=\mathfrak{K}^\infty.$$

Case 3. $|(\alpha)_1|=2$.

As $<{}_{p,q}\alpha>\in\mathfrak{H}^{<\xi}$, the induction assumption gives (a_0^p,a_1^p,r^p) with

$$\big(<{}_{0,q}\alpha>,a_0,a_1,a_0^0,a_1^0,r^0\big)\in\mathfrak{K}^\infty,$$

and $(<{}_{(p)_0+1,q}\alpha>,a_0,a_1,a_0^p,a_1^p,r^p)\in\mathfrak{K}^\infty$, for any $p\geq 1$. As in the proof of Lemma 6.2 we see that $\mathfrak{K}^\infty\in\Pi^1_1$. By Δ^1_1-selection, we may assume that the sequences (a_0^p), (a_1^p) and (r^p) are Δ^1_1. In particular, there is $c_i\in\Delta^1_1$ with $(c_i)_p=a_i^p$. We set $(s)_p:=r^p$, and

$$b_i:=\mathcal{A}\big((\alpha)_0,<a_i,(s)_1,(s)_2,...>\big).$$

The induction assumption gives d_0,d_1,r such that $(<{}_{0,q}\alpha>,b_0,b_1,d_0,d_1,r)\in\mathfrak{K}^\infty$. We are done since $(\alpha,a_0,a_1,b_0,b_1,r)\in\mathfrak{K}^\infty$. □

The next lemma is the crucial separation lemma.

Lemma 6.8 *Let $\vec{v}:=(\alpha,a_0,a_1,b_0,b_1,r)\in\mathfrak{K}^\infty$ with $\alpha\in\Delta^1_1$ normalized and $a_0,a_1\in\Delta^1_1$, Σ in $\Sigma^1_1(\mathcal{N}^d)$ with $(\neg\mathcal{U}_r)\cap\Sigma=\emptyset$. Then there are $m\in\omega$ and $\beta,\gamma\in\mathcal{N}$ such that $(\alpha,m\beta,\gamma)\in\mathfrak{J}^\infty$ and C_γ separates $A_1\cap\Sigma$ from $A_0\cap\Sigma$. In particular, $A_1\cap\Sigma$ is separable from $A_0\cap\Sigma$ by a $\Delta^1_1\cap\mathbf{\Gamma}_{c(\alpha)}(\tau_1)$ set.*

Proof. The last assertion comes from Lemma 6.7.(a). Let η be an ordinal with $\vec{v}\in\mathfrak{K}^\eta$. We argue by induction on η. So assume that $\vec{v}\in\mathfrak{K}^\eta\setminus\mathfrak{K}^{<\eta}$.

Case 1. $|(\alpha)_1|=0$.

We set $m:=0$, $\beta:=0^\infty$, and choose $\gamma\in\Delta^1_1\cap W$ with $C_\gamma=\emptyset$. We are done since $\emptyset=A_1\cap\Sigma$.

Case 2. $|(\alpha)_1|=1$.

As α is normalized, $|_j\alpha|=0$ for any j. We set $m:=1$, $\beta:=0^\infty$, and choose $\gamma\in\Delta^1_1\cap W$ with $C_\gamma=\mathcal{N}^d$. Then $\delta\in\Delta^1_1\cap W$ with $C_\delta=\emptyset$ is a witness for the fact that $(\alpha,m\beta,\gamma)\in\mathfrak{J}^\infty$. We are done since $r=a_0$.

Case 3. $|(\alpha)_1|=2$.

There are $c_0,c_1,s\in\Delta^1_1$ with $\big(<{}_{(p)_0+1,q}\alpha>,a_0,a_1,(c_0)_p,(c_1)_p,(s)_p\big)\in\mathfrak{K}^{<\eta}$, for any $p\geq 1$, and, for any $i\in 2$, $b_i=\mathcal{A}\big((\alpha)_0,<a_i,(s)_1,(s)_2,...>\big)$. Moreover, there are $d_0,d_1\in\Delta^1_1$ with $(<{}_{0,q}\alpha>,b_0,b_1,d_0,d_1,r)\in\mathfrak{K}^{<\eta}$.

By Lemma 6.7.(a), one of the goals is to build $C_\gamma\in\mathbf{\Gamma}_{c(\alpha)}(\tau_1)$. The proof of Lemma 6.7.(a) shows that $\mathbf{\Gamma}_{c(\alpha)}=S_{|(\alpha)_0|}(\bigcup_{p\geq 1}\ \mathbf{\Gamma}_{c(<{}_{p,q}\alpha>)},\mathbf{\Gamma}_{c(<{}_{0,q}\alpha>)})$. This means that we want to find some sequences $(C_p)_{p\geq 1}$, $(S_p)_{p\geq 1}$ and B such that $C_\gamma=\bigcup_{p\geq 1}\ (S_p\cap C_p)\cup(B\setminus\bigcup_{p\geq 1}\ C_p)$.

- Let us construct B.

The induction assumption gives $\beta^0,\gamma^0\in\mathcal{N}$ such that $(<{}_{0,q}\alpha>,\beta^0,\gamma^0)\in\mathfrak{J}^\infty$ and C_{γ^0} separates $\underline{A}_1\cap\Sigma$ from $\underline{A}_0\cap\Sigma$. We set $B:=C_{\gamma^0}$.

- Let us construct the C_p's.

We set $\xi:=|(\alpha)_0|$. Note that $b_i=A_i\cap\bigcap_{p\geq 1}\ \overline{\neg\mathcal{U}_{(s)_p}}^{\xi}$. This implies that

$$U:=\big(C_{\gamma^0}\cap A_0\cap\Sigma\big)\cup\big(\neg C_{\gamma^0}\cap A_1\cap\Sigma\big)\subseteq\bigcup_{p\geq 1}\ \neg\overline{\neg\mathcal{U}_{(s)_p}}^{\xi}.$$

As in the proof of Lemma 6.6 we see that the relation "$\vec{\delta}\notin\overline{\neg\mathcal{U}_{(s)_p}}^{|(\alpha)_0|}$" is Π^1_1 in $(p,\alpha,s,\vec{\delta})$. By Δ^1_1-selection there is a Δ^1_1-recursive map $f:\mathcal{N}^d\rightarrow\omega$ such that $f(\vec{\delta})\geq 1$ for any $\vec{\delta}\in\mathcal{N}^d$ and $\vec{\delta}\notin\overline{\neg\mathcal{U}_{(s)_{f(\vec{\delta})}}}^{\xi}$ for any $\vec{\delta}\in U$.

In particular, for any $\vec{\delta}\in U$ there is $P\in\Sigma^1_1\cap\mathbf{\Pi}^0_{<\xi}(\tau_1)$ such that $\vec{\delta}\in P\subseteq\mathcal{U}_{(s)_{f(\vec{\delta})}}$. Now P and $\neg\mathcal{U}_{(s)_{f(\vec{\delta})}}$ are disjoint Σ^1_1 sets, separable by a $\mathbf{\Pi}^0_{<\xi}(\tau_1)$ set. As $\alpha\in\Delta^1_1$, $1\leq|(\alpha)_0|<\omega_1^{\text{CK}}$. As in the proof of Lemma 6.7.(a) we get T_d and S. Theorem 4.2.2 gives $(\beta',\gamma')\in(\Delta^1_1\times\Delta^1_1)\cap V_{<\xi}$ with $P\subseteq C_{\gamma'}\subseteq\mathcal{U}_{(s)_{f(\vec{\delta})}}$.

By Lemma 4.2.3.(2).(a) the relation "(β',γ') is in $(\Delta^1_1\times\Delta^1_1)\cap V_{<\xi}$" is Π^1_1, so there is a Δ^1_1-recursive map $g:\mathcal{N}^d\rightarrow\omega\times(\mathcal{N}\times\mathcal{N})$ such that

$$\forall\vec{\delta}\in U\ \ g_0(\vec{\delta})=f(\vec{\delta})\wedge g_1(\vec{\delta})\in(\Delta^1_1\times\Delta^1_1)\cap V_{<\xi}\wedge\vec{\delta}\in C_{(g_1(\vec{\delta}))_1}\subseteq\mathcal{U}_{(s)_{f(\vec{\delta})}},$$

by Δ^1_1-selection. In particular, the Σ^1_1 set $g[U]$ is a subset of

$$\big\{\big(p,(\beta',\gamma')\big)\in\omega\times\big((\Delta^1_1\times\Delta^1_1)\cap V_{<\xi}\big)\mid C_{\gamma'}\subseteq\mathcal{U}_{(s)_p}\big\},$$

which is Π^1_1 and countable. The separation theorem gives $D\in\Delta^1_1$ between these two sets. As D is countable, there are $N,\tilde{\beta},\tilde{\gamma}\in\Delta^1_1$ with $D=\Big\{\Big(N(q),\big((\tilde{\beta})_q,(\tilde{\gamma})_q\big)\Big)\mid q\in\omega\Big\}$. Now we can define $C_p:=\bigcup_{q\in\omega,N(q)=p}\ C_{(\tilde{\gamma})_q}\setminus(\bigcup_{r<q}\ C_{(\tilde{\gamma})_r})$.

- We now study the properties of the C_p's. We can say that

$\circ$ The relation "$\vec{\delta}\in C_p$" is Δ^1_1 in $(p,\vec{\delta})$.

$\circ$ The C_p's are pairwise disjoint.

$\circ$ $C_p\in\mathbf{\Sigma}^0_\xi(\tau_1)$ since $C_{(\tilde{\gamma})_q}\in\mathbf{\Pi}^0_{<\xi}(\tau_1)\subseteq\mathbf{\Delta}^0_\xi(\tau_1)$, by Theorem 4.2.2.

$\circ$ We set $E:=\{(p,\vec{\delta})\in\omega\times\mathcal{N}^d\mid\exists q\in\omega\ \ N(q)=p\wedge\vec{\delta}\in C_{(\tilde{\gamma})_q}\}$, so that $E\in\Delta^1_1$ and $E_p\in\mathbf{\Sigma}^0_1(\tau_\xi)$ for any $p\geq 1$. Note that $C_p\subseteq E_p$.

$\circ$ $\bigcup_{p\geq 1}\ C_p=\bigcup_{p\geq 1}\ E_p$.

$\circ$ E_p separates $U\cap f^{-1}(\{p\})$ from $\neg\mathcal{U}_{(s)_p}$. In particular, U is a subset of the Δ^1_1 set $\bigcup_{p\geq 1}\ C_p$. Moreover, $\bigcap_{p\geq 1}\ \overline{\neg\mathcal{U}_{(s)_p}}^{\xi}\subseteq\neg(\bigcup_{p\geq 1}\ E_p)$.

- The induction assumption gives, for any $p\geq 1$, β^p,γ^p with $(<{}_{(p)_0+1,q}\alpha>,\beta^p,\gamma^p)\in\mathfrak{J}^\infty$ and C_{γ^p} separates $A_1\cap E_p$ from $A_0\cap E_p$. As in the proof of Lemma 6.7.(b) we may assume that the sequences (β^p) and (γ^p) are Δ^1_1. By Δ^1_1-selection again there is a Δ^1_1-recursive map $h:\omega\rightarrow\mathcal{N}\times\mathcal{N}$ such that $h(p)\in(\Delta^1_1\times\Delta^1_1)\cap V_\xi$ and $C_{h_1(p)}=\neg C_p$ for any $p\geq 1$. We set $\big((\beta)_p\big)_1:=h_0(p)$ and $\big((\delta)_p\big)_1:=h_1(p)$, so that $\Big((\alpha)_0,\big((\beta)_p\big)_1,\big((\delta)_p\big)_1\Big)\in Q$ for any $p\geq 1$.

We set $m:=2$, $(\beta)_0:=\beta^0$, and $\big((\beta)_p\big)_0:=\beta^p$ if $p\geq 1$, so that β is completely defined. Similarly, we define $(\delta)_0:=\gamma^0$, and $\big((\delta)_p\big)_0:=\gamma^p$ if $p\geq 1$. Finally, we choose $\gamma\in\Delta^1_1\cap W$ such that $C_\gamma=\bigcup_{p\geq 1}\left(C_{\gamma^p}\backslash C_{h_1(p)}\right)\cup(C_{(\delta)_0}\cap\bigcap_{p\geq 1}C_{h_1(p)})$, so that $(\alpha,m\beta,\gamma)\in\mathfrak{J}^\infty$ and C_γ separates $A_1\cap\Sigma$ from $A_0\cap\Sigma$. $\square$

The next result is the actual (effective) content of Theorem 1.10.(1). It is also the version of Theorem 4.4.1 for the non self-dual Wadge classes of Borel sets. Let $j_d:(d^\omega)^d\rightarrow\mathcal{N}$ be a continuous embedding (for example we can embed $(d^\omega)^d$ into $\mathcal{N}^d$ in the obvious way, and then use a bijection between $\mathcal{N}^d$ and $\mathcal{N}$).

Theorem 6.9 *Let T_d be a tree with Δ^1_1 suitable levels, α in Δ^1_1 normalized, β,γ in $\mathcal{N}$ such that $(\alpha,\beta,\gamma)\in\mathfrak{J}^\infty$, $S:=j_d^{-1}(C^{\mathcal{N}}_\gamma)\cap\lceil T_d\rceil$, and $a_0,a_1,b_0,b_1,r\in\mathcal{N}$ with $\vec{v}:=(\alpha,a_0,a_1,b_0,b_1,r)\in\mathfrak{K}^\infty$. Then one of the following holds:*

(a) $\neg\mathcal{U}_r=\emptyset$.

(b) The inequality $\big((\Pi''_i\lceil T_d\rceil)_{i\in d},S,\lceil T_d\rceil\setminus S\big)\leq\big((\mathcal{N})_{i\in d},A_0,A_1\big)$ holds.

Now we can state the version of Theorem 4.2.2 for the non self-dual Wadge classes of Borel sets.

Theorem 6.10 *Let T_d be a tree with Δ^1_1 suitable levels, α in Δ^1_1 normalized, β,γ in $\mathcal{N}$ such that $(\alpha,\beta,\gamma)\in\mathfrak{J}^\infty$, $S:=j_d^{-1}(C^{\mathcal{N}}_\gamma)\cap\lceil T_d\rceil$, and $a_0,a_1,b_0,b_1,r\in\mathcal{N}$ with $\vec{v}:=(\alpha,a_0,a_1,b_0,b_1,r)\in\mathfrak{K}^\infty$. We assume that S is not separable from $\lceil T_d\rceil\backslash S$ by a pot($\check{\mathbf{\Gamma}}_{c(\alpha)}$) set. Then the following are equivalent:*

(a) The set A_0 is not separable from A_1 by a pot($\check{\mathbf{\Gamma}}_{c(\alpha)}$) set.

(b) The set A_0 is not separable from A_1 by a $\Delta^1_1\cap$pot($\check{\mathbf{\Gamma}}_{c(\alpha)}$) set.

(c) $\neg\big(\exists\beta',\gamma'\in\mathcal{N}$ such that $(\alpha,\beta',\gamma')\in\mathfrak{J}^\infty$ and $A_1\subseteq C_{\gamma'}\subseteq\neg A_0\big)$.

(d) The set A_0 is not separable from A_1 by a $\check{\mathbf{\Gamma}}_{c(\alpha)}(\tau_1)$ set.

(e) $\neg\mathcal{U}_r\neq\emptyset$.

(f) The inequality $\big((d^\omega)_{i\in d},S,\lceil T_d\rceil\setminus S\big)\leq\big((\mathcal{N})_{i\in d},A_0,A_1\big)$ holds.

Proof. (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (d) are clear since $\Delta_{\mathcal{N}}$ is Polish.

(b) $\Rightarrow$ (c) This comes from Lemma 6.7.(a).

(b) $\Rightarrow$ (e), (c) $\Rightarrow$ (e) and (d) $\Rightarrow$ (e) This comes from Lemma 6.8.

(e) $\Rightarrow$ (f) This comes from Theorem 6.9 (as $\Pi''_i\lceil T_d\rceil$ is compact, we just have to compose with continuous retractions to get functions defined on d^ω).

(f) $\Rightarrow$ (a) If $P\in$pot($\check{\mathbf{\Gamma}}_{c(\alpha)}$) separates A_0 from A_1 and (f) holds, then $S\subseteq(\Pi_{i\in d}\ f_i)^{-1}(P)\subseteq\neg(\lceil T_d\rceil\backslash S)$. This implies that S is separable from $\lceil T_d\rceil\backslash S$ by a pot($\check{\mathbf{\Gamma}}_{c(\alpha)}$) set, by Lemma 4.4.7. But this contradicts the assumption on S. $\square$

Proof of Theorem 1.10.(1). Note first that (a) and (b) cannot hold simultaneously, as in the proof of Theorem 6.10.

We assume that (a) does not hold. This implies that the X_i's are not empty, since otherwise $A_0=A_1=\emptyset$, and $\emptyset\in\check{\mathbf{\Gamma}}$ unless $\mathbf{\Gamma}=\{\emptyset\}$. As in the proof of Theorem 4.1, we may assume that $X_i=\mathcal{N}$ for each $i\in d$, by Lemma 4.4.7. By Theorem 5.1.3 there is $u\in\mathcal{D}$ with $\mathbf{\Gamma}(\mathcal{N})=\mathbf{\Gamma}_u(\mathcal{N})$. If E is a zero-dimensional Polish space, then we also have $\mathbf{\Gamma}(E)=\mathbf{\Gamma}_u(E)$, by Theorem 4.1.3 in [Lo-SR2]. It follows that pot$(\mathbf{\Gamma})=$ pot$(\mathbf{\Gamma}_u)$. By Lemmas 6.2 and 6.4 we may assume that there is $\alpha\in\mathfrak{H}^\infty$ normalized with $c(\alpha)=u$.

By Theorem 4.1.3 in [Lo-SR2] there is $B\in\mathbf{\Gamma}(\mathcal{N})$ with $S=j_d^{-1}(B)\cap\lceil T_d\rceil$. To simplify the notation, we may assume that T_d has Δ^1_1 levels, $\alpha\in\Delta^1_1$, and $A_0,A_1\in\Sigma^1_1(\mathcal{N}^d)$. By Lemma 6.5 there are $\beta,\gamma\in\mathcal{N}$ such that $(\alpha,\beta,\gamma)\in\mathfrak{I}^\infty$ and $C^{\mathcal{N}}_\gamma=B$. Lemma 6.7.(b) gives b_0,b_1,r with $(\alpha,a_0,a_1,b_0,b_1,r)\in\mathfrak{K}^\infty$. Lemma 6.8 implies that $\neg\mathcal{U}_r\neq\emptyset$. So (b) holds, by Theorem 6.10. □

The sequel is devoted to the proof of Theorem 6.9. We have to introduce a lot of objects before we can do it. We will create some paragraphs to describe these objects. We start with a general notion. The idea is that, given a set S in $\mathbf{\Gamma}_{c(\alpha)}(\lceil T_d\rceil)$, and with the help of the tree $\mathfrak{T}(\alpha)$, we will keep all the $\mathbf{\Sigma}^0_\xi$ (or equivalently $\mathbf{\Pi}^0_\xi$, if we pass to complements) sets used to build S in mind. We will represent these $\mathbf{\Pi}^0_\xi$ sets, on most sequences s of $\mathfrak{T}(\alpha)$, by induction on $|s|$, applying the Debs-Saint Raymond theorem. At each induction step, some $\mathbf{\Pi}^0_\xi$ sets of the level become closed, but we also partially simplify the $\mathbf{\Pi}^0_\xi$ sets to come. This is the reason why the ordinal subtraction is involved (recall the definition of the ordinal subtraction after Theorem 5.1.3).

Definition 6.11 *Let X be a set, $A\subseteq X$, $\mathcal{B}$ be a countable family of subsets of X, and $\mathbf{\Gamma}$ be a Borel class. We say that $A\in\mathbf{\Gamma}(\mathcal{B})$ if $A\in\mathbf{\Gamma}(X,\tau)$ for any topology τ on X containing $\mathcal{B}$.*

Proposition 6.12 *Let X be a topological space.*

(a) Let $A\subseteq X$, $\mathcal{B}$ be a countable family of open subsets of X, and $\mathbf{\Gamma}$ be a Borel class. Then $A\in\mathbf{\Gamma}(X)$ if $A\in\mathbf{\Gamma}(\mathcal{B})$.

(b) Let Y be a set, $B\subseteq Y$, $f:X\to Y$ be a bijection, $\mathcal{B}$ be a countable family of subsets of Y, and $\mathbf{\Gamma}$ be a Borel class. Then $f^{-1}(B)\in\mathbf{\Gamma}(\{f^{-1}(D)\mid D\in\mathcal{B}\})$ if $B\in\mathbf{\Gamma}(\mathcal{B})$.

(c) Let $1\leq\eta\leq\xi$ and $A\in\mathbf{\Pi}^0_\xi(X)$. We assume that X is metrizable. Then there is $\mathcal{B}\subseteq\mathbf{\Pi}^0_\eta(X)$ countable such that $A\in\mathbf{\Pi}^0_{1+(\xi-\eta)}(\check{\mathcal{B}})$, where $\check{\mathcal{B}}:=\{\neg B\mid B\in\mathcal{B}\}$.

In practice, X will be the metrizable space $[R]$, for some tree relation R, and f will be the canonical map given by the Debs-Saint Raymond theorem.

Proof. (a) The topology τ is simply the topology of X.

(b) Let τ be a topology on X containing $\{f^{-1}(D)\mid D\in\mathcal{B}\}$. Then $\sigma:=\{f[A]\mid A\in\tau\}$ is a topology on Y containing $\mathcal{B}$. Thus $B\in\mathbf{\Gamma}(Y,\sigma)$ since $B\in\mathbf{\Gamma}(\mathcal{B})$. Therefore $f^{-1}(B)\in\mathbf{\Gamma}(X,\tau)$ since $f:(X,\tau)\to(Y,\sigma)$ is continuous.

(c) We argue by induction on $\xi-\eta$. The result is clear if $\xi-\eta=0$. So assume that $\xi-\eta\geq 1$. Write $A=\bigcap_{n\in\omega}\ \neg A_n$, where $\eta_n<\xi$ and $A_n\in\mathbf{\Pi}^0_{\eta_n}(X)$. As X is metrizable, we may assume that $\eta\leq\eta_n$. The induction assumption gives $\mathcal{B}_n\subseteq\mathbf{\Pi}^0_\eta(X)$ countable such that $A_n\in\mathbf{\Pi}^0_{1+(\eta_n-\eta)}(\check{\mathcal{B}}_n)$. It remains to set $\mathcal{B}:=\bigcup_{n\in\omega}\ \mathcal{B}_n$. □

(A) The witnesses

Notation. We first define a map producing witnesses for the fact that $\vec{v}\in\mathfrak{K}^\infty$. More precisely, we define a map $\mathcal{V}:\mathfrak{K}^\infty\rightarrow\mathfrak{K}^\infty\cup(\mathfrak{K}^\infty)^\omega$. Let $\vec{v}:=(\alpha,a_0,a_1,b_0,b_1,r)\in\mathfrak{K}^\xi\setminus\mathfrak{K}^{<\xi}$. If $|(\alpha)_1|=0$, then we set $\mathcal{V}(\vec{v}):=\vec{v}$. If $|(\alpha)_1|=1$, then, using the definition of $\mathfrak{K}$, we set

$$\mathcal{V}(\vec{v}):=(<{}_j\alpha>,a_0,a_1,b_0,b_1,a_1).$$

Note that $\mathcal{V}(\vec{v})\in\mathfrak{K}^{<\xi}$. If $|(\alpha)_1|=2$, then we set

$$\mathcal{V}(\vec{v})(p):=\begin{cases}\big(<_{0,q}\alpha>,a_0,a_1,(c_0)_0,(c_1)_0,(s)_0\big)\text{ if }p=0,\\ \\ \big(<_{(p)_0+1,q}\alpha>,a_0,a_1,(c_0)_p,(c_1)_p,(s)_p\big)\text{ if }p\geq 1.\end{cases}$$

Here again, $\mathcal{V}(\vec{v})(p)\in\mathfrak{K}^{<\xi}$.

Similarly, we define a map $\mathcal{W}$ producing witnesses for the fact that $\vec{w}\in\mathfrak{I}^\infty$. Moreover, we keep δ in mind. More precisely, we define a map $\mathcal{W}:\mathfrak{I}^\infty\rightarrow\mathfrak{I}^\infty\cup(\mathcal{N}\times\mathfrak{I}^\infty)\cup\big(\mathcal{N}\times(\mathfrak{I}^\infty)^\omega\big)$. Let $\vec{w}:=(\alpha,m\beta,\gamma)$ be in $\mathfrak{I}^\xi\setminus\mathfrak{I}^{<\xi}$. If $|(\alpha)_1|=0$, then we set $\mathcal{W}(\vec{w}):=\vec{w}$. If $|(\alpha)_1|=1$, then, using the definition of $\mathfrak{I}$ and choosing δ, we set $\mathcal{W}(\vec{w}):=\big(\delta,(<{}_j\alpha>,\beta,\delta)\big)$. If $|(\alpha)_1|=2$, then we set $\mathcal{W}(\vec{w}):=\big(\delta,\mathcal{Y}(\vec{w})\big)$, where

$$\mathcal{Y}(\vec{w})(p):=\begin{cases}\big(<_{0,q}\alpha>,(\beta)_0,(\delta)_0\big)\text{ if }p=0,\\ \\ \big(<_{(p)_0+1,q}\alpha>,\big((\beta)_p\big)_0,\big((\delta)_p\big)_0\big)\text{ if }p\geq 1.\end{cases}$$

(B) The trees associated with the codes for the non self-dual Wadge classes of Borel sets

• Recall the definition of $\mathfrak{T}(\alpha)$ after Lemma 6.2. Similarly, we define $\mathfrak{T}:\mathfrak{I}^\infty\rightarrow\{\text{trees on }\omega\times\mathfrak{I}^\infty\}$ as follows. Let $\vec{w}:=(\alpha,\beta,\gamma)\in\mathfrak{I}^\xi\setminus\mathfrak{I}^{<\xi}$. We set

$$\mathfrak{T}(\vec{w}):=\begin{cases}\{\emptyset\}\cup\{<(0,\vec{w})>\}\text{ if }|(\alpha)_1|=0,\\ \{\emptyset\}\cup\big\{(0,\vec{w})^\frown s\mid s\in\mathfrak{T}\big(\mathcal{Y}(\vec{w})\big)\big\}\text{ if }|(\alpha)_1|=1,\\ \{\emptyset\}\cup\bigcup_{p\in\omega}\ \big\{(p,\vec{w})^\frown s\mid s\in\mathfrak{T}\big(\mathcal{Y}(\vec{w})(p)\big)\big\}\text{ if }|(\alpha)_1|=2.\end{cases}$$

Here again $\mathfrak{T}(\vec{w})$ is a countable well founded tree containing the sequence $<(0,\vec{w})>$. The set of maximal sequences in $\mathfrak{T}(\vec{w})$ is $\mathcal{M}_{\vec{w}}:=\{s\in\mathfrak{T}(\vec{w})\mid\forall t\in\mathfrak{T}(\vec{w})\ \ s\subseteq t\Rightarrow s=t\}$.

• Fix $\vec{w}:=(\alpha,\beta,\gamma)\in\mathfrak{I}^\infty$ with $\alpha\in\varDelta^1_1$ normalized. In the sequel, it will be convenient to set, for $s\in\mathfrak{T}(\vec{w})\setminus\mathcal{M}_{\vec{w}}$,

$$s_1(|s|):=\begin{cases}\vec{w}\text{ if }s=\emptyset,\\ \mathcal{Y}\big(s_1(|s|-1)\big)\text{ if }s\neq\emptyset\wedge\big|\big(s_1(|s|-1)(0)\big)_1\big|=1,\\ \mathcal{Y}\big(s_1(|s|-1)\big)\big(s_0(|s|-1)\big)\text{ if }s\neq\emptyset\wedge\big|\big(s_1(|s|-1)(0)\big)_1\big|=2.\end{cases}$$

• Let $s\in\mathfrak{T}(\vec{w})$. We set $B_s:=\{i<|s|\mid |\big(s_1(i)(0)\big)_1|=2\}$. As α is normalized, B_s is a natural number. Note that $B_s\leq |s|$. If moreover $s\in\mathfrak{T}(\vec{w})\setminus\mathcal{M}_{\vec{w}}$, then we set $C_s:=\{i\leq |s|\mid |\big(s_1(i)(0)\big)_1|=2\}$.

• The ordinals $|(\alpha)_0|$, for $\alpha\in\Delta^1_1\cap\mathfrak{H}^\infty$, will be of particular importance in the sequel. We define a function $\mathcal{Z}:\mathfrak{T}(\vec{w})\setminus\mathcal{M}_{\vec{w}}\rightarrow(\omega_1^{\text{CK}})^{<\omega}$ satisfying $|\mathcal{Z}(s)|=|s|+1$. The sequence s codes some $\mathbf{\Pi}^0_\xi$ sets, and the role of $\mathcal{Z}(s)$ is to give these ξ's. We set $\mathcal{Z}(s)(i):=\big|\big(s_1(i)(0)\big)_0\big|$ if $i\leq |s|$. We can easily check the following properties of $\mathcal{Z}(s)$:

- $\mathcal{Z}(s)(i)$ depends only on $s|i$.
- $\mathcal{Z}(s)\subseteq\mathcal{Z}(t)$ if $s\subseteq t$.
- $\mathcal{Z}(s)(i+1)\geq\mathcal{Z}(s)(i)$ or $\mathcal{Z}(s)(i+1)=0$ if $i<|s|$.
- $\mathcal{Z}(s)(i+1)=0$ if $\mathcal{Z}(s)(i)=0$ and $i<|s|$.
- $\big(\mathcal{Z}(s)(i)\big)_{i\in C_s}$ is an increasing sequence of recursive ordinals different from zero.

(C) The resolution families

• Fix $\vec{w}:=(\alpha,\beta,\gamma)\in\mathfrak{I}^\infty$ with $\alpha\in\Delta^1_1$ normalized, and $p\geq 1$. We set

$$\mathcal{Q}^{\vec{w}}_p:=\begin{cases}\mathcal{N}\text{ if }|(\alpha)_1|\leq 1,\\ C^{\mathcal{N}}_{((\mathcal{W}_0(\vec{w}))_p)_1}\text{ if }|(\alpha)_1|=2.\end{cases}$$

Note that $\mathcal{Q}^{\vec{w}}_p\in\mathbf{\Pi}^0_{|(\alpha)_0|}(\mathcal{N})$ if $|(\alpha)_1|=2$, by Lemma 6.1.

• Recall the finite sets $c_l\subseteq d^d$ defined at the end of the proof of Proposition 2.2 (we only used the fact that T_d has finite levels to see that they are finite). We put $c:=\bigcup_{l\in\omega}\ c_l$, so that c is countable. This will be the countable set c mentioned in Definition 4.3.1.

• Recall the embedding j_d defined before Theorem 6.9. We set $\mathcal{P}^{\vec{w}}_p:=h[j_d^{-1}(\mathcal{Q}^{\vec{w}}_p)\cap c^\omega]$, so that the union $\mathcal{P}^{\vec{w}}_p\cup\mathcal{P}^{\vec{w}}_q=[\subseteq]$ if $p\neq q\geq 1$. Moreover, $\mathcal{P}^{s_1(i)}_p\in\mathbf{\Pi}^0_{\mathcal{Z}(s)(i)}([\subseteq])$ if $s\in\mathfrak{T}(\vec{w})\setminus\mathcal{M}_{\vec{w}}$ and $i\in C_s$.

• If T is a tree and $s\in T$, then $T_s:=\{t\in T\mid s\subseteq t\}$.

• Fix $\vec{w}:=(\alpha,\beta,\gamma)\in\mathfrak{I}^\infty$ with $\alpha\in\Delta^1_1$ normalized and $|(\alpha)_1|=2$. We say that $s\in\mathfrak{T}(\vec{w})$ is $extendable$ if there is $t\in\mathfrak{T}(\vec{w})_s$ such that $|s|<B_t$ (which implies that $s\notin\mathcal{M}_{\vec{w}}$). We will construct, for each s extendable, a resolution family $(R^\rho_s)_{\rho\leq\eta_s}$. We construct simultaneously some ordinals ξ_s and θ_s. If θ is an ordinal, then we set

$$\theta^*:=\begin{cases}\eta\text{ if }\theta=\eta+1,\\ \theta\text{ otherwise}\end{cases}$$

(this is what appears in the Debs-Saint Raymond theorem). The following will hold: $\eta_s=\theta^*_s$, $\xi_s=\mathcal{Z}(s)(|s|)$ and

$$\theta_s:=\begin{cases}\xi_s=\mathcal{Z}(s)(0)=|(\alpha)_0|\text{ if }s=\emptyset,\\ 1+(\xi_s-\xi_{s^-})\text{ if }s\neq\emptyset.\end{cases}$$

We want the resolution family to satisfy the following conditions.

- The family $(R_s^\rho)_{\rho\leq\eta_s}$ is uniform if θ_s is a limit ordinal.
- $R_\emptyset^0 = \subseteq$, and $R_{s^-}^{\eta_{s^-}} = R_s^0$ if $s \neq \emptyset$.
- $\Pi_s : [R_s^{\eta_s}] \rightarrow [R_s^0]$ is a continuous bijection.
- We set ${}_s\Pi := \Pi_{s|0} \circ \Pi_{s|1} \circ ... \circ \Pi_s$. Then ${}_s\Pi^{-1}(\mathcal{P}_p^{s_1(|s|)}) \in \mathbf{\Pi}_1^0([R_s^{\eta_s}])$ if $p \geq 1$.
- ${}_s\Pi^{-1}(\mathcal{P}_p^{t_1(j+1)}) \in \mathbf{\Pi}^0_{1+(\mathcal{Z}(t)(j+1)-\xi_s)}([R_s^{\eta_s}])$ if $p \geq 1$, $t \in \mathfrak{T}(\vec{w})_s \setminus \mathcal{M}_{\vec{w}}$ and $|s| < j+1 \in C_t$.

• The construction is by induction on $|s|$. Assume that $s = \emptyset$, $p \geq 1$, $t \in \mathfrak{T}(\vec{w}) \setminus \mathcal{M}_{\vec{w}}$ and $j+1 \in C_t$. Proposition 6.12.(c) gives $\mathcal{B}_p^{t,j} \subseteq \mathbf{\Pi}^0_{\theta_\emptyset}([\subseteq])$ countable such that $\mathcal{P}_p^{t_1(j+1)} \in \mathbf{\Pi}^0_{1+(\mathcal{Z}(t)(j+1)-\theta_\emptyset)}(\check{\mathcal{B}}_p^{t,j})$. This implies that $u_\emptyset := \{\mathcal{P}_p^{\vec{w}} \mid p \geq 1\} \cup \bigcup_{p\geq 1, t\in\mathfrak{T}(\vec{w})\setminus\mathcal{M}_{\vec{w}}, j+1\in C_t} \mathcal{B}_p^{t,j}$ is countable and made of $\mathbf{\Pi}^0_{\theta_\emptyset}([\subseteq])$ sets. Theorems 4.3.4 and 4.4.4 give a family $(R_\emptyset^\rho)_{\rho\leq\eta_\emptyset}$, uniform if $\theta_\emptyset$ is a limit ordinal, such that

- $R_\emptyset^0 = \subseteq$,
- $\Pi_\emptyset : [R_\emptyset^{\eta_\emptyset}] \rightarrow [R_\emptyset^0]$ is a continuous bijection,
- $\Pi_\emptyset^{-1}(Q) \in \mathbf{\Pi}_1^0([R_\emptyset^{\eta_\emptyset}])$ for each $Q \in u_\emptyset$.

This family is suitable, by Proposition 6.12.

• Assume now that $s \neq \emptyset$ is extendable, and that the construction is done for the strict predecessors of s. Note that ${}_{s^-}\Pi^{-1}(\mathcal{P}_p^{s_1(|s|)}) \in \mathbf{\Pi}^0_{\theta_s}([R_{s^-}^{\eta_{s^-}}])$. Assume that $p \geq 1$, $t \in \mathfrak{T}(\vec{w})_s \setminus \mathcal{M}_{\vec{w}}$ and $|s| < j+1 \in C_t$. Then Proposition 6.12.(c) gives a countable family $\mathcal{C}_p^{t,j} \subseteq \mathbf{\Pi}^0_{\theta_s}([R_{s^-}^{\eta_{s^-}}])$ such that ${}_{s^-}\Pi^{-1}(\mathcal{P}_p^{t_1(j+1)})$ is in $\mathbf{\Pi}^0_{1+(\mathcal{Z}(t)(j+1)-\xi_s)}(\check{\mathcal{C}}_p^{t,j})$. This implies that

$$u_s := \{{}_{s^-}\Pi^{-1}(\mathcal{P}_p^{s_1(|s|)}) \mid p \geq 1\} \cup \bigcup_{p\geq 1, t\in\mathfrak{T}(\vec{w})_s\setminus\mathcal{M}_{\vec{w}}, |s|<j+1\in C_t} \mathcal{C}_p^{t,j}$$

is countable and made of $\mathbf{\Pi}^0_{\theta_s}([R_{s^-}^{\eta_{s^-}}])$ sets. Theorems 4.3.4 and 4.4.4 give a resolution family $(R_s^\rho)_{\rho\leq\eta_s}$, uniform if θ_s is a limit ordinal, such that

- $R_s^0 = R_{s^-}^{\eta_{s^-}}$,
- $\Pi_s : [R_s^{\eta_s}] \rightarrow [R_s^0]$ is a continuous bijection,
- $\Pi_s^{-1}(Q) \in \mathbf{\Pi}_1^0([R_s^{\eta_s}])$ for each $Q \in u_s$.

This family is suitable, by Proposition 6.12. This completes the construction of the families.

(D) The subsets of T_d

We now build some subsets of T_d that will play the role that D and $T_d \setminus D$ played in the proof of Theorem 4.4.1. Fix $\vec{w} := (\alpha, \beta, \gamma) \in \mathfrak{I}^\infty$ with $\alpha \in \varDelta_1^1$ normalized and $|(\alpha)_1| = 2$. We will define a family of subsets of T_d as follows. Assume that $s \in \mathfrak{T}(\vec{w})$ is extendable.

We set, for $q\geq 1$,

$$P_0(s):=\Big\{\vec s\in T_d \mid \vec s=\vec\emptyset \vee \forall p\geq 1\ \ \exists\mathcal{B}_p\in{}_s\Pi^{-1}(\mathcal{P}_p^{s_1(|s|)})\ \ \vec s\in\mathcal{B}_p\Big\},$$

$$P_q(s):=\Big\{\vec s\in T_d \mid \vec s\neq\vec\emptyset \wedge \forall\mathcal{B}_q\in{}_s\Pi^{-1}(\mathcal{P}_q^{s_1(|s|)})\ \ \vec s\notin\mathcal{B}_q \wedge$$
$$\forall p\in\omega\setminus\{0,q\}\ \ \exists\mathcal{B}_p\in{}_s\Pi^{-1}(\mathcal{P}_p^{s_1(|s|)})\ \ \vec s\in\mathcal{B}_p\Big\}.$$

Note that the $P_q(s)$'s are pairwise disjoint. We set, for $s\in\mathfrak{T}(\vec w)$ and $i\leq |s|$, $\mathcal{I}_{i,s}:=\bigcap_{j<i}\ P_{s(j)(0)}(s|j)$. If $i=|s|$, then we write $\mathcal{I}_s$ instead of $\mathcal{I}_{i,s}$. The next lemma associates to each $\vec t\in T_d$ a sequence $s(\vec t\,)$ in $\mathfrak{T}(\vec w)$ specifying in which $P_q(s)$'s the sequence $\vec t$ is.

Proposition 6.13 *Let $\vec w:=(\alpha,\beta,\gamma)\in\mathfrak{I}^\infty$ with $\alpha\in\varDelta^1_1$ normalized and $|(\alpha)_1|=2$, and $\vec t\in T_d$. Then there are $l\in\omega$ and $s(\vec t\,)\in\mathfrak{T}(\vec w)$ of length l satisfying the following statements.*
(a) $\vec t\in\mathcal{I}_{s(\vec t\,)}$.
(b) If $s(\vec t\,)$ is extendable by t, then $\vec t\notin P_{t(l)(0)}(t|l)$.

Proof. We actually construct, for $j\in\omega$, a sequence $s_j\in\mathfrak{T}(\vec w)$. We will have $s_j\subseteq s_{j+1}$, $|s_j|=j$ if $j\leq l$, $s_j=s_l$ if $j>l$, and $\vec t\in\mathcal{I}_{s_j}$. At the end, $s(\vec t\,)$ will be s_l. The definition of s_j is by induction on j. Assume that $(s_k)_{k\leq j}$ are constructed and satisfy these properties, which is the case for $j=0$. We may assume that $|s_j|=j$.

If s_j is not extendable or $\vec t\notin\mathcal{B}$ for each $\mathcal{B}\in[R^{\eta_{s_j}}_{s_j}]$, then we set $s_{j+1}:=s_j$. If $\vec t\in\mathcal{B}$ for some $\mathcal{B}\in[R^{\eta_{s_j}}_{s_j}]$, then there is a unique natural number q such that $\vec t\in P_q(s_j)$ since

$${}_{s_j}\Pi^{-1}(\mathcal{P}_p^{(s_j)_1(j)})\cup{}_{s_j}\Pi^{-1}(\mathcal{P}_q^{(s_j)_1(j)})=[R^{\eta_{s_j}}_{s_j}]$$

if $p\neq q\geq 1$. We will have $|s_{j+1}|=j+1$, and $s_{j+1}(j)(0):=q$. Moreover,

$$s_{j+1}(j)(1):=\begin{cases}\vec w \text{ if } j=0,\\ \mathcal{Y}\big(s_j(j-1)(1)\big)\big(s_j(j-1)(0)\big) \text{ if } j\geq 1.\end{cases}$$

This completes the construction of the s_j's, and they are in $\mathfrak{T}(\vec w)$. The well-foundedness of $\mathfrak{T}(\vec w)$ proves the existence of l, and $s(\vec t\,)$ is suitable. □

Notation. Proposition 6.13 associates $s(\vec t\,)\in\mathfrak{T}(\vec w)$ to $\vec t\in T_d$. Under the same conditions, we can associate $S(\vec t\,)\in\mathcal{M}_{\vec w}$ to $\vec t$. In order to do this, we need the following lemma:

Lemma 6.14 *Let $\vec w:=(\alpha,\beta,\gamma)\in\mathfrak{I}^\infty$ with $\alpha\in\varDelta^1_1$ normalized and $|(\alpha)_1|=2$, and $s\in\mathfrak{T}(\vec w)$. Then there is $S\in\mathcal{M}_{\vec w}$ extending s such that $S_0(i)=0$ if $|s|\leq i<|S|$.*

Proof. If $s=\emptyset$, then we set $S(0):=(0,\vec w)$. If $\mathcal{W}\big(S_1(i)\big)\neq S_1(i)$, then we set

$$S(i+1):=\begin{cases}\Big(0,\mathcal{Y}\big(S(i)\big)\Big) \text{ if } \mathcal{Y}\big(S(i)\big)\in\mathfrak{I}^\infty,\\ \\ \Big(0,\mathcal{Y}\big(S(i)\big)(0)\Big) \text{ if } \mathcal{Y}\big(S(i)\big)\in(\mathfrak{I}^\infty)^\omega.\end{cases}$$

By induction, we see that $S|(i+1)\in\mathfrak{T}(\vec w)$ for each $i<|S|$, which proves that the length of S is finite since $\mathfrak{T}(\vec w)$ is well-founded. Thus $S\in\mathcal{M}_{\vec w}$.

If $s\neq\emptyset$, then $S(|s|-1)$ is defined. We argue similarly. The only thing to change is that

$$S(|s|):=\Big(0,\mathcal{Y}\big(s(|s|-1)\big)\big(s_0(|s|-1)\big)\Big)$$

if $\mathcal{W}\big(s_1(|s|-1)\big)\neq s_1(|s|-1)$ and $\mathcal{Y}\big(s(|s|-1)\big)\in(\mathfrak{I}^\infty)^\omega$. □

We now associate a maximal extension $S(\vec{t}\,)$ of $s(\vec{t}\,)$ to any $\vec{t}$ in T_d.

Remark. There is $S(\vec{\emptyset})\in\mathcal{M}_{\vec{w}}$ with $\big(S(\vec{\emptyset})\big)_0(i)=0$ if $i<|S(\vec{\emptyset})|$. Note that $s(\vec{\emptyset})\subseteq S(\vec{\emptyset})$. If $\vec{\emptyset}\neq\vec{t}\in T_d$, then we define $S(\vec{t}\,)$ by induction on $|\vec{t}|$:

- If $s(\vec{t}\,)=\emptyset$, then $\vec{t}\neq\emptyset$ since $\vec{\emptyset}\in P_0(\emptyset)$, and $S(\vec{t}\,):=S(\vec{t}^{\,\eta_\emptyset}_{\emptyset})$.
- If $s(\vec{t}\,)\neq\emptyset$ and $\vec{t}^{\,\eta_{s(\vec{t}\,)^-}}_{s(\vec{t}\,)^-}\in\mathcal{I}_{s(\vec{t}\,)}$, then $S(\vec{t}\,):=S(\vec{t}^{\,\eta_{s(\vec{t}\,)^-}}_{s(\vec{t}\,)^-})$.
- If $s(\vec{t}\,)\neq\emptyset$ and $\vec{t}^{\,\eta_{s(\vec{t}\,)^-}}_{s(\vec{t}\,)^-}\notin\mathcal{I}_{s(\vec{t}\,)}$, then $S(\vec{t}\,)$ is the extension of $s(\vec{t}\,)$ given by Lemma 6.14 applied to $s:=s(\vec{t}\,)$.

Note that $S(\vec{t}\,)\in\mathcal{M}_{\vec{w}}$ and is an extension of $s(\vec{t}\,)$, by induction on $|\vec{t}|$. This comes from the fact that $s(\vec{t}\,)\subseteq s(\vec{t}^{\,\eta_{s(\vec{t}\,)^-}}_{s(\vec{t}\,)^-})$ in the second case.

(E) The tuples

We now keep the tuples $(\alpha,a_0,a_1,b_0,b_1,r)$ along the elements of $\mathfrak{T}(\vec{w})$ in mind, using the witness map $\mathcal{V}$. Fix $\vec{w}:=(\alpha,\beta,\gamma)\in\mathfrak{I}^\infty$ and $\vec{v}:=(\alpha,a_0,a_1,b_0,b_1,r)\in\mathfrak{K}^\infty$. In the sequel, we will say that $(\vec{w},\vec{v}\,)$ is *standard* if $\alpha\in\varDelta^1_1$ is normalized and $|(\alpha)_1|=2$. Assume that $(\vec{w},\vec{v}\,)$ is standard. We will define a map $V:\mathfrak{T}(\vec{w})\rightarrow(\mathfrak{K}^\infty)^{<\omega}$ such that $|V(s)|=|s|$, $V^{s,i}:=(V^{s,i}_j)_{j\leq 5}:=V(s)(i)$ depends only on $s|i$ as follows. We set, for $i<|s|$,

$$V^{s,i}:=\left\{\begin{array}{l}\vec{v}\text{ if }i=0,\\ \mathcal{V}(V^{s,i-1})\text{ if }i\geq 1\wedge|(V^{s,i-1}_0)_1|\leq 1,\\ \mathcal{V}(V^{s,i-1})\big(s_0(i-1)\big)\text{ if }i\geq 1\wedge|(V^{s,i-1}_0)_1|=2.\end{array}\right.$$

Lemma 6.15 *Let $(\vec{w},\vec{v}\,)$ be standard, $s\in\mathfrak{T}(\vec{w})$, and $i<|s|$. Then $V^{s,i}_0=s_1(i)(0)$. In particular, $s\notin\mathcal{M}_{\vec{w}}$ and $i\leq|s|$ imply that $\mathcal{Z}(s)(i)=|(V^{s,i}_0)_0|$.*

Proof. The last assertion clearly comes from the first one. The proof is by induction on i. The assertion is clear for $i=0$ since $V^{s,0}_0=s_1(0)(0)=\alpha$. Assume that it holds for $i<|s|-1$.

- If $i\notin B_s$, then $|(V^{s,i}_0)_1|=|\big(s_1(i)(0)\big)_1|=1$. Thus
$$V^{s,i+1}_0=\mathcal{V}(V^{s,i})(0)=<_jV^{s,i}_0>=<_js_1(i)(0)>=s_1(i+1)(0).$$
- If $i\in B_s$, then $|(V^{s,i}_0)_1|=|\big(s_1(i)(0)\big)_1|=2$. If moreover $s_0(i)=0$, then

$$V^{s,i+1}_0=<_{0,q}V^{s,i}_0>=<_{0,q}s_1(i)(0)>=s_1(i+1)(0).$$

The argument is similar if $s_0(i)\geq 1$. □

The next lemma is a preparation for Lemma 6.21, which is the crucial step for proving a version of the claim in the proof of Theorem 4.4.1 for the non self-dual Wadge classes of Borel sets.

Lemma 6.16 *Let* $(\vec{w},\vec{v}\ )$ *be standard,* $t\in\mathfrak{T}(\vec{w})$*, and* $i\in B_t$*.*
(a) If $t_0(i)=0$*, then* $\neg\mathcal{U}_{V_5^{t,i}}\subseteq\neg\mathcal{U}_{V_5^{t,i+1}}$*.*
(b) The inclusion $\neg\mathcal{U}_{V_5^{t,i}}\subseteq\overline{\neg\mathcal{U}_{V_5^{t,i+1}}}^{\xi_{t|i}}$ *holds.*

Proof. (a) Note that $V^{t,i+1}=\mathcal{V}(V^{t,i})(0)$, by Lemma 6.15. Thus $V_5^{t,i+1}=\mathcal{V}(V^{t,i})(0)(5)=(s)_0$ for some s for which $\neg\mathcal{U}_{V_5^{t,i}}\subseteq\neg\mathcal{U}_{(s)_0}$, by the 2nd and the 4th remarks after the definition of $\mathfrak{K}$.

(b) We may assume that $t_0(i)\geq 1$, so that $V_5^{t,i+1}=(s)_{t_0(i)}$, and $\neg\mathcal{U}_{V_5^{t,i}}\subseteq\overline{\neg\mathcal{U}_{V_5^{t,i+1}}}^{|(V_0^{t,i})_0|}$ by the 5th remark after the definition of $\mathfrak{K}$ and the definition of $\mathcal{A}$. We are done, by Lemma 6.15. □

(F) The sequences of natural numbers

Let $s\in\mathfrak{T}(\vec{w})$. We have to keep the natural numbers $s_0(i)$ in mind. We will consider an ordering of these finite sequences of natural numbers that will help us to prove the claim we just mentioned.

Notation. Fix $(\vec{w},\vec{v}\ )$ standard and $s,u\in\mathfrak{T}(\vec{w})$.

• If s and u are not compatible, then we denote $s\wedge u:=s|i=u|i$, where i is minimal with $s(i)\neq u(i)$. Note that $|s\wedge u|\in B_s$.

• We define $O(s)\in\omega^{|s|}$: we set $O(s)(i):=s_0(i)$.

• We also define a partial order on $\omega^{<\omega}$ as follows:

$$O\sqsubseteq P\ \Leftrightarrow\ O=P\vee\exists i<\text{min}(|O|,|P|)\ \big(O|i=P|i\wedge O(i)=0<P(i)\big).$$

Lemma 6.17 *Let* $(\vec{w},\vec{v}\ )$ *be standard and* $s,u\in\mathfrak{T}(\vec{w})$ *be incompatible. We assume that* $\vec{s}$ *is in* $\mathcal{I}_{|s\wedge u|+1,s}$*,* $\vec{t}\in\mathcal{I}_{|s\wedge u|+1,u}$ *and* $\vec{s}\ R_{s||s\wedge u|}^{\eta_{s||s\wedge u|}}\ \vec{t}$*. Then* $O(s)\sqsubseteq O(u)$*.*

Proof. As $s(|s\wedge u|)\neq u(|s\wedge u|)$ and $s_1(|s\wedge u|)=u_1(|s\wedge u|)$, $s_0(|s\wedge u|)\neq u_0(|s\wedge u|)$. Recall the definition of the $P_q(s)$'s. Note the following facts. Assume that $i\in B_s$ and $\vec{s}\ R_{s|i}^{\eta_{s|i}}\ \vec{t}$.
- If $s_0(i)=0$ and $\vec{t}\in P_0(s|i)$, then $\vec{s}\in P_0(s|i)$ too.
- If $s_0(i)\geq 1$ and $\vec{t}\in P_{s_0(i)}(s|i)$, then $\vec{s}\in P_0(s|i)\cup P_{s_0(i)}(s|i)$.

These facts imply that $s_0(|s\wedge u|)=0<u_0(|s\wedge u|)$. Therefore $O(s)\sqsubseteq O(u)$. □

(G) The ranges

The goal of this paragraph is to define the analytic sets $r\big(S(\vec{t}\ )\big)$ that will contain $U_{\vec{t}}$ in the proof of Theorem 6.9. They will play the role that $\overline{A_0}^{\xi}\cap A_1$ and A_0 played in the proof of Theorem 4.4.1 (see Conditions (4)-(5)).

Notation. Fix $(\vec{w}, \vec{v}\ )$ standard and $s \in \mathfrak{T}(\vec{w}) \setminus \{\emptyset\}$. We set

$$i^s := \begin{cases} |s|-1 \text{ if } \forall j < |s| \ \ s_0(j) \geq 1, \\ \min\{i < |s| \mid s_0(i) = 0\} \text{ otherwise,} \end{cases}$$

$$I^s := \begin{cases} |s|-1 \text{ if } s_0(|s|-1) \geq 1, \\ \min\{i < |s| \mid \forall j \geq i \ \ s_0(j) = 0\} \text{ otherwise.} \end{cases}$$

Note that $i^s \leq I^s \leq B_s$. We associate $b_0^{s,i}, b_1^{s,i}, r^{s,i} \in \mathcal{N}$ with each $i^s \leq i < |s|$. The definition is by induction on i. We set $b_\varepsilon^{s,i^s} := b_\varepsilon(V_0^{s,i^s}, a_0, a_1)$, $r^{s,i^s} := r(V_0^{s,i^s}, a_0, a_1) = V_5^{s,i^s}$. Then

$$b_\varepsilon^{s,i+1} := \begin{cases} b_\varepsilon^{s,i} \text{ if } s_0(i+1) \geq 1, \\ b_\varepsilon(V_0^{s,i+1}, b_0^{s,i}, b_1^{s,i}) \text{ if } s_0(i+1) = 0, \end{cases}$$

$$r^{s,i+1} := \begin{cases} r^{s,i} \text{ if } s_0(i+1) \geq 1, \\ r(V_0^{s,i+1}, b_0^{s,i}, b_1^{s,i}) \text{ if } s_0(i+1) = 0. \end{cases}$$

The $range$ of s is $r(s) := \neg\mathcal{U}_{r^{s,I^s}}$.

Lemma 6.18 *Assume that $(\vec{w}, \vec{v}\ )$ is standard, $s \in \mathfrak{T}(\vec{w}) \setminus \{\emptyset\}$, and $i^s \leq i < B_s - 1$ satisfies $s_0(i) = 0$. Then $r^{s,i} = r^{s,i+1}$.*

Proof. We may assume that $s_0(i+1) = 0$. Assume first that $i = i^s$. Then

$$\begin{aligned} r^{s,i^s} &= r(V_0^{s,i^s}, a_0, a_1) \\ &= r\big(\mathcal{V}(V^{s,i^s})(0)(0), b_0(V_0^{s,i^s}, a_0, a_1), b_1(V_0^{s,i^s}, a_0, a_1)\big) \\ &= r\big(\mathcal{V}(V^{s,i^s})\big(s_0(i^s)\big)(0), b_0(V_0^{s,i^s}, a_0, a_1), b_1(V_0^{s,i^s}, a_0, a_1)\big) \\ &= r\big(V_0^{s,i^s+1}, b_0(V_0^{s,i^s}, a_0, a_1), b_1(V_0^{s,i^s}, a_0, a_1)\big) \\ &= r(V_0^{s,i^s+1}, b_0^{s,i^s}, b_1^{s,i^s}) \\ &= r^{s,i^s+1}. \end{aligned}$$

The argument is similar if $i > i^s$. □

Lemma 6.19 *Let $(\vec{w}, \vec{v}\ )$ be standard. Then there is $S(\vec{\emptyset}) \in \mathcal{M}_{\vec{w}}$ such that $\vec{\emptyset} \in \mathcal{I}_{B_{S(\vec{\emptyset})}, S(\vec{\emptyset})}$ and $\neg\mathcal{U}_r \subseteq r\big(S(\vec{\emptyset})\big)$.*

Proof. We set $s := S(\vec{\emptyset})$. We already saw that $s \in \mathcal{M}_{\vec{w}}$, $\vec{\emptyset} \in \mathcal{I}_{B_s, s}$, and $s_0(i) = 0$ for each $i < |s|$ after Lemma 6.14. Note that $i^s = I^s = 0$. Thus

$$\neg\mathcal{U}_r = \neg\mathcal{U}_{V_5^{s,0}} = \neg\mathcal{U}_{V_5^{s,i^s}} = \neg\mathcal{U}_{r^{s,i^s}} = \neg\mathcal{U}_{r^{s,I^s}} = r(s).$$

This finishes the proof. □

The role of the next objects is to determine whether we go to the A_0 side or the A_1 side in the proof of Theorem 6.9.

Notation. Let $\vec{w} := (\alpha, \beta, \gamma) \in \mathfrak{I}^\infty$ with $\alpha \in \Delta_1^1$ normalized and $|(\alpha)_1| = 2$, and $s \in \mathcal{M}_{\vec{w}}$. We set $\varepsilon_s := 0$ if $B_s < |s| - 1$, $\varepsilon_s := 1$ otherwise, i.e., if $B_s = |s| - 1$.

Lemma 6.20 *Let $(\vec{w},\vec{v}\ )$ be standard and $s\in\mathcal{M}_{\vec{w}}$. Then $r(s)\subseteq\neg\mathcal{U}_{a_{\varepsilon_s}}$.*

Proof. Note first that $\neg\mathcal{U}_{b^{s,i}_\varepsilon}\subseteq\neg\mathcal{U}_{a_\varepsilon}$, by induction on i and the 2nd remark after the definition of $\mathfrak{K}$. This implies that $\neg\mathcal{U}_{r^{s,I^s}}\subseteq\neg\mathcal{U}_{r(V^{s,I^s}_0,a_0,a_1)}=\neg\mathcal{U}_{V^{s,I^s}_5}$, by the 4th remark after the definition of $\mathfrak{K}$. Thus $r(s)=\neg\mathcal{U}_{r^{s,I^s}}\subseteq\neg\mathcal{U}_{V^{s,I^s}_5}$. Lemma 6.16 implies that $\neg\mathcal{U}_{V^{s,I^s}_5}\subseteq\neg\mathcal{U}_{V^{s,B_s}_5}$. But $V^{s,B_s}_5=a_{\varepsilon_s}$, by Lemma 6.15. □

The next lemma is crucial for proving the claim mentioned before Lemma 6.16.

Lemma 6.21 *Let $(\vec{w},\vec{v}\ )$ be standard, and $s,t\in\mathfrak{T}(\vec{w})$ with $O(s)\neq O(t)$ and $O(s)\sqsubseteq O(t)$. Then $r(s)\subseteq\overline{r(t)}^{\xi_{s||s\wedge t|}}$.*

Proof. We can write $O(s):=0^{j_0}m_0...0^{j_{l-1}}m_{l-1}0^{j_l}$, with $l,j_i\in\omega$, and $m_i\geq 1$. Similarly, we write $O(t):=0^{k_0}n_0...0^{k_{q-1}}n_{q-1}0^{k_q}$. The assumption implies that $q\geq 1$, and also the existence of $p<q$ with $(j_i,m_i)=(k_i,n_i)$ if $i<p$ and $k_j<j_p$. Lemma 6.14 shows the existence of $l_{p+1}\geq 1$ and $u\in\mathcal{M}_{\vec{w}}$ with $O(u)=0^{k_0}n_0...0^{k_{p-1}}n_{p-1}0^{k_p}n_p0^{l_{p+1}}$ if $p<q-1$. If $p=q-1$, then we set $u:=t$. Note that $O(s)\neq O(u)$, $O(s)\sqsubseteq O(u)$, and $O(u)\sqsubseteq O(t)$. Moreover, $O(u)\neq O(t)$ and $|s\wedge t|=|s\wedge u|<|t\wedge u|$ if $p<q-1$. It is enough to prove that $r(s)\subseteq\overline{r(u)}^{\xi_{s||s\wedge u|}}$. This means that we may assume that $(j_i,m_i)=(k_i,n_i)$ if $i<q-1$ and $k_{q-1}<j_{q-1}$. Thus $I^t\geq 1$, $|s\wedge t|=I^t-1$, $s|(I^t-1)=t|(I^t-1)$, $s_0(I^t-1)=0<t_0(I^t-1)$ and $i^s\leq I^t-1$.

Case 1. $i^s=I^s$ and $i^t=I^t$.

Note that $r(s)=\neg\mathcal{U}_{r^{s,I^s}}=\neg\mathcal{U}_{r^{s,i^s}}=\neg\mathcal{U}_{V^{s,i^s}_5}=\neg\mathcal{U}_{V^{t,I^s}_5}$. Lemma 6.16 implies that

$$r(s)=\neg\mathcal{U}_{V^{t,I^s}_5}\subseteq\neg\mathcal{U}_{V^{t,I^t-1}_5}\subseteq\overline{\neg\mathcal{U}_{V^{t,I^t}_5}}^{\xi_{t|(I^t-1)}}=\overline{r(t)}^{\xi_{s||s\wedge t|}}.$$

Case 2. $i^s=I^s$ and $i^t<I^t$.

Note that $i^s=i^t<I^t-1$. Lemma 6.18 implies that $r(s)=\neg\mathcal{U}_{r^{s,I^s}}=\neg\mathcal{U}_{r^{s,I^t-1}}$. Thus

$$\begin{aligned}r(s)&=\neg\mathcal{U}_{r(V^{s,I^t-1}_0,b^{s,I^t-2}_0,b^{s,I^t-2}_1)}=\neg\mathcal{U}_{r(V^{t,I^t-1}_0,b^{t,I^t-2}_0,b^{t,I^t-2}_1)}\\&=\neg\mathcal{U}_{r(V^{t,I^t-1}_0,b^{t,I^t-1}_0,b^{t,I^t-1}_1)}\subseteq\overline{\neg\mathcal{U}_{r(V^{t,I^t}_0,b^{t,I^t-1}_0,b^{t,I^t-1}_1)}}^{\xi_{t|(I^t-1)}}=\overline{r(t)}^{\xi_{s||s\wedge t|}},\end{aligned}$$

by Lemma 6.16.

Case 3. $i^s<I^s<I^t$.

We argue as in Case 2.

Case 4. $i^s<I^s$ and $I^t\leq I^s$, which implies that $I^t<I^s$.

The 5th remark after the definition of $\mathfrak{K}$ gives $\varepsilon\in 2$ with $r(s)=\neg\mathcal{U}_{r^{s,I^s}}\subseteq\neg\mathcal{U}_{b^{s,I^s-1}_\varepsilon}$. Thus $r(s)\subseteq\neg\mathcal{U}_{b^{s,I^s-1}_\varepsilon}\subseteq...\subseteq\neg\mathcal{U}_{b^{s,I^t-1}_\varepsilon}$.

If $I^t\geq 2$, then

$$\neg\mathcal{U}_{b_\varepsilon^{s,I^t-1}}=\neg\mathcal{U}_{a_\varepsilon(V_0^{t,I^t-1},b_0^{t,I^t-2},b_1^{t,I^t-2})}\subseteq\overline{\neg\mathcal{U}_{r(V_0^{t,I^t},b_0^{t,I^t-2},b_1^{t,I^t-2})}}^{\xi_{s||s\wedge t|}}$$
$$=\overline{\neg\mathcal{U}_{r(V_0^{t,I^t},b_0^{t,I^t-1},b_1^{t,I^t-1})}}^{\xi_{s||s\wedge t|}}=\overline{r(t)}^{\xi_{s||s\wedge t|}}.$$

Otherwise, $I^t=1$, $i^s=0$, $i^t=I^t$ and $\neg\mathcal{U}_{b_\varepsilon^{s,0}}=\neg\mathcal{U}_{a_\varepsilon(V_0^{t,0},a_0,a_1)}\subseteq\overline{\neg\mathcal{U}_{r(V_0^{t,1},a_0,a_1)}}^{\xi_{s||s\wedge t|}}=\overline{r(t)}^{\xi_{s||s\wedge t|}}$.
This finishes the proof. □

(H) The maximal sequences

We now associate a maximal sequence to a pair $(\vec{\beta},\vec{w})$ with $\vec{\beta}\in\lceil T_d\rceil$. Its construction is similar to that of the $s(\vec{t}\,)$'s, but is about infinite sequences instead of finite ones.

• Let $\vec{w}:=(\alpha,\beta,\gamma)\in\mathfrak{I}^\infty$ with $\alpha\in\Delta^1_1$ normalized and $|(\alpha)_1|=2$, and $\vec{\beta}\in\lceil T_d\rceil$. We will define $s(\vec{\beta},\vec{w})\in\mathcal{M}_{\vec{w}}$. Recall the definition of $\mathcal{Q}_p^{\vec{w}}$. We set, for $s\in\mathcal{M}_{\vec{w}}$ and $i\in B_s$,

$$E_i^s:=\begin{cases}\bigcap_{p\geq 1}\ \mathcal{Q}_p^{s(i)(1)} \text{ if } s(i)(0)=0,\\ \neg\mathcal{Q}_{s(i)(0)}^{s(i)(1)} \text{ if } s(i)(0)\geq 1.\end{cases}$$

We define $s(\vec{\beta},\vec{w})$ in such a way that $j_d(\vec{\beta})\in\bigcap_{i\in B_{s(\vec{\beta},\vec{w})}}\ E_i^{s(\vec{\beta},\vec{w})}$. Let ξ be an ordinal such that $\vec{w}\in\mathfrak{I}^\xi\setminus\mathfrak{I}^{<\xi}$. The definition of $s(\vec{\beta},\vec{w})$ is by induction on ξ.

Case 1. $|(\alpha)_1|=0$.

We set $s(\vec{\beta},\vec{w}):=<(0,\vec{w})>$.

Case 2. $|(\alpha)_1|=1$.

We set $s(\vec{\beta},\vec{w}):=(0,\vec{w})^\frown s\big(\vec{\beta},\mathcal{Y}(\vec{w})\big)$.

Case 3. $|(\alpha)_1|=2$.

We set $s(\vec{\beta},\vec{w}):=\begin{cases}(0,\vec{w})^\frown s\big(\vec{\beta},\mathcal{Y}(\vec{w})(0)\big) \text{ if } j_d(\vec{\beta})\in\bigcap_{p\geq 1}\ \mathcal{Q}_p^{\vec{w}},\\ (p,\vec{w})^\frown s\big(\vec{\beta},\mathcal{Y}(\vec{w})(p)\big) \text{ if } j_d(\vec{\beta})\notin\mathcal{Q}_p^{\vec{w}}\wedge p\geq 1.\end{cases}$

• We set $(\vec{\beta}|j_k)_{k\in\omega}:={}_{s(\vec{\beta},\vec{w})|(B_{s(\vec{\beta},\vec{w})}-1)}\Pi^{-1}\big(h(\vec{\beta})\big)$.

Recall the definition of ε_s before Lemma 6.20.

Lemma 6.22 *Let $\vec{w}:=(\alpha,\beta,\gamma)\in\mathfrak{I}^\infty$ with $\alpha\in\Delta^1_1$ normalized and $|(\alpha)_1|=2$, and $\vec{\beta}\in\lceil T_d\rceil$.*
(a) There is $k_0\in\omega$ such that $\vec{\beta}|j_k\in\mathcal{I}_{B_{s(\vec{\beta},\vec{w})},s(\vec{\beta},\vec{w})}$ if $k\geq k_0$. In this case, the sequence $s(\vec{\beta}|j_k)$ given by Proposition 6.13 is $s(\vec{\beta},\vec{w})|B_{s(\vec{\beta},\vec{w})}$, and is not extendable.
(b) The sequence $j_d(\vec{\beta})$ is in $C_\gamma^{\mathcal{N}}$ if and only if $\varepsilon_{s(\vec{\beta},\vec{w})}=0$.

Proof. We set $s := s(\vec\beta, \vec w)$ for simplicity.

(a) In order to define k_0, we will define, for $i < B_s$, $k_0^i \in \omega$, and we will set $k_0 := \max\{k_0^i \mid i < B_s\}$. In order to do this, we set $(\vec\beta|j_k^i)_{k\in\omega} := {}_{s|i}\Pi^{-1}\big(h(\vec\beta)\big)$, so that $(\vec\beta|j_k^{i+1})_{k\in\omega}$ is a subsequence of $(\vec\beta|j_k^i)_{k\in\omega}$ if $i < B_s - 1$. By the choice of the E_i^s's we get, for $i < B_s$,

$$h(\vec\beta) \in \begin{cases} \bigcap_{p\geq 1} \mathcal{P}_p^{s_1(i)} \text{ if } s_0(i) = 0, \\ \neg\mathcal{P}_{s_0(i)}^{s_1(i)} \text{ if } s_0(i) \geq 1, \end{cases}$$

$$(\vec\beta|j_k^i)_{k\in\omega} \in \begin{cases} \bigcap_{p\geq 1} {}_{s|i}\Pi^{-1}(\mathcal{P}_p^{s_1(i)}) \text{ if } s_0(i) = 0, \\ \neg_{s|i}\Pi^{-1}(\mathcal{P}_{s_0(i)}^{s_1(i)}) \text{ if } s_0(i) \geq 1. \end{cases}$$

Note the existence of $\mathcal{B}_p^i$ in ${}_{s|i}\Pi^{-1}(\mathcal{P}_p^{s_1(i)})$ such that $\vec\beta|j_k^i \in \mathcal{B}_p^i$ if $s_0(i) = 0$, $k \in \omega$ and $p \geq 1$. If $s_0(i) \geq 1$ and $p \in \omega \setminus \{0, s_0(i)\}$, then $(\vec\beta|j_k^i)_{k\in\omega} \in {}_{s|i}\Pi^{-1}(\mathcal{P}_p^{s_1(i)})$ since $\mathcal{P}_p^{s_1(i)} \cup \mathcal{P}_{s_0(i)}^{s_1(i)} = [\subseteq]$. This implies the existence of $\mathcal{B}_p^i \in {}_{s|i}\Pi^{-1}(\mathcal{P}_p^{s_1(i)})$ such that $\vec\beta|j_k^i \in \mathcal{B}_p^i$ if $k \in \omega$. As ${}_{s|i}\Pi^{-1}(\mathcal{P}_{s_0(i)}^{s_1(i)}) \in \mathbf{\Pi}_1^0([R_{s|i}^{\eta_{s|i}}])$, there is $k_0^i \geq 1$ such that $\vec\beta|j_k^i \notin \mathcal{B}_{s_0(i)}^i$ if $s_0(i) \geq 1$, $\mathcal{B}_{s_0(i)}^i \in {}_{s|i}\Pi^{-1}(\mathcal{P}_{s_0(i)}^{s_1(i)})$ and $k \geq k_0^i$. This defines k_0^i and k_0. It remains to check that $\vec\beta|j_k \in P_{s(i)(0)}(s|i)$ if $i < B_s$ and $k \geq k_0$. This comes from the fact that $j_k = j_k^{B_s-1} = j_{K(k)}^i$ for some $K(k) \geq k \geq k_0 \geq k_0^i$. The last assertion comes from the construction of $s(\vec t\,)$.

(b) We define, for $i < |s|$, $\varepsilon_s^i \in 2$. The definition is by induction on i. We first set $\varepsilon_s^0 := 1$. Then $\varepsilon_s^{i+1} := 0$ if $|s| - i - 2 \notin B_s$, $\varepsilon_s^{i+1} := \varepsilon_s^i$ otherwise. Note that $\varepsilon_s = \varepsilon_s^{|s|-1}$ (ε_s is defined before Lemma 6.20). We have to see that $j_d(\vec\beta)$ is in $C_{s_1(0)(2)}^{\mathcal{N}}$ if and only if $\varepsilon_s^{|s|-1} = 0$. We prove the following stronger fact: $j_d(\vec\beta) \in C_{s_1(|s|-i-1)(2)}^{\mathcal{N}}$ is equivalent to $\varepsilon_s^i = 0$ if $i < |s|$. Here again we argue by induction on i. The result is clear for $i = 0$ since $C_{s_1(|s|-1)(2)}^{\mathcal{N}} = \emptyset$. So assume that the result is true for $i < |s| - 1$.

If $|s| - i - 2 \notin B_s$, then we are done since $\varepsilon_s^{i+1} = 1 - \varepsilon_s^i$ and $C_{s_1(|s|-i-2)(2)}^{\mathcal{N}} = \neg C_{s_1(|s|-i-1)(2)}^{\mathcal{N}}$. If $|s| - i - 2 \in B_s$, then $\varepsilon_s^{i+1} = \varepsilon_s^i$ and

$$C_{s_1(|s|-i-2)(2)}^{\mathcal{N}} = \bigcup_{p\geq 1} \big(C_{((\mathcal{W}_0(s_1(|s|-i-2)))_p)_0}^{\mathcal{N}} \setminus C_{((\mathcal{W}_0(s_1(|s|-i-2)))_p)_1}^{\mathcal{N}}\big) \cup \big(C_{(\mathcal{W}_0(s_1(|s|-i-2)))_0}^{\mathcal{N}} \cap \bigcap_{p\geq 1} C_{((\mathcal{W}_0(s_1(|s|-i-2)))_p)_1}^{\mathcal{N}}\big).$$

If $s_0(|s| - i - 2) = 0$, then $j_d(\vec\beta) \in \bigcap_{p\geq 1} \mathcal{Q}_p^{s_1(|s|-i-2)} = \bigcap_{p\geq 1} C_{((\mathcal{W}_0(s_1(|s|-i-2)))_p)_1}^{\mathcal{N}}$. We can say that $j_d(\vec\beta) \in C_{s_1(|s|-i-2)(2)}^{\mathcal{N}}$ is equivalent to $j_d(\vec\beta) \in C_{(\mathcal{W}_0(s_1(|s|-i-2)))_0}^{\mathcal{N}} = C_{s_1(|s|-i-1)(2)}^{\mathcal{N}}$, and we are done by induction assumption. We argue similarly when $s_0(|s| - i - 2) \geq 1$. $\square$

Remark. Recall the definition of an extendable sequence at the beginning of the construction of the resolution families. If s is not extendable, then s admits a unique extension $M(s)$ in $\mathcal{M}_{\vec w}$. In particular, in Lemma 6.22.(a), $M\big(s(\vec\beta|j_k)\big) = s(\vec\beta, \vec w) = S(\vec\beta|j_k)$. In Lemma 6.19, $s(\vec\emptyset) = s|B_s$ is not extendable and $M\big(s(\vec\emptyset)\big) = S(\vec\emptyset)$.

Notation. Recall the construction of the resolution families, and also the proof of Theorem 4.4.5, especially the definition of $\eta(\vec{t}\,)$. If θ_s is a limit ordinal, then we consider some ordinals $\eta_s(\vec{t}\,)$'s, as in the proof of Theorem 4.4.5. We set $\rho(s,\vec{s}\,):=\begin{cases}\eta_s \text{ if } \theta_s \text{ is a successor ordinal,}\\ \eta_s(\vec{s}\,) \text{ if } \theta_s \text{ is a limit ordinal.}\end{cases}$

The next lemma is the final preparation for proving the claim mentioned before Lemme 6.16.

Lemma 6.23 *Let $\vec{w}:=(\alpha,\beta,\gamma)\in\mathfrak{I}^\infty$ with $\alpha\in\Delta^1_1$ normalized and $|(\alpha)_1|=2$, $s\in\mathfrak{T}(\vec{w})$, and $i<B_s$. Then $\big(\Sigma_{j\leq i}\ \rho(s|j,v)\big)+1\leq\xi_{s|i}$.*

Proof. We argue by induction on i. Note first that $\rho(s|0,v)+1\leq\theta_{s|0}=\xi_{s|0}$. Then, inductively,

$$\begin{aligned}\big(\Sigma_{j\leq i+1}\ \rho(s|j,v)\big)+1 &\leq \big(\Sigma_{j\leq i}\ \rho(s|j,v)\big)+\theta_{s|(i+1)}\\ &\leq \big(\Sigma_{j\leq i}\ \rho(s|j,v)\big)+1+(\xi_{s|(i+1)}-\xi_{s|i})\\ &\leq \xi_{s|i}+(\xi_{s|(i+1)}-\xi_{s|i})\\ &\leq \xi_{s|(i+1)}\end{aligned}$$

This finishes the proof. □

Proof of Theorem 6.9. Let ξ be an ordinal with $\vec{w}:=(\alpha,\beta,\gamma)\in\mathfrak{I}^\xi$. We argue by induction on ξ. So assume that $\vec{w}\in\mathfrak{I}^\xi\setminus\mathfrak{I}^{<\xi}$.

Case 1. $|(\alpha)_1|=0$.

Lemma 6.5 implies that $C^{\mathcal{N}}_\gamma\in\mathbf{\Gamma}_{c(\alpha)}=\mathbf{\Gamma}_{0^\infty}=\{\emptyset\}$, so that $S=\emptyset$. Note also that $r=a_1$. Assume that (a) does not hold. Then $A_1\neq\emptyset$, so it contains some $\vec{\alpha}$. We just have to set $f_i(\beta_i):=\alpha_i$.

Case 2. $|(\alpha)_1|=1$.

The fact that $\vec{w}\in\mathfrak{I}^\xi$ gives $\delta\in\mathcal{N}$ with $(<{}_j\alpha>,\beta,\delta)\in\mathfrak{I}^{<\xi}$ and $C^{\mathcal{N}}_\gamma=\neg C^{\mathcal{N}}_\delta$ (see the definition of $\mathfrak{I}$). As α is normalized, $C^{\mathcal{N}}_\delta=\emptyset$, so that $S=\lceil T_d\rceil$. Note also that $r=a_0$. Assume that (a) does not hold. Then $A_0\neq\emptyset$, and we argue as in Case 1.

Case 3. $|(\alpha)_1|=2$.

Assume that (a) does not hold. We construct $(\alpha^i_s)_{i\in d,s\in\Pi''_iT_d}$, $(O^i_s)_{i\leq|s|,i\in d,s\in\Pi''_iT_d}$, $(U_{\vec{s}})_{\vec{s}\in T_d}$, as in the proof of Theorems 4.4.1 and 4.4.5.

We want these objects to satisfy the following conditions.

$$(1)\ \alpha^i_s\in O^i_s\subseteq\Omega_{\mathcal{N}}\wedge(\alpha^i_{s_i})_{i\in d}\in U_{\vec{s}}\subseteq\Omega_{\mathcal{N}^d},$$

$$(2)\ O^i_{sq}\subseteq O^i_s,$$

$$(3)\ \text{diam}_{d_{\mathcal{N}}}(O^i_s)\leq 2^{-|s|}\wedge\text{diam}_{d_{\mathcal{N}^d}}(U_{\vec{s}})\leq 2^{-|\vec{s}|},$$

$$(4)\ \vec{t}\in T_d\ \Rightarrow\ U_{\vec{t}}\subseteq r\big(S(\vec{t}\,)\big),$$

$$(5)\ \left(\begin{array}{c}\vec{s},\vec{t}\in\bigcap_{j<i,\eta_{s|j}\geq 1}\ P_{s_0(j)}(s|j)\\ 1\leq\rho\leq\rho(s|i,\vec{s})\\ \vec{s}\ R^\rho_{s|i}\ \vec{t}\end{array}\right)\ \Rightarrow\ U_{\vec{t}}\subseteq\overline{U_{\vec{s}}}^{(\Sigma_{j<i}\ \rho(s|j,\vec{s}))+\rho},$$

$$(6)\ \Big(\vec{s}\in\mathcal{I}_{s(\vec{t}\,)}\wedge\vec{s}\ R^{\eta_{s(\vec{t}\,)^-}}_{s(\vec{t}\,)^-}\ \vec{t}\Big)\Rightarrow\ U_{\vec{t}}\subseteq U_{\vec{s}}.$$

• Let us prove that this construction is sufficient to get the theorem.

- Fix $\vec{\beta}\in\lceil T_d\rceil$ and set $\sigma:=s(\vec{\beta},\vec{w})$. Lemma 6.22 gives $k_0\in\omega$ such that $\vec{\beta}|j_k\in\mathcal{I}_{B_\sigma,\sigma}$ for each $k\geq k_0$. Proposition 6.13 gives $s(\vec{\beta}|j_k)\in\mathfrak{T}(\vec{w})$ with $\vec{\beta}|j_k\in\mathcal{I}_{s(\vec{\beta}|j_k)}$, and Lemma 6.22.(a) implies that $s(\vec{\beta}|j_k)=\sigma|B_\sigma$. This implies that $(U_{\vec{\beta}|j_k})_{k\geq k_0}$ is decreasing since $\vec{\beta}|j_k\ R^{\eta_{\sigma|(B_\sigma-1)}}_{\sigma|(B_\sigma-1)}\ \vec{\beta}|j_{k+1}$ for each natural number k, by Condition (6). As in the proof of Theorem 4.4.1 we define $F(\vec{\beta})$ and f_i continuous with $F(\vec{\beta})=(\Pi_{i\in d}\ f_i)(\vec{\beta})$. Note that $S\subseteq(\Pi_{i\in d}\ f_i)^{-1}(A_0)$ and $\lceil T_d\rceil\backslash S\subseteq(\Pi_{i\in d}\ f_i)^{-1}(A_1)$, by Lemmas 6.20 and 6.22, since $r(\sigma)\subseteq A_{\varepsilon_\sigma}$.

• So let us prove that the construction is possible.

- As $\neg\mathcal{U}_r$ is nonempty and $\varSigma^1_1$, we can choose $(\alpha^i_\emptyset)_{i\in d}\in\neg\mathcal{U}_r\cap\Omega_{\mathcal{N}^d}$. Then we choose a $\varSigma^1_1$ subset $U_{\vec{\emptyset}}$ of $\mathcal{N}^d$, with $d_{\mathcal{N}^d}$-diameter at most 1, such that $(\alpha^i_\emptyset)_{i\in d}\in U_{\vec{\emptyset}}\subseteq\neg\mathcal{U}_r\cap\Omega_{\mathcal{N}^d}$. We choose a $\varSigma^1_1$ subset $O^0_\emptyset$ of $\Omega_{\mathcal{N}}$, with $d_{\mathcal{N}}$-diameter at most 1, with $\alpha^0_\emptyset\in O^0_\emptyset\subseteq\Omega_{\mathcal{N}}$, which is possible since $\Omega_{\mathcal{N}^d}\subseteq\Omega^d_{\mathcal{N}}$. Assume that $(\alpha^i_s)_{|s|\leq l}$, $(O^i_s)_{|s|\leq l}$ and $(U_{\vec{s}})_{|s_0|\leq l}$ satisfying conditions (1)-(6) have been constructed, which is the case for $l=0$ by Lemma 6.19.

- Let $v:=\overrightarrow{tm}\in T_d\cap(d^{l+1})^d$. We define $X_i:=O^i_{t_i}$ if $i\leq l$, and $\mathcal{N}$ if $i>l$.

Claim. *Assume that $s\in\mathfrak{T}(\vec{w})$, $i<B_s$, $v^{\eta_{s|i}}_{s|i},v\in\mathcal{I}_{i,s}$, and $i_0\leq i$ is minimal with $\eta_{s|i_0}\geq 1$.*
(a) The set

$$U_{v^{\rho(s|i,v)}_{s|i}}\cap\bigcap_{1\leq\rho<\rho(s|i,v)}\ \overline{U_{v^\rho_{s|i}}}^{(\Sigma_{j<i}\ \rho(s|j,v))+\rho}\cap\bigcap_{j<i}\ \bigcap_{1\leq\rho\leq\rho(s|j,v)}\ \overline{U_{v^\rho_{s|j}}}^{(\Sigma_{k<j}\ \rho(s|k,v))+\rho}\cap(\Pi_{i\in d}\ X_i)$$

is τ_1-dense in $\overline{U_{v^1_{s|i_0}}}^1\cap(\Pi_{i\in d}\ X_i)$.

(b) Assume moreover that $u\in\mathfrak{T}(\vec{w})$, s and u are incompatible, $i:=|s\wedge u|$, $v\in P_{u0(i)}(u|i)$, and $v_{s|i}^{\eta_{s|i}}\in P_{s0(i)}(s|i)$. Then $r\big(S(v)\big)\cap\bigcap_{j\leq i}\ \bigcap_{1\leq\rho\leq\rho(s|j,v)}\ \overline{U_{v_{s|j}^\rho}}^{(\Sigma_{k<j}\ \rho(s|k,v))+\rho}\cap(\Pi_{i\in d}\ X_i)$ is τ_1-dense in $\overline{U_{v_{s|i_0}^1}}^1\cap(\Pi_{i\in d}\ X_i)$.

(a) Assume first that $i_0=0$. Note that $v_\emptyset^{\rho+1}\ R_\emptyset^{\rho+1}\ v_\emptyset^\rho\ R_\emptyset^\rho\ v$ if $1\leq\rho<\rho(\emptyset,v)$, by Lemma 4.3.2. As in the proof of Claim 2 in Theorem 4.4.5, this implies that $U_{v_\emptyset^\rho}\subseteq\overline{U_{v_\emptyset^{\rho+1}}}^{\rho+1}$. By assumption, $v_{s|i}^{\eta_{s|i}},v\in\mathcal{I}_{i,s}$. Note that $v_{s|(j+1)}^\rho\in P_{s0(k)}(s|k)$ if $k\leq j<i$ and $\rho\leq\eta_{s|(j+1)}$. Indeed, this comes from the fact that $v_{s|i}^{\eta_{s|i}}\ R_{s|k}^{\eta_{s|k}}\ v_{s|(j+1)}^\rho\ R_{s|k}^{\eta_{s|k}}\ v$. As in the proof of Claim 2 in Theorem 4.4.5 again, this implies that $U_{v_{s|(j+1)}^\rho}\subseteq\overline{U_{v_{s|(j+1)}^{\rho+1}}}^{(\Sigma_{k<j+1}\ \rho(s|k,v))+\rho+1}$ if $\rho<\rho(s|(j+1),v)$. Note that $v_{s|(j+1)}^0=v_{s|j}^{\eta_{s|j}}=v_{s|j}^{\rho(s|j,v)}$. This implies the result. We argue similarly if $i_0>0$.

(b) By (a) and Lemma 6.22, it is enough to see that $U:=U_{v_{s|i}^{\rho(s|i,v)}}\subseteq\overline{r\big(S(v)\big)}^{\xi_{s|i}}$. The induction assumption implies that $U\subseteq r\big(S(v_{s|i}^{\eta_{s|i}})\big)$. So let us prove that $r\big(S(v_{s|i}^{\eta_{s|i}})\big)\subseteq\overline{r\big(S(v)\big)}^{\xi_{s|i}}$. Note that $s|(i+1)\subseteq s(v_{s|i}^{\eta_{s|i}})\subseteq S(v_{s|i}^{\eta_{s|i}})$ and, similarly, $u|(i+1)\subseteq S(v)$. Now $O\big(S(v_{s|i}^{\eta_{s|i}})\big)\sqsubseteq O\big(S(v)\big)$, by Lemma 6.17, and the beginning of its proof shows that $O\big(S(v_{s|i}^{\eta_{s|i}})\big)\neq O\big(S(v)\big)$. It remains to apply Lemma 6.21. $\diamond$

- Let $\mathcal{X}:=d^{l+1}$. The map $\Psi:\mathcal{X}^d\rightarrow\varSigma_1^1(\mathcal{N}^d)$ is defined on $\mathcal{T}^{l+1}$ by

$$\Psi(v):=\begin{cases}
r\big(S(v)\big)\cap\bigcap_{1\leq\rho\leq\rho(\emptyset,v)}\ \overline{U_{v_\emptyset^\rho}}^\rho\cap(\Pi_{i\in d}\ X_i)\cap\Omega_{\mathcal{N}^d}\ \text{if}\ s(v)=\emptyset,\\[2ex]
U_{v_{s(v)^-}^{\rho(s(v)^-,v)}}\cap\bigcap_{1\leq\rho<\rho(s(v)^-,v)}\ \overline{U_{v_{s(v)^-}^\rho}}^{(\Sigma_{j<|s(v)|-1}\ \rho(s|j,v))+\rho}\\
\qquad\cap\bigcap_{j<|s(v)|-1}\ \bigcap_{1\leq\rho\leq\rho(s|j,v)}\ \overline{U_{v_{s|j}^\rho}}^{(\Sigma_{k<j}\ \rho(s|k,v))+\rho}\cap(\Pi_{i\in d}\ X_i)\\
\qquad\text{if}\ s(v)\neq\emptyset\wedge v_{s(v)^-}^{\eta_{s(v)^-}}\in\mathcal{I}_{s(v)}\wedge\exists i_0<|s(v)|\ \ \eta_{s(v)|i_0}\geq 1,\\[2ex]
r\big(S(v)\big)\cap\bigcap_{j\leq i}\ \bigcap_{1\leq\rho\leq\rho(s(v)|j,v)}\ \overline{U_{v_{s(v)|j}^\rho}}^{(\Sigma_{k<j}\ \rho(s(v)|k,v))+\rho}\cap(\Pi_{i\in d}\ X_i)\cap\Omega_{\mathcal{N}^d}\\
\qquad\text{if}\ s(v)\neq\emptyset\wedge v_{s(v)^-}^{\eta_{s(v)^-}}\notin\mathcal{I}_{s(v)}\\
\qquad\wedge\ i<|s(v)|\ \text{is maximal with}\ v_{s(v)|i}^{\eta_{s(v)|i}}\in\mathcal{I}_{i,s(v)}\wedge\ \exists i_0\leq i\ \ \eta_{s(v)|i_0}\geq 1,\\[2ex]
U_{\vec{t}}\cap(\Pi_{i\in d}\ X_i)\ \text{if}\ s(v)\neq\emptyset\wedge v_{s(v)^-}^{\eta_{s(v)^-}}\in\mathcal{I}_{s(v)}\wedge\ \forall i_0<|s(v)|\ \ \eta_{s(v)|i_0}=0,\\[2ex]
r\big(S(v)\big)\cap(\Pi_{i\in d}\ X_i)\cap\Omega_{\mathcal{N}^d}\ \ \text{if}\ \ s(v)\neq\emptyset\wedge\ v_{s(v)^-}^{\eta_{s(v)^-}}\notin\mathcal{I}_{s(v)}\\
\qquad\wedge\ i<|s(v)|\ \text{is maximal with}\ v_{s(v)|i}^{\eta_{s(v)|i}}\in\mathcal{I}_{i,s(v)}\wedge\ \forall i_0\leq i\ \ \eta_{s(v)|i_0}=0.
\end{cases}$$

By the claim, $\Psi(v)$ is τ_1-dense in $\overline{U_{v_{s(v)|i_0}^1}}^1\cap(\Pi_{i\in d}\ X_i)$ in the second and the third cases.

In these cases, as $v^1_{s(v)|i_0}\subseteq\vec{t}\subseteq v$ and $R^1_{s(v)|i_0}$ is distinguished in $R^0_{s(v)|i_0}=\subseteq$, $v^1_{s(v)|i_0}\ R^1_{s(v)|i_0}\ \vec{t}$ and $U_{\vec{t}}\subseteq\overline{U_{v^1_{s(v)|i_0}}}^1$, by induction assumption. Therefore $U_{\vec{t}}\cap(\Pi_{i\in d}\ X_i)\subseteq\overline{U_{v^1_{s(v)|i_0}}}^1\cap(\Pi_{i\in d}\ X_i)\subseteq\overline{\Psi}(v)$. Similarly, one can prove that this also holds in the last two cases.

Let us look at the first case. If $\eta_\emptyset\geq 1$, then $U_{v_\emptyset^{\rho(\emptyset,v)}}\cap\bigcap_{1\leq\rho<\rho(\emptyset,v)}\ \overline{U_{v^\rho_\emptyset}}^\rho\cap(\Pi_{i\in d}\ X_i)$ is τ_1-dense in $\overline{U_{v^1_\emptyset}}^1\cap(\Pi_{i\in d}\ X_i)$, as in the claim. Now $U_{v_\emptyset^{\rho(\emptyset,v)}}\subseteq r\big(S(v^{\eta_\emptyset}_\emptyset)\big)=r\big(S(v)\big)$ and we can repeat the previous argument since $i_0=0$. If $\eta_\emptyset=0$, then $v^{\eta_\emptyset}_\emptyset=\vec{t}$,

$$U_{\vec{t}}\cap(\Pi_{i\in d}\ X_i)\subseteq r\big(S(\vec{t}\,)\big)\cap(\Pi_{i\in d}\ X_i)=r\big(S(v)\big)\cap(\Pi_{i\in d}\ X_i)$$

and we are done.

Now we can write $(\alpha^i_{t_i})_{i\in d}\in U_{\vec{t}}\cap(\Pi_{i\in d}\ X_i)\subseteq\overline{\Psi}(v)$, and we conclude as in the proof of Theorem 4.4.1. $\square$

The rest of this section is devoted to the proof of Theorem 1.10.(2) when $\Delta(\mathbf{\Gamma})$ is a Wadge class, and also to the proof of Theorem 1.5. Recall Theorem 5.2.8. We will say that $\alpha\in\Delta^1_1\cap\mathfrak{H}^\infty$ is $suitable$ if $\Delta(\mathbf{\Gamma}_{c(\alpha)})$ is a Wadge class and one of the following holds:

(1) There is $\overline{\alpha}\in\Delta^1_1\cap\mathfrak{H}^\infty$ normalized with

$$\mathbf{\Gamma}_{c(\alpha)}=\Big\{(A_0\cap C_0)\cup(A_1\cap C_1)\mid A_0,\neg A_1\in\mathbf{\Gamma}_{c(\overline{\alpha})}\wedge C_0,C_1\in\mathbf{\Sigma}^0_1\wedge C_0\cap C_1=\emptyset\Big\}.$$

(2) There is $\alpha'\in\Delta^1_1$ such that $(\alpha')_p\in\mathfrak{H}^\infty$ is normalized for each $p\geq 1$, $\big(\mathbf{\Gamma}_{c((\alpha')_p)}\big)_{p\geq 1}$ is strictly increasing, and $\mathbf{\Gamma}_{c(\alpha)}=\Big\{\bigcup_{p\geq 1}\ (A_p\cap C_p)\mid A_p\in\mathbf{\Gamma}_{c((\alpha')_p)}\wedge C_p\in\mathbf{\Sigma}^0_1\wedge C_p\cap C_q=\emptyset\text{ if }p\neq q\Big\}$.

Assume that α is suitable and $a_0,a_1\in\Delta^1_1$ satisfy $A_0\cap A_1=\emptyset$. Then Lemma 6.7.(b) gives $r(\overline{\alpha},a_0,a_1)$ and $r(\overline{\alpha},a_1,a_0)$, or $r\big((\alpha')_p,a_0,a_1\big)$. We set $R(\overline{\alpha},a_0,a_1):=\neg\mathcal{U}_{r(\overline{\alpha},a_0,a_1)}$ in the same fashion as before, and

$$R'(\alpha,a_0,a_1):=\begin{cases}\overline{R(\overline{\alpha},a_0,a_1)}^1\cap\overline{R(\overline{\alpha},a_1,a_0)}^1\text{ if we are in Case (1),}\\ \\ \bigcap_{p\geq 1}\ \overline{R\big((\alpha')_p,a_0,a_1\big)}^1\text{ if we are in Case (2).}\end{cases}$$

We now give the self-dual version of Lemma 6.8.

Lemma 6.24 *Let α be suitable, and $a_0,a_1\in\Delta^1_1$ such that $A_0\cap A_1=\emptyset$. We assume that $R'(\alpha,a_0,a_1)$ is empty. Then A_0 is separable from A_1 by a $\Delta^1_1\cap\Delta(\mathbf{\Gamma}_{c(\alpha)})(\tau_1)$ set.*

Proof. (1) As $\overline{R(\overline{\alpha},a_0,a_1)}^1\cap\overline{R(\overline{\alpha},a_1,a_0)}^1=\emptyset$, there is $C\in\mathbf{\Delta}^0_1(\tau_1)$ separating $R(\overline{\alpha},a_0,a_1)$ from $R(\overline{\alpha},a_1,a_0)$. As $R(\overline{\alpha},a_0,a_1)$ and $R(\overline{\alpha},a_1,a_0)$ are Σ^1_1, we may assume that $C\in\Delta^1_1$, by Theorem 4.2.2. A double application of Lemmas 6.7.(b) and 6.8 gives some sets $B_0,B_1\in\Delta^1_1\cap\mathbf{\Gamma}_{c(\overline{\alpha})}(\tau_1)$ such that B_0 (resp., B_1) separates $A_0\cap C$ (resp., $A_1\setminus C$) from $A_1\cap C$ (resp., $A_0\setminus C$). Now the set $(B_0\cap C)\cup(\neg B_1\cap\neg C)$ is suitable.

(2) The proof is similar, but we have to use the Δ^1_1-selection principle. As $\mathfrak{K}^\infty$ is Π^1_1 and the sequence $r\big((\alpha')_p,a_0,a_1\big)$ is Δ^1_1 and completely determined by $(\alpha')_p$, a_0 and a_1, $\Big(r\big((\alpha')_p,a_0,a_1\big)\Big)_{p\geq 1}$ is Δ^1_1. As $\bigcap_{p\geq 1}\ \overline{R\big((\alpha')_p,a_0,a_1\big)}^{1}=\emptyset$, there is a Δ^1_1-recursive map $f:\mathcal{N}^d\rightarrow\omega$ such that $f(\vec{\alpha})\geq 1$ and $\vec{\alpha}\notin\overline{R\big((\alpha')_{f(\vec{\alpha})},a_0,a_1\big)}^{1}$ for each $\vec{\alpha}\in\mathcal{N}^d$.

We set $U_p:=f^{-1}(\{p\})$, so that U_p and $R\big((\alpha')_p,a_0,a_1\big)$ are disjoint Σ^1_1 and separable by a τ_1-open set. By Theorem 4.2.2, there is $V_p\in\Delta^1_1\cap\mathbf{\Sigma}^0_1(\tau_1)$ separating them. Moreover, we may assume that the sequence (V_p) is Δ^1_1. We reduce the sequence (V_p), which gives a Δ^1_1-sequence (C_p) of $\Delta^1_1\cap\mathbf{\Sigma}^0_1(\tau_1)$ sets. Note that (C_p) is a partition of $\mathcal{N}^d$ into $\mathbf{\Delta}^0_1(\tau_1)$ sets. As $R\big((\alpha')_p,a_0,a_1\big)\cap C_p=\emptyset$, Lemma 6.8 gives $\beta',\gamma'\in\mathcal{N}$ such that $\big((\alpha')_p,(\beta')_p,(\gamma')_p\big)\in\mathfrak{I}^\infty$ and $C_{(\gamma')_p}$ separates $A_1\cap C_p$ from $A_0\cap C_p$ for each $p\geq 1$. Moreover, we may assume that $\beta',\gamma'\in\Delta^1_1$. Now the set $\bigcup_{p\geq 1}\ (\neg C_{(\gamma')_p}\cap C_p)$ is suitable. □

We now give the self-dual version of Theorem 6.9.

Theorem 6.25 *Let T_d be a tree with Δ^1_1 suitable levels, α be suitable, $\beta_\varepsilon,\gamma_\varepsilon\in\mathcal{N}$ be such that $(\alpha,\beta_\varepsilon,\gamma_\varepsilon)\in\mathfrak{I}^\infty$, $S_\varepsilon:=j_d^{-1}(C^{\mathcal{N}}_{\gamma_\varepsilon})\cap\lceil T_d\rceil$, and $a_0,a_1,b_0,b_1,r\in\mathcal{N}$ with $\vec{v}:=(\alpha,a_0,a_1,b_0,b_1,r)\in\mathfrak{K}^\infty$. We assume that S_0 and S_1 are disjoint. Then one of the following holds:*
(a) $R'(\alpha,a_0,a_1)=\emptyset$.
(b) The inequality $\big((\Pi''_i\lceil T_d\rceil)_{i\in d},S_0,S_1\big)\leq\big((\mathcal{N})_{i\in d},A_0,A_1\big)$ holds.

Now we can state the version of Theorem 4.2.2 for the self-dual Wadge classes of Borel sets.

Theorem 6.26 *Let T_d be a tree with Δ^1_1 suitable levels, α be suitable, $\beta_\varepsilon,\gamma_\varepsilon\in\mathcal{N}$ be such that $(\alpha,\beta_\varepsilon,\gamma_\varepsilon)\in\mathfrak{I}^\infty$, $S_\varepsilon:=j_d^{-1}(C^{\mathcal{N}}_{\gamma_\varepsilon})\cap\lceil T_d\rceil$, and $a_0,a_1,b_0,b_1,r\in\mathcal{N}$ with $\vec{v}:=(\alpha,a_0,a_1,b_0,b_1,r)\in\mathfrak{K}^\infty$. We assume that S_0, S_1 are disjoint and not separable by a pot$\big(\Delta(\mathbf{\Gamma}_{c(\alpha)})\big)$ set. Then the following are equivalent:*
(a) The set A_0 is not separable from A_1 by a pot$\big(\Delta(\mathbf{\Gamma}_{c(\alpha)})\big)$ set.
(b) The set A_0 is not separable from A_1 by a $\Delta^1_1\cap$pot$\big(\Delta(\mathbf{\Gamma}_{c(\alpha)})\big)$ set.
(c) The set A_0 is not separable from A_1 by a $\Delta(\mathbf{\Gamma}_{c(\alpha)})(\tau_1)$ set.
(d) $R'(\alpha,a_0,a_1)\neq\emptyset$.
(e) The inequality $\big((d^\omega)_{i\in d},S_0,S_1\big)\leq\big((\mathcal{N})_{i\in d},A_0,A_1\big)$ holds.

Proof. We argue as in the proof of Theorem 6.10, using Lemma 6.24 (resp., Theorem 6.25) instead of Lemma 6.8 (resp., Theorem 6.9). □

Proof of Theorem 1.10.(2). We argue as in the proof of Theorem 1.8.(1). Theorem 5.2.8 gives $\overline{u}$ or $\big((u')_p\big)_{p\geq 1}$. The equalities in Theorem 5.2.8 hold in $\mathcal{N}$, and also in any zero-dimensional Polish space (we argue as in Lemma 5.2.2 to see it). Using Definition 5.1.2, we can build $u\in\mathcal{D}$ with $\mathbf{\Gamma}=\mathbf{\Gamma}_u$. Lemmas 6.2 and 6.4 give $\alpha\in\mathfrak{H}^\infty$ normalized with $\mathbf{\Gamma}_{c(\alpha)}=\mathbf{\Gamma}_u$, and $\overline{\alpha}\in\mathfrak{H}^\infty$ (resp., $\alpha'\in\mathfrak{H}^\infty$ such that $(\alpha')_p$ is) normalized with $\mathbf{\Gamma}_{\overline{u}}=\mathbf{\Gamma}_{c(\overline{\alpha})}$ (resp., $\mathbf{\Gamma}_{(u')_p}=\mathbf{\Gamma}_{c((\alpha')_p)}$).

By Theorem 4.1.3 in [Lo-SR2] there is $B_\varepsilon\in\mathbf{\Gamma}(\mathcal{N})$ with $S_\varepsilon=j_d^{-1}(B_\varepsilon)\cap\lceil T_d\rceil$. In order to simplify the notation, we may assume that T_d has Δ^1_1 levels, α, as well as $\overline{\alpha}$ (or α'), are Δ^1_1, and A_0,A_1 are Σ^1_1.

By Lemma 6.5 there are $\beta_\varepsilon, \gamma_\varepsilon \in \mathcal{N}$ such that $(\alpha, \beta_\varepsilon, \gamma_\varepsilon) \in \mathfrak{I}^\infty$ and $\mathcal{C}^{\mathcal{N}}_{\gamma_\varepsilon} = B_\varepsilon$. Lemma 6.7.(b) gives b_0, b_1, r with $(\alpha, a_0, a_1, b_0, b_1, r) \in \mathfrak{K}^\infty$. Lemma 6.24 implies that $R'(\alpha, a_0, a_1) \neq \emptyset$. So (b) holds, by Theorem 6.26. $\square$

Proof of Theorem 6.25. (1) Let $C^\varepsilon_{\varepsilon'} \in \mathbf{\Sigma}^0_1(\lceil T_d \rceil)$, $A^\varepsilon_0 \in \mathbf{\Gamma}_{c(\overline{\alpha})}(\lceil T_d \rceil)$, $A^\varepsilon_1 \in \check{\mathbf{\Gamma}}_{c(\overline{\alpha})}(\lceil T_d \rceil)$ such that $S_\varepsilon = (A^\varepsilon_0 \cap C^\varepsilon_0) \cup (A^\varepsilon_1 \cap C^\varepsilon_1)$. We reduce $(C^0_0, C^0_1, C^1_0, C^1_1)$. This gives a family $(O^0_0, O^0_1, O^1_0, O^1_1)$ of open subsets of $\lceil T_d \rceil$. Note that $S_\varepsilon \subseteq T^\varepsilon := (A^\varepsilon_0 \cap O^\varepsilon_0) \cup (A^\varepsilon_1 \cap O^\varepsilon_1) \cup (\neg A^{1-\varepsilon}_0 \cap O^{1-\varepsilon}_0) \cup (\neg A^{1-\varepsilon}_1 \cap O^{1-\varepsilon}_1)$. We will in fact ensure that $\big((\Pi''_i \lceil T_d \rceil)_{i \in d}, T^0, T^1\big) \leq \big((\mathcal{N})_{i \in d}, A_0, A_1\big)$ if (a) does not hold, which will be enough.

Subcase 1. $|(\alpha)_0| = 0$.

We set $o^\varepsilon_{\varepsilon'} := h[\lceil T_d \rceil \setminus O^\varepsilon_{\varepsilon'}]$, so that $o^\varepsilon_{\varepsilon'} \in \mathbf{\Pi}^0_1([\subseteq])$. We also set

$$D := \{\vec{s} \in T_d \mid \vec{s} = \vec{\emptyset} \vee \forall (\varepsilon, \varepsilon') \in 2^2 \ \ \exists \mathcal{B} \in o^\varepsilon_{\varepsilon'} \ \ \vec{s} \in \mathcal{B}\},$$

$$D^\varepsilon_{\varepsilon'} := \{\vec{s} \in T_d \mid \vec{s} \neq \vec{\emptyset} \wedge \forall \mathcal{B} \in o^\varepsilon_{\varepsilon'} \ \ \vec{s} \notin \mathcal{B} \ \wedge \forall (\varepsilon'', \varepsilon''') \in 2^2 \setminus \{(\varepsilon, \varepsilon')\} \ \ \exists \mathcal{B} \in o^{\varepsilon''}_{\varepsilon'''} \ \ \vec{s} \in \mathcal{B}\},$$

so that $(D, D^0_0, D^0_1, D^1_0, D^1_1)$ is a partition of T_d. The proof is very similar to the proof of Theorem 4.4.2 when $\xi = 1$. The changes to make in the conditions (1)-(7) are as follows:

(4) $U_{\vec{s}} \subseteq R'(\alpha, a_0, a_1) = \overline{A_0}^1 \cap \overline{A_1}^1$ if $\vec{s} \in D$,

(5) $U_{\vec{s}} \subseteq A_0$ if $\vec{s} \in D^0_1 \cup D^1_0$,

(6) $U_{\vec{s}} \subseteq A_1$ if $\vec{s} \in D^0_0 \cup D^1_1$,

(7) $(\vec{s}, \vec{t} \in D \vee \vec{s}, \vec{t} \in D^\varepsilon_{\varepsilon'}) \Rightarrow U_{\vec{t}} \subseteq U_{\vec{s}}$.

We conclude as in the proof of Theorem 4.4.2.

Subcase 2. $|(\alpha)_0| \geq 1$.

We will have the same kind of construction as in the proof of Theorem 6.9. As long as $\vec{t} \in D$, the inclusion $U_{\vec{t}} \subseteq R'(\alpha, a_0, a_1)$ will hold. If $\vec{t} \in D^\varepsilon_{\varepsilon'}$, then all the extensions of $\vec{t}$ will stay in $D^\varepsilon_{\varepsilon'}$, and we will copy the construction in the proof of Theorem 6.9, since inside the clopen set defined by $\vec{t}$ we want to reduce a pair $(\tilde{S}_0, \tilde{S}_1)$ to (A_0, A_1).

As $A^\varepsilon_0 \in \mathbf{\Gamma}_{c(\overline{\alpha})}(\lceil T_d \rceil)$, there is $B^\varepsilon_0 \in \mathbf{\Gamma}_{c(\overline{\alpha})}(\mathcal{N})$ with $A^\varepsilon_0 = j_d^{-1}(B^\varepsilon_0) \cap \lceil T_d \rceil$. As $\overline{\alpha} \in \Delta^1_1 \cap \mathfrak{H}^\infty$, Lemma 6.5.(b) gives $\beta^\varepsilon_0, \gamma^\varepsilon_0 \in \mathcal{N}$ such that $(\overline{\alpha}, \beta^\varepsilon_0, \gamma^\varepsilon_0) \in \mathfrak{I}^\infty$ and $\mathcal{C}^{\mathcal{N}}_{\gamma^\varepsilon_0} = B^\varepsilon_0$. Similarly, there are $\beta^\varepsilon_1, \gamma^\varepsilon_1 \in \mathcal{N}$ such that $(\overline{\alpha}, \beta^\varepsilon_1, \gamma^\varepsilon_1) \in \mathfrak{I}^\infty$ and $A^\varepsilon_1 = j_d^{-1}(\neg \mathcal{C}^{\mathcal{N}}_{\gamma^\varepsilon_1}) \cap \lceil T_d \rceil$.

We can associate to any $(\varepsilon, \varepsilon') \in 2^2$ the objects we met before, among which the function $\mathcal{Z}^{\varepsilon, \varepsilon'}$, the ordinals $\eta^{\varepsilon, \varepsilon'}_s$, the resolution families $(R^\rho_{\varepsilon, \varepsilon', s})_{\rho \leq \eta^{\varepsilon, \varepsilon'}_s}$, and the ordinals $\rho(\varepsilon, \varepsilon', s, \vec{s}\,)$.

Instead of considering the set $P_q(s)$, we will consider $P_q^{\varepsilon,\varepsilon'}(s)\cap D^\varepsilon_{\varepsilon'}$. If $\vec{t}\in D^\varepsilon_{\varepsilon'}$, then we set $\vec{w}(\vec{t}\,):=\vec{w}^\varepsilon_{\varepsilon'}$. This allows us to define $s(\vec{t}\,)\in\mathfrak{T}\big(\vec{w}(\vec{t}\,)\big)$ and $S(\vec{t}\,)\in\mathcal{M}_{\vec{w}(\vec{t})}$. We also set

$$\vec{v}(\vec{t}\,):=\left\{\begin{array}{l}(\overline{\alpha},a_0,a_1,b_0,b_1,r)\text{ if }\vec{t}\in D^0_0\cup D^1_1,\\(\overline{\alpha},a_1,a_0,b_0,b_1,r)\text{ if }\vec{t}\in D^0_1\cup D^1_0.\end{array}\right.$$

The other modifications to make in the conditions (1)-(6) are as follows. In condition (4), we ask for the inclusion $U_{\vec{t}}\subseteq R\big(S(\vec{t}\,)\big)$ only if $\vec{t}\notin D$. If $\vec{t}\in D$, then we want that $U_{\vec{t}}\subseteq R'(\alpha,a_0,a_1)$. Condition (6) was described when $\vec{s},\vec{t}\in D^\varepsilon_{\varepsilon'}$. If $\vec{s},\vec{t}\in D$, then we also want that $U_{\vec{t}}\subseteq U_{\vec{s}}$.

The sequence $F(\vec{\beta})$ is defined if $\beta\in C^0_0\cup C^0_1\cup C^1_0\cup C^1_1$. If $\beta\notin C^0_0\cup C^0_1\cup C^1_0\cup C^1_1$, then $\vec{\beta}|k\in D$ for each natural number k, and $F(\vec{\beta})$ is also defined. The definition of $\vec{v}(\vec{t}\,)$ ensures that $T^\varepsilon\subseteq(\Pi_{i\in d}\ f_i)^{-1}(A_\varepsilon)$.

The definition of $\Psi(v)$ is done if $v\notin D$. If $v\in D$, then we simply set $\Psi(v):=U_{\vec{t}}\cap(\Pi_{i\in d}\ X_i)$. Then we conclude as in the proof of Theorem 6.9.

(2) Let $C^\varepsilon_p\in\mathbf{\Sigma}^0_1(\lceil T_d\rceil)$ and $A^\varepsilon_p\in\mathbf{\Gamma}_{c((\alpha')_p)}(\lceil T_d\rceil)$ be such that $S_\varepsilon=\bigcup_{p\geq 1}\ (A^\varepsilon_p\cap C^\varepsilon_p)$. We reduce the family $(C^0_1,C^0_2,...,C^1_1,C^1_2,...)$. This gives a family $(O^0_1,O^0_2,...,O^1_1,O^1_2,...)$ of open subsets of $\lceil T_d\rceil$. Note that $S_\varepsilon\subseteq T^\varepsilon:=(A^\varepsilon_1\cap O^\varepsilon_1)\cup\bigcup_{p\geq 1}\ \big((\neg A^{1-\varepsilon}_p\cap O^{1-\varepsilon}_p)\cup(A^\varepsilon_{p+1}\cap O^\varepsilon_{p+1})\big)$. We will in fact ensure that $\big((\Pi''_i\lceil T_d\rceil)_{i\in d},T^0,T^1\big)\leq\big((\mathcal{N})_{i\in d},A_0,A_1\big)$ if (a) does not hold, which will be enough.

The proof is similar. We can assume that $\big|\big((\alpha')_p\big)_0\big|\geq 1$ for each $p\geq 1$, since $(\mathbf{\Gamma}_{c((\alpha')_p)})_{p\geq 1}$ is strictly increasing. So there is no Subcase 1. We set

$$\vec{v}(\vec{t}\,):=\left\{\begin{array}{l}(\overline{\alpha},a_0,a_1,b_0,b_1,r)\text{ if }\vec{t}\in\bigcup_{p\geq 1}\ D^0_p,\\(\overline{\alpha},a_1,a_0,b_0,b_1,r)\text{ if }\vec{t}\in\bigcup_{p\geq 1}\ D^1_p.\end{array}\right.$$

We conclude as in Case 1. □

It remains to prove Theorem 1.5.

Proof of Theorem 1.5. Theorem 1.3 gives $\mathbb{S}_0,\mathbb{S}_1\subseteq N_0\times N_1$.

Case 1. $C=$ graphs.

We set $\mathbb{R}_\varepsilon:=\mathbb{S}_\varepsilon\cup(\mathbb{S}_\varepsilon)^{-1}$. As $\mathbb{R}_0\cup\mathbb{R}_1=\mathbb{S}_0\cup\mathbb{S}_1\cup(\mathbb{S}_0\cup\mathbb{S}_1)^{-1}$, $\mathbb{R}_0,\mathbb{R}_0\cup\mathbb{R}_1\in C$. Let X be a Polish space and R be a Borel subset of X^2 in C. If (a) and (b) hold, then $\mathbb{R}_0$ is separable from $\mathbb{R}_1$ by a pot$(\mathbf{\Gamma})$ set S. Thus $\mathbb{S}_0=\mathbb{R}_0\cap(N_0\times N_1)$ is separable from $\mathbb{S}_1=\mathbb{R}_1\cap(N_0\times N_1)$ by S, which is absurd. So assume that R is not in pot$(\mathbf{\Gamma})$. Theorem 1.3 gives $f_0,f_1:\mathcal{C}\to X$ continuous such that $\mathbb{S}_0\subseteq(f_0\times f_1)^{-1}(R)$ and $\mathbb{S}_1\subseteq(f_0\times f_1)^{-1}(\neg R)$. We set $f(i\alpha):=f_i(i\alpha)$, so that f is continuous. Note that $\mathbb{S}_0\subseteq(f\times f)^{-1}(R)$, so that $\mathbb{R}_0\subseteq(f\times f)^{-1}(R)$. Similarly, $\mathbb{R}_1\subseteq(f\times f)^{-1}(\neg R)$.

Case 2. $C=$ oriented graphs.

We set $\mathbb{R}_\varepsilon:=\mathbb{S}_\varepsilon$, and argue as in Case 1.

Case 3. $C=$ quasi orders or $C=$ partial orders.

We set $\mathbb{R}_0:=\mathbb{S}_0\cup\Delta(\mathcal{C})$, $\mathbb{R}_1:=\mathbb{S}_1$, and argue as in Case 1. □

7 Injectivity complements

In the introduction, we saw that G. Debs proved that we can have the f_i's one-to-one in Theorem 1.3 when $d=2$, $\mathbf{\Gamma}\in\{\mathbf{\Pi}^0_\xi,\mathbf{\Sigma}^0_\xi\}$ and $\xi\geq 3$.

• This cannot be extended to higher dimensions, even if we replace $(d^\omega)^d$ with $\Pi_{i\in d}\ P_i$, where P_i is a sequence of Polish spaces.

Indeed, we argue by contradiction. Recall the proof of Theorem 3.1. We saw that there is $\mathbf{C}_\xi$ in $\mathbf{\Sigma}^0_\xi(\mathcal{C})\backslash\mathbf{\Pi}^0_\xi$ such that $\mathbb{S}_\xi:=\{\vec\alpha\in\lceil T_3\rceil\mid\mathcal{S}(\alpha_0\Delta\alpha_1)\in\mathbf{C}_\xi\}$ is not separable from $\lceil T_3\rceil\backslash\mathbb{S}_\xi$ by a pot($\mathbf{\Pi}^0_\xi$) set. We set

$$\begin{aligned}B_0&:=\{\vec\alpha\in 3^\omega\times 3^\omega\times 1\mid\mathcal{S}(\alpha_0\Delta\alpha_1)\in\mathbf{C}_\xi\},\\ B_1&:=\{\vec\alpha\in 3^\omega\times 1\times 3^\omega\mid\mathcal{S}(\alpha_0\Delta\alpha_2)\in\mathbf{C}_\xi\},\\ B_2&:=\{\vec\alpha\in 1\times 3^\omega\times 3^\omega\mid\mathcal{S}(\alpha_1\Delta\alpha_2)\in\mathbf{C}_\xi\}.\end{aligned}$$

Let $O:3^\omega\rightarrow 1$. As $\mathbb{S}_\xi:=(\text{Id}_{3^\omega}\times\text{Id}_{3^\omega}\times O)^{-1}(B_0)\cap\lceil T_3\rceil$, $B_0\notin$ pot($\mathbf{\Pi}^0_\xi$). Similarly, $B_1,B_2\notin$ pot($\mathbf{\Pi}^0_\xi$). This implies that the P_i's have cardinality at most one, and $\mathbb{S}_0\in\mathbf{\Delta}^0_1$. Thus $\mathbb{S}_0$ is separable from $\mathbb{S}_1$ by a pot($\mathbf{\Pi}^0_\xi$) set, which is absurd.

• If $d=\omega$, $\mathbf{\Gamma}=\mathbf{\Pi}^0_\xi$ and $\xi\geq 3$, then we cannot ensure that at least two of the f_i's are one-to-one. Indeed, we argue by contradiction again. Consider $X_i:=\omega$, and $B_\xi\in\mathbf{\Sigma}^0_\xi(\mathcal{N})\setminus\mathbf{\Pi}^0_\xi$. Then B_ξ is not pot($\mathbf{\Pi}^0_\xi$) since the topology on ω is discrete. This implies that two of the P_i's at least are countable, say P_0,P_1 for example. Consider now $A_0:=\mathbb{S}_\xi$ and $A_1:=\lceil T_\omega\rceil\backslash\mathbb{S}_\xi$. Then $(f_i\circ\Pi_i)[\mathbb{S}_0]$ is countable for each $i\in 2$. Thus $C:=(\Pi_{i\in d}\ f_i)[\mathbb{S}_0]\subseteq\mathbb{S}_\xi\subseteq\lceil T_\omega\rceil$ is countable since an element of $\lceil T_\omega\rceil$ is completely determined by two of its coordinates. Thus $C\in$ pot($\mathbf{\Sigma}^0_2$) $\subseteq$ pot($\mathbf{\Pi}^0_\xi$). Therefore $(\Pi_{i\in d}\ f_i)^{-1}(C)$ is a pot($\mathbf{\Pi}^0_\xi$) set separating $\mathbb{S}_0$ from $\mathbb{S}_1$, which is absurd.

• However, if $\mathbf{\Gamma}\in\{\mathbf{\Pi}^0_\xi,\mathbf{\Sigma}^0_\xi,\mathbf{\Delta}^0_\xi\}$ and $\xi\geq 3$, then we can ensure that $(\Pi_{i\in d}\ f_i)_{|\mathbb{S}_0\cup\mathbb{S}_1}$ is one-to-one, using G. Debs's proof and some additional arguments. This remains true if $\mathbf{\Gamma}=\mathbf{\Gamma}_u$ is a non self-dual Wadge class of Borel sets with $u(0)\geq 3$. This leads to the following notation. Let $(P_i)_{i\in d}$, $(X_i)_{i\in d}$ be sequences of Polish spaces, and S_0, S_1 (resp., A_0, A_1) be disjoint analytic subsets of $\Pi_{i\in d}\ P_i$ (resp., $\Pi_{i\in d}\ X_i$). Then

$$\big((P_i)_{i\in d},S_0,S_1\big)\sqsubseteq\big((X_i)_{i\in d},A_0,A_1\big)\ \Leftrightarrow\ \forall i\in d\ \ \exists f_i:P_i\rightarrow X_i\ \text{ continuous such that }\ (\Pi_{i\in d}\ f_i)_{|S_0\cup S_1}\ \text{ is one-to-one and }\ \forall\varepsilon\in 2\ \ S_\varepsilon\subseteq(\Pi_{i\in d}\ f_i)^{-1}(A_\varepsilon).$$

Theorem 7.1 *There is no tuple* $\big((P_i)_{i\in 2},S_0,S_1\big)$*, where the* P_i*'s are Polish spaces and* S_0*,* S_1 *disjoint analytic subsets of* $\Pi_{i\in 2}\ P_i$*, such that for any tuple* $\big((X_i)_{i\in 2},A_0,A_1\big)$ *of the same type exactly one of the following holds:*

(a) The set A_0 *is separable from* A_1 *by a pot(*$\mathbf{\Pi}^0_1$*) set.*

(b) The inequality $\big((P_i)_{i\in 2},S_0,S_1\big)\sqsubseteq\big((X_i)_{i\in 2},A_0,A_1\big)$ *holds.*

One can prove this result with the Borel digraph $A_0:=\bigcup_{n\in\omega}\ \text{Gr}(g_{n|\mathcal{C}\backslash M})$ considered in [L5] (see Section 3), which has countable vertical sections but is not locally countable. We give here another proof which moreover shows that we cannot hope for a positive result, even if A_0 is locally countable. This has to be noticed, since the locally countable sets have been considered a lot during the last decades.

Lemma 7.2 *Let* $\mathbf{\Gamma}$ *be a Borel class, and* $\big((P_i)_{i\in 2},S_0,S_1\big)$ *be as in the statement of Theorem 7.1 such that* S_0 *is not separable from* S_1 *by a pot*$(\mathbf{\Gamma})$ *set. Then* $S_0\cap(\Pi''_0 S_1\times\Pi''_1 S_1)$ *is not separable from* S_1 *by a pot*$(\mathbf{\Gamma})$ *set. Moreover,* S_0 *is not separable from* $S_1\cap(\Pi''_0 S_0\times\Pi''_1 S_0)$ *by a pot*$(\mathbf{\Gamma})$ *set.*

Proof. We prove the first assertion by contradiction, which gives $P\in$ pot$(\mathbf{\Gamma})$. The first reflection theorem gives Borel sets B_0,B_1 such that $\Pi''_i S_1\subseteq B_i$ and $S_0\cap(B_0\times B_1)\subseteq P$. Now

$$S_0\subseteq P\cup(\neg B_0\times P_1)\cup(P_0\times\neg B_1)\subseteq\neg S_1,$$

which contradicts the fact that S_0 is not separable from S_1 by a pot$(\mathbf{\Gamma})$ set.

We prove the second assertion using the first one (we pass to complements). □

Lemma 7.3 *Let* $\big((P_i)_{i\in 2},S_0,S_1\big)$ *and* $\big((X_i)_{i\in 2},A_0,A_1\big)$ *be as in the statement of Theorem 7.1 such that* $\big((P_i)_{i\in 2},S_0,S_1\big)\sqsubseteq\big((X_i)_{i\in 2},A_0,A_1\big)$*,* $(f_i)_{i\in 2}$ *be witnesses for this inequality, and* $\epsilon\in 2$ *be such that* A_ϵ *is Borel locally countable. Then* $f_{i|\Pi''_i S_\epsilon}$ *is countable-to-one for any* $i\in 2$ *and* S_ϵ *is locally countable.*

Proof. The inequality $\big((P_i)_{i\in 2},S_0,S_1\big)\sqsubseteq\big((X_i)_{i\in 2},A_0,A_1\big)$ gives $f_i:P_i\rightarrow X_i$ continuous such that $(\Pi_{i\in 2}\ f_i)_{|S_0\cup S_1}$ is one-to-one, and also $S_\varepsilon\subseteq(\Pi_{i\in 2}\ f_i)^{-1}(A_\varepsilon)$ for each $\varepsilon\in 2$.

• By the Lusin-Novikov theorem and Lemma 2.4.(a) in [L2] we can find Borel one-to-one partial functions b_n with Borel domain such that $A_\epsilon=\bigcup_{n\in\omega}\ \mathrm{Gr}(b_n)$. We set $R_n:=S_\epsilon\cap(\Pi_{i\in 2}\ f_i)^{-1}\big(\mathrm{Gr}(b_n)\big)$. Let us prove that $f_{i|\Pi''_i R_n}$ is one-to-one for each $i\in 2$.

Assume for example that $i=0$. Let $z\neq z'\in\Pi''_0 R_n$, and $y,y'\in P_1$ such that $(z,y),(z',y')\in R_n$. As $(z,y)\neq(z',y')$, $\big(f_0(z),f_1(y)\big)\neq\big(f_0(z'),f_1(y')\big)$. But $b_n\big(f_0(z)\big)=f_1(y)$, $b_n\big(f_0(z')\big)=f_1(y')$, so that $f_0(z)\neq f_0(z')$ since b_n is a partial function. If $i=1$, then we use the fact that b_n is one-to-one to see that $f_{i|\Pi''_i R_n}$ is also one-to-one.

• This proves that $f_{i|\Pi''_i S_\epsilon}$ is countable-to-one since $S_\epsilon=\bigcup_{n\in\omega}\ R_n$.

• Now S_ϵ is locally countable since $S_\epsilon\subseteq(\Pi_{i\in 2}\ f_{i|\Pi''_i S_\epsilon})^{-1}(A_\epsilon)$, A_ϵ is locally countable and $f_{i|\Pi''_i S_\epsilon}$ is countable-to-one for any $i\in 2$. □

Lemma 7.4 *Let* Y *be a Polish space,* B *be a Borel subset of* Y *and* $(f_n)_{n\in\omega}$ *be a sequence of Borel partial functions from a Borel subset of* B *into* B*. We assume that* $F:=\bigcup_{n\in\omega}\ Gr(f_n)$ *is disjoint from* $\Delta(B)$*, but not separable from* $\Delta(B)$ *by a pot*$(\mathbf{\Pi}^0_1)$ *set. Then there are natural numbers* $n<p$ *and* $y\in B$ *such that* $f_n(y)$ *and* $f_n\big(f_p(y)\big)$ *are defined.*

Proof. We may assume that Y is recursively presented and B,F and the f_n's are Δ^1_1. We put

$$V:=\bigcup\{D\in\Delta^1_1(Y)\mid D^2\cap F\text{ has finite vertical sections}\}.$$

Then $V\in\Pi^1_1(Y)$.

Case 1. $V = Y$.

We can find a sequence $(D_n)_{n\in\omega}$ of Δ^1_1 subsets of Y such that $Y = \bigcup_{n\in\omega} D_n$ and $D_n^2 \cap F$ has finite vertical sections. By Theorem 3.6 in [Lo2], $D_n^2 \cap F$ is pot($\mathbf{\Pi}^0_1$), so that $D_n^2 \setminus F$ is pot($\mathbf{\Sigma}^0_1$). Thus $\Delta(B) \subseteq \bigcup_{n\in\omega} D_n^2 \setminus F \subseteq \neg F$ and $\Delta(B)$ is separable from F by a pot($\mathbf{\Sigma}^0_1$) set, which is absurd.

Case 2. $V \neq Y$.

The first reflection theorem proves that for each nonempty Σ^1_1 subset S of Y contained in $\neg V$ there is $y \in S$ such that $(S^2 \cap F)_y$ is infinite. So there is a natural number n such that $(Y \setminus V)^2$ meets Gr(f_n). In particular, $S := (Y \setminus V) \cap f_n^{-1}(Y \setminus V)$ is a nonempty Σ^1_1 subset of Y, which gives $y \in S$ such that $(S^2 \cap F)_y$ is infinite. This proves the existence of $p > n$ such that $\big(y, f_p(y)\big) \in S^2$. Note that $y \in B$ since $Y \setminus B \subseteq V$. Now it is clear that n, p and y are suitable. □

Lemma 7.5 *Let Y_0, Y_1 be Polish spaces, B_ε be a Borel subset of Y_ε (for $\varepsilon \in 2$), $i : B_0 \to B_1$ be a Borel isomorphism, $(c_n)_{n\in\omega}$ be a sequence of Borel partial one-to-one functions with Borel domain from Y_0 into Y_1, and $C := \bigcup_{n\in\omega}$ Gr(c_n). We assume that $C \cap (B_0 \times B_1)$ is disjoint from Gr(i), but not separable from Gr(i) by a pot($\mathbf{\Pi}^0_1$) set. Then there are natural numbers $n < p$ and $y \in Y_0$ such that $(ic_n^{-1}c_p)(y)$ and $(ic_n^{-1}i)(y)$ are defined and different.*

Proof. We set $d_n := c_{n|B_0 \cap c_n^{-1}(B_1)}$, so that $C \cap (B_0 \times B_1) = \bigcup_{n\in\omega}$ Gr(d_n). We also set

$$e_n := d_n \circ i^{-1}_{|i[B_0 \cap c_n^{-1}(B_1)]},$$

so that e_n is a Borel one-to-one partial function with Borel domain. Now we consider the pre-images $\Delta(B_1) = (i^{-1} \times \text{Id}_{B_1})^{-1}\big(\text{Gr}(i)\big)$ and Gr$(e_n) = (i^{-1} \times \text{Id}_{B_1})^{-1}\big(\text{Gr}(d_n)\big)$. Note that $E := \bigcup_{n\in\omega}$ Gr(e_n) is not separable from $\Delta(B_1)$ by a pot($\mathbf{\Pi}^0_1$) set. This implies that $\bigcup_{n\in\omega}$ Gr(e_n^{-1}) is not separable from $\Delta(B_1)$ by a pot($\mathbf{\Pi}^0_1$) set.

By Lemma 7.4 there are $n < p$ and $z \in B_1$ such that $(e_n)^{-1}(z)$ and $e_n^{-1}\big(e_p^{-1}(z)\big)$ are defined. We set $y := d_p^{-1}(z)$, so that $(id_n^{-1}d_p)(y)$ and $(id_n^{-1}i)(y)$ are defined and equal respectively to $(ic_n^{-1}c_p)(y)$ and $(ic_n^{-1}i)(y)$. Now note that $z \neq e_p^{-1}(z)$ for each z in the range of e_p. This implies that $(ic_n^{-1}c_p)(y) \neq (ic_n^{-1}i)(y)$. □

Lemma 7.6 *Let i be a continuous open partial function from $\mathcal{C}$ into $\mathcal{C}$ with open domain, $(c_n)_{n\in\omega}$ be a sequence of such functions, and $U_\varepsilon := \bigcup_{n\in\omega}$ Gr$(c_{2n+\varepsilon})$ (for $\varepsilon \in 2$). We assume that U_0 is disjoint from $U_1 \cup$ Gr(i), but $\emptyset \neq$ Gr$(i) \subseteq \overline{U_0} \cap \overline{U_1}$. Then U_0 is not separable from U_1 by a pot($\mathbf{\Delta}^0_1$) set, and U_0 is not separable from Gr(i) by a pot($\mathbf{\Pi}^0_1$) set. If moreover the Dom(c_n)'s are dense, then $U_0 \cap (\bigcap_{n\in\omega}$ Dom$(c_n) \times \mathcal{C})$ is not separable from $U_1 \cap (\bigcap_{n\in\omega}$ Dom$(c_n) \times \mathcal{C})$ by a pot($\mathbf{\Delta}^0_1$) set.*

Proof. We argue by contradiction, which gives $P \in$ pot($\mathbf{\Delta}^0_1$). Let G_i be a dense G_δ subset of $\mathcal{C}$ such that $P \cap (G_0 \times G_1) \in \mathbf{\Delta}^0_1(G_0 \times G_1)$. The proof of Lemma 3.5 in [L1] shows the inclusion Gr$(i) \subseteq \overline{\text{Gr}(i) \cap (G_0 \times G_1)}$, and similarly with c_n. Thus

$$\begin{aligned}\text{Gr}(i) &\subseteq \overline{\overline{U_0} \cap \overline{U_1} \cap (G_0 \times G_1)} \subseteq \overline{\overline{U_0 \cap (G_0 \times G_1)} \cap \overline{U_1 \cap (G_0 \times G_1)} \cap (G_0 \times G_1)}\\ &\subseteq \overline{\big(P \cap (G_0 \times G_1)\big) \setminus \big(P \cap (G_0 \times G_1)\big)} = \emptyset,\end{aligned}$$

which is absurd. The last assertion follows since we may assume that $G_0 \subseteq \bigcap_{n\in\omega}$ Dom(c_n). The proof of the second assertion is similar and simpler. □

Lemma 7.7 *There is a tuple* $\big((Y_i)_{i\in 2},B_0,B_1\big)$ *such that*

(a) Y_0 *and* Y_1 *are Polish spaces.*

(b) $B_0\!=\!\bigcup_{n\in\omega}\ Gr(c_n)\!\subseteq\!\Pi_{i\in 2}\ Y_i$*, for some Borel one-to-one partial functions* c_n *with Borel domain.*

(c) $B_1\!=\!Gr(i)$*, for some Borel function* $i\!:\!Y_0\!\rightarrow\!Y_1$*.*

(d) B_0 *is disjoint from* B_1*, but not separable from* B_1 *by a pot(*$\mathbf{\Pi}^0_1$*) set.*

(e) We set $C_\varepsilon:=\big(\bigcup_{n\in\omega}\ Gr(c_{2n+\varepsilon})\big)\cap(\bigcap_{n\in\omega}\ Dom(c_n)\!\times\! Y_1)$*, for* $\varepsilon\!\in\!2$*. Then* C_0 *is disjoint from* C_1*, but not separable from* C_1 *by a pot(*$\mathbf{\Delta}^0_1$*) set, and* $\overline{C_0}\cap\overline{C_1}\cap(\bigcap_{n\in\omega}\ Dom(c_n)\!\times\! Y_1)\!\subseteq\! Gr(i)$*.*

(f) The equality $(ic_n^{-1}c_p)(y)\!=\!(ic_n^{-1}i)(y)$ *holds as soon as the two members of the equality are defined and* $n\!<\!p$*.*

Proof. We set $Y_i\!:=\!\mathcal{C}$ and $i(\alpha)(k)\!:=\!\alpha(2k)$.

• We first build an increasing sequence $(S_n)_{n\in\omega}$ of co-infinite subsets of ω, a sequence $(\psi_n)_{n\in\omega}$ of bijections with $\psi_n:\neg S_n\twoheadrightarrow\neg 2S_n$, and a sequence $(h_n)_{n\in\omega}$ of homeomorphisms from $\mathcal{C}$ onto itself. We do it by induction on n. We set $S_0:=\emptyset$, $\psi_0:=\mathrm{Id}_\omega$ and $h_0:=\mathrm{Id}_{\mathcal{C}}$. Assume that $(S_q)_{q\leq n}$, $(\psi_q)_{q\leq n}$ and $(h_q)_{q\leq n}$ are constructed, which is the case for $n\!=\!0$. We define a map $\varphi_n\!:\!\omega\!\rightarrow\!\omega$ by

$$\varphi_n(k):=\begin{cases}\psi_n^{-1}(k)\text{ if }k\notin 2S_n,\\ \\ \frac{k}{2}\text{ if }k\in 2S_n.\end{cases}$$

Note that φ_n is a bijection. We set $S_{n+1}:=\varphi_n[2\omega]\cup(n+1)$, which is co-infinite. The sequence $(S_n)_{n\in\omega}$ is increasing since $S_n=\varphi_n[2S_n]\subseteq S_{n+1}$. As S_{n+1} is co-infinite we can build the bijection $\psi_{n+1}:\neg S_{n+1}\rightarrow\neg 2S_{n+1}$ in such a way that $\psi_{n+1}(k)\neq\psi_q(k)$ for infinitely many $k\notin S_{n+1}$, for any $q\leq n$. We set

$$h_{n+1}(\alpha)(k):=\begin{cases}i(\alpha)(k)\text{ if }k\in S_{n+1},\\ \\ \alpha\big(\psi_{n+1}(k)\big)\text{ if }k\notin S_{n+1}.\end{cases}$$

As h_{n+1} permutes the coordinates, it is an homeomorphism.

• We set $D_n:=\{\alpha\in\mathcal{C}\mid i(\alpha)\neq h_n(\alpha)\wedge\forall q<n\ \ h_n(\alpha)\neq h_q(\alpha)\}$, so that D_n is an open subset of $\mathcal{C}$. We set $c_n:=h_{n|D_n}$, so that c_n is an homeomorphism from D_n onto its open range, B_0 is disjoint from B_1, and C_0 is disjoint from C_1.

Let us prove that D_n is dense for any natural number n. Note that

$$D_0=\{\alpha\in\mathcal{C}\mid\exists k\in\omega\ \ \alpha(2k)\neq\alpha(k)\},$$

which is clearly dense. Now D_{n+1} contains

$$\{\alpha\in\mathcal{C}\mid\exists k\notin S_{n+1}\ \ \alpha(2k)\neq\alpha\big(\psi_{n+1}(k)\big)\}\cap\bigcap_{q<n}\{\alpha\in\mathcal{C}\mid\exists k\notin S_{n+1}\ \ \alpha\big(\psi_{n+1}(k)\big)\neq\alpha\big(\psi_q(k)\big)\}.$$

The set $\{\alpha\in\mathcal{C}\mid\exists k\notin S_{n+1}\ \ \alpha(2k)\neq\alpha\big(\psi_{n+1}(k)\big)\}$ is open dense since the odd natural numbers are in $\psi_{n+1}[\neg S_{n+1}]$. The set $\{\alpha\in\mathcal{C}\mid\exists k\notin S_{n+1}\ \ \alpha\big(\psi_{n+1}(k)\big)\neq\alpha\big(\psi_q(k)\big)\}$ is open dense by construction of ψ_{n+1}. This proves that D_{n+1} is dense.

• Note that $\text{Gr}(i)\subseteq\overline{C_0}\cap\overline{C_1}$ since $i(\alpha)|n=h_n(\alpha)|n$, D_n is dense and i is continuous. Lemma 7.6 proves the non-separation assertions. Note also that $\overline{C_0}\cap\overline{C_1}\cap(\bigcap_{n\in\omega}\ \text{Dom}(c_n)\times\mathcal{C})\subseteq\text{Gr}(i)$ since $i(\alpha)|n=h_n(\alpha)|n$ and c_n is continuous.

• Now it is enough to prove that $ih_n^{-1}h_p=ih_n^{-1}i$ if $n<p$. Note that

$$h_n^{-1}(\beta)(j):=\begin{cases}\beta(k)\text{ if }j=2k\in 2S_n,\\[2ex] \beta\big(\psi_n^{-1}(j)\big)\text{ if }j\notin 2S_n.\end{cases}$$

Thus

$$(ih_n^{-1}i)(\alpha)(k)=i\big((h_n^{-1}i)(\alpha)\big)(k)=(h_n^{-1}i)(\alpha)(2k)=\begin{cases}i(\alpha)(k)\text{ if }k\in S_n,\\[2ex] i(\alpha)\big(\psi_n^{-1}(2k)\big)\text{ if }k\notin S_n.\end{cases}$$

Similarly,

$$(ih_n^{-1}h_p)(\alpha)(k)=\begin{cases}h_p(\alpha)(k)\text{ if }k\in S_n,\\[2ex] h_p(\alpha)\big(\psi_n^{-1}(2k)\big)\text{ if }k\notin S_n.\end{cases}$$

Note that $S_n\subseteq S_p$. Thus $(ih_n^{-1}h_p)(\alpha)(k)=(ih_n^{-1}i)(\alpha)(k)$ if $k\in S_n$. If $k\notin S_n$, then $2k\notin 2S_n$ and $\varphi_n(2k)=\psi_n^{-1}(2k)\in S_{n+1}\subseteq S_p$. Thus

$$(ih_n^{-1}h_p)(\alpha)(k)=h_p(\alpha)\big(\psi_n^{-1}(2k)\big)=i(\alpha)\big(\psi_n^{-1}(2k)\big)=(ih_n^{-1}i)(\alpha)(k).$$

This finishes the proof. □

Proof of Theorem 7.1. We argue by contradiction. Note that S_0 is not separable from S_1 by a pot($\mathbf{\Pi}^0_1$) set since (b) holds. By Lemma 7.2 we may assume that $S_1\subseteq\Pi''_0S_0\times\Pi''_1S_0$.

• Recall the digraph A_1 in [L5], that we will call A_0. If we take $X_i:=\mathcal{C}$ and $A_1:=\Delta(\mathcal{C})$, then by Corollary 12 in [L5], A_0 is Borel locally countable, not pot($\mathbf{\Pi}^0_1$), and $A_1=\overline{A_0}\setminus A_0$. It follows that A_0 is not separable from A_1 by a pot($\mathbf{\Pi}^0_1$) set Q, since otherwise we would have $A_0=Q\cap\overline{A_0}\in\text{pot}(\mathbf{\Pi}^0_1)$. This implies that $\big((X_i)_{i\in 2},A_0,A_1\big)$ satisfies condition (b) in Theorem 7.1. By Lemma 7.3, $f_{i|\Pi''_iS_0}$ is countable-to-one for any $i\in 2$ and S_0 is locally countable.

• Lemma 7.7 gives a tuple $\big((Y_i)_{i\in 2},B_0,B_1\big)$. Note that $\big((Y_i)_{i\in 2},B_0,B_1\big)$ satisfies condition (b) in Theorem 7.1, which gives $g_i:P_i\rightarrow Y_i$. Lemma 7.3 implies that $g_{i|\Pi''_iS_0}$ is countable-to-one for any $i\in 2$. The first reflection theorem gives a Borel set $O_i\supseteq\Pi''_iS_0$ such that $f_{i|O_i}$ and $g_{i|O_i}$ are countable-to-one, for any $i\in 2$. By Lemma 2.4.(a) in [L2] we can find a partition $(O^i_n)_{n\in\omega}$ of O_i into Borel sets such that $f_{i|O^i_n}$ and $g_{i|O^i_n}$ are one-to-one, for any $i\in 2$.

• We set $R_\varepsilon:=(\Pi_{i\in 2}\ f_{i|O_i})^{-1}(A_\varepsilon)\cap(\Pi_{i\in 2}\ g_i)^{-1}(B_\varepsilon)$, for any $\varepsilon\in 2$, so that R_ε is a Borel subset of $\Pi_{i\in 2}\ P_i$ containing S_ε. In particular, R_0 is not separable from R_1 by a pot($\mathbf{\Pi}^0_1$) set. We choose natural numbers n_0 and n_1 such that $R_0\cap(\Pi_{i\in 2}\ O^i_{n_i})$ is not separable from $R_1\cap(\Pi_{i\in 2}\ O^i_{n_i})$ by a pot($\mathbf{\Pi}^0_1$) set. We set $D_\varepsilon:=(\Pi_{i\in 2}\ f_i)[R_\varepsilon\cap(\Pi_{i\in 2}\ O^i_{n_i})]$, so that D_0 is a Borel subset of A_0 which is not separable from D_1 by a pot($\mathbf{\Pi}^0_1$) set.

Note that D_1 is a Borel subset of $A_1=\Delta(\mathcal{C})$. In particular, there is a Borel subset D of $\mathcal{C}$ such that $D_1=\Delta(D)$. By Lemma 7.2, $D_0\cap D^2$ is not separable from D_1 by a pot($\mathbf{\Pi}^0_1$) set. Let $h_i:D\to Y_i$ be defined by $h_i(\alpha):=(g_i\circ {f_i}^{-1})(\alpha)$. Then h_i is Borel, one-to-one, and $D_\varepsilon\cap D^2\subseteq A_\varepsilon\cap(\Pi_{i\in 2}\ h_i)^{-1}(B_\varepsilon)$.

• Note that $(\Pi_{i\in 2}\ h_i)[\Delta(D)]$ is a Borel subset of B_1, which proves the existence of a Borel subset B of Y_0 such that $(\Pi_{i\in 2}\ h_i)[\Delta(D)]=\text{Gr}(i_{|B})$. If $y\neq z\in B$, then $\big(y,i(y)\big)=\big(h_0(\alpha),h_1(\alpha)\big)$ and

$$\big(z,i(z)\big)=\big(h_0(\beta),h_1(\beta)\big)$$

for some $\alpha\neq\beta\in D$. As h_1 is one-to-one we get $i(y)\neq i(z)$, $i_{|B}$ is one-to-one and $i''B$ is Borel.

As $D_0\cap D^2\subseteq(\Pi_{i\in 2}\ h_i)^{-1}(B_0)$ and $D_1\subseteq(\Pi_{i\in 2}\ h_i)^{-1}\big(\text{Gr}(i_{|B})\big)$, B_0 is not separable from $\text{Gr}(i_{|B})$ by a pot($\mathbf{\Pi}^0_1$) set. By Lemma 7.2, $B_0\cap(B\times i''B)$ is not separable from $\text{Gr}(i_{|B})$ by a pot($\mathbf{\Pi}^0_1$) set.

• By Lemma 7.5 applied to $C_0:=B$ and $C_1:=i''B$ there are $n<p$ and $y\in Y_0$ such that $(ic_n^{-1}c_p)(y)$ and $(ic_n^{-1}i)(y)$ are defined and different, which contradicts Lemma 7.7.(f). $\square$

Remark. We recover the algebraic relation "$g_n=g_n\circ g_p$ if $n<p$" that was already present in Section 3 in [L5] mentioned just after the statement of Theorem 7.1.

Theorem 7.8 *There is no tuple $\big((P_i)_{i\in 2},S_0,S_1\big)$, where the P_i's are Polish spaces and S_0, S_1 disjoint analytic subsets of $\Pi_{i\in 2}\ P_i$, such that for any tuple $\big((X_i)_{i\in 2},A_0,A_1\big)$ of the same type exactly one of the following holds:*

(a) The set A_0 is separable from A_1 by a pot($\mathbf{\Delta}^0_1$) set.

(b) The inequality $\big((P_i)_{i\in 2},S_0,S_1\big)\sqsubseteq\big((X_i)_{i\in 2},A_0,A_1\big)$ holds.

Proof. Let us indicate the differences with the proof of Theorem 7.1. This time, S_0 is not separable from S_1 by a pot($\mathbf{\Delta}^0_1$) set.

• Note that $A_0=\bigcup_{n\in\omega}\ \text{Gr}(H_n)$, where $H_n:N_{s_n0}\to N_{s_n1}$ is a partial homeomorphism with clopen domain and range. The crucial properties of $(s_n)_{n\in\omega}\subseteq 2^{<\omega}$ is that it is dense and $|s_n|=n$. We can easily ensure this in such a way that $(s_{2n})_{n\in\omega}$ and $(s_{2n+1})_{n\in\omega}$ are dense. We set $U_\varepsilon:=\bigcup_{n\in\omega}\ \text{Gr}(H_{2n+\varepsilon})$. The previous remark implies that $\Delta(\mathcal{C})=\overline{U_\varepsilon}\setminus U_\varepsilon$. By Lemma 7.6, U_0 is not separable from U_1 by a pot($\mathbf{\Delta}^0_1$) set. So here again $f_{i|\Pi''_iS_0}$ is countable-to-one for any $i\in 2$, and S_0, S_1 are locally countable by Lemma 7.3.

• Lemma 7.7 gives a tuple $\Big(\big(\bigcap_{n\in\omega}\ \text{Gr}(c_n),\mathcal{C}\big),C_0,C_1\Big)$. Note that $\Big(\big(\bigcap_{n\in\omega}\ \text{Gr}(c_n),\mathcal{C}\big),C_0,C_1\Big)$ satisfies condition (b) in Theorem 7.8.

• We change the topology on $\mathcal{C}$ into a finer Polish topology τ so that the sets $f''_iO^i_{n_i}$ become clopen and the maps $f^{-1}_{i|O^i_{n_i}}$ become continuous. Now

$$\overline{D_0}^{\tau^2}\cap\overline{D_1}^{\tau^2}\subseteq\overline{U_0}\cap\overline{U_1}=\big(U_0\cup\Delta(\mathcal{C})\big)\cap\big(U_1\cup\Delta(\mathcal{C})\big)=\Delta(\mathcal{C}).$$

So there is a Borel subset D of $\mathcal{C}$ such that $\overline{D_0}^{\tau^2}\cap\overline{D_1}^{\tau^2}=\Delta(D)$, and $D\subseteq\bigcap_{i\in 2}\ f''_iO^i_{n_i}$.

• Let us prove that $D_0\cap D^2$ is not separable from $D_1\cap D^2$ by a pot($\mathbf{\Delta}^0_1$) set.

We argue by contradiction, which gives $P\in$ pot($\mathbf{\Delta}^0_1$) such that $D_0\cap D^2\subseteq P\subseteq D^2\setminus D_1$. The sets $\overline{D_0}^{\tau^2}\cap(\neg D\times\mathcal{C})$ and $\overline{D_1}^{\tau^2}\cap(\neg D\times\mathcal{C})$ are disjoint, pot($\mathbf{\Pi}^0_1$), so that they are separable by Δ_l in pot($\mathbf{\Delta}^0_1$). Similarly, there is $\Delta_r\in$pot($\mathbf{\Delta}^0_1$) which separates $\overline{D_0}^{\tau^2}\cap(\mathcal{C}\times\neg D)$ from $\overline{D_1}^{\tau^2}\cap(\mathcal{C}\times\neg D)$. Now we can write

$$D_0\subseteq P\cup\big(D_0\cap(\neg D\times\mathcal{C})\big)\cup\big(D_0\cap(\mathcal{C}\times\neg D)\big)\subseteq P\cup\big(\Delta_l\cap(\neg D\times\mathcal{C})\big)\cup\big(\Delta_r\cap(\mathcal{C}\times\neg D)\big)\subseteq\neg D_1,$$

which is absurd since $P\cup\big(\Delta_l\cap(\neg D\times\mathcal{C})\big)\cup\big(\Delta_r\cap(\mathcal{C}\times\neg D)\big)\in$pot($\mathbf{\Delta}^0_1$).

• Let us prove that $D_0\cap D^2$ is not separable from $\Delta(D)$ by a pot($\mathbf{\Pi}^0_1$) set.

We argue by contradiction, which gives $Q\in$pot($\mathbf{\Pi}^0_1$) such that $D_0\cap D^2\subseteq Q\subseteq D^2\setminus\Delta(D)$. The sets Q and $\Delta(D)$ are disjoint, pot($\mathbf{\Pi}^0_1$), so that there is R in pot($\mathbf{\Delta}^0_1$) such that $Q\subseteq R\subseteq D^2\setminus\Delta(D)$. The sets $\overline{D_0}^{\tau^2}\cap R$ and $\overline{D_1}^{\tau^2}\cap R$ are disjoint, pot($\mathbf{\Pi}^0_1$), so that there is S in pot($\mathbf{\Delta}^0_1$) such that $\overline{D_0}^{\tau^2}\cap R\subseteq S\subseteq R\setminus\overline{D_1}^{\tau^2}$. But S separates $D_0\cap D^2$ from $D_1\cap D^2$, which contradicts the previous point.

• Note that $(\Pi_{i\in 2}\ h_i)[\Delta(D)]\subseteq\overline{C_0}\cap\overline{C_1}\cap(\bigcap_{n\in\omega}\ \text{Dom}(c_n)\times\mathcal{C})\subseteq\text{Gr}(i)$. We conclude as in the proof of Theorem 7.1. $\square$

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