On small analytic relations

Dominique LECOMTE

May 9, 2023

• Sorbonne Université and Université Paris Cité, CNRS,
Institut de Mathématiques de Jussieu-Paris Rive Gauche, F-75005 Paris, France

dominique.lecomte@upmc.fr

• Université de Picardie, I.U.T. de l'Oise, site de Creil,
13, allée de la faïencerie, 60 100 Creil, France

Abstract. We study the class of analytic binary relations on Polish spaces, compared by the notions of continuous reducibility or injective continuous reducibility. In particular, we characterize when a locally countable Borel relation is $\boldsymbol{\Sigma}^0_\xi$ (or $\boldsymbol{\Pi}^0_\xi$), when $\xi \geq 3$, by providing a concrete finite antichain basis. We give a similar characterization for arbitrary relations when $\xi = 1$. When $\xi = 2$, we provide a concrete antichain of size continuum made of locally countable Borel relations minimal among non-$\boldsymbol{\Sigma}^0_2$ (or non-$\boldsymbol{\Pi}^0_2$) relations. The proof of this last result allows us to strengthen a result due to Baumgartner in topological Ramsey theory on the space of rational numbers. We prove that positive results hold when $\xi = 2$ in the acyclic case. We give a general positive result in the non-necessarily locally countable case, with another suitable acyclicity assumption. We provide a concrete finite antichain basis for the class of uncountable analytic relations. Finally, we deduce from our positive results some antichain basis for graphs, of small cardinality (most of the time 1 or 2).

2010 Mathematics Subject Classification. Primary: 03E15, Secondary: 28A05, 54H05

Keywords and phrases. acyclic, analytic relation, antichain basis, Borel class, countable, descriptive complexity, continuous reducibility, graph, Ramsey

Acknowledgments. The author wishes to thank the anonymous referee for the careful reading of the article, the suggestions that improved the presentation of this work, and the simplification of the proof of Lemma 6.4.

1 Introduction

This article presents a continuation of the work in [L5], in which the descriptive complexity of Borel equivalence relations on Polish spaces was studied (recall that a topological space is **Polish** if it is separable and completely metrizable). These relations are compared using the notion of **continuous reducibility**, which is as follows. Recall that if X, Y are topological spaces and $A \subseteq X^2$, $B \subseteq Y^2$,

$$(X, A) \leq_c (Y, B) \Leftrightarrow \exists f : X \to Y \text{ continuous with } A = (f \times f)^{-1}(B)$$

(we say that f **reduces** A to B). When the function f can be injective, we write $\sqsubseteq_c$ instead of $\leq_c$. Sometimes, when the space is clear for instance, we will talk about A instead of (X, A). The motivation for considering these quasi-orders is as follows (recall that a **quasi-order** is a reflexive and transitive relation). A standard way of comparing the descriptive complexity of subsets of zero-dimensional Polish spaces is the **Wadge quasi-order** (see [W]; recall that a topological space is **zero-dimensional** if there is a basis for its topology made of clopen, i.e., closed and open, sets). If S, Z are zero-dimensional Polish spaces and $C \subseteq S$, $D \subseteq Z$,

$$(S, C) \leq_W (Z, D) \Leftrightarrow \exists g : S \to Z \text{ continuous with } C = g^{-1}(D).$$

However, the pre-image of a graph by an arbitrary continuous map is not in general a graph, for instance. Note that the classes of reflexive relations, irreflexive relations, symmetric relations, transitive relations are closed under pre-images by a square map. Moreover, the class of antisymmetric relations is closed under pre-images by the square of an injective map. This is the reason why square maps are considered to compare graphs, equivalence relations... The most common way of comparing Borel equivalence relations is the notion of Borel reducibility (see, for example, [G], [Ka]). However, very early in the theory, injective continuous reducibility was considered, for instance in Silver's theorem (see [S]).

The most classical hierarchy of topological complexity in descriptive set theory is the one given by the Borel classes. If $\mathbf{\Gamma}$ is a class of subsets of the metrizable spaces, then $\check{\mathbf{\Gamma}} := \{\neg S \mid S \in \mathbf{\Gamma}\}$ is its **dual class**, and $\Delta(\mathbf{\Gamma}) := \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$. Recall that the Borel hierarchy is the inclusion from left to right in the following picture:

	$\mathbf{\Sigma}^0_1 = \text{open}$		$\mathbf{\Sigma}^0_2 = F_\sigma$			$\mathbf{\Sigma}^0_\xi = (\bigcup_{\eta<\xi} \mathbf{\Pi}^0_\eta)_\sigma$	
$\mathbf{\Delta}^0_1 = \text{clopen}$		$\mathbf{\Delta}^0_2 = \mathbf{\Sigma}^0_2 \cap \mathbf{\Pi}^0_2$		$\cdots$	$\mathbf{\Delta}^0_\xi = \mathbf{\Sigma}^0_\xi \cap \mathbf{\Pi}^0_\xi$		$\cdots$
	$\mathbf{\Pi}^0_1 = \text{closed}$		$\mathbf{\Pi}^0_2 = G_\delta$			$\mathbf{\Pi}^0_\xi = \check{\mathbf{\Sigma}}^0_\xi$	

This hierarchy is strict in uncountable Polish spaces, in which the non self-dual classes are those of the form $\mathbf{\Sigma}^0_\xi$ or $\mathbf{\Pi}^0_\xi$. In the sequel, by non self-dual Borel class, we mean exactly those classes. A class $\mathbf{\Gamma}$ of subsets of the zero-dimensional Polish spaces is a **Wadge class** if there is a zero-dimensional space Z and a subset D of Z in $\mathbf{\Gamma}$ such that a subset C of a zero-dimensional space S is in $\mathbf{\Gamma}$ exactly when $(S, C) \leq_W (Z, D)$. The hierarchy of the Wadge classes of Borel sets, compared with the inclusion, refines greatly the hierarchy of the non self-dual Borel classes, and is the finest hierarchy of topological complexity considered in descriptive set theory (see [Lo-SR2]).

We are interested in the descriptive complexity of Borel relations on Polish spaces. In order to approach this problem, it is useful to consider invariants for the considered quasi-order. A natural invariant for Borel reducibility has been studied, the notion of potential complexity (see, for example, [L2], [L3], and [Lo2] for the definition). A Borel relation R on a Polish space X is **potentially** in a Wadge class $\mathbf{\Gamma}$ if we can find a finer Polish topology τ on X such that R is in $\mathbf{\Gamma}$ in the product $(X,\tau)^2$. This is an invariant in the sense that any relation which is Borel reducible to a relation potentially in $\mathbf{\Gamma}$ has also to be potentially in $\mathbf{\Gamma}$. Along similar lines, any relation which is continuously reducible to a relation in $\mathbf{\Gamma}$ has also to be in $\mathbf{\Gamma}$.

We already mentioned the equivalence relations. A number of other interesting relations can be considered on a Polish space X, in the descriptive set theoretic context. Let us mention

- the **digraphs** (which do not meet the **diagonal** $\Delta(X):=\{(x,x)\mid x\in X\}$ of X),
- the **graphs** (i.e., the symmetric digraphs),
- the **oriented graphs** (i.e., the antisymmetric digraphs),
- the **quasi-orders**, strict or not,
- the **partial orders** (i.e., the antisymmetric quasi-orders), strict or not.

For instance, we refer to [Lo3], [L1], [K-Ma]. For **locally countable** relations (i.e., relations with countable horizontal and vertical sections), we refer to [K2] in the case of equivalence relations. An important subclass of the class of locally countable Borel equivalence relations is the class of treeable locally countable Borel equivalence relations, generated by an acyclic locally countable Borel graph. More generally, the locally countable digraphs have been widely considered, not necessarily to study equivalence relations (see [K-Ma]). All this motivates the work in the present paper, mostly devoted to the study of the descriptive complexity of arbitrary locally countable or/and acyclic Borel relations on Polish spaces.

We are looking for characterizations of the relations in a fixed Borel class $\mathbf{\Gamma}$. So we will consider the continuous and injective continuous reducibilities. In other words, we want to give answers to the following very simple question: when is a relation $\mathbf{\Sigma}^0_\xi$ (or $\mathbf{\Pi}^0_\xi$)? We are looking for characterizations of the following form: a relation is either simple, or more complicated than a typical complex relation. So we need to introduce, for some Borel classes $\mathbf{\Gamma}$, examples of complex relations.

Notation. Let $\mathbf{\Gamma}$ be a non self-dual Borel class, i.e., a class of the form $\mathbf{\Sigma}^0_\xi$ or $\mathbf{\Pi}^0_\xi$ for some countable ordinal $\xi>0$, called the **rank** of $\mathbf{\Gamma}$.

If the rank of $\mathbf{\Gamma}$ is one (i.e., if $\mathbf{\Gamma}\in\{\mathbf{\Sigma}^0_1,\mathbf{\Pi}^0_1\}$), then we set $\mathbb{K}:=\{0\}\cup\{2^{-k}\mid k\in\omega\}\subseteq\mathbb{R}$, $\mathbb{C}:=\{0\}$ if $\mathbf{\Gamma}=\mathbf{\Sigma}^0_1$, and $\mathbb{C}:=\mathbb{K}\setminus\{0\}$ if $\mathbf{\Gamma}=\mathbf{\Pi}^0_1$. The particular choice of $\mathbb{K}$ and $\mathbb{C}$ provides the injectivity part in Theorem 1.5 to come.

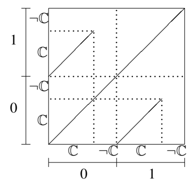
If the rank of $\mathbf{\Gamma}$ is at least two, then we set $\mathbb{K}:=2^\omega$, and $\mathbb{C}\cap N_s\in\check{\mathbf{\Gamma}}(N_s)\setminus\mathbf{\Gamma}(N_s)$ for each $s\in 2^{<\omega}$, where $N_s:=\{\alpha\in 2^\omega\mid s\subseteq\alpha\}$ is the standard basic clopen subset of 2^ω associated with s (the existence of $\mathbb{C}$ comes from Lemma 4.5 in [L5]). In particular, $\mathbb{C}$ is dense and co-dense in 2^ω. We set

$$\mathbb{C}:=I_2:=\{\alpha\in 2^\omega\mid\exists^\infty n\in\omega\ \ \alpha(n)=1\}$$

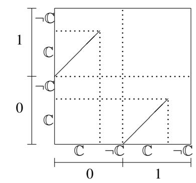
if $\mathbf{\Gamma}=\mathbf{\Sigma}^0_2$, and $\mathbb{C}:=Q_2:=\{\alpha\in 2^\omega\mid\forall^\infty n\in\omega\ \ \alpha(n)=0\}$ if $\mathbf{\Gamma}=\mathbf{\Pi}^0_2$. The particular choice of $\mathbb{K}$ and $\mathbb{C}$ provides the injectivity part in Theorems 1.3 and 1.6-1.8 to come.

In the sequel, we will say that $(\mathbb{K},\mathbb{C})$ is **$\mathbf{\Gamma}$-good** if either the rank of $\mathbf{\Gamma}$ is at most two and $(\mathbb{K},\mathbb{C})$ corresponds to the particular choices just above, or the rank of $\mathbf{\Gamma}$ is at least three, $\mathbb{K}=2^\omega$, and $\mathbb{C}\cap N_s\in\check{\mathbf{\Gamma}}(N_s)\setminus\mathbf{\Gamma}(N_s)$ for each $s\in 2^{<\omega}$. In particular, $\mathbb{K}$ is a metrizable compact space and $\mathbb{C}\in\check{\mathbf{\Gamma}}(\mathbb{K})\setminus\mathbf{\Gamma}(\mathbb{K})$ if $(\mathbb{K},\mathbb{C})$ is $\mathbf{\Gamma}$-good.

Examples. Let $\mathbf{\Gamma}$ be a non self-dual Borel class, and $(\mathbb{K},\mathbb{C})$ be $\mathbf{\Gamma}$-good. In [L5], the equivalence relation $\mathbb{E}_3^{\mathbf{\Gamma}}$ on $\mathbb{D}:=2\times\mathbb{K}$ is defined as follows: $(\varepsilon,x)\ \mathbb{E}_3^{\mathbf{\Gamma}}\ (\eta,y)\Leftrightarrow(\varepsilon,x)=(\eta,y)\vee(x=y\in\mathbb{C})$. The graph $\mathbb{G}_m^{\mathbf{\Gamma}}:=\mathbb{E}_3^{\mathbf{\Gamma}}\setminus\Delta(\mathbb{D})$, obtained from $\mathbb{E}_3^{\mathbf{\Gamma}}$ by removing the diagonal, will be very important in the sequel (the letter "m" expresses the minimality).



$\mathbb{E}_3^{\mathbf{\Gamma}}$



$\mathbb{G}_m^{\mathbf{\Gamma}}$

The main result in [L5] is as follows. Most of our results will hold in analytic spaces and not only in Polish spaces. Recall that a separable metrizable space is an **analytic space** if it is homeomorphic to an analytic subset of a Polish space.

Theorem 1.1 *Let $\mathbf{\Gamma}$ be a non self-dual Borel class of rank at least three. Then $\mathbb{E}_3^{\mathbf{\Gamma}}$ is the $\sqsubseteq_c$-minimum among non-$\mathbf{\Gamma}$ locally countable Borel equivalence relations on an analytic space.*

In fact, this result is also valid for equivalence relations with $\mathbf{\Sigma}_2^0$ classes, i.e., $\mathbf{\Sigma}_2^0$ sections. Recall that if $(Q,\leq)$ is a quasi-ordered class, then a **basis** is a subclass B of Q such that any element of Q is $\leq$-above an element of B. We are looking for basis as small as possible, so in fact for antichains (an **antichain** is a subclass of Q made of pairwise $\leq$-incomparable elements). As we will see, the solution of our problem heavily depends on the rank of the non self-dual Borel class considered. The next result solves our problem for the classes of rank at least three.

Theorem 1.2 *(1) Let $\mathbf{\Gamma}$ be a non self-dual Borel class of rank at least three. Then there is a concrete 34 element $\sqsubseteq_c$ and $\leq_c$-antichain basis for the class of non-$\mathbf{\Gamma}$ locally countable Borel relations on an analytic space.*
(2) $\mathbb{G}_m^{\mathbf{\Gamma}}$ is the $\sqsubseteq_c$-minimum among non-$\mathbf{\Gamma}$ locally countable Borel graphs on an analytic space.
(3) These results also hold when $\mathbf{\Gamma}$ has rank two for relations whose sections are in $\Delta(\mathbf{\Gamma})$.

The next surprising result shows that this complexity assumption on the sections is useful for the classes of rank two, for which any basis must have size continuum.

Theorem 1.3 *Let $\mathbf{\Gamma}$ be a non self-dual Borel class of rank two. Then there is a concrete $\leq_c$-antichain of size continuum made of locally countable Borel relations on 2^ω which are $\leq_c$ and $\sqsubseteq_c$-minimal among non-$\mathbf{\Gamma}$ relations on an analytic space.*

Similar results hold for graphs (see Corollary 3.13 and Theorem 3.15). Our analysis of the rank two also provides a basis for the class of non-$\mathbf{\Sigma}_2^0$ locally countable Borel relations on an analytic space (see Theorem 2.15). The proof of Theorem 1.3 strengthens Theorem 1.1 in [B] (see also Theorem 6.31 in [T], in topological Ramsey theory on the space $\mathbb{Q}$ of rational numbers).

Theorem 1.4 *There is an onto coloring $c:\mathbb{Q}^{[2]}\rightarrow\omega$ with the property that, for any $H\subseteq\mathbb{Q}$ homeomorphic to $\mathbb{Q}$, there is $h:\mathbb{Q}\rightarrow H$, homeomorphism onto $h[\mathbb{Q}]$, for which $c(\{x,y\})=c\big(\{h(x),h(y)\}\big)$ if $x,y\in\mathbb{Q}$. In particular, c takes all the values from ω on $H^{[2]}$.*

Question. For the classes of rank two, we saw that any basis must have size continuum. Is there an antichain basis?

The next result solves our problem for the classes of rank one.

Theorem 1.5 *Let $\mathbf{\Gamma}$ be a non self-dual Borel class of rank one. Then there is a concrete 7360 element $\leq_c$-antichain basis, made of relations on a countable metrizable compact space, for the class of non-$\mathbf{\Gamma}$ relations on a first countable topological space. A similar result holds for $\sqsubseteq_c$, with 2 more elements in the antichain basis.*

Note that in Theorem 1.5, the fact of assuming that X is analytic or that R is locally countable Borel does not change the result since the elements of the antichain basis satisfy these stronger assumptions. The "first countable" assumption ensures that closures and sequential closures coincide. Here again, similar results hold for graphs, with much smaller antichain basis (of cardinality $\leq_c$-5, $\sqsubseteq_c$-6 for $\mathbf{\Pi}^0_1$, and 10 for $\mathbf{\Sigma}^0_1$). We will not give the proof of Theorem 1.5 since it is elementary. We simply describe the different antichain basis in Section 5.

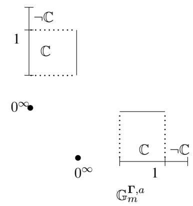
Remark. Theorem 1.5 provides a finite antichain basis for the class of non-closed Borel relations on a Polish space, for $\sqsubseteq_c$ and $\leq_c$. This situation is very different for the class $\mathcal{C}$ of non-potentially closed Borel relations on a Polish space. Indeed, [L1] provides an antichain of size continuum made of minimal elements of $\mathcal{C}$, for any of these two quasi-orders. It also follows from [L-M] that in fact there is no antichain basis in $\mathcal{C}$, for any of these two quasi-orders again.

The works in [K-S-T], [L-M], [L-Z], [L4] and also [C-L-M] show that an acyclicity assumption can give positive dichotomy results (see, for example, Theorem 1.9 in [L4]). This is a way to fix our problem with the rank two. If A is a binary relation on a set X, then $A^{-1}:=\{(x,y)\in X^2\mid (y,x)\in A\}$. The **symmetrization** of A is $s(A):=A\cup A^{-1}$. An A-**path** is a finite sequence $(x_i)_{i\leq n}$ of points of X such that $(x_i,x_{i+1})\in A$ if $i<n$. We say that A is **acyclic** if there is no injective A-path $(x_i)_{i\leq n}$ with $n\geq 2$ and $(x_n,x_0)\in A$. In practice, we will consider acyclicity only for symmetric relations. We will say that A is **s-acyclic** if $s(A)$ is acyclic.

Theorem 1.6 *(1) There is a concrete 34 element $\sqsubseteq_c$ and $\leq_c$-antichain basis for the class of non-$\mathbf{\Sigma}^0_2$ s-acyclic locally countable Borel relations on an analytic space.*

(2) $\mathbb{G}_m^{\mathbf{\Sigma}^0_2}$ is the $\sqsubseteq_c$-minimum among non-$\mathbf{\Sigma}^0_2$ acyclic locally countable Borel graphs on an analytic space.

Notation. For the class $\mathbf{\Pi}^0_2$, we need some more examples since the complexity of a locally countable relation can come from the complexity of one of its sections in this case. Let $\mathbf{\Gamma}$ be a non self-dual Borel class of rank at least two, and $(2^\omega,\mathbb{C})$ be $\mathbf{\Gamma}$-good. We set $\mathbb{S}:=\{0^\infty\}\cup N_1$ (where N_1 is the basic clopen set $\{\alpha\in 2^\omega\mid\alpha(0)=1\}$), and $\mathbb{G}_m^{\mathbf{\Gamma},a}:=s(\{(0^\infty,1\alpha)\mid\alpha\in\mathbb{C}\})$.



Theorem 1.7 *(1) There is a concrete 52 element $\sqsubseteq_c$ and $\leq_c$-antichain basis for the class of non-$\mathbf{\Pi}^0_2$ s-acyclic locally countable Borel relations on an analytic space.*

(2) $\{(\mathbb{D}, \mathbb{G}_m^{\mathbf{\Pi}^0_2}), (\mathbb{S}, \mathbb{G}_m^{\mathbf{\Pi}^0_2,a})\}$ is a $\sqsubseteq_c$ and $\leq_c$-antichain basis for the class of non-$\mathbf{\Pi}^0_2$ acyclic locally countable Borel graphs on an analytic space.

Theorems 1.6 and 1.7 are consequences of the following, which does not involve local countability.

Theorem 1.8 *Let $\mathbf{\Gamma}$ be a non self-dual Borel class of rank two.*

(1) There is a concrete 76 element $\sqsubseteq_c$ and $\leq_c$-antichain basis for the class of non-$\mathbf{\Gamma}$ s-acyclic Borel relations on an analytic space.

(2) $\{(\mathbb{D}, \mathbb{G}_m^{\mathbf{\Gamma}}), (\mathbb{S}, \mathbb{G}_m^{\mathbf{\Gamma},a})\}$ is a $\sqsubseteq_c$ and $\leq_c$-antichain basis for the class of non-$\mathbf{\Gamma}$ acyclic Borel graphs on an analytic space.

This result can be extended, with a suitable acyclicity assumption. In [L4], it is shown that the containment in an s-acyclic $\mathbf{\Sigma}^0_2$ relation allows some positive reducibility results (see, for example Theorem 4.1 in [L4]). A natural way to ensure this is to have an s-acyclic closure (recall Theorem 1.9 in [L4]). In spaces of infinite sequences like the Baire space ω^ω, having s-acyclic levels is sufficient to ensure this (see Proposition 2.7 in [L4]). Moreover, there is an s-acyclic closed relation on 2^ω containing Borel relations of arbitrarily high complexity (even potential complexity), by Proposition 3.17 in [L4]. The next result unifies the classes of rank at least two.

Theorem 1.9 *Let $\mathbf{\Gamma}$ be a non self-dual Borel class of rank at least two.*

(1) There is a concrete 76 element $\sqsubseteq_c$ and $\leq_c$-antichain basis for the class of non-$\mathbf{\Gamma}$ Borel relations on an analytic space contained in an s-acyclic $\mathbf{\Sigma}^0_2$ relation.

(2) $\{(\mathbb{D}, \mathbb{G}_m^{\mathbf{\Gamma}}), (\mathbb{S}, \mathbb{G}_m^{\mathbf{\Gamma},a})\}$ is a $\sqsubseteq_c$ and $\leq_c$-antichain basis for the class of non-$\mathbf{\Gamma}$ Borel graphs on an analytic space contained in an acyclic $\mathbf{\Sigma}^0_2$ graph.

Questions. For the classes of rank at least three, we gave finite antichain basis for small relations. Is there an antichain basis if we do not assume smallness? If yes, is it finite? Countable? Is it true that any basis must have size continuum? The graph $\big(2^\omega, \mathbb{C}^2 \setminus \Delta(2^\omega)\big)$ shows that we cannot simply erase the acyclicity assumptions in Theorems 1.8.(2) and 1.9.(2) (in order to see that, an argument by contradiction provides a continuous reduction f, we use the density of $\mathbb{C}$, and we discuss the constantness of f).

Theorems 1.2, 1.3-1.4, 1.6-1.9 are proved in Sections 2, 3, 4 respectively. In Section 6, we close this study of $\sqsubseteq_c$ by providing an antichain basis for the class of uncountable analytic relations on a Hausdorff topological space, which gives a perfect set theorem for binary relations. We extend the notation $\mathbb{G}_m^{\mathbf{\Gamma}}, \mathbb{G}_m^{\mathbf{\Gamma},a}$ to the class $\mathbf{\Gamma} = \{\emptyset\}$.

Theorem 1.10 *(1) There is a concrete 13 element $\sqsubseteq_c$-antichain basis for the class of uncountable analytic relations on a Hausdorff topological space.*

(2) $\{(\mathbb{D},\mathbb{G}_m^{\{\emptyset\}}),(\mathbb{S},\mathbb{G}_m^{\{\emptyset\},a}),(2^\omega,\neq)\}$ is a $\sqsubseteq_c$-antichain basis for the class of uncountable analytic graphs on a Hausdorff topological space.

As $(2^\omega,\neq)$ is not acyclic, we recover the basis met in Theorems 1.8.(2) and 1.9.(2) in the acyclic case. Also, $(\mathbb{S},\mathbb{G}_m^{\{\emptyset\},a})$ and $(2^\omega,\neq)$ are not locally countable. In conclusion, [L5] and the present study show that, when our finite antichain basis exist, they are small in the cases of equivalence relations and graphs.

We saw that there is no antichain basis in the class of non-potentially closed Borel relations on a Polish space. However, it follows from the main results in [L2] and [L3] that this problem can be fixed if we allow partial reductions, on a closed relation (which in fact is suitable for any non self-dual Borel class). This solution involves the following quasi-order, less considered than $\sqsubseteq_c$ and $\leq_c$. Let X, Y be topological spaces, and $A_0,A_1\subseteq X^2$ (resp., $B_0,B_1\subseteq Y^2$) be disjoint. Then we set

$$(X,A_0,A_1)\leq(Y,B_0,B_1)\Leftrightarrow\exists f\!:\!X\to Y\text{ continuous with }\forall\varepsilon\in 2\ \ A_\varepsilon\subseteq(f\times f)^{-1}(B_\varepsilon).$$

A similar result holds here, for the Borel classes instead of the potential Borel classes. We define $\mathbb{O}_m^{\mathbf{\Gamma}}:=\big\{\big((0,x),(1,x)\big)\mid x\in\mathbb{C}\big\}$. Note that $\mathbb{G}_m^{\mathbf{\Gamma}}=s(\mathbb{O}_m^{\mathbf{\Gamma}})$.

Theorem 1.11 *Let $\mathbf{\Gamma}$ be a non self-dual Borel class.*

(1) Let X be an analytic space, and R be a Borel relation on X. Exactly one of the following holds:

(a) the relation R is a $\mathbf{\Gamma}$ subset of X^2,

(b) $(\mathbb{D},\mathbb{O}_m^{\mathbf{\Gamma}},\mathbb{O}_m^{\{\emptyset\}}\setminus\mathbb{O}_m^{\mathbf{\Gamma}})\leq(X,R,X^2\setminus R)$.

(2) A similar statement holds for graphs, with $(\mathbb{G}_m^{\mathbf{\Gamma}},\mathbb{G}_m^{\{\emptyset\}}\setminus\mathbb{G}_m^{\mathbf{\Gamma}})$ instead of $(\mathbb{O}_m^{\mathbf{\Gamma}},\mathbb{O}_m^{\{\emptyset\}}\setminus\mathbb{O}_m^{\mathbf{\Gamma}})$.

This last result can be extended to any non self-dual Wadge class of Borel sets. Note that there is no injectivity in Theorem 1.11. For instance, if $(\mathbb{D},\mathbb{G}_m^{\mathbf{\Gamma}},\mathbb{G}_m^{\{\emptyset\}}\setminus\mathbb{G}_m^{\mathbf{\Gamma}})\leq(\mathbb{S},\mathbb{G}_m^{\mathbf{\Gamma},\mathbf{a}},\mathbb{S}^2\setminus\mathbb{G}_m^{\mathbf{\Gamma},\mathbf{a}})$ with an injective witness f, then we can find $\varepsilon\in 2$ and $l\in\omega$ such that $f(\varepsilon,1^l\alpha)(0)=0$ and thus $f(\varepsilon,1^l\alpha)=0^\infty$ for any $\alpha\in 2^\omega$, contradicting the injectivity of f.

2 The general case

2.1 Preliminary results

We first extend Lemma 4.1 in [L5].

Lemma 2.1 *Let $\mathbf{\Gamma}$ be a class of sets closed under continuous pre-images, Y,Z be topological spaces, and R,S be a relation on Y,Z respectively.*

(a) If R is in $\mathbf{\Gamma}$, then the sections of R are also in $\mathbf{\Gamma}$.

(b) If S has vertical (resp., horizontal) sections in $\mathbf{\Gamma}$ and $(Y,R)\leq_c(Z,S)$, then the vertical (resp., horizontal) sections of R are also in $\mathbf{\Gamma}$.

(c) If S has countable vertical (resp., horizontal) sections and $(Y,R)\sqsubseteq_c(Z,S)$, then the vertical (resp., horizontal) sections of R are also countable.

Proof. (a) comes from the fact that if $y \in Y$, then the maps $i_y : y' \mapsto (y, y')$, $j_y : y' \mapsto (y', y)$ are continuous and satisfy $R_y = i_y^{-1}(R)$, $R^y = j_y^{-1}(R)$. The statements (b), (c) come from the facts that $R_y = f^{-1}(S_{f(y)})$, $R^y = f^{-1}(S^{f(y)})$. $\square$

We now extend Theorem 4.3 in [L5].

Notation. Let R be a relation on $\mathbb{D}$. We set, for $\varepsilon, \eta \in 2$, $R_{\varepsilon,\eta} := \big\{(\alpha, \beta) \in \mathbb{K}^2 \mid \big((\varepsilon, \alpha), (\eta, \beta)\big) \in R\big\}$.

Theorem 2.2 *Let $\mathbf{\Gamma}$ be a non self-dual Borel class of rank at least two, $(2^\omega, \mathbb{C})$ be $\mathbf{\Gamma}$-good, X be an analytic space, and R be a Borel relation on X. Exactly one of the following holds:*

(1) the relation R is a $\mathbf{\Gamma}$ subset of X^2,

(2) one of the following holds:

(a) the relation R has at least one section not in $\mathbf{\Gamma}$,

(b) there is a relation $\mathbb{R}$ on 2^ω such that $\mathbb{R} \cap \Delta(2^\omega) = \Delta(\mathbb{C})$ and $(2^\omega, \mathbb{R}) \sqsubseteq_c (X, R)$,

(c) there is a relation $\mathbb{R}$ on $\mathbb{D}$ such that $\mathbb{R}_{0,1} \cap \Delta(2^\omega) = \Delta(\mathbb{C})$ and $(\mathbb{D}, \mathbb{R}) \sqsubseteq_c (X, R)$.

Proof. We first note that (1) and (2) cannot hold simultaneously. Indeed, we argue by contradiction, which successively implies that R has sections in $\mathbf{\Gamma}$ by Lemma 2.1.(a), $\mathbb{R} \in \mathbf{\Gamma}\big((2^\omega)^2\big)$, $\mathbb{R} \in \mathbf{\Gamma}(\mathbb{D}^2)$, and $\mathbb{R} \cap \Delta(2^\omega) \in \mathbf{\Gamma}\big(\Delta(2^\omega)\big)$, $\mathbb{R}_{0,1} \cap \Delta(2^\omega) \in \mathbf{\Gamma}\big(\Delta(2^\omega)\big)$. Thus $\mathbb{C} \in \mathbf{\Gamma}(2^\omega)$, which is absurd.

Assume now that (1) and (2).(a) do not hold. Theorem 1.9 in [L5] gives an $f := (f_0, f_1) : 2^\omega \to X^2$ continuous with injective coordinates such that $\mathbb{C} = f^{-1}(R)$.

Case 1. $f[\neg\mathbb{C}] \subseteq \Delta(X)$.

Note that $f[2^\omega] \subseteq \Delta(X)$, by the choice of $\mathbb{C}$, which implies that $f_0 = f_1$. We next define $\mathbb{R} := (f_0 \times f_0)^{-1}(R)$. Note that $\mathbb{R} \cap \Delta(2^\omega) = \Delta(\mathbb{C})$ and $(2^\omega, \mathbb{R}) \sqsubseteq_c (X, R)$, with witness f_0.

Case 2. $f[\neg\mathbb{C}] \not\subseteq \Delta(X)$.

We may assume that f_0 and f_1 have disjoint ranges, by the choice of $\mathbb{C}$. We define $g : \mathbb{D} \to X$ by $g(\varepsilon, \alpha) := f_\varepsilon(\alpha)$. Note that g is injective continuous, $\big\{\big((0, \alpha), (1, \alpha)\big) \mid \alpha \in \mathbb{C}\big\} \subseteq (g \times g)^{-1}(R)$ and $\big\{\big((0, \alpha), (1, \alpha)\big) \mid \alpha \notin \mathbb{C}\big\} \subseteq (g \times g)^{-1}(\neg R)$. It remains to set $\mathbb{R} := (g \times g)^{-1}(R)$. $\square$

The following property is crucial in the sequel, as well as in [L5]. It is strongly related to condition (2) in Theorem 2.2.

Definition 2.3 *Let $f : \mathbb{K} \to \mathbb{K}$ be a function, and $\mathbb{C} \subseteq \mathbb{K}$. We say that f* **preserves** *$\mathbb{C}$ if $\mathbb{C} = f^{-1}(\mathbb{C})$.*

Lemma 2.4 *Let $f : \mathbb{K} \to \mathbb{K}$ be a function, $\mathbb{C} \subseteq \mathbb{K}$, and R be a relation on $\mathbb{K}$ with $R \cap \Delta(\mathbb{K}) = \Delta(\mathbb{C})$. Then the following are equivalent:*

(1) f preserves $\mathbb{C}$.

(2) $(f \times f)^{-1}(R) \cap \Delta(\mathbb{K}) = \Delta(\mathbb{C})$.

Proof. We just apply the definitions. $\square$

Finally, we will make a strong use of the following result (see page 433 in [Lo-SR1]). When $\xi = 2$, this result strengthens Hurewicz's theorem characterizing when a Borel subset of a Polish space is $\mathbf{\Sigma}^0_2$ (see 21.18 in [K1]). Indeed, B does not have to be the complement of A in what follows, which will be used several times. Moreover, this allows applications in analytic spaces.

Theorem 2.5 *(Louveau-Saint Raymond) Let $\xi \geq 1$ be a countable ordinal, $(\mathbb{K},\mathbb{C})$ be $\mathbf{\Sigma}^0_\xi$-good (or simply $\mathbb{C}\in\mathbf{\Pi}^0_\xi(2^\omega)\setminus\mathbf{\Sigma}^0_\xi(2^\omega)$ if $\xi\geq 3$), X be a Polish space, and A,B be disjoint analytic subsets of X. Then exactly one of the following holds:*

(a) the set A is separable from B by a $\mathbf{\Sigma}^0_\xi$ set,

(b) we can find an injective continuous $f:\mathbb{K}\to X$ such that $\mathbb{C}\subseteq f^{-1}(A)$ and $\neg\mathbb{C}\subseteq f^{-1}(B)$.

A first consequence of this is Theorem 1.11 (these two results extend to the non self-dual Wadge classes of Borel sets, see Theorem 5.2 in [Lo-SR2]).

Proof of Theorem 1.11. (1) As $\mathbb{C}\notin\mathbf{\Gamma}$, $\mathbb{O}^{\mathbf{\Gamma}}_m$ is not separable from $\mathbb{O}^{\{\emptyset\}}_m\setminus\mathbb{O}^{\mathbf{\Gamma}}_m$ by a set in $\mathbf{\Gamma}$, and (a), (b) cannot hold simultaneously. So assume that (a) does not hold. As X is separable metrizable, we may assume that X is a subset of the Polish space $[0,1]^\omega$ (see 4.14 in [K1]). Note that the analytic set R is not separable from the analytic set $X^2\setminus R$ by a $\mathbf{\Gamma}$ subset of $([0,1]^\omega)^2$. Theorem 2.5 provides a continuous $h:\mathbb{K}\to([0,1]^\omega)^2$ such that $\mathbb{C}\subseteq h^{-1}(R)$ and $\neg\mathbb{C}\subseteq h^{-1}(X^2\setminus R)$. Note that h takes values in X^2. We define $f:\mathbb{D}\to X$ by $f(\varepsilon,x):=h(x)(\varepsilon)$. Note that f is continuous. Moreover,

$$\big((0,x),(1,x)\big)\in\mathbb{O}^{\mathbf{\Gamma}}_m \Leftrightarrow x\in\mathbb{C}\Leftrightarrow h(x)=\big(h(x)(0),h(x)(1)\big)\in R\Leftrightarrow\big(f(0,x),f(1,x)\big)\in R,$$

which implies that (b) holds.

(2) We just have to consider the symmetrizations of the relations appearing in (1). □

2.2 Simplifications for the rank two

We will see that, for classes of rank two, the basic examples are contained in $\Delta(2^\omega)\cup Q^2_2$. The next proof is the first of our two proofs using effective descriptive set theory (see also Theorem 4.6).

Lemma 2.6 *Let R be a Borel relation on I_2 whose sections are separable from Q_2 by a $\mathbf{\Sigma}^0_2$ set. Then we can find a sequence $(R_n)_{n\in\omega}$ of relations closed in $I_2\times 2^\omega$ and $2^\omega\times I_2$, as well as $f:2^\omega\to 2^\omega$ injective continuous preserving Q_2 such that $(f\times f)^{-1}(R)\subseteq\bigcup_{n\in\omega}\ R_n$.*

Proof. In order to simplify the notation, we assume that R is a Δ^1_1 relation on 2^ω. Recall that we can find Π^1_1 sets $W\subseteq 2^\omega\times\omega$ and $C\subseteq 2^\omega\times\omega\times 2^\omega$ such that $\Delta^1_1(\alpha)(2^\omega)=\{C_{\alpha,n}\mid(\alpha,n)\in W\}$ for each $\alpha\in 2^\omega$ and $\{(\alpha,n,\beta)\in 2^\omega\times\omega\times 2^\omega\mid(\alpha,n)\in W\ \wedge\ (\alpha,n,\beta)\notin C\}$ is a Π^1_1 subset of $2^\omega\times\omega\times 2^\omega$ (see Section 2 in [Lo1]). Intuitively, W_α is the set of codes for the $\Delta^1_1(\alpha)$ subsets of 2^ω. We set $W_2:=\{(\alpha,n)\in W\mid C_{\alpha,n}$ is a $\mathbf{\Pi}^0_2\cap\Delta^1_1(\alpha)$ subset of $2^\omega\}$. Intuitively, $(W_2)_\alpha$ is the set of codes for the $\mathbf{\Pi}^0_2\cap\Delta^1_1(\alpha)$ subsets of 2^ω. By Section 2 in [Lo1], the set W_2 is Π^1_1. We set

$$P:=\{(\alpha,n)\in 2^\omega\times\omega\mid(\alpha,n)\in W_2\wedge R_\alpha\subseteq\neg C_{\alpha,n}\subseteq I_2\}.$$

Note that P is Π^1_1. Moreover, for each $\alpha\in I_2$, there is an $n\in\omega$ such that $(\alpha,n)\in P$, by Theorem 2.B' in [Lo1]. The Δ-selection principle provides a Δ^1_1-recursive map $f:2^\omega\to\omega$ such that $\big(\alpha,f(\alpha)\big)\in P$ if $\alpha\in I_2$ (see 4B.5 in [Mo]). We set $B:=\big\{(\alpha,\beta)\in I_2\times 2^\omega\mid\big(\alpha,f(\alpha),\beta\big)\notin C\big\}$. Note that B is a Δ^1_1 set with vertical sections in $\mathbf{\Sigma}^0_2$, and $R_\alpha\subseteq B_\alpha\subseteq I_2$ for each $\alpha\in I_2$. Theorem 3.6 in [Lo1] provides a finer Polish topology τ on 2^ω such that $B\in\mathbf{\Sigma}^0_2\big((2^\omega,\tau)\times 2^\omega\big)$. Note that the identity map from $(2^\omega,\tau)$ into 2^ω is a continuous bijection. By 15.2 in [K1], it is a Borel isomorphism.

By 11.5 in [K1], its inverse is Baire measurable. By 8.38 in [K1], there is a dense G_δ subset G of 2^ω on which τ coincides with the usual topology on 2^ω. In particular, $B\cap(G\times 2^\omega)\in\mathbf{\Sigma}^0_2(G\times 2^\omega)$ and we may assume that $G\subseteq I_2$. Note that G is not separable from Q_2 by a set in $\mathbf{\Sigma}^0_2$, by Baire's theorem. Theorem 2.5 provides $g:2^\omega\rightarrow 2^\omega$ injective continuous such that $I_2\subseteq g^{-1}(G)$ and $Q_2\subseteq g^{-1}(Q_2)$. Note that $(g\times g)^{-1}(B)$ is $\mathbf{\Sigma}^0_2$ in $I_2\times 2^\omega$ and contained in I_2^2. So, replacing B with $(g\times g)^{-1}(B)$ if necessary, we may assume that R is contained in a Borel set B which is $\mathbf{\Sigma}^0_2$ in $I_2\times 2^\omega$ and contained in I_2^2. Similarly, we may assume that R is contained in a Borel set D which is $\mathbf{\Sigma}^0_2$ in $2^\omega\times I_2$ and contained in I_2^2. Let $(B_p)_{p\in\omega}, (D_q)_{q\in\omega}$ be sequences of closed relations on 2^ω with $B=\bigcup_{p\in\omega}\ B_p\cap(I_2\times 2^\omega)$, $D=\bigcup_{q\in\omega}\ D_q\cap(2^\omega\times I_2)$ respectively. We set $R_{p,q}:=B_p\cap D_q\cap I_2^2$. Note that $R_{p,q}$ is closed in $I_2\times 2^\omega$ and $2^\omega\times I_2$ since, for example, $R_{p,q}:=B_p\cap D_q\cap(I_2\times 2^\omega)=B_p\cap D_q\cap(2^\omega\times I_2)$, and $R\subseteq\bigcup_{p,q\in\omega}\ R_{p,q}$. $\square$

Notation. We define a well-order $\leq_l$ of order type ω on $2^{<\omega}$ by $s\leq_l t\Leftrightarrow |s|<|t|\vee(|s|=|t|\wedge s\leq_{\text{lex}} t)$ and, as usual for linear orders, set $s<_l t\ \Leftrightarrow\ s\leq_l t\wedge s\neq t$. Let $b:(\omega,\leq)\rightarrow(2^{<\omega},\leq_l)$ be the increasing bijection, $\alpha_{n+1}:=b(n)10^\infty$, $\alpha_0:=0^\infty$. Note that $Q_2=\{\alpha_n\mid n\in\omega\}$. We then set $Q:=\{\emptyset\}\cup\{u1\mid u\in 2^{<\omega}\}$. Note that $Q_2=\{t0^\infty\mid t\in Q\}$.

Definition 2.7 *A* **Cantor set with dense finites** *is a copy C of 2^ω in 2^ω such that $Q_2\cap C$ is dense in C.*

Note that if C is a Cantor set with dense finites, then $Q_2\cap C$ is countable dense, and also co-dense, in C, which implies that $Q_2\cap C$ is $\mathbf{\Sigma}^0_2$ and not $\mathbf{\Pi}^0_2$ in C, by Baire's theorem.

Conventions. In the rest of Sections 2 and 3, we will perform a number of Cantor-like constructions. The following will always hold. We fix a Cantor set with dense finites C, and we want to construct $f:2^\omega\rightarrow C$ injective continuous preserving Q_2. We inductively construct a sequence $(n_t)_{t\in 2^{<\omega}}$ of positive natural numbers, and a sequence $(U_t)_{t\in 2^{<\omega}}$ of basic clopen subsets of C, satisfying the following conditions.

(1) $U_{t\varepsilon}\subseteq U_t$
(2) $\alpha_{n_t}\in U_t$
(3) $\text{diam}(U_t)\leq 2^{-|t|}$
(4) $U_{t0}\cap U_{t1}=\emptyset$
(5) $n_{t0}=n_t$
(6) $U_{t1}\cap\{\alpha_n\mid n\leq|t|\}=\emptyset$

Assume that this is done. Using (1)-(3), we define $f:2^\omega\rightarrow C$ by $\{f(\beta)\}:=\bigcap_{n\in\omega}\ U_{\beta|n}$, and f is injective continuous by (4). If $t\in Q$ and $\alpha=t0^\infty$, then $f(\alpha)=\alpha_{n_t}$ by (5), which implies that $Q_2\subseteq f^{-1}(Q_2)$. Condition (6) ensures that $I_2\subseteq f^{-1}(I_2)$, which implies that f preserves Q_2. For the first step of the induction, we choose $n_\emptyset\geq 1$ in such a way that $\alpha_{n_\emptyset}\in C$, and a basic clopen neighbourhood $U_\emptyset$ of $\alpha_{n_\emptyset}$. Condition (5) defines n_{t0}. It will also be convenient to set $s_t:=b(n_t-1)1$.

The next result is in the style of the Mycielski-Kuratowski Theorem (see 19.1 in [K1]).

Lemma 2.8 *Let $(R_n)_{n\in\omega}$ be a sequence of relations on I_2 which are closed in $I_2\times 2^\omega$ and in $2^\omega\times I_2$. Then there is $f:2^\omega\rightarrow 2^\omega$ injective continuous preserving Q_2 such that $\big(f(\alpha),f(\beta)\big)\notin\bigcup_{n\in\omega}\ R_n$ if $\alpha\neq\beta\in I_2$.*

Proof. We ensure (1)-(6) with $C=2^\omega$ and

$$(7)\ (U_{s1}\times U_{t\varepsilon})\cap(\bigcup_{n\leq l}R_n)=\emptyset \text{ if } s\neq t\in 2^l$$

Assume that this is done. If $\alpha\neq\beta\in I_2$, then we can find l with $\alpha|l\neq\beta|l$, and a strictly increasing sequence $(l_k)_{k\in\omega}$ of natural numbers bigger than l such that $\alpha(l_k)=1$ for each k. Condition (7) ensures that $U_{\alpha|(l_k+1)}\times U_{\beta|(l_k+1)}$ does not meet $\bigcup_{n\leq l_k}R_n$, which implies that $\big(f(\alpha),f(\beta)\big)\notin\bigcup_{n\in\omega}R_n$.

So it is enough to prove that the construction is possible. We first set $n_\emptyset:=1$ and $U_\emptyset:=2^\omega$. We choose $n_1\geq 1$ such that $\alpha_{n_1}\neq\alpha_{n_0}$, and U_0,U_1 disjoint with diameter at most 2^{-1} such that $\alpha_{n_\varepsilon}\in U_\varepsilon\subseteq 2^\omega\setminus\{\alpha_0\}$. Assume that $(n_t)_{|t|\leq l}$ and $(U_t)_{|t|\leq l}$ satisfying (1)-(7) have been constructed for some $l\geq 1$, which is the case for $l=1$. We set $F:=\{\alpha_{n_t}\mid t\in 2^l\}\cup\{\alpha_n\mid n\leq l\}$ and $L:=\bigcup_{n\leq l}R_n$.

Claim. *Let $U,U_0,\cdots,U_m$ be nonempty open subsets of 2^ω, $\gamma_i\in Q_2\cap U_i$, for each $i\leq m$, and F be a finite subset of Q_2 containing $\{\gamma_i\mid i\leq m\}$. Then we can find $\gamma\in Q_2\cap U$ and clopen subsets $V,V_0,\cdots,V_m$ of 2^ω with diameter at most 2^{-l-1} such that $\gamma\in V\subseteq U\setminus(F\cup V_0)$, $\gamma_i\in V_i\subseteq U_i$ and*

$$\big((V\times V_i)\cup(V_i\times V)\big)\cap L=\emptyset$$

for each $i\leq m$.

Indeed, fix $i\leq m$. Note that $I_2\times\{\gamma_i\}\subseteq I_2\times Q_2$ and $L\subseteq I_2^2$ are disjoint closed subsets of the zero-dimensional space $I_2\times 2^\omega$. This gives a clopen subset ${}_iC$ of $I_2\times 2^\omega$ with $I_2\times\{\gamma_i\}\subseteq {}_iC\subseteq\neg L$ (see 22.16 in [K1]). So for each $\beta\in I_2$ we can find a clopen subset ${}_{i,\beta}O$ of I_2 and a clopen subset ${}_{i,\beta}D$ of 2^ω such that $(\beta,\gamma_i)\in{}_{i,\beta}O\times{}_{i,\beta}D\subseteq{}_iC$ and ${}_{i,\beta}D\subseteq U_i$. As $I_2=\bigcup_{\beta\in I_2}{}_{i,\beta}O$, we can find a sequence $(\beta_p)_{p\in\omega}$ of points of I_2 with the property that $I_2=\bigcup_{p\in\omega}{}_{i,\beta_p}O$. We set ${}_{i,p}O:={}_{i,\beta_p}O\setminus(\bigcup_{q<p}{}_{i,\beta_q}O)$. Note that $({}_{i,p}O)_{p\in\omega}$ is a partition of I_2 into clopen sets. We set ${}_{i,p}D:={}_{i,\beta_p}D$. Note that ${}_{i,p}O\times{}_{i,p}D\subseteq{}_iC$. Let ${}_{i,p}W$ be an open subset of 2^ω with ${}_{i,p}O=I_2\cap{}_{i,p}W$. By 22.16 in [K1], there is a sequence $({}_{i,p}U)_{p\in\omega}$ of pairwise disjoint open subsets of 2^ω such that ${}_{i,p}U\subseteq{}_{i,p}W$ and $\bigcup_{p\in\omega}{}_{i,p}U=\bigcup_{p\in\omega}{}_{i,p}W$. Note that ${}_{i,p}O=I_2\cap{}_{i,p}U$.

Similarly, let C_i be a clopen subset of $2^\omega\times I_2$ with $\{\gamma_i\}\times I_2\subseteq C_i\subseteq\neg L$, $(O_{i,q})_{q\in\omega}$ be a partition of I_2 into clopen sets, $(D_{i,q})_{q\in\omega}$ be a sequence of clopen subsets of 2^ω, and $(U_{i,q})_{q\in\omega}$ be a sequence of pairwise disjoint open subsets of 2^ω such that $D_{i,q}\times O_{i,q}\subseteq C_i$, $\gamma_i\in D_{i,q}\subseteq U_i$ and $O_{i,q}=I_2\cap U_{i,q}$. Now pick $\beta\in I_2\cap U$, $p_i\in\omega$ with $\beta\in{}_{i,p_i}O$, and $q_i\in\omega$ with $\beta\in O_{i,q_i}$. We set

$$U':=U\cap\bigcap_{i\leq m}({}_{i,p_i}U\cap U_{i,q_i})\setminus F.$$

As U' is an open subset of 2^ω containing β, we can choose $\gamma\in Q_2\cap U'$, and a clopen subset V of 2^ω with diameter at most 2^{-l-1} such that $\gamma\in V\subseteq U'$. If $i\leq m$, then we choose a clopen neighbourhood V_i of γ_i with diameter at most 2^{-l-1} such that $V_i\subseteq{}_{i,p_i}D\cap D_{i,q_i}$. As $\gamma\neq\gamma_0$, we can ensure that V does not meet V_0. For example, note that $(I_2\cap V)\times V_i\subseteq(I_2\cap{}_{i,p_i}U)\times{}_{i,p_i}D\subseteq{}_iC\subseteq\neg L$ since $I_2\cap{}_{i,p_i}U={}_{i,p_i}O$. $\diamond$

We put a linear order on the set 2^l of binary sequences of length l, which implies that 2^l is enumerated injectively by $\{t_j\mid j<2^l\}$. Let $j<2^l$. Note that (5) defines n_{t_j0}.

We construct, by induction on $j<2^l$, n_{t_j1}, and sequences $(U^j_{t0})_{t\in 2^l}$, $(U^j_{t_i1})_{i\leq j}$ of clopen subsets of 2^ω satisfying the following:

(a) $U^{j+1}_{t0}\subseteq U^j_{t0}\subseteq U_t \wedge U^{j+1}_{t_i1}\subseteq U^j_{t_i1}\subseteq U_{t_i}$
(b) $\alpha_{n_t}\in U^j_{t0} \wedge \alpha_{n_{t_i1}}\in U^j_{t_i1}$
(c) diam(U^j_{t0}), diam$(U^j_{t_i1})\leq 2^{-l-1}$
(d) $U^j_{t_j0}\cap U^j_{t_j1}=\emptyset$
(e) $U^j_{t_j1}\cap F=\emptyset$
(f) $\Big(\big(U^j_{t_j1}\times(U^j_{t0}\cup U^j_{t_i1})\big)\cup\big((U^j_{t0}\cup U^j_{t_i1})\times U^j_{t_j1}\big)\Big)\cap L=\emptyset$ if $i<j$

In order to do this, we first apply the claim to $U:=U_{t_0}$ and a family $(\gamma_i)_{i\leq m}$ of elements of Q_2 enumerating $\{\alpha_{n_t}\mid t\in 2^l\}$ in such a way that $\gamma_0=\alpha_{n_{t_0}}$. The corresponding family of open sets enumerates $\{U_t\mid t\in 2^l\}$. The claim provides $\alpha_{n_{t_01}}\in Q_2\cap U_{t_0}$ and clopen subsets $U^0_{t_01}, U^0_{t0}$ of 2^ω with diameter at most 2^{-l-1} such that $\alpha_{n_{t_01}}\in U^0_{t_01}\subseteq U_{t_0}\setminus(F\cup U^0_{t_00})$, $\alpha_{n_t}\in U^0_{t0}\subseteq U_t$ and

$$\big((U^0_{t_01}\times U^0_{t0})\cup(U^0_{t0}\times U^0_{t_01})\big)\cap L=\emptyset$$

for each $t\in 2^l$. Assume then that $j<2^l-1$ and $(n_{t_k1})_{k\leq j}$, $(U^k_{t0})_{t\in 2^l,k\leq j}$ and $(U^k_{t_i1})_{i\leq k\leq j}$ satisfying (a)-(f) have been constructed, which is the case for $j=0$.

We now apply the claim to $U:=U_{t_{j+1}}$ and a family $(\gamma_i)_{i\leq m}$ of elements of Q_2 enumerating $\{\alpha_{n_t}\mid t\in 2^l\}\cup\{\alpha_{n_{t_i1}}\mid i\leq j\}$ in such a way that $\gamma_0=\alpha_{n_{t_{j+1}}}$. The corresponding family of open sets enumerates $\{U^j_{t0}\mid t\in 2^l\}\cup\{U^j_{t_i1}\mid i\leq j\}$. The claim provides $\alpha_{n_{t_{j+1}1}}\in Q_2\cap U_{t_{j+1}}$ and clopen subsets $U^{j+1}_{t_{j+1}1}, U^{j+1}_{t0}, U^{j+1}_{t_i1}$ of 2^ω of diameter at most 2^{-l-1} with the properties that $\alpha_{n_{t_{j+1}1}}$ is in $U^{j+1}_{t_{j+1}1}\subseteq U_{t_{j+1}}\setminus(F\cup U^{j+1}_{t_{j+1}0})$, $\alpha_{n_t}\in U^{j+1}_{t0}\subseteq U^j_{t0}$, $\alpha_{n_{t_i1}}\in U^{j+1}_{t_i1}\subseteq U^j_{t_i1}$ and, when $i\leq j$,

$$\Big(\big(U^{j+1}_{t_{j+1}1}\times(U^{j+1}_{t0}\cup U^{j+1}_{t_i1})\big)\cup\big((U^{j+1}_{t0}\cup U^{j+1}_{t_i1})\times U^{j+1}_{t_{j+1}1}\big)\Big)\cap L=\emptyset.$$

It remains to set $U_{t\varepsilon}:=U^{2^l-1}_{t\varepsilon}$. □

Corollary 2.9 *Let R be a locally countable Borel relation on 2^ω. Then we can find $f:2^\omega\to 2^\omega$ injective continuous preserving Q_2 such that $R':=(f\times f)^{-1}(R)\subseteq\Delta(2^\omega)\cup Q_2^2$.*

Proof. As Q_2 is countable and R is locally countable, the set $C:=\bigcup_{\alpha\in Q_2}(R_\alpha\cup R^\alpha)$ is countable. We set $G:=I_2\setminus C$. Note that $G\subseteq I_2$ is a non-meager subset of 2^ω having the Baire property. Lemma 7.2 in [L5] provides $f:2^\omega\to 2^\omega$ injective continuous such that $f[I_2]\subseteq G$ and $f[Q_2]\subseteq Q_2$. This proves that we may assume that $R\cap\big((I_2\times Q_2)\cup(Q_2\times I_2)\big)=\emptyset$. By Lemma 2.6 applied to $R\cap I_2^2$, we may assume that there is a sequence $(R_n)_{n\in\omega}$ of relations closed in $I_2\times 2^\omega$ and $2^\omega\times I_2$ such that $R\cap I_2^2\subseteq\bigcup_{n\in\omega}R_n$. It remains to apply Lemma 2.8. □

Corollary 2.9 leads to the following.

Definition 2.10 *Let $\mathbf{\Gamma}$ be a non self-dual Borel class of rank two, and $(2^\omega,\mathbb{C})$ be $\mathbf{\Gamma}$-good. A relation $\mathcal{R}$ on 2^ω is* **diagonally complex** *if it it satisfies the following:*

(1) $\mathcal{R}\cap\Delta(2^\omega)=\Delta(\mathbb{C})$,

(2) $\mathcal{R}\subseteq\Delta(\mathbb{C})\cup Q_2^2$.

Note that a diagonally complex relation is not in $\mathbf{\Gamma}$ by (1), and locally countable Borel by (2).

Notation. We set, for any digraph $\mathcal{D}$ on Q, $\mathcal{R}_{\mathcal{D}} := \Delta(\mathbb{C}) \cup \{(s0^\infty, t0^\infty) \mid (s,t) \in \mathcal{D}\}$. Note that any diagonally complex relation is of the form $\mathcal{R}_{\mathcal{D}}$, for some digraph $\mathcal{D}$ on Q.

In our future Cantor-like constructions, the definition of n_{t1} will be by induction on $\leq_l$, except where indicated. Corollary 2.9 simplifies the structure of the locally countable Borel relations on 2^ω. Some further simplification is possible when the sections are nowhere dense.

Lemma 2.11 *Let R be a relation on 2^ω with nowhere dense sections. Then there is $f : 2^\omega \to 2^\omega$ injective continuous preserving Q_2 such that $\big(f(\alpha), f(\beta)\big) \notin R$ if $\alpha \neq \beta \in Q_2$.*

Proof. We ensure (1)-(6) with $C = 2^\omega$ and

$$(7)\ (\alpha_{n_s}, \alpha_{n_t}), (\alpha_{n_t}, \alpha_{n_s}) \notin R \text{ if } t \in Q \wedge s \in 2^{|t|} \wedge s <_{\text{lex}} t$$

Assume that this is done. If $\alpha \neq \beta \in Q_2$, then, without loss of generality, $\alpha <_{\text{lex}} \beta$ and there are initial segments s, t of α, β respectively satisfying the assumption in (7). Condition (7) ensures that $\big(f(\alpha), f(\beta)\big) \notin R$.

So it is enough to prove that the construction is possible. We first set $n_\emptyset := 1$ and $U_\emptyset := 2^\omega$. Assume that $(n_t)_{|t| \leq l}$ and $(U_t)_{|t| \leq l}$ satisfying (1)-(7) have been constructed, which is the case for $l = 0$. Fix $t \in 2^l$. We choose $n_{t1} \geq 1$ such that

$$\alpha_{n_{t1}} \in U_t \setminus (\{\alpha_{n_t}\} \cup \{\alpha_n \mid n \leq l\} \cup \bigcup_{s \in 2^{l+1}, s <_{\text{lex}} t1} \overline{R_{\alpha_{n_s}}} \cup \overline{R^{\alpha_{n_s}}}),$$

which exists since R has nowhere dense sections. We then choose a clopen neighbourhood with small diameter $U_{t\varepsilon}$ of $\alpha_{n_{t\varepsilon}}$ contained in U_t, ensuring (4) and (6). □

Corollary 2.12 *Let $\mathcal{R}$ be a diagonally complex relation.*

(1) If $\mathcal{R}$ has nowhere dense sections, then $(2^\omega, \mathcal{R}_\emptyset) \sqsubseteq_c (2^\omega, \mathcal{R})$.

(2) If $Q_2^2 \setminus \mathcal{R}$ has nowhere dense sections, then $(2^\omega, \mathcal{R}_{\neq}) \sqsubseteq_c (2^\omega, \mathcal{R})$.

Proof. Lemma 2.11 gives $f : 2^\omega \to 2^\omega$ injective continuous preserving Q_2 such that $\big(f(\alpha), f(\beta)\big)$ is not in $\mathcal{R}$ (resp., in $\mathcal{R}$) if $\alpha \neq \beta \in Q_2$. Note that $(f \times f)^{-1}(\mathcal{R}) = \mathcal{R}_\emptyset$ (resp., $(f \times f)^{-1}(\mathcal{R}) = \mathcal{R}_{\neq}$). □

Theorem 1.2 provides a basis. Theorem 2.15 to come provides another one, and is a consequence of Corollary 2.9.

Notation. Let $\mathbf{\Gamma}$ be a non self-dual Borel class of rank at least two, and $(2^\omega, \mathbb{C})$ be $\mathbf{\Gamma}$-good. We set $S_0 := \mathbb{C}$, $S_1 := \emptyset$, $S_2 := 2^\omega \setminus \mathbb{C}$, $S_3 := 2^\omega$. The next result motivates the introduction of these sets.

Lemma 2.13 *Let $\mathbf{\Gamma}$ be a non self-dual Borel class of rank at least two, $(2^\omega, \mathbb{C})$ be $\mathbf{\Gamma}$-good, and B be a Borel subset of 2^ω. Then we can find $j \in 4$ and $f : 2^\omega \to 2^\omega$ injective continuous preserving $\mathbb{C}$ such that $f^{-1}(B) = S_j$.*

Proof. Note that since $\mathbb{C} \in \check{\mathbf{\Gamma}}(2^\omega) \setminus \mathbf{\Gamma}(2^\omega)$, either $\mathbb{C} \setminus B$ is not separable from $\neg\mathbb{C}$ by a set in $\mathbf{\Gamma}$, or $\mathbb{C} \cap B$ is not separable from $\neg\mathbb{C}$ by a set in $\mathbf{\Gamma}$, because $\mathbf{\Gamma}$ is closed under finite unions. Assume, for example, that the first case occurs.

Similarly, either $\mathbb{C}\setminus B$ is not separable from $(\neg\mathbb{C})\cap(\neg B)$ by a set in $\mathbf{\Gamma}$, or $\mathbb{C}\setminus B$ is not separable from $(\neg\mathbb{C})\cap B$ by a set in $\mathbf{\Gamma}$, because $\mathbf{\Gamma}$ is closed under finite intersections. This shows that one of the following cases occurs:

- $\mathbb{C}\setminus B$ is not separable from $(\neg\mathbb{C})\cap(\neg B)$ by a set in $\mathbf{\Gamma}$,
- $\mathbb{C}\setminus B$ is not separable from $(\neg\mathbb{C})\cap B$ by a set in $\mathbf{\Gamma}$,
- $\mathbb{C}\cap B$ is not separable from $(\neg\mathbb{C})\cap(\neg B)$ by a set in $\mathbf{\Gamma}$,
- $\mathbb{C}\cap B$ is not separable from $(\neg\mathbb{C})\cap B$ by a set in $\mathbf{\Gamma}$.

Assume, for example, that we are in the first of these four cases. By Theorem 2.5, there is $f:2^\omega\to 2^\omega$ injective continuous such that $\mathbb{C}\subseteq f^{-1}(\mathbb{C}\setminus B)$ and $\neg\mathbb{C}\subseteq f^{-1}\big((\neg\mathbb{C})\cap(\neg B)\big)$. In this case, $f^{-1}(B)=\emptyset=S_1$. In the other cases, $f^{-1}(B)=S_2,S_0,S_3$, respectively. So we proved that $f^{-1}(B)=S_j$ for some $j\in 4$. □

Corollary 2.14 *Let $\mathbf{\Gamma}$ be a non self-dual Borel class of rank at least two, $(2^\omega,\mathbb{C})$ be $\mathbf{\Gamma}$-good, and R be a Borel relation on $\mathbb{D}$. Then there is $f:2^\omega\to 2^\omega$ injective continuous preserving $\mathbb{C}$ such that $(f\times f)^{-1}(R_{\varepsilon,\eta})\cap\Delta(2^\omega)\in\{\Delta(S_j)\mid j\leq 3\}$ for each $\varepsilon,\eta\in 2$.*

Proof. We set, for $\varepsilon,\eta\in 2$, $E_{\varepsilon,\eta}:=\{\alpha\in 2^\omega\mid(\alpha,\alpha)\in R_{\varepsilon,\eta}\}$. Note that $E_{\varepsilon,\eta}$ is a Borel subset of 2^ω and $R_{\varepsilon,\eta}\cap\Delta(2^\omega)=\Delta(E_{\varepsilon,\eta})$. Now fix $\varepsilon,\eta\in 2$. Lemma 2.13 provides $j\in 4$ and $g:2^\omega\to 2^\omega$ injective continuous preserving $\mathbb{C}$ such that $(g\times g)^{-1}(E_{\varepsilon,\eta})=S_j$. We just have to apply this for each $\varepsilon,\eta\in 2$.□

Theorem 2.15 *Let $\mathbf{\Gamma}$ be a non self-dual Borel class of rank two, $(2^\omega,\mathbb{C})$ be $\mathbf{\Gamma}$-good, X be an analytic space, and R be a locally countable Borel relation on X whose sections are in $\mathbf{\Gamma}$. Exactly one of the following holds.*

(1) the relation R is a $\mathbf{\Gamma}$ subset of X^2,

(2) one of the following holds:

(a) there is a diagonally complex relation $\mathbb{R}$ on 2^ω such that $(2^\omega,\mathbb{R})\sqsubseteq_c(X,R)$,

(b) there is a relation $\mathbb{R}$ on $\mathbb{D}$ such that, for each $\varepsilon,\eta\in 2$, if $\mathbb{R}_{\varepsilon,\eta}\cap\Delta(2^\omega)=\Delta(E_{\varepsilon,\eta})$, then

(i) $E_{0,1}=\mathbb{C}=S_0$, $E_{1,0}\in\{S_j\mid j\leq 3\}$, and $E_{\varepsilon,\varepsilon}\in\{S_j\mid 1\leq j\leq 3\}$,

(ii) $\mathbb{R}_{\varepsilon,\eta}\subseteq\Delta(E_{\varepsilon,\eta})\cup Q_2^2$ (in particular, $\mathbb{R}_{0,1}$ is diagonally complex),

and $(\mathbb{D},\mathbb{R})\sqsubseteq_c(X,R)$.

Proof. By Theorem 2.2, (1) and (2) cannot hold simutaneously. Assume that (1) does not hold. By Theorem 2.2 again, one of the following holds.

(a) There is a relation $\mathbb{R}$ on 2^ω such that $\mathbb{R}\cap\Delta(2^\omega)=\Delta(\mathbb{C})$ and $(2^\omega,\mathbb{R})\sqsubseteq_c(X,R)$. By Corollary 2.9 we may assume that $\mathbb{R}\subseteq\Delta(2^\omega)\cup Q_2^2$, which implies that $\mathbb{R}$ is a diagonally complex relation.

(b) There is a relation $\mathbb{R}'$ on $\mathbb{D}$ such that $\mathbb{R}'_{0,1}\cap\Delta(2^\omega)=\Delta(\mathbb{C})$ and $(\mathbb{D},\mathbb{R}')\sqsubseteq_c(X,R)$. Note that $\mathbb{R}'_{\varepsilon,\eta}$ is a locally countable Borel relation on 2^ω, for each $\varepsilon,\eta\in 2$. Corollary 2.9 provides $g':2^\omega\to 2^\omega$ injective continuous preserving $\mathbb{C}$ such that $(g'\times g')^{-1}(\bigcup_{\varepsilon,\eta\in 2}\mathbb{R}'_{\varepsilon,\eta})\subseteq\Delta(2^\omega)\cup Q_2^2$. We define $h:\mathbb{D}\to\mathbb{D}$ by $h(\varepsilon,\alpha):=\big(\varepsilon,g'(\alpha)\big)$ and set $\mathbb{R}=(h\times h)^{-1}(\mathbb{R}')$. Note that $\mathbb{R}_{\varepsilon,\eta}\subseteq\Delta(2^\omega)\cup Q_2^2$. By Corollary 2.14, we may assume that $\mathbb{R}_{\varepsilon,\eta}\cap\Delta(2^\omega)\in\{\Delta(S_j)\mid j\leq 3\}$. We are done since we are reduced to Case (a) if $\mathbb{R}_{\varepsilon,\varepsilon}\cap\Delta(2^\omega)=\Delta(\mathbb{C})$. □

2.3 Proof of Theorem 1.2

We now introduce a first antichain basis.

Notation. Let $\mathbf{\Gamma}$ be a non self-dual Borel class of rank at least two, and $(2^\omega,\mathbb{C})$ be $\mathbf{\Gamma}$-good. We set

$$P:=\big\{t\in 4^{(2^2)}\mid t(0,0),t(1,1)\neq 0\wedge t(0,1)=0\wedge \big(t(1,0)=0\Rightarrow t(0,0)\leq t(1,1)\big)\big\}$$

and, for $t\in P$, $\mathbb{R}^{\mathbf{\Gamma}}_t:=\big\{\big((\varepsilon,x),(\eta,x)\big)\in\mathbb{D}^2\mid x\in S_{t(\varepsilon,\eta)}\big\}$. We order 2^2 lexicographically, which implies that, for example, $\mathbb{E}^{\mathbf{\Gamma}}_3=\mathbb{R}^{\mathbf{\Gamma}}_{3,0,0,3}$. Note that $\mathbb{G}^{\mathbf{\Gamma}}_m=\mathbb{R}^{\mathbf{\Gamma}}_{1,0,0,1}$. Finally,

$$\mathcal{A}^{\mathbf{\Gamma}}:=\big\{\big(2^\omega,\Delta(\mathbb{C})\big)\big\}\cup\{(\mathbb{D},\mathbb{R}^{\mathbf{\Gamma}}_t)\mid t\in P\}.$$

Note that the sections of the elements of $\mathcal{A}^{\mathbf{\Gamma}}$ have cardinality at most two, and are in particular closed.

Lemma 2.16 *Let $\mathbf{\Gamma}$ be a non self-dual Borel class of rank at least two. Then $\mathcal{A}^{\mathbf{\Gamma}}$ is a 34 element $\leq_c$-antichain.*

Proof. Let $(\mathbb{X},\mathbb{R})\neq(\mathbb{X}',\mathbb{R}')$ in $\mathcal{A}^{\mathbf{\Gamma}}$. We argue by contradiction, which gives $f:\mathbb{X}\to\mathbb{X}'$ continuous. Assume first that $(\mathbb{X},\mathbb{R}),(\mathbb{X}',\mathbb{R}')$ are of the form $(\mathbb{D},\mathbb{R}^{\mathbf{\Gamma}}_t),(\mathbb{D},\mathbb{R}^{\mathbf{\Gamma}}_{t'})$ respectively, which implies that $f(\varepsilon,\alpha)$ is of the form $\big(f_0(\varepsilon,\alpha),f_1(\varepsilon,\alpha)\big)\in 2\times 2^\omega$.

Let us prove that $f_0(0,\alpha)\neq f_0(1,\alpha)$ if $\alpha\in\mathbb{C}$. We argue by contradiction, which gives $l\in\omega$ such that $f_0(0,\beta)=f_0(1,\beta)=:\varepsilon$ if $\beta\in N_{\alpha|l}$, by the continuity of f_0. This also gives continuous maps $g_\eta:N_{\alpha|l}\to 2^\omega$ such that $f_1(\eta,\beta)=g_\eta(\beta)$ if $\beta\in N_{\alpha|l}$. If $\beta\in\mathbb{C}\cap N_{\alpha|l}$, then $\big((0,\beta),(1,\beta)\big)\in\mathbb{R}$, which implies that $\big(f(0,\beta),f(1,\beta)\big)=\Big(\big(\varepsilon,g_0(\beta)\big),\big(\varepsilon,g_1(\beta)\big)\Big)\in\mathbb{R}'$, $g_0(\beta)=g_1(\beta)$ and $g_0=g_1=:g$ by the continuity of g_0,g_1. Note that there is $j\in\{1,2,3\}$ such that $\mathbb{C}\cap N_{\alpha|l}=g^{-1}(S_j)\cap N_{\alpha|l}$, which contradicts the choice of $\mathbb{C}$.

Fix $\alpha\in\mathbb{C}$. Note that there is $l\in\omega$ such that $\varepsilon_0:=f_0(0,\beta)\neq f_0(1,\beta)$ if $\beta\in N_{\alpha|l}$, by the continuity of f_0. There are $g_\eta:N_{\alpha|l}\to 2^\omega$ continuous such that $f_1(\eta,\beta)=g_\eta(\beta)$ if $\beta\in N_{\alpha|l}$. If $\beta\in\mathbb{C}\cap N_{\alpha|l}$, then $\big((0,\beta),(1,\beta)\big)\in\mathbb{R}$, which implies that $\big(f(0,\beta),f(1,\beta)\big)=\Big(\big(\varepsilon_0,g_0(\beta)\big),\big(1-\varepsilon_0,g_1(\beta)\big)\Big)\in\mathbb{R}'$, $g_0(\beta)=g_1(\beta)$ and $g_0=g_1=:g$ by the continuity of g_0,g_1. Note that

$$\mathbb{C}\cap N_{\alpha|l}=g^{-1}(S_{t'(\varepsilon_0,1-\varepsilon_0)})\cap N_{\alpha|l},$$

which implies that $t'(\varepsilon_0,1-\varepsilon_0)=0$ by the choice of $\mathbb{C}$ and $\mathbb{C}\cap N_{\alpha|l}=g^{-1}(\mathbb{C})\cap N_{\alpha|l}$. If $\varepsilon_0=0$, then

$$S_{t(\varepsilon,\eta)}\cap N_{\alpha|l}=g^{-1}(S_{t'(\varepsilon,\eta)})\cap N_{\alpha|l}=S_{t'(\varepsilon,\eta)}\cap N_{\alpha|l}$$

for $\varepsilon,\eta\in 2$, which implies that $t=t'$ by the choice of $\mathbb{C}$. Thus $\varepsilon_0=1$ and

$$S_{t(\varepsilon,\eta)}\cap N_{\alpha|l}=g^{-1}(S_{t'(1-\varepsilon,1-\eta)})\cap N_{\alpha|l}=S_{t'(1-\varepsilon,1-\eta)}\cap N_{\alpha|l}$$

for $\varepsilon,\eta\in 2$, which implies that $t(\varepsilon,\eta)=t'(1-\varepsilon,1-\eta)$ if $\varepsilon,\eta\in 2$, by the choice of $\mathbb{C}$. In particular, note that $t'(1,0)=t(0,1)=0=t'(0,1)=t(1,0)$, $t'(0,0),t'(1,1)\neq 0$,

$$t(1,1)=t'(0,0)\leq t'(1,1)=t(0,0),$$

$t(0,0),t(1,1)\neq 0$ and $t'(1,1)=t(0,0)\leq t(1,1)=t'(0,0)$, which implies that $t=t'$ again.

We now have to consider $\big(2^\omega,\Delta(\mathbb{C})\big)$. Assume first that $\big(2^\omega,\Delta(\mathbb{C})\big)\leq_c(\mathbb{D},\mathbb{R}_t^{\mathbf{\Gamma}})$, with $(\mathbb{D},\mathbb{R}_t)$ in $\mathcal{A}^{\mathbf{\Gamma}}$ and witness f. Let $\alpha\in\mathbb{C}$. Then we can find $\varepsilon_0\in 2$ with $f_0(\alpha)=\varepsilon_0$, and $l\in\omega$ such that $f_0(\beta)=\varepsilon_0$ if $\beta\in N_{\alpha|l}$, by the continuity of f_0. Note that $\mathbb{C}\cap N_{\alpha|l}=f_1^{-1}(S_{t(\varepsilon_0,\varepsilon_0)})\cap N_{\alpha|l}$, which implies that $t(\varepsilon_0,\varepsilon_0)=0$ by the choice of $\mathbb{C}$, which contradicts the definition of $\mathcal{A}^{\mathbf{\Gamma}}$.

Assume now that $(\mathbb{D},\mathbb{R}_t^{\mathbf{\Gamma}})\leq_c\big(2^\omega,\Delta(\mathbb{C})\big)$, with $(\mathbb{D},\mathbb{R}_t)$ in $\mathcal{A}^{\mathbf{\Gamma}}$ and witness f. Let $\alpha\in\mathbb{C}$. Then $\big((0,\alpha),(1,\alpha)\big)\in\mathbb{R}_t^{\mathbf{\Gamma}}$, which implies that $f(0,\alpha)=f(1,\alpha)\in\mathbb{C}$. Then $g(\beta):=f(0,\beta)=f(1,\beta)$ for each $\beta\in 2^\omega$, by the choice of $\mathbb{C}$. We set, for $\varepsilon,\eta\in 2$, $\mathbb{R}_{\varepsilon,\eta}:=\big\{(\alpha,\beta)\in 2^\omega\times 2^\omega\mid\big((\varepsilon,\alpha),(\eta,\beta)\big)\in\mathbb{R}_t^{\mathbf{\Gamma}}\big\}$. Note that

$$(\alpha,\beta)\in\mathbb{R}_{\varepsilon,\eta}\Leftrightarrow\big(g(\alpha),g(\beta)\big)\in\Delta(\mathbb{C}),$$

which implies that $\mathbb{R}_{0,0}=\mathbb{R}_{0,1}$, which contradicts the definition of $\mathcal{A}^{\mathbf{\Gamma}}$. □

The next result provides the basis part of Theorem 1.2.

Lemma 2.17 *Let $\mathbf{\Gamma}$ be a non self-dual Borel class of rank at least two, X be an analytic space, and R be a locally countable Borel relation on X whose sections are in $\Delta(\mathbf{\Gamma})$. Exactly one of the following holds:*

(a) the relation R is a $\mathbf{\Gamma}$ subset of X^2,

(b) there is $(\mathbb{X},\mathbb{R})\in\mathcal{A}^{\mathbf{\Gamma}}$ such that $(\mathbb{X},\mathbb{R})\sqsubseteq_c(X,R)$.

Proof. By Theorem 2.2, (a) and (b) cannot hold simutaneously. Assume that (a) does not hold. By Theorem 2.2 again, one of the following holds.

(1) There is a relation $\mathbb{R}$ on 2^ω such that $\mathbb{R}\cap\Delta(2^\omega)=\Delta(\mathbb{C})$ and $(2^\omega,\mathbb{R})\sqsubseteq_c(X,R)$. As R is locally countable Borel, so is $\mathbb{R}$ by Lemma 2.1, which implies that we can apply Corollary 3.3 in [L5] if the rank of $\mathbf{\Gamma}$ is at least three. This gives $g:2^\omega\to 2^\omega$ injective continuous such that g preserves $\mathbb{C}$, and $\big(g(\alpha),g(\beta)\big)\notin\mathbb{R}$ if $\alpha\neq\beta$. Note that g is a witness for the fact that $\big(2^\omega,\Delta(\mathbb{C})\big)\sqsubseteq_c(2^\omega,\mathbb{R})$, which implies that $\big(2^\omega,\Delta(\mathbb{C})\big)\sqsubseteq_c(X,R)$. If the rank of $\mathbf{\Gamma}$ is two, then by Corollary 2.9 we may assume that $\mathbb{R}\subseteq\Delta(2^\omega)\cup Q_2^2$. As the sections of R are countable and $\mathbf{\Pi}_2^0$, so are those of $\mathbb{R}$. In particular, $\mathbb{R}$ has nowhere dense sections. By Corollary 2.12, we may assume that $\mathbb{R}=\Delta(\mathbb{C})$, which implies that, here again, $\big(2^\omega,\Delta(\mathbb{C})\big)\sqsubseteq_c(X,R)$.

(2) There is a relation $\mathbb{R}$ on $\mathbb{D}$ such that $\mathbb{R}_{0,1}\cap\Delta(2^\omega)=\Delta(\mathbb{C})$ and $(\mathbb{D},\mathbb{R})\sqsubseteq_c(X,R)$. Note that $\mathbb{R}_{\varepsilon,\eta}$ is a locally countable Borel relation on 2^ω. If the rank of $\mathbf{\Gamma}$ is at least three, then Corollary 3.3 in [L5] provides $g:2^\omega\to 2^\omega$ injective continuous such that g preserves $\mathbb{C}$, and $\big(g(\alpha),g(\beta)\big)\notin\bigcup_{\varepsilon,\eta\in 2}\ \mathbb{R}_{\varepsilon,\eta}$ if $\alpha\neq\beta$. If the rank of $\mathbf{\Gamma}$ is two, then Corollary 2.9 provides $g'':2^\omega\to 2^\omega$ injective continuous preserving $\mathbb{C}$ such that $\mathbb{R}'':=(g''\times g'')^{-1}(\bigcup_{\varepsilon,\eta\in 2}\ \mathbb{R}_{\varepsilon,\eta})\subseteq\Delta(2^\omega)\cup Q_2^2$. As the sections of R are countable and $\mathbf{\Pi}_2^0$, so are those of $\mathbb{R}''$. In particular, $\mathbb{R}''$ has nowhere dense sections. By Lemma 2.11, we may assume that $\mathbb{R}''\subseteq\Delta(2^\omega)$. So we may assume that g exists in both cases.

We define $h:\mathbb{D}\to\mathbb{D}$ by $h(\varepsilon,\alpha):=\big(\varepsilon,g(\alpha)\big)$. Note that h is injective and continuous. We then set $\mathbb{R}':=(h\times h)^{-1}(\mathbb{R})$. Repeating the notation above, $\mathbb{R}'_{\varepsilon,\eta}\subseteq\Delta(2^\omega)$ by the property of g, and h is a witness for the fact that $(\mathbb{D},\mathbb{R}')\sqsubseteq_c(\mathbb{D},\mathbb{R})$. This means that we may assume that $\mathbb{R}_{\varepsilon,\eta}\subseteq\Delta(2^\omega)$, for $\varepsilon,\eta\in 2$, and that $\mathbb{R}_{0,1}=\Delta(\mathbb{C})=\Delta(S_0)$. By Corollary 2.14, we may assume that $\mathbb{R}_{\varepsilon,\eta}\in\{\Delta(S_j)\mid j\leq 3\}$. Note that if $\mathbb{R}_{\varepsilon,\varepsilon}=\Delta(S_0)$ for some $\varepsilon\in 2$, then $\big(2^\omega,\Delta(\mathbb{C})\big)\sqsubseteq_c(X,R)$. So we may assume that $\mathbb{R}_{\varepsilon,\varepsilon}=\Delta(S_j)$ for some $j\in\{1,2,3\}$. Finally, $(\mathbb{D},\mathbb{R}_{i,0,0,j}^{\mathbf{\Gamma}})\sqsubseteq_c(\mathbb{D},\mathbb{R}_{j,0,0,i}^{\mathbf{\Gamma}})$ if $i>j>0$ with witness $(\varepsilon,\alpha)\mapsto(1-\varepsilon,\alpha)$. So (b) holds. □

Proof of Theorem 1.2. For (1) and (3), we apply Lemmas 2.16 and 2.17. For (2), we use the closure properties of $\sqsubseteq_c$ and the fact that $\mathbb{G}_m^{\mathbf{\Gamma}}$ is the only graph in $\mathcal{A}^{\mathbf{\Gamma}}$. $\square$

3 Rank two

In this section, $\mathbf{\Gamma}$ is a non self-dual Borel class of rank two.

3.1 Diagonally complex relations: minimality and comparability

Up to restrictions to Cantor sets, two comparable diagonally complex relations are bi-reducible.

Lemma 3.1 *Let $\mathcal{R},\mathcal{R}_*$ be diagonally complex, and K be a Cantor set with dense finites and the property that $(K,\mathcal{R}_*\cap K^2)\leq_c(2^\omega,\mathcal{R})$. Then $(C,\mathcal{R}\cap C^2)\sqsubseteq_c(K,\mathcal{R}_*\cap K^2)$ for some C with dense finites.*

Proof. Let f be a witness for the fact that $(K,\mathcal{R}_*\cap K^2)\leq_c(2^\omega,\mathcal{R})$. Note that f preserves $\mathbb{C}$ since $\mathcal{R}_*,\mathcal{R}$ agree with $\Delta(\mathbb{C})$ on $\Delta(2^\omega)$. This implies that f is nowhere dense-to-one. Let us prove that there is $g:2^\omega\rightarrow K$ continuous such that g preserves $\mathbb{C}$ and $f\big(g(\alpha)\big)<_{\text{lex}}f\big(g(\beta)\big)$ if $\alpha<_{\text{lex}}\beta$. We ensure (1)-(6) with $C=K$ and

$$(4)\ f(\alpha)<_{\text{lex}}f(\beta)\text{ if }\alpha\in U_{t0}\text{ and }\beta\in U_{t1}$$

Assume that this is done. If $\alpha<_{\text{lex}}\beta$, then Condition (4) ensures that $f\big(g(\alpha)\big)<_{\text{lex}}f\big(g(\beta)\big)$.

So it is enough to prove that the construction is possible. We set $U_\emptyset:=K$. Assume that $(n_t)_{|t|\leq l}$ and $(U_t)_{|t|\leq l}$ satisfying (1)-(6) have been constructed, which is the case for $l=0$. Fix $t\in 2^l$. Note that α_{n_t} is in $U_t\cap Q_2$, which implies that $f(\alpha_{n_t})\in Q_2$. This gives $s\in 2^{<\omega}$ such that $f(\alpha_{n_t})=s0^\infty$. In particular, there is a clopen neighbourhood $N\subseteq U_t$ of α_{n_t} such that $f(\beta)\in N_s$ if $\beta\in N$. We choose $n_{t1}\geq 1$ such that $\alpha_{n_{t1}}\in N\setminus\Big(\{\alpha_n\mid n\leq l\}\cup f^{-1}\big(\{f(\alpha_{n_t})\}\big)\Big)$, which is possible since f is nowhere dense-to-one. Note that $f(\alpha_{n_t})<_{\text{lex}}f(\alpha_{n_{t1}})$. It remains to choose a small enough clopen neighbourhood $U_{t\varepsilon}$ of $\alpha_{n_{t\varepsilon}}$ to finish the construction, using the continuity of f.

Note then that $f_{|g[2^\omega]}$ is a homeomorphism onto $\tilde{C}:=f\big[g[2^\omega]\big]$. Moreover,

$$\big(g(\alpha),g(\beta)\big)\in\mathcal{R}_*\Leftrightarrow\Big(f\big(g(\alpha)\big),f\big(g(\beta)\big)\Big)\in\mathcal{R},$$

which implies that $f^{-1}_{|g[2^\omega]}$ is a witness for the fact that $(\tilde{C},\mathcal{R}\cap\tilde{C}^2)\sqsubseteq_c(K,\mathcal{R}_*\cap K^2)$, and $\mathcal{R}\cap\tilde{C}^2$ is not in $\mathbf{\Gamma}$ since the map $\alpha\mapsto\Big(f\big(g(\alpha)\big),f\big(g(\alpha)\big)\Big)$ reduces $\mathbb{C}$ to $\mathcal{R}\cap\tilde{C}^2$. Thus $\tilde{C}\cap\mathbb{C}$ is not separable from $\tilde{C}\backslash\mathbb{C}$ by a $\mathbf{\Gamma}$ set. Using the Hurewicz theorem, we get $m:2^\omega\rightarrow\tilde{C}$ injective continuous such that $\mathbb{C}=m^{-1}(\mathbb{C})$. It remains to set $C:=m[2^\omega]$. $\square$

The minimality of diagonally complex relations can be seen on restrictions to Cantor sets.

Lemma 3.2 *Let $\mathcal{R}$ be a diagonally complex relation, X be an analytic space, and R be a non-$\mathbf{\Pi}^0_2$ relation on X such that $(X,R)\leq_c(2^\omega,\mathcal{R})$, with witness f. Then there is $g:2^\omega\rightarrow X$ injective continuous such that $Q_2=g^{-1}\big(f^{-1}(Q_2)\big)$.*

Proof. Note that $\mathcal{R}=\Delta(S)\cup(\mathcal{R}\cap Q_2^2)$, where $S=I_2$ if $\mathbf{\Gamma}=\mathbf{\Sigma}^0_2$, and $S=\emptyset$ if $\mathbf{\Gamma}=\mathbf{\Pi}^0_2$. As $Q_2^2\backslash\mathcal{R}$ is countable, it is a $\mathbf{\Sigma}^0_2$ subset of Q_2^2, and $\mathcal{R}\cap Q_2^2$ is a $\mathbf{\Pi}^0_2$ subset of Q_2^2, which provides $G\in\mathbf{\Pi}^0_2(2^\omega\times 2^\omega)$ such that $\mathcal{R}\cap Q_2^2=G\cap Q_2^2$. As $R=(f\times f)^{-1}(\mathcal{R})=(f\times f)^{-1}\big(\Delta(S)\big)\cup\big((f\times f)^{-1}(G)\cap f^{-1}(Q_2)^2\big)$, $f^{-1}(Q_2)$ is not $\mathbf{\Pi}^0_2$. Theorem 2.5 provides g as desired. □

Corollary 3.3 *Let $\mathcal{R}$ be a diagonally complex relation, X be an analytic space, and R be a non-$\mathbf{\Gamma}$ relation on X with $(X,R)\leq_c(2^\omega,\mathcal{R})$. Then there is a Cantor set C with dense finites such that $(C,\mathcal{R}\cap C^2)\sqsubseteq_c(X,R)$.*

Proof. Assume first that $\mathbf{\Gamma}=\mathbf{\Pi}^0_2$, and let f be a witness for the fact that $(X,R)\leq_c(2^\omega,\mathcal{R})$. Lemma 3.2 provides $g:2^\omega\to X$ injective continuous with the property that $Q_2=g^{-1}\big(f^{-1}(Q_2)\big)$. We set $\mathcal{R}':=(g\times g)^{-1}(R)$. Note that, for each $\alpha\in 2^\omega$,

$$(\alpha,\alpha)\in\mathcal{R}'\Leftrightarrow\big(g(\alpha),g(\alpha)\big)\in R\Leftrightarrow\Big(f\big(g(\alpha)\big),f\big(g(\alpha)\big)\Big)\in\mathcal{R}\Leftrightarrow f\big(g(\alpha)\big)\in Q_2\Leftrightarrow\alpha\in Q_2.$$

In other words, $\mathcal{R}'\cap\Delta(2^\omega)=\Delta(Q_2)$. This argument also shows that $\mathcal{R}'\subseteq Q_2^2$. In other words, $\mathcal{R}'$ is diagonally complex, $(2^\omega,\mathcal{R}')\sqsubseteq_c(X,R)$ and $(2^\omega,\mathcal{R}')\leq_c(2^\omega,\mathcal{R})$. Lemma 3.1 provides a Cantor set C with dense finites such that $(C,\mathcal{R}\cap C^2)\sqsubseteq_c(2^\omega,\mathcal{R}')$.

Assume now that $\mathbf{\Gamma}=\mathbf{\Sigma}^0_2$. As $\mathcal{R}$ is diagonally complex, the vertical sections of $\mathcal{R}$ are in $\mathbf{\Sigma}^0_2$, which implies that the vertical sections of R are also in $\mathbf{\Sigma}^0_2$. By Theorem 2.2, one of the following holds:

(1) there is a relation $\mathbb{R}_*$ on 2^ω such that $\mathbb{R}_*\cap\Delta(2^\omega)=\Delta(I_2)$ and $(2^\omega,\mathbb{R}_*)\sqsubseteq_c(X,R)$.

(2) there is a relation $\mathbb{R}_*$ on $\mathbb{D}$ such that $D_\infty:=\big\{\big((0,\alpha),(1,\alpha)\big)\mid\alpha\in I_2\big\}\subseteq\mathbb{R}_*$,

$$D_f:=\big\{\big((0,\alpha),(1,\alpha)\big)\mid\alpha\in Q_2\big\}\subseteq\neg\mathbb{R}_*$$

and $(\mathbb{D},\mathbb{R}_*)\sqsubseteq_c(X,R)$. We set $D:=D_\infty\cup D_f$.

Assume that (2) holds, which gives $f:\mathbb{D}\to 2^\omega$ continuous such that $\mathbb{R}_*=(f\times f)^{-1}(\mathcal{R})$. Note that

- $(f\times f)[D_\infty]\subseteq\mathcal{R}$ is not separable from $(f\times f)[D_f]$ by a $\mathbf{\Sigma}^0_2$ set,
- $I_\infty:=(f\times f)[D_\infty]\cap\Delta(2^\omega)$ is not separable from $(f\times f)[D_f]$ by a $\mathbf{\Sigma}^0_2$ set since $\mathcal{R}\backslash\Delta(2^\omega)\subseteq Q_2^2$ is $\mathbf{\Sigma}^0_2$,
- I_∞ is not separable from $I_f:=(f\times f)[D_f]\cap\Delta(2^\omega)$ by a $\mathbf{\Sigma}^0_2$ set,
- $R_\infty:=D\cap(f\times f)^{-1}(I_\infty)$ is not separable from $R_f:=D\cap(f\times f)^{-1}(I_f)$ by a $\mathbf{\Sigma}^0_2$ set (otherwise $R_\infty\subseteq S\subseteq\neg R_f$ and I_∞ is separable from I_f by the K_σ set $(f\times f)[S\cap D]$),
- $C_\infty:=\big\{\alpha\in 2^\omega\mid\big((0,\alpha),(1,\alpha)\big)\in R_\infty\big\}$ is not separable from $C_f:=\big\{\alpha\in 2^\omega\mid\big((0,\alpha),(1,\alpha)\big)\in R_f\big\}$ by a $\mathbf{\Sigma}^0_2$ set.

Theorem 2.5 provides $g:2^\omega\to 2^\omega$ injective continuous with $I_2\subseteq g^{-1}(C_\infty)$ and $Q_2\subseteq g^{-1}(C_f)$. We define a map $c:2^\omega\to\{0\}\times 2^\omega$ by $c(\alpha):=\big(0,g(\alpha)\big)$, and set $c':=f\circ c$,

$$\mathbb{R}':=(c'\times c')^{-1}(\mathcal{R})=(c\times c)^{-1}(\mathbb{R}_*).$$

As c is injective continuous, $\mathbb{R}'$ is a Borel relation on 2^ω and $(2^\omega,\mathbb{R}')\sqsubseteq_c(\mathbb{D},\mathbb{R}_*),(X,R)$. If $\alpha\in I_2$, then $\Big(\big(0,g(\alpha)\big),\big(1,g(\alpha)\big)\Big)\in R_\infty$, and $\Big(f\big(0,g(\alpha)\big),f\big(1,g(\alpha)\big)\Big)\in I_\infty\subseteq\Delta(2^\omega)$.

This implies that $\big(f\big(0,g(\alpha)\big),f\big(0,g(\alpha)\big)\big)\in\mathcal{R}$ and $(\alpha,\alpha)\in\mathbb{R}'$. Similarly, if $\alpha\in Q_2$, then $(\alpha,\alpha)\notin\mathbb{R}'$. Thus $\mathbb{R}'\cap\Delta(2^\omega)=\Delta(I_2)$, and $\mathbb{R}'$ is a witness for the fact that (1) also holds. So (1) holds in any case.

This implies that $(2^\omega,\mathbb{R}_*)\leq_c(2^\omega,\mathcal{R})$, with witness f'. By Lemma 2.4, f' preserves Q_2. In particular, $\mathbb{R}_*\cap\big((Q_2\times I_2)\cup(I_2\times Q_2)\big)=\emptyset$ since $\mathcal{R}$ is diagonally complex. Note that $\mathbb{R}_*$ has $\mathbf{\Sigma}^0_2$ sections since so does $\mathcal{R}$. This implies that $\mathbb{R}_*\cap I_2^2$ is a Borel relation on I_2 whose sections are separable from Q_2 by a $\mathbf{\Sigma}^0_2$ set. Lemmas 2.6 and 2.8 provide $g':2^\omega\rightarrow 2^\omega$ injective continuous preserving Q_2 such that $(g'\times g')^{-1}(\mathbb{R}_*\cap I_2^2)\subseteq\Delta(I_2)$. Thus we may assume that $\mathbb{R}_*$ is diagonally complex. We now apply Lemma 3.1. □

A consequence of Lemma 3.1 is a characterisation of the minimality of diagonally complex relations.

Corollary 3.4 *Let $\mathcal{R}$ be a diagonally complex relation. The following are equivalent:*

(a) $\mathcal{R}$ is $\leq_c$ and $\sqsubseteq_c$-minimal among non-$\mathbf{\Gamma}$ relations on an analytic space,

(b) $(2^\omega,\mathcal{R})\sqsubseteq_c(C,\mathcal{R}\cap C^2)$ if C is a Cantor set with dense finites.

Proof. (a) ⇒ (b) As $Q_2\cap C$ is dense in C, $\mathcal{R}\cap\Delta(C)$ and $\mathcal{R}\cap C^2$ are not in $\mathbf{\Gamma}$. As

$$(C,\mathcal{R}\cap C^2)\sqsubseteq_c(2^\omega,\mathcal{R})$$

and (a) holds, (b) holds.

(b) ⇒ (a) Let X be an analytic space and R be a non-$\mathbf{\Gamma}$ relation on X with $(X,R)\leq_c(2^\omega,\mathcal{R})$. Corollary 3.3 provides a Cantor set C with dense finites and the property that $(C,\mathcal{R}\cap C^2)\sqsubseteq_c(X,R)$. By (b), $(2^\omega,\mathcal{R})\sqsubseteq_c(X,R)$. □

Another consequence of Lemma 3.1 is about the comparison of minimal diagonally complex relations.

Notation. If $\mathcal{R},\mathcal{R}'$ are relation on 2^ω, then we set

$$(2^\omega,\mathcal{R}')\equiv_c(2^\omega,\mathcal{R})\Leftrightarrow(2^\omega,\mathcal{R}')\sqsubseteq_c(2^\omega,\mathcal{R})\wedge(2^\omega,\mathcal{R})\sqsubseteq_c(2^\omega,\mathcal{R}').$$

Corollary 3.5 *Let $\mathcal{R},\mathcal{R}'$ be diagonally complex relations, $\leq_c$ and $\sqsubseteq_c$-minimal among among non-$\mathbf{\Gamma}$ relations on an analytic space, with $(2^\omega,\mathcal{R})\leq_c(2^\omega,\mathcal{R}')$. Then $(2^\omega,\mathcal{R}')\equiv_c(2^\omega,\mathcal{R})$.*

Proof. Lemma 3.1 provides a Cantor set with dense finites C such that $(C,\mathcal{R}'\cap C^2)\sqsubseteq_c(2^\omega,\mathcal{R})$. By Corollary 3.4, $(2^\omega,\mathcal{R}')\sqsubseteq_c(2^\omega,\mathcal{R})$. An application of this fact implies that $(2^\omega,\mathcal{R})\sqsubseteq_c(2^\omega,\mathcal{R}')$. □

3.2 Proof of Theorems 1.3 and 1.4

We now introduce our antichain of size continuum.

Notation. We define $i:Q^2\rightarrow\omega$ as follows. We want to ensure that $i(z,t)=i(s_z,s_t)$, where s_t is defined before Lemma 2.8. The definition of i is partly inspired by the oscillation map osc defined in [T] after Theorem 6.33 as follows. The elements of $2^{<\omega}$ are identified with finite subsets of ω, through the characteristic function.

The oscillation between z and t describes the behaviours of the symmetric difference $z\Delta t$. The equivalence relation $\sim_{zt}$ is defined on $z\Delta t$ by

$$j \sim_{zt} k \Leftrightarrow [\min(j,k),\max(j,k)]\cap(z\setminus t)=\emptyset \vee [\min(j,k),\max(j,k)]\cap(t\setminus z)=\emptyset.$$

Then $\text{osc}(z,t):=|z\Delta t/\sim_{zt}|$. For i, we work on $z\cup t$ instead of $z\Delta t$, and the definition depends more heavily on the initial segments of z and t, in particular on their lexicographic ordering. If $z\in 2^{<\omega}\setminus\{\emptyset\}$, then we set $z^-:=z|\max\{l<|z| \mid z|l\in Q\}$. We also set, for $\iota\in\{<,>\}$,

$$\perp_\iota:=\{(z,t)\in Q^2 \mid \exists i<\min(|z|,|t|)\ \ z|i=t|i\wedge z(i)\ \iota\ t(i)\}.$$

The definition of $i(z,t)$ is by induction on $\max(|z|,|t|)$. We set

$$i(z,t):=\begin{cases} 0 \text{ if } z=t,\\ i(z,t^-) \text{ if } |z|<|t^-| \vee \big(|z|=|t^-| \wedge (z,t^-)\in\perp_<\big),\\ i(z,t^-)+1 \text{ if } (|z|<|t| \wedge |z|>|t^-|) \vee \big(|z|=|t^-| \wedge (z,t^-)\notin\perp_<\big),\\ i(z^-,t) \text{ if } |t|<|z^-| \vee \big(|t|=|z^-| \wedge (t,z^-)\in\perp_<\big),\\ i(z^-,t)+1 \text{ if } (|t|<|z| \wedge |t|>|z^-|) \vee \big(|t|=|z^-| \wedge (t,z^-)\notin\perp_<\big),\\ i(z^-,t^-)+1 \text{ if } |z|=|t|\wedge\big((|z^-|<|t^-|\wedge t^-<_{\text{lex}} z^-)\vee(|t^-|<|z^-|\wedge z^-<_{\text{lex}} t^-)\big),\\ i(z^-,t^-)+2 \text{ if } |z|=|t|\wedge\big((|z^-|<|t^-|\wedge z^-<_{\text{lex}} t^-)\vee(|t^-|<|z^-|\wedge t^-<_{\text{lex}} z^-)\vee\\ \hfill (|z^-|=|t^-|\wedge z^-\neq t^-)\big).\end{cases}$$

Note that $i(z,t)=i(t,z)$ if $z,t\in Q$.

Lemma 3.6 *Let $(s_t)_{t\in 2^{<\omega}}$ be a sequence of elements of Q with*

(a) $|s_z|<|s_t|$ if $z<_l t$ are in Q
(b) $s_t\subsetneqq s_{t1}$ if t is in $2^{<\omega}$

Then $i(z,t)=i(s_z,s_t)$ if $z,t\in Q$.

Proof. We argue by induction on $\max(|z|,|t|)$. As $i(z,t)=i(t,z)$ if $z,t\in Q$, we may assume that $z<_l t$. In particular, $t\neq\emptyset$ is of the form $t'1$, and $s_{t^-}=s_{t'}\subsetneqq s_{t'1}=s_t$ since $n_{t^-}=n_{t'}$. We go through the cases of the definition of i.

If $|z|<|t^-|$, then $|s_z|<|s_{t^-}|\leq\big|(s_t)^-\big|$ since $s_{t^-}\subsetneqq s_t$, and

$$i(z,t)=i(z,t^-)=i(s_z,s_{t^-})=\dots=i\big(s_z,(s_t)^-\big)=i(s_z,s_t).$$

If $|z|=|t^-|$ and $(z,t^-)\in\perp_<$, then $|s_z|<|s_{t^-}|\leq|(s_t)^-|$ again and we conclude as above.

If $|z|<|t|$ and $|z|>|t^-|$, then $|s_z|<|s_t|$ and $|s_z|>|s_{t^-}|$. Let $s\in Q$ with $s\subseteq s_t$ and $|s_z|<|s|$ be of minimal length. Note that $i(z,t)=i(z,t^-)+1=i(s_z,s_{t^-})+1=i(s_z,s)=i(s_z,s_t)$.

If $|z|=|t^-|$ and $(z,t^-)\notin\perp_<$, then either $z=t^-$, or $(z,t^-)\in\perp_>$, which implies that

$$i(z,t)=i(z,t^-)+1=i(s_z,s_{t^-})+1=\dots=i\big(s_z,(s_t)^-\big)=i(s_z,s_t).$$

If $|z|=|t|$, $|t^-|<|z^-|$ and $z^-<_{\text{lex}} t^-$, then $|s_{t^-}|<|s_{z^-}|<|s_z|<|s_t|$,

$$i(z,t)=i(z^-,t^-)+1=i(s_{z^-},s_{t^-})+1=i(s_z,s_{t^-})+1=i(s_z,s_t).$$

If $|z|=|t|$, $|z^-|<|t^-|$ and $z^- <_{\text{lex}} t^-$, then $|s_{z^-}|<|s_{t^-}|<|s_z|<|s_t|$,

$$i(z,t)=i(z^-,t^-)+2=i(s_{z^-},s_{t^-})+2=i(s_z,s_{t^-})+1=i(s_z,s_t).$$

If $|z|=|t|$, $|z^-|=|t^-|$ and $z^-\neq t^-$, then $(z^-,t^-)\in\perp_<$, $|s_{z^-}|<|s_{t^-}|<|s_z|<|s_t|$, and we conclude as above. Thus $i(z,t)=i(s_z,s_t)$ in any case, as desired. □

The next result is the key lemma to prove Theorems 1.3 and 1.4.

Lemma 3.7 *Let $H\subseteq Q_2$ be homeomorphic to Q_2. Then there is $f:2^\omega\to 2^\omega$ injective continuous satisfying the following properties:*

(i) $f[Q_2]\subseteq H$; in particular, for each $t\in Q$ there is $n_t\geq 1$ with $f(t0^\infty)=s_t0^\infty$,

(ii) $f[I_2]\subseteq I_2$,

(iii) $i(z,t)=i(s_z,s_t)$ if $z,t\in Q$.

Proof. We ensure (1)-(6) with $C:=\overline{H}$, that U_t is of the form $N_{z_t}\cap C$, and

(2) $\alpha_{n_t}\in N_{z_t}\cap H$ if t is in $2^{<\omega}$
(4) $z_{t0} <_{\text{lex}} z_{t1}$ and $|z_{t0}|=|z_{t1}|$ if t is in $2^{<\omega}$
(7) $|s_z|<|s_t|$ if $z <_l t$ are in Q
(8) $s_t \subsetneqq s_{t1}$ if t is in $2^{<\omega}$
(9) $i(z,t)=i(s_z,s_t)$ if $z,t\in Q$

It is enough to prove that the construction is possible. We choose $n_\emptyset\geq 1$ with $s_\emptyset 0^\infty\in H$, and set $z_\emptyset:=s_\emptyset$. Assume that $(n_t)_{|t|\leq l}$ and $(z_t)_{|t|\leq l}$ satisfying (1)-(9) have been constructed, which is the case for $l=0$.

Fix $t\in 2^l$. As $\alpha_{n_t}\in U_t=N_{z_t}\cap C$, $z_t\subseteq s_t0^\infty$. We choose $s\subseteq s_t0^\infty$ extending z_t and s_t, in such a way that $N_s\cap(\{\alpha_n\mid n\leq l\}\setminus\{\alpha_{n_t}\})=\emptyset$ and $|s_z|<|s|$ if $z<_l t1$. Let $n_{t1}\geq 1$ such that $\alpha_{n_{t1}}\in N_s\cap H\setminus\{\alpha_{n_t}\}$, which is possible by density of H in the perfect space C. We set $z_{t1}=s_{t1}$ and $z_{t0}=\alpha_{n_t}||z_{t1}|$. Note that (1) holds. (2) holds by definition. By our extensions, (3) holds. (4) holds because of the choice of s and n_{t1}. (5)-(8) hold by construction. By Lemma 3.6, (9) holds. □

Proof of Theorem 1.4. We may replace $\mathbb{Q}$ with its topological copy Q_2. We set, for $z,t\in Q$, $c(\{z0^\infty,t0^\infty\}):=i(z,t)$, which is well-defined by symmetry of i. As $i\big((10)^k0^\infty,(01)^k0^\infty\big)=2k$ and $i\big((01)^k0^\infty,(10)^{k+1}0^\infty\big)=2k+1$, c is onto. If $H\subseteq Q_2$ is homeomorphic to Q_2, then Lemma 3.7 provides $f:2^\omega\to 2^\omega$. We set $h:=f_{|Q_2}$. If $z,t\in Q$, then

$$c\big(\{h(z0^\infty),h(t0^\infty)\}\big)=c(\{s_z0^\infty,s_t0^\infty\})=i(s_z,s_t)=i(z,t)=c(\{z0^\infty,t0^\infty\}).$$

In particular, c takes all the values from ω on $H^{[2]}$. □

Notation. We now set, when $(2^\omega,\mathbb{C})$ is $\mathbf{\Gamma}$-good and $\beta\in 2^\omega$,

$$\mathbb{R}_\beta:=\Delta(\mathbb{C})\cup\bigcup_{\beta(p)=1}\{(z0^\infty,t0^\infty)\mid z\neq t\in Q\wedge\ i(z,t)=p\}.$$

Note that $\mathbb{R}_\beta$ is symmetric.

Corollary 3.8 *Let $\beta \in 2^\omega$. Then $\mathbb{R}_\beta$ is $\leq_c$ and $\sqsubseteq_c$-minimal among non-$\mathbf{\Gamma}$ relations on an analytic space.*

Proof. Note that $\mathbb{R}_\beta$ is a diagonally complex relation, and therefore not in $\mathbf{\Gamma}$. Let C be a Cantor set with dense finites. By Corollary 3.4, it is enough to see that the inequality $(2^\omega, \mathbb{R}_\beta) \sqsubseteq_c (C, \mathbb{R}_\beta \cap C^2)$ holds. We apply Lemma 3.7 to $H := C \cap Q_2$. As C is a Cantor set with dense finites, H is nonempty, countable, metrizable, dense-in-itself, and therefore homeomorphic to Q_2 (see 7.12 in [K]). Lemma 3.7 provides a witness $f : 2^\omega \to 2^\omega$ for the fact that $(2^\omega, \mathbb{R}_\beta) \sqsubseteq_c (C, \mathbb{R}_\beta \cap C^2)$. □

Lemma 3.9 *The family $(\mathbb{R}_\beta)_{\beta \in N_0}$ is a $\leq_c$-antichain.*

Proof. Assume that $\beta, \beta' \in N_0$ and $(2^\omega, \mathbb{R}_\beta) \leq_c (2^\omega, \mathbb{R}_{\beta'})$, with witness f. By Lemma 2.4, f preserves $\mathbb{C}$.

Claim. *Let $u_0 \in 2^{<\omega}$. Then we can find $u'_0 \in Q$ and $u_1, u'_1 \in Q \setminus \{\emptyset\}$ such that $f(u_0 0^\infty) = u'_0 0^\infty$ and $f(u_0 u_1 0^\infty) = u'_0 u'_1 0^\infty$.*

Indeed, let $u'_0 \in Q$ with $f(u_0 0^\infty) = u'_0 0^\infty$. We can find a sequence $(\beta_k)_{k \in \omega}$ of points of I_2 converging to $u_0 0^\infty$. As f preserves $\mathbb{C}$, $f(u_0 0^\infty) \neq f(\beta_k)$, and we can find $(n_k)_{k \in \omega}$ stricly increasing such that $\alpha_{n_k} | k = \beta_k | k$ and $f(u_0 0^\infty) \neq f(\alpha_{n_k})$, which implies that $(\alpha_{n_k})_{k \in \omega}$ is a sequence of points different from $u_0 0^\infty$ converging to $u_0 0^\infty$. Let $v_k, v'_k \in Q$ with $\alpha_{n_k} = v_k 0^\infty$ and $f(\alpha_{n_k}) = v'_k 0^\infty$. We may assume that $u_0 \subsetneqq v_k$. As f is continuous, we may assume that $f(\alpha_{n_k}) \in N_{u'_0}$ for each k, which implies that $u'_0 \subsetneqq v'_k$. It remains to choose $u_1, u'_1 \in Q \setminus \{\emptyset\}$ such that $v_1 = u_0 u_1$ and $v'_1 = u'_0 u'_1$. ◇

Assume that $t_0, \cdots, t_{2k+2} \in 2^{<\omega}$ are not initial segments of 0^∞. Then

$$(t_0 t_1 0^{|t_2|} \cdots t_{2k-1} 0^{|t_{2k}|} t_{2k+1} 0^\infty, t_0 0^{|t_1|} t_2 \cdots 0^{|t_{2k+1}|} t_{2k+2} 0^\infty)$$

is in $\{(z0^\infty, t0^\infty) \mid z \neq t \in Q \wedge i(z,t) = 2k+2\}$. Similarly,

$$(t_0 0^{|t_1|} t_2 \cdots 0^{|t_{2k-1}|} t_{2k} 0^\infty, t_0 t_1 0^{|t_2|} \cdots t_{2k-1} 0^{|t_{2k}|} t_{2k+1} 0^\infty)$$

is in $\{(z0^\infty, t0^\infty) \mid z \neq t \in Q \wedge i(z,t) = 2k+1\}$. We first apply the claim to $u_0 := t_0 \in Q$, which gives $t'_0 \in Q$ and $t_1, t'_1 \in Q \setminus \{\emptyset\}$ with the properties that $f(t_0 0^\infty) = t'_0 0^\infty$ and $f(t_0 t_1 0^\infty) = t'_0 t'_1 0^\infty$. The continuity of f provides $k_1 \geq |t_1|$ such that $f[N_{t_0 0^{k_1}}] \subseteq N_{t'_0 0^{|t'_1|}}$. We next apply the claim to $u_0 := t_0 0^{k_1}$, which gives $\tilde{t}_2, \tilde{t}'_2$ in $Q \setminus \{\emptyset\}$ such that $f(t_0 0^{k_1} \tilde{t}_2 0^\infty) = t'_0 \tilde{t}'_2 0^\infty$. Note that $0^{|t'_1|} \subseteq \tilde{t}'_2$. We set $t_2 := 0^{k_1 - |t_1|} \tilde{t}_2$ and $t'_2 := \tilde{t}'_2 - 0^{|t'_1|}$. Note that t_2, t'_2 are not initial segments of 0^∞ and

$$f(t_0 0^{|t_1|} t_2 0^\infty) = t'_0 0^{|t'_1|} t'_2 0^\infty.$$

This argument shows that we can find sequences of finite binary sequences $(t_j)_{j \in \omega}$ and $(t'_j)_{j \in \omega}$ which are not initial segments of 0^∞ and satisfy $f(t_0 0^{|t_1|} t_2 \cdots 0^{|t_{2k-1}|} t_{2k} 0^\infty) = t'_0 0^{|t'_1|} t'_2 \cdots 0^{|t'_{2k-1}|} t'_{2k} 0^\infty$ and $f(t_0 t_1 0^{|t_2|} \cdots t_{2k-1} 0^{|t_{2k}|} t_{2k+1} 0^\infty) = t'_0 t'_1 0^{|t'_2|} \cdots t'_{2k-1} 0^{|t'_{2k}|} t'_{2k+1} 0^\infty$ for each $k \in \omega$. By the remark after the claim, $\beta = \beta'$. □

Proof of Theorem 1.3. We apply Corollary 3.8 and Lemma 3.9. □

Theorem 1.3 shows that if $\leq$ is in $\{\leq_c, \sqsubseteq_c\}$, then the class of non-$\mathbf{\Pi}^0_2$ countable relations on analytic spaces, equipped with $\leq$, contains antichains of size continuum made of minimal relations.

3.3 Graphs

Theorem 1.3 shows that, for the classes of rank two, any basis must have size continuum. It is natural to ask whether it is also the case for graphs. We will see that it is indeed the case.

Notation. We define, for $\beta\in 2^\omega$, $\mathbb{G}_\beta:=s\Big(\big\{\big((0,\alpha),(1,\gamma)\big)\in\mathbb{D}^2\mid(\alpha,\gamma)\in\mathbb{R}_\beta\big\}\Big)$. Note that $\mathbb{G}_\beta$ is a locally countable graph.

Lemma 3.10 *Let $\mathbf{\Gamma}:=\mathbf{\Sigma}^0_2$, $(2^\omega,\mathbb{C})$ be $\mathbf{\Gamma}$-good, $\beta\in 2^\omega$, and $\mathbb{R}$ be a relation on $\mathbb{D}$ with the properties that $\mathbb{R}_{0,1}\cap\Delta(2^\omega)=\Delta(\mathbb{C})$ and $(\mathbb{D},\mathbb{R})\sqsubseteq_c(\mathbb{D},\mathbb{G}_\beta)$. Then $(2^\omega,\mathbb{R}_\beta)\sqsubseteq_c(2^\omega,\mathbb{R}_{0,1})$ with witness h having the property that $(\mathbb{R}_{0,0}\cup\mathbb{R}_{1,1})\cap h[2^\omega]^2=\emptyset$.*

Proof. Let f be a witness for the fact that $(\mathbb{D},\mathbb{R})\sqsubseteq_c(\mathbb{D},\mathbb{G}_\beta)$.

We set, for $\varepsilon\in 2$, $X_\varepsilon:=\{z\in\mathbb{D}\mid f_0(z)=\varepsilon\}$, which defines a partition of $\mathbb{D}$ into clopen sets. The definition of $\mathbb{G}_\beta$ shows that $\mathbb{R}\subseteq(X_0\times X_1)\cup(X_1\times X_0)$. If $\alpha_0\in\mathbb{C}$, then $\big((0,\alpha_0),(1,\alpha_0)\big)\in\mathbb{R}$, which gives $\varepsilon_0\in 2$ and $l\in\omega$ such that $(0,\alpha)\in X_{\varepsilon_0}$ and $(1,\alpha)\in X_{1-\varepsilon_0}$ if $\alpha\in N_{\alpha_0|l}$. This shows that $\mathbb{R}\cap(2\times N_{\alpha_0|l})^2\subseteq s\Big(\big\{\big((\varepsilon_0,\alpha),(1-\varepsilon_0,\gamma)\big)\mid\alpha,\gamma\in 2^\omega\big\}\Big)$. Note that $\mathbb{R}_{\varepsilon,\eta}$ is a relation on 2^ω. As $\mathbb{R}_{0,1}\cap\Delta(2^\omega)=\Delta(\mathbb{C})$, $\mathbb{R}_{0,1}$ is not in $\mathbf{\Gamma}$. Moreover, by symmetry of $\mathbb{R}_\beta$,

$$(\alpha,\gamma)\in\mathbb{R}_{0,1}\Leftrightarrow\big((0,\alpha),(1,\gamma)\big)\in\mathbb{R}\Leftrightarrow\big(f(0,\alpha),f(1,\gamma)\big)\in\mathbb{G}_\beta\Leftrightarrow\big(f_1(0,\alpha),f_1(1,\gamma)\big)\in\mathbb{R}_\beta$$

if $\alpha\in N_{\alpha_0|l}$.

Claim. *$f_1(0,\alpha)=f_1(1,\alpha)$ if $\alpha\in\mathbb{C}\cap N_{\alpha_0|l}$.*

Indeed, note that $\alpha\in\mathbb{C}\Leftrightarrow\big((0,\alpha),(1,\alpha)\big)\in\mathbb{R}\Leftrightarrow\big(f_1(0,\alpha),f_1(1,\alpha)\big)\in\mathbb{R}_\beta$ if $\alpha\in N_{\alpha_0|l}$. We argue by contradiction, which gives $\alpha_0\in\mathbb{C}\cap N_{\alpha_0|l}$, $t\in 2^{<\omega}$, $\eta\in 2$ and $l'\geq l$ with the properties that $f_1(0,\alpha)\in N_{t\eta}$ and $f_1(1,\alpha)\in N_{t(1-\eta)}$ if $\alpha\in N_{\alpha_0|l'}$. In particular,

$$\mathbb{C}\cap N_{\alpha_0|l'}=\big\{\alpha\in N_{\alpha_0|l'}\mid\big(f_1(0,\alpha),f_1(1,\alpha)\big)\in\mathbb{R}_\beta\cap(N_{t\eta}\times N_{t(1-\eta)})\big\},$$

which shows that $\mathbb{C}\cap N_{\alpha_0|l'}\in\mathbf{\Gamma}=\mathbf{\Sigma}^0_2$ since $\mathbb{R}_\beta\setminus\Delta(2^\omega)$ is countable, contradicting the choice of $\mathbb{C}$.$\diamond$

By the claim, continuity of f and density of $\mathbb{C}$, $f_1(0,\alpha)=f_1(1,\alpha)$ if $\alpha\in N_{\alpha_0|l}$. This shows that $(N_{\alpha_0|l},\mathbb{R}_{0,1}\cap N^2_{\alpha_0|l})\leq_c(2^\omega,\mathbb{R}_\beta)$. By the proof of Corollary 3.4 and Corollary 3.8,

$$(2^\omega,\mathbb{R}_\beta)\sqsubseteq_c(N_{\alpha_0|l},\mathbb{R}_{0,1}\cap N^2_{\alpha_0|l}),$$

and we are done. $\square$

Lemma 3.11 *Let $\mathbf{\Gamma}:=\mathbf{\Sigma}^0_2$, $(2^\omega,\mathbb{C})$ be $\mathbf{\Gamma}$-good, and $\beta\in 2^\omega$. Then $\mathbb{G}_\beta$ is $\sqsubseteq_c$-minimal among non-$\mathbf{\Gamma}$ relations on an analytic space.*

Proof. Let X be an analytic space, and R be a non-$\mathbf{\Gamma}$ relation on X with $(X,R)\sqsubseteq_c(\mathbb{D},\mathbb{G}_\beta)$. As $\mathbb{G}_\beta$ has sections in $\mathbf{\Gamma}=\mathbf{\Sigma^0_2}$, R too. By Theorem 2.2, one of the following holds:

(1) there is a relation $\mathbb{R}$ on 2^ω such that $\mathbb{R}\cap\Delta(2^\omega)=\Delta(\mathbb{C})$ and $(2^\omega,\mathbb{R})\sqsubseteq_c(X,R)$,

(2) there is a relation $\mathbb{R}$ on $\mathbb{D}$ such that $\mathbb{R}_{0,1}\cap\Delta(2^\omega)=\Delta(\mathbb{C})$ and $(\mathbb{D},\mathbb{R})\sqsubseteq_c(X,R)$.

As $\mathbb{G}_\beta$ is a digraph, R and $\mathbb{R}$ are digraphs too. So (1) cannot hold. By Lemma 3.10,

$$(2^\omega, \mathbb{R}_\beta) \sqsubseteq_c (2^\omega, \mathbb{R}_{0,1}),$$

with witness h having the property that $(\mathbb{R}_{0,0} \cup \mathbb{R}_{1,1}) \cap h[2^\omega]^2 = \emptyset$. We define $k : \mathbb{D} \to \mathbb{D}$ by $k(\varepsilon, \alpha) := \big(\varepsilon, h(\alpha)\big)$. Note that k is injective, continuous, and

$$\big((0,\alpha),(1,\gamma)\big) \in \mathbb{G}_\beta \Leftrightarrow \big(h(\alpha), h(\gamma)\big) \in \mathbb{R}_{0,1} \Leftrightarrow \big(k(0,\alpha), k(1,\gamma)\big) \in \mathbb{R},$$

showing that $(\mathbb{D}, \mathbb{G}_\beta) \sqsubseteq_c (X, R)$ as desired. □

Lemma 3.12 *Let $\boldsymbol{\Gamma} := \boldsymbol{\Sigma}^0_2$, and $(2^\omega, \mathbb{C})$ be $\boldsymbol{\Gamma}$-good. Then $(\mathbb{G}_\beta)_{\beta \in N_0}$ is a $\sqsubseteq_c$-antichain.*

Proof. Let $\beta, \beta' \in N_0$. Assume that $(\mathbb{D}, \mathbb{G}_\beta) \sqsubseteq_c (\mathbb{D}, \mathbb{G}_{\beta'})$. Lemma 3.10 shows that

$$(2^\omega, \mathbb{R}_{\beta'}) \sqsubseteq_c \big(2^\omega, (\mathbb{G}_\beta)_{0,1}\big) = (2^\omega, \mathbb{R}_\beta),$$

which implies that $\beta = \beta'$ by Lemma 3.9. □

Corollary 3.13 *There is a concrete $\sqsubseteq_c$-antichain of size continuum made of locally countable Borel graphs on $\mathbb{D}$ which are $\sqsubseteq_c$-minimal among non-$\boldsymbol{\Sigma}^0_2$ graphs on an analytic space.*

Proof. We apply Lemmas 3.11 and 3.12. □

We now turn to the study of the class $\boldsymbol{\Pi}^0_2$.

Lemma 3.14 *Assume that $\boldsymbol{\Gamma} = \boldsymbol{\Pi}^0_2$ and $\beta \in N_0 \setminus \{0^\infty\}$. Then $\mathbb{R}_\beta$ has a non-$\boldsymbol{\Pi}^0_2$ section.*

Proof. We argue by contradiction, which implies that $\mathbb{R}_\beta$ has nowhere dense sections since it is locally countable. By Corollary 2.12, $\big(2^\omega, \Delta(\mathbb{C})\big) \sqsubseteq_c (2^\omega, \mathbb{R}_\beta)$. Note that $(2^\omega, \mathbb{R}_\beta) \sqsubseteq_c \big(2^\omega, \Delta(\mathbb{C})\big)$ by Theorem 1.2 and Corollary 3.5, which contradicts the injectivity since $\beta \in N_0 \setminus \{0^\infty\}$. □

Remark. Lemma 3.14 shows that a locally countable relation can have a non-$\boldsymbol{\Pi}^0_2$ section, and therefore be non-$\boldsymbol{\Pi}^0_2$ because of that. This is not the case for the class $\boldsymbol{\Sigma}^0_2$. This is the reason why we cannot argue for the class $\boldsymbol{\Pi}^0_2$ as we did in Corollary 3.13 for the class $\boldsymbol{\Sigma}^0_2$. More precisely, note that $\mathbb{G}_m^{\boldsymbol{\Pi}^0_2,a}$ is a (locally) countable graph with a non-$\boldsymbol{\Pi}^0_2$ section. Moreover, $(\mathbb{S}, \mathbb{G}_m^{\boldsymbol{\Pi}^0_2,a}) \sqsubseteq_c (\mathbb{D}, \mathbb{G}_\beta)$ for any $\beta \in N_0 \setminus \{0^\infty\}$. Indeed, Lemma 3.14 gives $\alpha_0 \in 2^\omega$ such that $(\mathbb{R}_\beta)_{\alpha_0}$ is not $\boldsymbol{\Pi}^0_2$ since $\mathbb{R}_\beta$ is symmetric. By the Hurewicz theorem, there is $g : 2^\omega \to 2^\omega$ injective continuous such that $Q_2 = g^{-1}\big((\mathbb{R}_\beta)_{\alpha_0}\big)$. We define $f : \mathbb{S} \to \mathbb{D}$ by $f(0^\infty) := (0, \alpha_0)$ and $f(1\alpha) := \big(1, g(\alpha)\big)$. Note that f is injective continuous and

$$\begin{aligned}
(\varepsilon\alpha, \eta\gamma) \in \mathbb{G}_m^{\boldsymbol{\Pi}^0_2,a} &\Leftrightarrow (\varepsilon = 0 \wedge \alpha = 0^\infty \wedge \eta = 1 \wedge \gamma \in Q_2) \vee (\varepsilon = 1 \wedge \alpha \in Q_2 \wedge \eta = 0 \wedge \gamma = 0^\infty) \\
&\Leftrightarrow \Big(\big(f(\varepsilon\alpha), f(\eta\gamma)\big) = \Big((0, \alpha_0), \big(1, g(\gamma)\big)\Big) \wedge \big(\alpha_0, g(\gamma)\big) \in \mathbb{R}_\beta \Big) \vee \\
&\qquad \Big(\big(f(\varepsilon\alpha), f(\eta\gamma)\big) = \Big(\big(1, g(\alpha)\big), (0, \alpha_0)\Big) \wedge \big(\alpha_0, g(\alpha)\big) \in \mathbb{R}_\beta \Big) \\
&\Leftrightarrow \big(f(\varepsilon\alpha), f(\eta\gamma)\big) \in \mathbb{G}_\beta.
\end{aligned}$$

So we need to find other examples for the class $\boldsymbol{\Pi}^0_2$.

Theorem 3.15 *There is a concrete $\sqsubseteq_c$-antichain of size continuum made of locally countable Borel graphs on 2^ω which are $\sqsubseteq_c$-minimal among non-$\mathbf{\Pi}^0_2$ graphs on an analytic space.*

Proof. We set, for $\beta\in N_0\setminus\{0^\infty\}$, $\mathbb{H}_\beta:=\mathbb{R}_\beta\setminus\Delta(2^\omega)$. As $\mathbb{R}_\beta$ is symmetric, $\mathbb{H}_\beta$ is a graph. In particular, we can speak of proj$[\mathbb{H}_\beta]$. By Lemma 3.14, $\mathbb{R}_\beta$ has a non-$\mathbf{\Pi}^0_2$ section. A consequence of this is that $\mathbb{H}_\beta$ is not $\mathbf{\Pi}^0_2$. Let X be an analytic space and R be a non-$\mathbf{\Pi}^0_2$ relation on X with $(X,R)\sqsubseteq_c(2^\omega,\mathbb{H}_\beta)$, with witness f. By the injectivity of f, we get $\big(X,R\cup(f\times f)^{-1}\big(\Delta(\mathbb{C})\big)\big)\sqsubseteq_c(2^\omega,\mathbb{R}_\beta)$. As $\Delta(\mathbb{C})\in\mathbf{\Sigma}^0_2$ is disjoint from $\mathbb{H}_\beta$, $R\cup(f\times f)^{-1}\big(\Delta(\mathbb{C})\big)$ is not $\mathbf{\Pi}^0_2$. By Corollary 3.8 and the injectivity of f, $(2^\omega,\mathbb{R}_\beta)\sqsubseteq_c\big(X,R\cup(f\times f)^{-1}\big(\Delta(\mathbb{C})\big)\big)$. By the injectivity of f again, $(2^\omega,\mathbb{H}_\beta)\sqsubseteq_c(X,R)$, showing the minimality of $\mathbb{H}_\beta$.

Assume now that $\beta,\beta'\in N_0\setminus\{0^\infty\}$ and $(2^\omega,\mathbb{H}_\beta)\sqsubseteq_c(2^\omega,\mathbb{H}_{\beta'})$, with witness f.

Claim. *Let $u_0\in 2^{<\omega}$ with $u_0 0^\infty\in$ proj$[\mathbb{H}_\beta]$. Then we can find $u_0'\in Q$ and $u_1,u_1'\in Q\setminus\{\emptyset\}$ such that $f(u_0 0^\infty)=u_0'0^\infty$, $u_0u_1 0^\infty\in$ proj$[\mathbb{H}_\beta]$ and $f(u_0u_1 0^\infty)=u_0'u_1'0^\infty$.*

Indeed, as $u_0 0^\infty\in$ proj$[\mathbb{H}_\beta]$, there is γ such that $(u_0 0^\infty,\gamma)$ is in $\mathbb{H}_\beta$, and $\big(f(u_0 0^\infty),f(\gamma)\big)$ is therefore in $\mathbb{H}_{\beta'}$. In particular, $f(u_0 0^\infty)\in Q_2$ and there is $u_0'\in Q$ with $f(u_0 0^\infty)=u_0'0^\infty$. As f is continuous, there is k_0 such that $f[N_{u_0 0^{k_0}}]\subseteq N_{u_0'}$.

Let us prove that proj$[\mathbb{H}_\beta]$ is dense in 2^ω. Let $s\in 2^{<\omega}$, and $p:=2k+\varepsilon\geq 1$ with $\beta(p)=1$. We set $(z,t):=\big(s1(10)^{k+\varepsilon}1,s1(01)^k\big)$. Note that $z\neq t\in Q$, $i(z,t)=p$, $(z0^\infty,t0^\infty)\in\mathbb{H}_\beta\cap N_s^2$ and proj$[\mathbb{H}_\beta]$ meets N_s as desired.

Let $\alpha\in$ proj$[\mathbb{H}_\beta]\cap N_{u_0 0^{k_0}1}$, and δ such that (α,δ) is in $\mathbb{H}_\beta$. Then $\big(f(\alpha),f(\delta)\big)$ is in $\mathbb{H}_{\beta'}$, and $f(\alpha)\in Q_2\cap N_{u_0'}\setminus\{u_0'0^\infty\}$ by injectivity of f. This gives $u_1,u_1'\in Q\setminus\{\emptyset\}$ with $\alpha=u_0u_1 0^\infty$ and $f(\alpha)=u_0'u_1'0^\infty$. $\diamond$

The end of the proof is now similar to that of Lemma 3.9. Let us indicate the differences. We first apply the claim to $u_0:=t_0\in Q$ such that $t_0 0^\infty\in$ proj$[\mathbb{H}_\beta]$, which gives $t_0'\in Q$ and $t_1,t_1'\in Q\setminus\{\emptyset\}$ such that $f(t_0 0^\infty)=t_0'0^\infty$, $t_0t_1 0^\infty\in$ proj$[\mathbb{H}_\beta]$ and $f(t_0t_1 0^\infty)=t_0't_1'0^\infty$. The continuity of f provides $k_1\geq|t_1|$ such that $f[N_{t_0 0^{k_1}}]\subseteq N_{t_0'0^{|t_1'|}}$. We next apply the claim to $u_0:=t_0 0^{k_1}$, which gives $\tilde t_2,\tilde t_2'$ in $Q\setminus\{\emptyset\}$ such that $t_0 0^{k_1}\tilde t_2 0^\infty\in$ proj$[\mathbb{H}_\beta]$ and $f(t_0 0^{k_1}\tilde t_2 0^\infty)=t_0'\tilde t_2'0^\infty$. This provides sequences $(t_j)_{j\in\omega}$ and $(t_j')_{j\in\omega}$ as in the proof of Lemma 3.9. By the remark after the claim in the proof of Lemma 3.9, $\beta=\beta'$. $\square$

4 Acyclicity

Remark. Assume that $\beta\in N_0\setminus\{0^\infty\}$. Then $\mathbb{R}_\beta$ is not s-acyclic. Indeed, $(0^\infty,10^\infty,1^2 0^\infty)$ is an $s(\mathbb{R}_\beta)$-cycle if $\beta(1)=1$, $(10^\infty,010^\infty,0^2 10^\infty)$ is an $s(\mathbb{R}_\beta)$-cycle if $\beta(2)=1$,

$$(101^{k+1}0^\infty,1^{k+3}0^\infty,0101^k 0^\infty)$$

is an $s(\mathbb{R}_\beta)$-cycle if $\beta(2k+3)=1$, and $(0101^2 0(110)^k 0^\infty,1010^3(10^2)^k 0^\infty,010^2 1^2(101)^k 0^\infty)$ is an $s(\mathbb{R}_\beta)$-cycle if $\beta(2k{+}4)=1$. We will see that Theorem 1.2 can be extended, under a suitable acyclicity assumption. We need to introduce new examples.

Notation. Let $\mathbf{\Gamma}$ be a non self-dual Borel class of rank at least two, and $(2^\omega,\mathbb{C})$ be $\mathbf{\Gamma}$-good. We set $A:=\big\{t\in 4^{(2^2)}\mid t(0,0)\in 2\wedge\big(t(0,1)=0\vee t(1,0)=0\big)\wedge t(1,1)\neq 0\big\}$. We then set, for $t\in A$,

$$\mathbb{R}_t^{\mathbf{\Gamma},a}:=\{(0^\infty,0^\infty)\mid t(0,0)=1\}\cup\{(0^\infty,1\alpha)\mid\alpha\in S_{t(0,1)}\}\cup\{(1\alpha,0^\infty)\mid\alpha\in S_{t(1,0)}\}\cup\{(1\alpha,1\alpha)\mid\alpha\in S_{t(1,1)}\}.$$

Note that $\mathbb{G}_m^{\mathbf{\Gamma},a}=\mathbb{R}_{0,0,0,1}^{\mathbf{\Gamma},a}$. Finally $\mathcal{B}^{\mathbf{\Gamma}}:=\big\{(\mathbb{S},\mathbb{R}_t^{\mathbf{\Gamma},a})\mid t\in A\big\}$.

Lemma 4.1 *Let $\mathbf{\Gamma}$ be a non self-dual Borel class of rank at least two. Then $\mathcal{A}^{\mathbf{\Gamma}}\cup\mathcal{B}^{\mathbf{\Gamma}}$ is a 76 element $\leq_c$-antichain.*

Proof. Assume that $t,t'\in A$ and $\mathbb{R}_t^{\mathbf{\Gamma},a}$ is $\leq_c$-below $\mathbb{R}_{t'}^{\mathbf{\Gamma},a}$ with witness $f:\mathbb{S}\to\mathbb{S}$. If $\mathbb{R}_{t'}^{\mathbf{\Gamma},a}$ is symmetric, then $\mathbb{R}_t^{\mathbf{\Gamma},a}$ is too, and t' is of the form $(\varepsilon,0,0,\varepsilon')$. As $\mathbb{R}_t^{\mathbf{\Gamma},a},\mathbb{R}_{t'}^{\mathbf{\Gamma},a}$ have only one vertical section not in $\mathbf{\Gamma}$, $f(0^\infty)=0^\infty$, $f(1,\alpha)(0)=1$ by the choice of $\mathbb{C}$, and the function $f_1(1,.)$ defined by $1f_1(1,\alpha)=f(1\alpha)$ preserves $\mathbb{C}$. Thus $t=t'$. So we may assume that $\mathbb{R}_{t'}^{\mathbf{\Gamma},a}$ is not symmetric, i.e., t' is of the form $(\varepsilon,\varepsilon',\varepsilon'',\varepsilon''')$ with $\varepsilon'\neq\varepsilon''$, and $\varepsilon'=0$ or $\varepsilon''=0$. If, for example, $\varepsilon'\neq 0$, then $\mathbb{R}_{t'}^{\mathbf{\Gamma},a}$ has vertical sections in $\mathbf{\Gamma}$, as well as $\mathbb{R}_t^{\mathbf{\Gamma},a}$, which implies that t is of the form $(\eta,\eta',0,\eta''')$ with $\eta'\neq 0$. Note that $\mathbb{R}_t^{\mathbf{\Gamma},a},\mathbb{R}_{t'}^{\mathbf{\Gamma},a}$ have only one horizontal section not in $\mathbf{\Gamma}$, $f(0^\infty)=0^\infty$, $f(1,\alpha)(0)=1$ by the choice of $\mathbb{C}$, and $f_1(1,.)$ preserves $\mathbb{C}$. Thus $t=t'$, even if $\varepsilon''\neq 0$.

As the elements of $\mathcal{B}^{\mathbf{\Gamma}}$ have a section not in $\mathbf{\Gamma}$ and the elements of $\mathcal{A}^{\mathbf{\Gamma}}$ have closed sections, an element of $\mathcal{B}^{\mathbf{\Gamma}}$ is not $\leq_c$-below an element of $\mathcal{A}^{\mathbf{\Gamma}}$. By Theorem 1.2, it remains to see that an element $\mathbb{R}$ of $\mathcal{A}^{\mathbf{\Gamma}}$ is not $\leq_c$-below an element $\mathbb{R}_t^{\mathbf{\Gamma},a}$ of $\mathcal{B}^{\mathbf{\Gamma}}$. We argue by contradiction, which provides $f:2^\omega\to\mathbb{S}$ or $f:\mathbb{D}\to\mathbb{S}$. In the first case, note first that $f(\alpha)(0)=1$, since otherwise $f(\beta)(0)=0$ if β is in a clopen neighbourhood C of α. If $\beta\neq\gamma\in C\cap\mathbb{C}$, then $f(\beta)=f(\gamma)=0^\infty$, which implies that $(0^\infty,0^\infty)\in\mathbb{R}_t^{\mathbf{\Gamma},a}$ and $(\beta,\gamma)\in\Delta(\mathbb{C})$, which is absurd. This shows that $\Delta(\mathbb{C})\in\mathbf{\Gamma}$ since $(\mathbb{R}_t^{\mathbf{\Gamma},a})_{1,1}\in\mathbf{\Gamma}$, which is absurd. In the second case, note first that $f(0,\alpha)(0)=1$, since otherwise $f(0,\beta)(0)=0$ if β is in a clopen neighbourhood C of α. We may assume that there is $\varepsilon_0\in 2$ with the property that $f(1,\beta)(0)=\varepsilon_0$ if $\beta\in C$. If $\beta\neq\gamma\in C\cap\mathbb{C}$, then either $\varepsilon_0=0$, $\big(f(0,\beta),f(1,\gamma)\big)=(0^\infty,0^\infty)\in\mathbb{R}_t^{\mathbf{\Gamma},a}$ and $\big((0,\beta),(1,\gamma)\big)\in\mathbb{R}$, or $\varepsilon_0=1$, $\big(f(0,\beta),f(1,\beta)\big),\big(f(0,\gamma),f(1,\gamma)\big)\in\mathbb{R}_t^{\mathbf{\Gamma},a}$, $f(1,\beta),f(1,\gamma)$ are in $\{1\alpha\mid\alpha\in\mathbb{C}\}$, $\big(f(0,\beta),f(1,\gamma)\big)\in\mathbb{R}_t^{\mathbf{\Gamma},a}$ and $\big((0,\beta),(1,\gamma)\big)\in\mathbb{R}$, which is absurd. Similarly, $f(1,\alpha)(0)=1$. This shows that $\mathbb{R}_{0,1}\in\mathbf{\Gamma}$ since $(\mathbb{R}_t^{\mathbf{\Gamma}})_{1,1}\in\mathbf{\Gamma}$, which is absurd. □

We now study the rank two case.

Lemma 4.2 *Let R be an s-acyclic Borel relation on I_2. Then we can find a sequence $(R_n)_{n\in\omega}$ of relations closed in $I_2\times 2^\omega$ and $2^\omega\times I_2$, as well as $f:2^\omega\to 2^\omega$ injective continuous preserving Q_2 such that $(f\times f)^{-1}(R)\subseteq\bigcup_{n\in\omega}R_n$.*

Proof. Assume first that R_{α_0} is not separable from Q_2 by a set in $\mathbf{\Sigma}_2^0$, for some $\alpha_0\in I_2$. Theorem 2.5 provides $h:2^\omega\to 2^\omega\setminus\{\alpha_0\}$ injective continuous such that $I_2\subseteq h^{-1}(R_{\alpha_0})$ and $Q_2\subseteq h^{-1}(Q_2)$. We set $R':=(h\times h)^{-1}(R)$. If $\alpha,\beta,\gamma\in I_2$ are pairwise distinct and $\beta,\gamma\in R'_\alpha$, then $\big(h(\beta),h(\alpha),h(\gamma),\alpha_0\big)$ is an $s(R)$-cycle, which is absurd. Thus R' has vertical sections of cardinality at most two. So, replacing R with R' if necessary, we may assume that R_α is separable from Q_2 by a set in $\mathbf{\Sigma}_2^0$ for each $\alpha\in 2^\omega$. Similarly, we may assume that R^α is separable from Q_2 by a set in $\mathbf{\Sigma}_2^0$ for each $\alpha\in 2^\omega$. It remains to apply Lemma 2.6. □

Lemma 4.3 *(a) Let R be an s-acyclic subrelation of Q_2^2. Then there is $f:2^\omega\to 2^\omega$ injective continuous preserving Q_2 such that $(f\times f)^{-1}(R)\subseteq\Delta(Q_2)$.*

(b) Let R be an s-acyclic subrelation of $(2\times Q_2)^2$. Then there is $f:2^\omega\to 2^\omega$ injective continuous preserving Q_2 such that $(f\times f)^{-1}(\bigcup_{\varepsilon,\eta\in 2}\ R_{\varepsilon,\eta})\subseteq\Delta(Q_2)$.

Proof. (a) Assume that R_{α_0} is not nowhere dense for some $\alpha_0\in Q_2$. This gives $s\in 2^{<\omega}$ with the property that $N_s\subseteq\overline{R_{\alpha_0}}$. Note that $N_s\cap R_{\alpha_0}\setminus\{\alpha_0\}$ is not separable from $N_s\cap I_2$ by a $\mathbf{\Pi}^0_2$ set, by Baire's theorem. Theorem 2.5 provides $h:2^\omega\to N_s\setminus\{\alpha_0\}$ injective continuous such that $Q_2\subseteq h^{-1}(R_{\alpha_0})$ and $I_2\subseteq h^{-1}(I_2)$. If $R':=(h\times h)^{-1}(R)$, $\alpha,\beta,\gamma\in Q_2$ are pairwise distinct and $\beta,\gamma\in R'_\alpha$, then $\big(h(\beta),h(\alpha),h(\gamma),\alpha_0\big)$ is an $s(R)$-cycle, which is absurd. This shows that, replacing R with R' if necessary, we may assume that R has vertical sections of cardinality at most two. In any case, we may assume that R has nowhere dense vertical sections. Similarly, we may assume that R has nowhere dense horizontal sections. It remains to apply Lemma 2.11.

(b) We argue similarly. Fix $\varepsilon,\eta\in 2$. We replace R with $R_{\varepsilon,\eta}$. We just have to note that the cycle $\big(h(\beta),h(\alpha),h(\gamma),\alpha_0\big)$ becomes $\Big(\big(\eta,h(\beta)\big),\big(\varepsilon,h(\alpha)\big),\big(\eta,h(\gamma)\big),\big(\varepsilon,\alpha_0\big)\Big)$. □

Lemma 4.4 *(a) Let R be an s-acyclic Borel subrelation of $(Q_2\times I_2)\cup(I_2\times Q_2)$. Then there is $f:2^\omega\to 2^\omega$ injective continuous preserving Q_2 such that $(f\times f)^{-1}(R)=\emptyset$.*

(b) Let R be an s-acyclic Borel subrelation of $\big((2\times Q_2)\times(2\times I_2)\big)\cup\big((2\times I_2)\times(2\times Q_2)\big)$. Then there is $f:2^\omega\to 2^\omega$ injective continuous preserving Q_2 such that $(f\times f)^{-1}(\bigcup_{\varepsilon,\eta\in 2}\ R_{\varepsilon,\eta})=\emptyset$.

Proof. (a) By symmetry, we may assume that $R\subseteq Q_2\times I_2$. Assume first that R_{α_0} is not meager for some $\alpha_0\in Q_2$. This gives $s\in 2^{<\omega}$ with the property that $N_s\cap R_{\alpha_0}$ is comeager in N_s. In particular, $N_s\cap Q_2\setminus\{\alpha_0\}$ is not separable from $N_s\cap R_{\alpha_0}$ by a $\mathbf{\Pi}^0_2$ set, by Baire's theorem. Theorem 2.5 provides $h:2^\omega\to N_s\setminus\{\alpha_0\}$ injective continuous with $Q_2\subseteq h^{-1}(Q_2)$ and $I_2\subseteq h^{-1}(R_{\alpha_0})$. If $R':=(h\times h)^{-1}(R)$, $\alpha\in Q_2$, $\beta\neq\gamma\in R'_\alpha$, then $\big(h(\beta),h(\alpha),h(\gamma),\alpha_0\big)$ is an $s(R)$-cycle, which is absurd. This shows that, replacing R with R' if necessary, we may assume that R has meager vertical sections. Let F be a meager $\mathbf{\Sigma}^0_2$ subset of 2^ω containing $\bigcup_{\alpha\in Q_2}\ R_\alpha$. Note that Q_2 is not separable from $I_2\setminus F$ by a $\mathbf{\Pi}^0_2$ set. Theorem 2.5 provides $f:2^\omega\to 2^\omega$ injective continuous with $Q_2\subseteq f^{-1}(Q_2)$ and $I_2\subseteq f^{-1}(I_2\setminus F)$. Thus $(f\times f)^{-1}(R)$ is empty.

(b) We argue as in the proof of Lemma 4.3.(b). □

Corollary 4.5 *(a) Let R be an s-acyclic Borel relation on 2^ω. Then there is $f:2^\omega\to 2^\omega$ injective continuous preserving Q_2 such that $(f\times f)^{-1}(R)\subseteq\Delta(2^\omega)$.*

(b) Let R be an s-acyclic Borel relation on $\mathbb{D}$. Then there is $f:2^\omega\to 2^\omega$ injective continuous preserving Q_2 such that $(f\times f)^{-1}(\bigcup_{\varepsilon,\eta\in 2}\ R_{\varepsilon,\eta})\subseteq\Delta(2^\omega)$.

Proof. (a) By Lemma 4.3, we may assume that $R\cap Q_2^2\subseteq\Delta(2^\omega)$. By Lemma 4.4, we may assume that $R\cap\big((Q_2\times I_2)\cup(I_2\times Q_2)\big)=\emptyset$. By Lemma 4.2, we may assume that $R\setminus\Delta(2^\omega)$ is contained in the union of a sequence $(R_n)_{n\in\omega}$ of relations on I_2 which are closed in $I_2\times 2^\omega$ and in $2^\omega\times I_2$. By Lemma 2.8 applied to $(R_n)_{n\in\omega}$, we may assume that $R\cap I_2^2\subseteq\Delta(2^\omega)$.

(b) By Lemma 4.3, we may assume that $\bigcup_{\varepsilon,\eta\in 2}\ R_{\varepsilon,\eta}\cap Q_2^2\subseteq\Delta(2^\omega)$. By Lemma 4.4, we may assume that $R\cap\Big(\big((2\times Q_2)\times(2\times I_2)\big)\cup\big((2\times I_2)\times(2\times Q_2)\big)\Big)=\emptyset$.

By Lemma 4.2, we may assume that $(\bigcup_{\varepsilon,\eta\in 2}\ R_{\varepsilon,\eta})\backslash\Delta(2^\omega)$ is contained in the union of a sequence $(R_n)_{n\in\omega}$ of relations on I_2 which are closed in $I_2\times 2^\omega$ and in $2^\omega\times I_2$. By Lemma 2.8 applied to $(R_n)_{n\in\omega}$, we may assume that $\bigcup_{\varepsilon,\eta\in 2}\ R_{\varepsilon,\eta}\cap I_2^2\subseteq\Delta(2^\omega)$. $\square$

Theorem 1.9 for classes of rank at least three is a consequence of Lemma 4.1 and the following result since the elements of $\mathcal{A}^{\mathbf{\Gamma}}$ are contained in either $\Delta(2^\omega)$, or in $\big\{\big((\varepsilon,x),(\eta,x)\big)\in\mathbb{D}^2\mid x\in 2^\omega\big\}$, and the elements of $\mathcal{B}^{\mathbf{\Gamma}}$ are contained in $\Delta(\mathbb{S})\cup(\{0^\infty\}\times N_1)\cup(N_1\times\{0^\infty\})$, which are s-acyclic and closed on the one side, and $\mathbb{G}_m^{\mathbf{\Gamma}}$ and $\mathbb{G}_m^{\mathbf{\Gamma},a}$ are the only graphs in $\mathcal{A}^{\mathbf{\Gamma}}\cup\mathcal{B}^{\mathbf{\Gamma}}$ on the other side. The next proof is the last one using effective descriptive set theory.

Notation. We set, for any relation R on $\mathbb{S}$, $R_{1,1}:=\{(\alpha,\beta)\in 2^\omega\times 2^\omega\mid(1\alpha,1\beta)\in R\}$.

Theorem 4.6 *Let $\mathbf{\Gamma}$ be a non self-dual Borel class of rank at least three, X be an analytic space, and R be a Borel relation on X contained in an s-acyclic Borel relation with $\mathbf{\Sigma}^0_2$ vertical sections. Exactly one of the following holds:*

(a) the relation R is a $\mathbf{\Gamma}$ subset of X^2,

(b) there is $(\mathbb{X},\mathbb{R})\in\mathcal{A}^{\mathbf{\Gamma}}\cup\mathcal{B}^{\mathbf{\Gamma}}$ such that $(\mathbb{X},\mathbb{R})\sqsubseteq_c(X,R)$.

Proof. By Theorem 1.2 and since the elements of $\mathcal{B}^{\mathbf{\Gamma}}$ have a section not in $\mathbf{\Gamma}$, (a) and (b) cannot hold simultaneously. Assume that (a) does not hold. By Theorem 2.2, one of the following holds:

(1) the relation R has at least one section not in $\mathbf{\Gamma}$,

(2) there is a relation $\mathcal{R}$ on 2^ω such that $\mathcal{R}\cap\Delta(2^\omega)=\Delta(\mathbb{C})$ and $(2^\omega,\mathcal{R})\sqsubseteq_c(X,R)$,

(3) there is a relation $\mathcal{R}$ on $\mathbb{D}$ such that $\mathbb{R}_{0,1}\cap\Delta(2^\omega)=\Delta(\mathbb{C})$ and $(\mathbb{D},\mathcal{R})\sqsubseteq_c(X,R)$.

(1) Let $x_0\in X$ such that, for example, R_{x_0} is not in $\mathbf{\Gamma}$, the other case being similar. Note that $R_{x_0}\backslash\{x_0\}$ is not separable from $X\backslash(R_{x_0}\cup\{x_0\})$ by a set in $\mathbf{\Gamma}$. Theorem 2.5 provides $h:2^\omega\to X\backslash\{x_0\}$ injective continuous with $\mathbb{C}=h^{-1}(R_{x_0})$. We define $g:\mathbb{S}\to X$ by $g(0^\infty):=x_0$ and $g(1\alpha):=h(\alpha)$. Note that g is injective continuous. Considering $(g\times g)^{-1}(R)$ if necessary, we may assume that $X=\mathbb{S}$, $x_0=0^\infty$ and $R_{x_0}\cap N_1=\{1\alpha\mid\alpha\in\mathbb{C}\}$. Let $\mathcal{A}$ be an s-acyclic Borel relation with $\mathbf{\Sigma}^0_2$ vertical sections containing R.

For the simplicity of the notation, we assume that the rank of $\mathbf{\Gamma}$ is less than ω_1^{CK}, and $\mathbb{C},\mathcal{A}$ are Δ^1_1. Theorem 3.5 in [Lo1] gives a sequence $(\mathcal{C}_n)$ of Δ^1_1 relations with closed vertical sections such that $\mathcal{A}=\bigcup_{n\in\omega}\ \mathcal{C}_n$. By Lemma 2.2.2 in [L5], $\Delta^1_1\cap 2^\omega$ is countable and Π^1_1, which implies that $V:=2^\omega\setminus(\Delta^1_1\cap 2^\omega)$ is Σ^1_1, disjoint from $\Delta^1_1\cap 2^\omega$, and $V\cap\mathbb{C}$ is not separable from $V\setminus\mathbb{C}$ by a set in $\mathbf{\Gamma}$. We will apply Theorem 3.2 in [L5], where the **Gandy-Harrington topology** Σ_{2^ω} on 2^ω generated by $\Sigma^1_1(2^\omega)$ is used. Let us prove that $\mathcal{A}_{1,1}\cap V^2$ is $(\Sigma_{2^\omega})^2$-meager in V^2. It is enough to see that $(\mathcal{C}_n)_{1,1}\cap V^2$ is $(\Sigma_{2^\omega})^2$-nowhere dense in V^2 for each n. By Lemma 3.1 in [L5], $(\mathcal{C}_n)_{1,1}$ is $(\Sigma_{2^\omega})^2$-closed. We argue by contradiction, which gives n and nonempty Σ^1_1 subsets S,T of 2^ω with the property that $S\times T\subseteq(\mathcal{C}_n)_{1,1}\cap V^2$. By the effective perfect set Theorem (see 4.F1 in [Mo]), S,T are uncountable. So pick $x,y\in S$ and $z,t\in T$ pairwise different. Then $(1x,1z,1y,1t)$ is an $s(\mathcal{A})$-cycle, which is absurd. Theorem 3.2 in [L5] provides $f:2^\omega\to 2^\omega$ injective continuous preserving $\mathbb{C}$ with the property that $\big(f(\alpha),f(\beta)\big)\notin\mathcal{A}_{1,1}$ if $\alpha\neq\beta$.

Considering the set $(f\times f)^{-1}(R)$ if necessary, we may assume that $R_{1,1}\subseteq\Delta(2^\omega)$. We set $E_{1,1}:=\{\alpha\in 2^\omega\mid(\alpha,\alpha)\in R_{1,1}\}$. Note that $E_{1,1}$ is a Borel subset of 2^ω and $R_{1,1}=\Delta(E_{1,1})$. By Lemma 2.13, we may assume that $E_{1,1}=S_j$ for some $j\in 4$. If $j=0$, then $\big(2^\omega,\Delta(\mathbb{C})\big)\sqsubseteq_c(X,R)$.

So we may assume that $j \neq 0$. Similarly, $\{\alpha \in 2^\omega \mid (1\alpha, 0^\infty) \in R\} = S_j$ for some $j \in 4$. This provides $t \in A$ with $(\mathbb{S}, \mathbb{R}_t^{\mathbf{\Gamma},a}) \sqsubseteq_c (X, R)$.

(2) We partially argue as in (1). Note that $\mathcal{R}$ is Borel and contained in an s-acyclic Borel relation with $\mathbf{\Sigma}^0_2$ vertical sections $\mathcal{A}$.

This time, $\mathcal{C}_n \in \Delta^1_1\big((2^\omega)^2\big)$. Let us prove that $\mathcal{C}_n \cap V^2$ is $(\Sigma_{2^\omega})^2$-nowhere dense in V^2 for each n. We argue by contradiction, which gives n and nonempty Σ^1_1 subsets S, T of 2^ω with $S \times T \subseteq \mathcal{C}_n \cap V^2$. Note that (x, z, y, t) is an $s(\mathcal{A})$-cycle, which is absurd. Theorem 3.2 in [L5] provides $f : 2^\omega \to 2^\omega$ injective continuous preserving $\mathbb{C}$ such that $\big(f(\alpha), f(\beta)\big) \notin \mathcal{A}$ if $\alpha \neq \beta$.

Considering $(f \times f)^{-1}(\mathcal{R})$ if necessary, we may assume that $\mathcal{R} \subseteq \Delta(2^\omega)$, which means that $\big(2^\omega, \Delta(\mathbb{C})\big) \sqsubseteq_c (X, R)$.

(3) We partially argue as in (2).

This time, $\mathcal{C}_n \in \Delta^1_1(\mathbb{D}^2)$. Fix $\varepsilon, \eta \in 2$. Let us prove that $(\mathcal{C}_n)_{\varepsilon,\eta} \cap V^2$ is $(\Sigma_{2^\omega})^2$-nowhere dense in V^2 for each n. We argue by contradiction, which gives n and nonempty Σ^1_1 subsets S, T of 2^ω with $S \times T \subseteq (\mathcal{C}_n)_{\varepsilon,\eta} \cap V^2$. Note that $\big((\varepsilon, x), (\eta, z), (\varepsilon, y), (\eta, t)\big)$ is an $s(\mathcal{A})$-cycle, which is absurd. Theorem 3.2 in [L5] provides $f : 2^\omega \to 2^\omega$ injective continuous preserving $\mathbb{C}$ such that $\big(f(\alpha), f(\beta)\big) \notin \bigcup_{\varepsilon,\eta \in 2} \mathcal{A}_{\varepsilon,\eta}$ if $\alpha \neq \beta$.

Considering $(f \times f)^{-1}(\mathcal{R})$ if necessary, we may assume that $\mathcal{R}_{\varepsilon,\eta} \subseteq \Delta(2^\omega)$ and $\mathcal{R}_{0,1} = \Delta(\mathbb{C})$. Theorem 1.2 provides $(\mathbb{X}, \mathbb{R}) \in \mathcal{A}^{\mathbf{\Gamma}}$ such that $(\mathbb{X}, \mathbb{R}) \sqsubseteq_c (X, R)$.

So (b) holds in any case. □

Theorem 1.8 is an immediate consequence of the following result.

Theorem 4.7 *Let $\mathbf{\Gamma}$ be a non self-dual Borel class of rank two, X be an analytic space, and R be an s-acyclic Borel relation on X. Exactly one of the following holds:*

(a) the relation R is a $\mathbf{\Gamma}$ subset of X^2,

(b) there is $(\mathbb{X}, \mathbb{R}) \in \mathcal{A}^{\mathbf{\Gamma}} \cup \mathcal{B}^{\mathbf{\Gamma}}$ such that $(\mathbb{X}, \mathbb{R}) \sqsubseteq_c (X, R)$.

Proof. We partially argue as in the proof of Theorem 4.6. For the case (1), recall that we may assume that $X = \mathbb{S}$, $x_0 = 0^\infty$ and $R_{x_0} \cap N_1 = \{1\alpha \mid \alpha \in \mathbb{C}\}$. Corollary 4.5 provides $f : 2^\omega \to 2^\omega$ injective continuous preserving $\mathbb{C}$ such that $\big(f(\alpha), f(\beta)\big) \notin R_{1,1}$ if $\alpha \neq \beta$. For the case (2), $\mathcal{R}$ is s-acyclic Borel, and by Corollary 4.5 we may assume that $\mathcal{R} \subseteq \Delta(2^\omega)$, which means that $\big(2^\omega, \Delta(\mathbb{C})\big) \sqsubseteq_c (X, R)$. For the case (3), by Corollary 4.5 we may assume that $\bigcup_{\varepsilon,\eta \in 2} R_{\varepsilon,\eta} \subseteq \Delta(2^\omega)$ and $\mathcal{R}_{0,1} = \Delta(\mathbb{C})$. □

Theorem 1.6 is an immediate consequence of the following result.

Corollary 4.8 *Let X be an analytic space, and R be an s-acyclic Borel relation on X whose sections are in $\mathbf{\Sigma}^0_2$. Exactly one of the following holds:*

(a) the relation R is a $\mathbf{\Sigma}^0_2$ subset of X^2,

(b) there is $(\mathbb{X}, \mathbb{R}) \in \mathcal{A}^{\mathbf{\Sigma}^0_2}$ such that $(\mathbb{X}, \mathbb{R}) \sqsubseteq_c (X, R)$.

In particular, $\mathcal{A}^{\mathbf{\Sigma}^0_2}$ is a 34 element $\sqsubseteq_c$ and $\leq_c$-antichain basis.

Proof. We apply Theorem 4.7 and use the fact that the elements of $\mathcal{B}^{\mathbf{\Gamma}}$ have a section not in $\mathbf{\Gamma}$. □

Notation. We set

$$\mathcal{C}^{\mathbf{\Pi}^0_2} := \mathcal{A}^{\mathbf{\Pi}^0_2} \cup \big\{(\mathbb{S}, \mathbb{R}^{\mathbf{\Pi}^0_2, a}_t) \mid t \in 4^{(2^2)} \wedge t(0,0), t(0,1), t(1,0) \in 2 \wedge \big(t(0,1)=0 \vee t(1,0)=0\big) \wedge t(1,1) \neq 0\big\}.$$

Theorem 1.7 is an immediate consequence of the following result.

Corollary 4.9 *Let X be an analytic space, and R be an s-acyclic locally countable Borel relation on X. Exactly one of the following holds:*

(a) the relation R is a $\mathbf{\Pi}^0_2$ subset of X^2,

(b) there is $(\mathbb{X}, \mathbb{R}) \in \mathcal{C}^{\mathbf{\Pi}^0_2}$ such that $(\mathbb{X}, \mathbb{R}) \sqsubseteq_c (X, R)$.

Moreover, $\mathcal{C}^{\mathbf{\Pi}^0_2}$ is a 52 element $\leq_c$-antichain (and thus a $\sqsubseteq_c$ and a $\leq_c$-antichain basis).

Proof. We apply Theorem 4.7 and use the fact that $\{\alpha \in 2^\omega \mid (1\alpha, 0^\infty) \in R\} = S_j$ for some $j \in 2$ since R is locally countable. This provides $(\mathbb{S}, \mathbb{R}^{\mathbf{\Pi}^0_2, a}_t)$ in $\mathcal{C}^{\mathbf{\Pi}^0_2}$ below (X, R). $\square$

5 Rank one

Notation. Let $\mathbb{K} := \{2^{-k} \mid k \in \omega\} \cup \{0\}$, and $\mathbb{C} := \{2^{-k} \mid k \in \omega\}$. We first set

$$\begin{aligned}
S_0 &:= \{(x,y) \in \mathbb{K}^2 \mid x, y \in \mathbb{C} \wedge x < y\},\\
S_1 &:= \{(x,y) \in \mathbb{K}^2 \mid x = y \in \mathbb{C}\},\\
S_2 &:= \{(x,y) \in \mathbb{K}^2 \mid x, y \in \mathbb{C} \wedge x > y\},\\
S_3 &:= \{(x,y) \in \mathbb{K}^2 \mid x \in \mathbb{C} \wedge y \notin \mathbb{C}\},\\
S_4 &:= \{(x,y) \in \mathbb{K}^2 \mid x \notin \mathbb{C} \wedge y \in \mathbb{C}\},\\
S_5 &:= \{(x,y) \in \mathbb{K}^2 \mid x, y \notin \mathbb{C}\}.
\end{aligned}$$

Note that $(S_j)_{j<6}$ is a partition of $\mathbb{K}^2$, and the vertical sections of S_2 and the horizontal sections of S_0 are infinite. We set $N := \big\{t \in 2^6 \mid \big(t \notin 1^6 \wedge t(5)=0\big) \vee \big(t(2)=1 \wedge t(3)=0\big) \vee \big(t(0)=1 \wedge t(4)=0\big)\big\}$. We first consider the class $\mathbf{\Pi}^0_1$ and set, for $t \in 2^6$, $\mathbb{R}^{\mathbf{\Pi}^0_1}_{\mathbb{K},t} := \bigcup_{j<6, t(j)=1} S_j$. $\{(\mathbb{K}, \mathbb{R}^{\mathbf{\Pi}^0_1}_{\mathbb{K},t}) \mid t \in N\}$ is a 45 element $\leq_c$-antichain since $(2^5-1)+2^{4-1}+(2^{4-1}-2^{4-1-2})=45$.

We next code relations having just one non-closed vertical section, with just one limit point on this vertical section, out of the diagonal. We set

$$V := \big\{t \in (2^6)^{2^2} \mid t(0,0) \in (\{0\}^5 \times 2) \wedge t(0,1) = (0,0,0,0,1,0) \wedge t(1,0) \in (\{0\}^3 \times 2 \times \{0\} \times 2) \wedge t(1,1) \notin N\big\},$$

and $\mathbb{L} := (\neg\mathbb{C}) \oplus \mathbb{K}$. We set, for $t \in V$,

$$\mathbb{R}^{\mathbf{\Pi}^0_1}_{\mathbb{L},t} := \bigcup_{(\varepsilon,\eta) \in 2^2, j<6, t(\varepsilon,\eta)(j)=1} \big\{\big((\varepsilon, x), (\eta, y)\big) \in \mathbb{L}^2 \mid (x,y) \in S_j\big\}.$$

$\{(\mathbb{L}, \mathbb{R}^{\mathbf{\Pi}^0_1}_{\mathbb{L},t}) \mid t \in V\}$ is a 152 element $\leq_c$-antichain since $2 \cdot 1 \cdot 2^2 \cdot (2^6 - 45) = 152$. Similarly, we code relations having closed vertical sections, and just one non-closed horizontal section, with just one limit point on this horizontal section, out of the diagonal.

We set

$$H := \big\{t\in(2^6)^{2^2} \mid t(0,0)\notin N \wedge t(0,1)=(0,0,0,1,0,0) \wedge\ t(1,0)\in(\{0\}^4\times 2^2)\setminus\{(0,0,0,0,1,0)\} \\ \wedge\, t(1,1)\in(\{0\}^5\times 2)\big\},$$

and $\mathbb{M}:=\mathbb{K}\oplus(\neg\mathbb{C})$. We set, for $t\in H$,

$$\mathbb{R}^{\mathbf{\Pi}^0_1}_{\mathbb{M},t} := \bigcup_{(\varepsilon,\eta)\in 2^2, j<6, t(\varepsilon,\eta)(j)=1} \big\{\big((\varepsilon,x),(\eta,y)\big)\in\mathbb{M}^2 \mid (x,y)\in S_j\big\}.$$

$\{(\mathbb{M},\mathbb{R}^{\mathbf{\Pi}^0_1}_{\mathbb{M},t}) \mid t\in H\}$ is a 114 element $\leq_c$-antichain since $(2^6-45)\cdot 1\cdot(2^2-1)\cdot 2=114$. We define a set of codes for relations on $\mathbb{K}$ with closed sections as follows:

$$C := \{t\in 2^6 \mid t(0)=1 \Rightarrow t(4)=1 \Rightarrow t(5)=1 \wedge t(2)=1 \Rightarrow t(3)=1 \Rightarrow t(5)=1\}.$$

We define a set of codes for the missing relations in our antichain basis. We set

$$S := \big\{t\in(2^6)^{2^2} \mid t(0,0),t(1,1)\notin N \wedge t(0,1)=(0,1,0,0,0,0) \wedge t(1,0)\in C \wedge \\ t(1,0)=(0,1,0,0,0,0) \Rightarrow t(0,0)\leq_{\text{lex}} t(1,1)\big\}.$$

We set, for $t\in S$, $\mathbb{R}^{\mathbf{\Pi}^0_1}_{\mathbb{D},t} := \bigcup_{(\varepsilon,\eta)\in 2^2, j<6, t(\varepsilon,\eta)(j)=1} \big\{\big((\varepsilon,x),(\eta,y)\big)\in\mathbb{D}^2 \mid (x,y)\in S_j\big\}$. Finally,

$$\mathcal{A}^{\mathbf{\Pi}^0_1} := \{(\mathbb{K},\mathbb{R}^{\mathbf{\Pi}^0_1}_{\mathbb{K},t}) \mid t\in N\} \cup \{(\mathbb{L},\mathbb{R}^{\mathbf{\Pi}^0_1}_{\mathbb{L},t}) \mid t\in V\} \cup \{(\mathbb{M},\mathbb{R}^{\mathbf{\Pi}^0_1}_{\mathbb{M},t}) \mid t\in H\} \cup \{(\mathbb{D},\mathbb{R}^{\mathbf{\Pi}^0_1}_{\mathbb{D},t}) \mid t\in S\}$$

is the 7360 element $\leq_c$-antichain basis mentioned in the statement of Theorem 1.5, since

$$45+152+114+(2^6-45)^2\cdot 1\cdot(20-1)+\frac{(2^6-45)\cdot(2^6-45+1)}{2}\cdot 1^2=7360.$$

We set, for $j\in 2^2$, $T_j := \{(2^{-2k-j(0)},2^{-2k-j(1)}) \mid k\in\omega\}$. We then define relations on $\mathbb{K}$ by $\mathbb{R}^{\mathbf{\Pi}^0_1}_0 := T_{(0,1)}$ and $\mathbb{R}^{\mathbf{\Pi}^0_1}_1 := T_{(0,1)}\cup T_{(1,0)}$. $(\mathbb{K},\mathbb{R}^{\mathbf{\Pi}^0_1}_0),(\mathbb{K},\mathbb{R}^{\mathbf{\Pi}^0_1}_1)$ are the 2 elements mentioned at the end of the statement of Theorem 1.5. For the class $\mathbf{\Sigma}^0_1$, we simply pass to complements.

For graphs, in the case of the class $\mathbf{\Pi}^0_1$, the 5-element $\leq_c$-antichain basis is described by the following codes:

- $(0,0,0,1,1,0)$ (acyclic) , $(1,0,1,0,0,0),(1,0,1,1,1,0)\in N$,
- $\big((0,0,0,0,0,0),(0,0,0,0,1,0),(0,0,0,1,0,0),(0,0,0,0,0,0)\big)\in V$ (acyclic),
- $\big((0,0,0,0,0,0),(0,1,0,0,0,0),(0,1,0,0,0,0),(0,0,0,0,0,0)\big)\in S$ (acyclic).

In order to get the 6-element $\sqsubseteq_c$-antichain basis, we just have to add $(\mathbb{K},\mathbb{R}^{\mathbf{\Pi}^0_1}_1)$ (which is acyclic). In the case of the class $\mathbf{\Sigma}^0_1$, the 10-element $\leq_c$ and $\sqsubseteq_c$-antichain basis is described as follows: take

- for $(1,1,1,0,0,1)\in N$, $\cup_{j<6,t(j)=0}\, S_j$ (which is acyclic),
- for

$$t\in\big\{\big((0,0,0,0,0,1),(0,0,0,0,1,0),(0,0,0,1,0,0),(\varepsilon_0,1,\varepsilon_0,\varepsilon_1,\varepsilon_1,1)\big)\in V \mid (\varepsilon_0,\varepsilon_1)\neq(1,0)\big\},$$

$\cup_{(\varepsilon,\eta)\in 2^2,j<6,t(\varepsilon,\eta)(j)=0}\, \big\{\big((\varepsilon,x),(\eta,y)\big)\in\mathbb{L}^2 \mid (x,y)\in S_j\big\}$ (which is acyclic when $(\varepsilon_0,\varepsilon_1)=(1,1)$),

- for

$$t\in\big\{\big((\varepsilon_0,1,\varepsilon_0,\varepsilon_1,\varepsilon_1,1),(0,1,0,0,0,0),(0,1,0,0,0,0),(\varepsilon_2,1,\varepsilon_2,\varepsilon_3,\varepsilon_3,1)\big)\in S \mid \\ (\varepsilon_0,\varepsilon_1),(\varepsilon_2,\varepsilon_3)\neq(1,0) \wedge (\varepsilon_0,1,\varepsilon_0,\varepsilon_1,\varepsilon_1,1)\leq_{\text{lex}}(\varepsilon_2,1,\varepsilon_2,\varepsilon_3,\varepsilon_3,1)\big\}.$$

$\cup_{(\varepsilon,\eta)\in 2^2,j<6,t(\varepsilon,\eta)(j)=0}\, \big\{\big((\varepsilon,x),(\eta,y)\big)\in\mathbb{D}^2 \mid (x,y)\in S_j\big\}$.

6 Uncountable analytic relations

Notation. We set $\mathbb{H} := N_1 \times \{0^\infty\}$, $\mathbb{V} := \{0^\infty\} \times N_1$, and $\mathbb{L} := \mathbb{H} \cup \mathbb{V}$. If $\mathbb{A} \in \{\mathbb{H}, \mathbb{V}, \mathbb{L}\}$, then $\mathbb{A}^+ := \mathbb{A} \cup \{(0^\infty, 0^\infty)\}$. Let $o : 2^\omega \to 2^\omega$ be defined by $o(\alpha)(n) := \alpha(n)$ exactly when $n > 0$. Then o is a homeomorphism and an involution. We set

$$\mathcal{A}^c := \Big\{ \big(2^\omega, (2^\omega)^2\big), (\mathbb{S}, \mathbb{H}), (\mathbb{S}, \mathbb{V}), (\mathbb{S}, \mathbb{L}), (\mathbb{S}, \mathbb{H}^+), (\mathbb{S}, \mathbb{V}^+), (\mathbb{S}, \mathbb{L}^+), (2^\omega, \neq), (2^\omega, <_{\text{lex}}), (2^\omega, =), (2^\omega, \leq_{\text{lex}}), \big(2^\omega, \text{Graph}(o)\big), \big(2^\omega, \text{Graph}(o_{|N_0})\big) \Big\}.$$

We enumerate $\mathcal{A}^c := \{\mathcal{E}_i \mid i \leq 12\}$ (in the previous order). The relation $\mathcal{E}_{11}$ is closely related to the relation $\mathbb{G}_m^{\{\emptyset\}}$ on $\mathbb{D} := 2 \times 2^\omega$ we met before Theorem 1.10. Indeed, the homeomorphism $h : 2^\omega \to \mathbb{D}$ defined by $h(\alpha) := \big(\alpha(0), (\alpha(1), \alpha(2), \cdots)\big)$ is a witness for the fact that $(2^\omega, \mathcal{E}_{11}) \equiv_c (\mathbb{D}, \mathbb{G}_m^{\{\emptyset\}})$.

Lemma 6.1 *$\mathcal{A}^c$ is an antichain.*

Proof. Note that if $(X, A) \sqsubseteq_c (Y, B)$ and B is in some Borel class $\mathbf{\Gamma}$, then A is in $\mathbf{\Gamma}$ too. The second coordinate of a member of $\mathcal{E}_0$-$\mathcal{E}_6$ (resp., $\mathcal{E}_7$-$\mathcal{E}_8$, $\mathcal{E}_9$-$\mathcal{E}_{12}$) is clopen (resp., open not closed, closed not open). This proves that no member of $\mathcal{E}_7$-$\mathcal{E}_{12}$ is reducible to a member of $\mathcal{E}_0$-$\mathcal{E}_6$, and that the members of $\mathcal{E}_7$-$\mathcal{E}_8$ are incomparable with the members of $\mathcal{E}_9$-$\mathcal{E}_{12}$. Note that the second coordinate of

- $\mathcal{E}_0$ and $\mathcal{E}_9$-$\mathcal{E}_{10}$ is reflexive.
- $\mathcal{E}_1$-$\mathcal{E}_3$, $\mathcal{E}_7$-$\mathcal{E}_8$ and $\mathcal{E}_{11}$-$\mathcal{E}_{12}$ is irreflexive.
- $\mathcal{E}_0$, $\mathcal{E}_3$, $\mathcal{E}_6$, $\mathcal{E}_7$, $\mathcal{E}_9$ and $\mathcal{E}_{11}$ is symmetric.
- $\mathcal{E}_1$-$\mathcal{E}_2$, $\mathcal{E}_4$-$\mathcal{E}_5$, $\mathcal{E}_8$-$\mathcal{E}_{10}$ and $\mathcal{E}_{12}$ is antisymmetric.
- $\mathcal{E}_0$-$\mathcal{E}_2$, $\mathcal{E}_4$-$\mathcal{E}_5$, $\mathcal{E}_8$-$\mathcal{E}_{10}$ and $\mathcal{E}_{12}$ is transitive.

Assume that $(X, A) \sqsubseteq_c (Y, B)$, and that P is one of the following properties of relations: reflexive, irreflexive, symmetric, antisymmetric, transitive. We already noticed that (X, A) has P if (Y, B) does. Note also that if (X, A) has P, then there is a copy C of X in Y such that $(C, B \cap C^2)$ has P. This implies that the only cases to consider are the following. In all these cases, we will prove a result of the form $(\mathbb{X}_j, \mathcal{E}_j) \not\sqsubseteq_c (\mathbb{X}_k, \mathcal{E}_k)$, $\mathcal{E}_j \not\sqsubseteq_c \mathcal{E}_k$ for short. We argue by contradiction, which gives $i : \mathbb{X}_j \to \mathbb{X}_k$ injective continuous with $\mathcal{E}_j = (i \times i)^{-1}(\mathcal{E}_k)$.

$\mathcal{E}_1 \not\sqsubseteq_c \mathcal{E}_2$: $i(10^\infty) = i(1^\infty) = 0^\infty$, which contradicts the injectivity of i. Similarly, $\mathcal{E}_2 \not\sqsubseteq_c \mathcal{E}_1$, $\mathcal{E}_4$ is incomparable with $\mathcal{E}_5$, $\mathcal{E}_1 \not\sqsubseteq_c \mathcal{E}_5$, $\mathcal{E}_2 \not\sqsubseteq_c \mathcal{E}_4$.

$\mathcal{E}_1 \not\sqsubseteq_c \mathcal{E}_4$: $i(0^\infty) = 0^\infty$, which implies that $(0^\infty, 0^\infty) \in \mathbb{H}$, which is absurd. Similarly, $\mathcal{E}_2 \not\sqsubseteq_c \mathcal{E}_5$ and $\mathcal{E}_3 \not\sqsubseteq_c \mathcal{E}_6$.

$\mathcal{E}_1 \not\sqsubseteq_c \mathcal{E}_8$: for example $i(10^\infty) <_{\text{lex}} i(1^\infty)$ and $(10^\infty, 1^\infty) \in \mathbb{H}$, which is absurd. Similarly, $\mathcal{E}_2 \not\sqsubseteq_c \mathcal{E}_8$.

$\mathcal{E}_1 \not\sqsubseteq_c \mathcal{E}_{12}$: $\big(i(10^\infty), i(0^\infty)\big) = (0\alpha, 1\alpha)$, $\big(i(1^\infty), i(0^\infty)\big) = (0\beta, 1\beta)$, which implies that $\alpha = \beta$, contradicting the injectivity of i. Similarly, $\mathcal{E}_2 \not\sqsubseteq_c \mathcal{E}_{12}$.

$\mathcal{E}_3 \not\sqsubseteq_c \mathcal{E}_7$: $i(10^\infty) \neq i(1^\infty)$, which implies that $(10^\infty, 1^\infty) \in \mathbb{L}$, which is absurd.

$\mathcal{E}_3 \not\sqsubseteq_c \mathcal{E}_{11}$: $\big(i(0^\infty), i(1^\infty)\big), \big(i(0^\infty), i(10^\infty)\big) \in \text{Graph}(o)$, which implies that $i(1^\infty) = i(10^\infty)$, contradicting the injectivity of i. $\square$

From now on, Y will be a Hausdorff topological space and B will be an uncountable analytic relation on Y. Note that $B\cap\Delta(Y)$ is analytic.

Lemma 6.2 *Assume that $B\cap\Delta(Y)$ is uncountable. Then $\big(2^\omega,(2^\omega)^2\big)\sqsubseteq_c(Y,B)$, $(2^\omega,=)\sqsubseteq_c(Y,B)$ or $(2^\omega,\leq_{lex})\sqsubseteq_c(Y,B)$.*

Proof. The perfect set theorem gives $j:2^\omega\rightarrow B\cap\Delta(Y)$ injective continuous. Note that

$$\pi:=\text{proj}_0\big[j[2^\omega]\big]=\text{proj}_1\big[j[2^\omega]\big]$$

is a copy of 2^ω, $\Delta(\pi)\subseteq B\cap\pi^2$ and $(\pi,B\cap\pi^2)\sqsubseteq_c(Y,B)$, which implies that we may assume that $Y=2^\omega$ and $\Delta(2^\omega)\subseteq B\in\mathbf{\Sigma}^1_1\big((2^\omega)^2\big)$. By 19.7 in [K], there is a copy P of 2^ω in 2^ω such that $<_{\text{lex}}\cap P^2\subseteq B$ or $<_{\text{lex}}\cap P^2\subseteq\neg B$. Similarly, there is a copy Q of 2^ω in P such that $>_{\text{lex}}\cap Q^2\subseteq B$ or $>_{\text{lex}}\cap Q^2\subseteq\neg B$.

Case 1. $<_{\text{lex}}\cap Q^2\subseteq B$ and $>_{\text{lex}}\cap Q^2\subseteq B$.

Note that $Q^2=B\cap Q^2$ and $\big(2^\omega,(2^\omega)^2\big)\sqsubseteq_c(Y,B)$.

Case 2. $<_{\text{lex}}\cap Q^2\subseteq\neg B$ and $>_{\text{lex}}\cap Q^2\subseteq\neg B$.

Note that $\Delta(Q)=B\cap Q^2$ and $(2^\omega,=)\sqsubseteq_c(Y,B)$.

Case 3. $<_{\text{lex}}\cap Q^2\subseteq B$ and $>_{\text{lex}}\cap Q^2\subseteq_c\neg B$.

Note that $\leq_{\text{lex}}\cap Q=B\cap Q^2$ and $(2^\omega,\leq_{\text{lex}})\sqsubseteq_c(Y,B)$.

Case 4. $<_{\text{lex}}\cap Q^2\subseteq\neg B$ and $>_{\text{lex}}\cap Q^2\subseteq B$.

Note that $\geq_{\text{lex}}\cap Q^2=B\cap Q^2$ and $(2^\omega,\geq_{\text{lex}})\sqsubseteq_c(Y,B)$. But $(2^\omega,\leq_{\text{lex}})\sqsubseteq_c(2^\omega,\geq_{\text{lex}})$, with witness i defined by $i(\alpha)(n):=1-\alpha(n)$. Thus $(2^\omega,\leq_{\text{lex}})\sqsubseteq_c(Y,B)$. □

Lemma 6.3 *Assume that there is $C\subseteq Y$ countable such that $B\subseteq(C\times Y)\cup(Y\times C)$. Then there is $1\leq i\leq 6$ such that $\mathcal{E}_i\sqsubseteq_c(Y,B)$.*

Proof. As B is uncountable, there is $y\in C$ such that B^y or B_y is uncountable. As B^y and B_y are analytic, there is a copy P of 2^ω in Y, disjoint from C, such that $P\times\{y\}\subseteq B$ or $\{y\}\times P\subseteq B$.

Case 1. $P\times\{y\}\subseteq B$.

Case 1.1. $(\{y\}\times P)\cap B$ is countable.

Note that there is a copy Q of 2^ω in P such that $\{y\}\times Q\subseteq\neg B$.

Case 1.1.1. $(y,y)\notin B$.

Note that $Q\times\{y\}=B\cap(\{y\}\cup Q)^2$ and $(\mathbb{S},\mathbb{H})\sqsubseteq_c(Y,B)$.

Case 1.1.2. $(y,y)\in B$.

Note that $Q\times\{y\}\cup\{(y,y)\}=B\cap(\{y\}\cup Q)^2$ and $(\mathbb{S},\mathbb{H}^+)\sqsubseteq_c(Y,B)$.

Case 1.2. $(\{y\}\times P)\cap B$ is uncountable.

Note that there is a copy Q of 2^ω in P such that $\{y\}\times Q\subseteq B$.

Case 1.2.1. $(y,y)\notin B$.

Note that $Q\times\{y\}\cup\{y\}\times Q=B\cap(\{y\}\cup Q)^2$ and $(\mathbb{S},\mathbb{L})\sqsubseteq_c(Y,B)$.

Case 1.2.2. $(y,y)\in B$.

Note that $Q\times\{y\}\cup\{y\}\times Q\cup\{(y,y)\}=B\cap(\{y\}\cup Q)^2$ and $(\mathbb{S},\mathbb{L}^+)\sqsubseteq_c(Y,B)$.

Case 2. $\{y\}\times P\subseteq B$.

Similarly, we show that $\mathcal{E}_2\sqsubseteq_c(Y,B)$, $\mathcal{E}_3\sqsubseteq_c(Y,B)$, $\mathcal{E}_5\sqsubseteq_c(Y,B)$ or $\mathcal{E}_6\sqsubseteq_c(Y,B)$. □

So from now on we will assume that $B\cap\Delta(Y)$ is countable, and that there is no countable subset C of Y such that $B\subseteq(C\times Y)\cup(Y\times C)$. In particular, we may assume that B is irreflexive. By Theorem 1 and Remark 2 in [P], there are $\varphi:2^\omega\to Y$ and $h:\varphi[2^\omega]\to Y$ injective continuous with Graph$(h)\subseteq B$. As B is irreflexive, we may assume that h has disjoint domain and range. We define $i:2^\omega\to Y$ by $i(0\alpha):=\varphi(\alpha)$ and $i(1\alpha):=h\big(\varphi(\alpha)\big)$. Note that i is injective continuous. We set $A:=(i\times i)^{-1}(B)$. Note that A is an analytic digraph on 2^ω, which contains Graph$(o_{|N_0})$, and that $(2^\omega,A)\sqsubseteq_c(Y,B)$. So from now on we will assume that $Y=2^\omega$, $B\in\mathbf{\Sigma}^1_1\big((2^\omega)^2\big)$ is a digraph, and Graph$(o_{|N_0})\subseteq B$.

Lemma 6.4 *Assume that B is meager. Then there is $11\leq i\leq 12$ such that $\mathcal{E}_i\sqsubseteq_c(Y,B)$.*

Proof. We first prove the following.

Claim. *It is enough to find a Cantor subset P of N_0 such that $B\cap(P\cup o[P])^2\subseteq$ Graph(o).*

Indeed, we distinguish two cases.

Case 1. Graph$(o_{|o[P]})\cap B$ is uncountable.

There is a copy Q of 2^ω in $o[P]$ with the property that Graph$(o_{|Q})\subseteq B$, which implies that Graph$(o_{|Q\cup o[Q]})=B\cap(Q\cup o[Q])^2$. Let $\psi:2^\omega\to Q$ be a homeomorphism. We define $j:2^\omega\to 2^\omega$ by the formulas $j(0\alpha):=\psi(\alpha)$ and $j(1\alpha):=o\big(\psi(\alpha)\big)$. Note that j is injective continuous. Also, Graph$(o)=(j\times j)^{-1}(B)$, which implies that $\big(2^\omega,\text{Graph}(o)\big)\sqsubseteq_c(Y,B)$.

Case 2. Graph$(o_{|o[P]})\cap B$ is countable.

There is a copy Q of 2^ω in $o[P]$ with the property that Graph$(o_{|Q})\subseteq\neg B$, which implies that Graph$(o_{|o[Q]})=B\cap(Q\cup o[Q])^2$. Let $\psi:2^\omega\to o[Q]$ be a homeomorphism. We define $j:2^\omega\to 2^\omega$ by the formulas $j(0\alpha):=\psi(\alpha)$ and $j(1\alpha):=o\big(\psi(\alpha)\big)$. Note that j is injective continuous. Also, Graph$(o_{|N_0})=(j\times j)^{-1}(B)$, which implies that $\big(2^\omega,\text{Graph}(o_{|N_0})\big)\sqsubseteq(Y,B)$. ⋄

We now give the end of the proof suggested by the anonymous referee, which simplifies the original one. Let $\mathcal{S}:2^\omega\to 2^\omega$ be the shift map, defined by $\mathcal{S}(\alpha)(n):=\alpha(n+1)$. As B is meager, so is $B':=(\mathcal{S}\times\mathcal{S})[B]$. The Mycielski-Kuratowski theorem provides a copy Q of 2^ω in 2^ω such that $B'\cap Q^2\subseteq\Delta(2^\omega)$ (see 19.1 in [K1]). We set $\tilde{P}:=\{0\}\times Q$. Note that P is a copy of 2^ω in N_0 and $B\cap(P\cup o[P])^2\subseteq$ Graph(o) since B is a digraph. □

So we may assume that B is not meager. The Baire property of B and 19.6 in [K] give a product of Cantor sets contained in B. This means that we may assume that $N_0\times N_1\subseteq B\subseteq\neg\Delta(2^\omega)$.

Proof of Theorem 1.10. (1) We distinguish several cases.

Case 1. $B\cap(N_1\times N_0)$ is not meager.

By 19.6 in [K], $B\cap(N_1\times N_0)$ contains a product of Cantor sets, which implies that we may assume that $(N_0\times N_1)\cup(N_1\times N_0)\subseteq B$.

Case 1.1. There is a Cantor subset of 2^ω which is B-discrete. Then $(\mathbb{S},\mathbb{L})\sqsubseteq_c(Y,B)$.

Indeed, assume for example that $Q\subseteq N_1$ is a Cantor B-discrete set. Let $h:2^\omega\to Q$ be a homeomorphism. We define $i:\mathbb{S}\to Y$ by $i(0^\infty):=0^\infty$ and $i(1\alpha):=h(\alpha)$. Note that i is injective continuous. Clearly $\mathbb{L}\subseteq(i{\times}i)^{-1}(B)$, and the converse holds since B is a digraph and Q is B-discrete.

Case 1.2. No Cantor subset of 2^ω is B-discrete. Then $(2^\omega,\neq)\sqsubseteq_c(Y,B)$ or $(2^\omega,<_{\text{lex}})\sqsubseteq_c(Y,B)$.

Indeed, as in the proof of Lemma 6.2 there is a Cantor subset Q of 2^ω with $<_{\text{lex}}\cap Q^2\subseteq B$ or $<_{\text{lex}}\cap Q^2\subseteq\neg B$, and $>_{\text{lex}}\cap Q^2\subseteq B$ or $>_{\text{lex}}\cap Q^2\subseteq\neg B$. As B is irreflexive and no Cantor subset of 2^ω is B-discrete, we cannot have $<_{\text{lex}}\cap Q^2\subseteq\neg B$ and $>_{\text{lex}}\cap Q^2\subseteq\neg B$.

Case 1.2.1. $<_{\text{lex}}\cap Q^2\subseteq B$ and $>_{\text{lex}}\cap Q^2\subseteq B$.

Note that $Q^2\setminus\Delta(Q)=B\cap Q^2$ and $(2^\omega,\neq)\sqsubseteq_c(Y,B)$.

Case 1.2.2. $<_{\text{lex}}\cap Q^2\subseteq B$ and $>_{\text{lex}}\cap Q^2\subseteq\neg B$.

Note that $<_{\text{lex}}\cap Q^2=B\cap Q^2$ and $(2^\omega,<_{\text{lex}})\sqsubseteq_c(Y,B)$.

Case 1.2.3. $<_{\text{lex}}\cap Q^2\subseteq\neg B$ and $>_{\text{lex}}\cap Q\subseteq B$.

Note that $>_{\text{lex}}\cap Q=B\cap Q$ and $(2^\omega,>_{\text{lex}})\sqsubseteq_c(Y,B)$. But $(2^\omega,<_{\text{lex}})\sqsubseteq_c(2^\omega,>_{\text{lex}})$, with witness i defined by $i(\alpha)(n):=1-\alpha(n)$. Thus $(2^\omega,<_{\text{lex}})\sqsubseteq_c(Y,B)$.

Case 2. $B\cap(N_1\times N_0)$ is meager.

By 19.6 in [K], $(\neg B)\cap(N_1\times N_0)$ contains a product of Cantor sets, which implies that we may assume that $N_0\times N_1\subseteq B\subseteq\neg(N_1\times N_0)$.

Case 2.1. There is a B-discrete Cantor subset of 2^ω. Then $(\mathbb{S},\mathbb{H})\sqsubseteq_c(Y,B)$ or $(\mathbb{S},\mathbb{V})\sqsubseteq_c(Y,B)$.

Indeed, assume for example that $Q\subseteq N_0$ is a Cantor B-discrete set. Then as in Case 1.1 we see that $(\mathbb{S},\mathbb{H})\sqsubseteq_c(Y,B)$. Similarly, if $Q\subseteq N_1$ is a Cantor B-discrete set, then $(\mathbb{S},\mathbb{V})\sqsubseteq_c(Y,B)$.

Case 2.2. No Cantor subset of 2^ω is B-discrete. Then $(2^\omega,\neq)\sqsubseteq_c(Y,B)$ or $(2^\omega,<_{\text{lex}})\sqsubseteq_c(Y,B)$.

Indeed, we argue as in Case 1.2.

(2) The indicated elements are the only graphs in $\mathcal{A}^c$, up to the isomorphism $(\varepsilon,\alpha)\mapsto\varepsilon\alpha$. $\square$

7 References

[B] J. E. Baumgartner, Partition relations for countable topological spaces, *J. Combin. Theory Ser. A* 43 (1986), 178-195

[C-L-M] J. D. Clemens, D. Lecomte and B. D. Miller, Essential countability of treeable equivalence relations, *Adv. Math.* 265 (2014), 1-31

[G] S. Gao, *Invariant Descriptive Set Theory,* Pure and Applied Mathematics, A Series of Monographs and Textbooks, 293, Taylor and Francis Group, 2009

[Ka] V. Kanovei, *Borel equivalence relations,* Amer. Math. Soc., 2008

[K1] A. S. Kechris, *Classical Descriptive Set Theory,* Springer-Verlag, 1995

[K2] A. S. Kechris, The theory of countable Borel equivalence relations, *preprint (see the url http://www.math.caltech.edu/ kechris/papers/lectures on CBER02.pdf)*

[K-Ma] A. S. Kechris, and A. S. Marks, Descriptive graph combinatorics, *preprint (see the url http://www.math.caltech.edu/ kechris/papers/combinatorics20.pdf)*

[K-S-T] A. S. Kechris, S. Solecki and S. Todorčević, Borel chromatic numbers, *Adv. Math.* 141 (1999), 1-44

[L1] D. Lecomte, On minimal non potentially closed subsets of the plane, *Topology Appl.* 154, 1 (2007) 241-262

[L2] D. Lecomte, How can we recognize potentially $\mathbf{\Pi}^0_\xi$ subsets of the plane?, *J. Math. Log.* 9, 1 (2009), 39-62

[L3] D. Lecomte, Potential Wadge classes, *Mem. Amer. Math. Soc.,* 221, 1038 (2013)

[L4] D. Lecomte, Acyclicity and reduction, *Ann. Pure Appl. Logic* 170, 3 (2019) 383-426

[L5] D. Lecomte, On the complexity of Borel equivalence relations with some countability property, *Trans. Amer. Math. Soc.* 373, 3 (2020), 1845-1883

[L-M] D. Lecomte and B. D. Miller, Basis theorems for non-potentially closed sets and graphs of uncountable Borel chromatic number, *J. Math. Log.* 8 (2008), 121-162

[L-Z] D. Lecomte and R. Zamora, Injective tests of low complexity in the plane, *Math. Logic Quart.* 65, 2 (2019), 134-169

[Lo1] A. Louveau, A separation theorem for Σ^1_1 sets, *Trans. Amer. Math. Soc.* 260 (1980), 363-378

[Lo2] A. Louveau, Ensembles analytiques et boréliens dans les espaces produit, *Astérisque (S. M. F.)* 78 (1980)

[Lo3] A. Louveau, Two Results on Borel Orders, *J. Symbolic Logic* 54, 3 (1989), 865-874

[Lo-SR1] A. Louveau and J. Saint Raymond, Borel classes and closed games: Wadge-type and Hurewicz-type results, *Trans. Amer. Math. Soc.* 304 (1987), 431-467

[Lo-SR2] A. Louveau and J. Saint Raymond, The strength of Borel Wadge determinacy, *Cabal Seminar 81-85, Lecture Notes in Math.* 1333 (1988), 1-30

[Mo] Y. N. Moschovakis, *Descriptive set theory,* North-Holland, 1980

[P] T. C. Przymusiński, On the notion of n-cardinality, *Proc. Amer. Math. Soc.* 69 (1978), 333-338

[S] J. H. Silver, Counting the number of equivalence classes of Borel and coanalytic equivalence relations, *Ann. Math. Logic* 18, 1 (1980), 1-28

[T] S. Todorčević, *Introduction to Ramsey spaces* Annals of Mathematics Studies, 174. Princeton University Press, Princeton, NJ, 2010. viii+287 pp

[W] W. W. Wadge, *Reducibility and determinateness on the Baire space, Ph. D. Thesis,* University of California, Berkeley, 1983