A separation result for countable unions of Borel rectangles

Dominique LECOMTE

February 19, 2019

• Université Paris 6, Institut de Mathématiques de Jussieu-Paris Rive Gauche,
Projet Analyse Fonctionnelle, Couloir 16-26, 4ème étage, Case 247,
4, place Jussieu, 75 252 Paris Cedex 05, France
dominique.lecomte@upmc.fr

• Université de Picardie, I.U.T. de l'Oise, site de Creil,
13, allée de la faïencerie, 60 107 Creil, France

Abstract. We provide dichotomy results characterizing when two disjoint analytic binary relations can be separated by a countable union of $\mathbf{\Sigma}^0_1\times\mathbf{\Sigma}^0_\xi$ sets, or by a $\mathbf{\Pi}^0_1\times\mathbf{\Pi}^0_\xi$ set.

2010 Mathematics Subject Classification. Primary: 03E15, Secondary: 54H05
Keywords and phrases. Borel class, countable Borel coloring, Borel rectangle
Acknowledgments. We would like to thank the two anonymous referees for their suggestions, that improved the presentation of this paper.

1 Introduction

The reader should see [K] for the standard descriptive set theoretic notation and material used in this paper. All our relations will be binary. The motivation for this work goes back to the following so called $\mathbb{G}_0$-dichotomy, essentially proved in [K-S-T].

Theorem 1.1 *(Kechris, Solecki, Todorčević) There is a Borel relation $\mathbb{G}_0$ on 2^ω such that, for any Polish space X and any analytic relation A on X, exactly one of the following holds:*

(a) there is $c\colon X\to\omega$ Borel such that $c(x)\neq c(y)$ if $(x,y)\in A$ (a **countable Borel coloring** *of A),*

(b) there is $f\colon 2^\omega\to X$ continuous such that $\mathbb{G}_0\subseteq (f\times f)^{-1}(A)$.

This result had a lot of developments since. For instance, Miller developed some techniques to recover many dichotomy results of descriptive set theory, without using effective descriptive set theory (see [M]). He replaces it with some versions of Theorem 1.1. In particular, he can prove Theorem 1.1 without effective descriptive set theory. In [L1], the author derives from Theorem 1.1 a dichotomy result characterizing when two disjoint analytic sets can be separated by a countable union of Borel rectangles. In order to state it, we give some notation that will also be useful to state our main results. It is about partial rectangular reduction.

Notation. Let, for $\varepsilon\in 2:=\{0,1\}$, $X_\varepsilon, Y_\varepsilon$ be Polish spaces, and $A_\varepsilon, B_\varepsilon$ be disjoint analytic subsets of $X_\varepsilon\times Y_\varepsilon$. We set

$$(X_0,Y_0,A_0,B_0)\leq (X_1,Y_1,A_1,B_1)\Leftrightarrow$$
$$\exists f\colon X_0\to X_1\ \ \exists g\colon Y_0\to Y_1 \text{ continuous with } A_0\subseteq (f\times g)^{-1}(A_1) \text{ and } B_0\subseteq (f\times g)^{-1}(B_1).$$

If X is a set, then the **diagonal** of X is $\Delta(X):=\{(x,x)\mid x\in X\}$.

Theorem 1.2 *Let X,Y be Polish spaces, and A,B be disjoint analytic subsets of $X\times Y$. Exactly one of the following holds:*

(a) the set A can be separated from B by a countable union of Borel rectangles,

(b) $\big(2^\omega,2^\omega,\Delta(2^\omega),\mathbb{G}_0\big)\leq (X,Y,A,B)$.

It is easy to check that Theorem 1.1 is also an easy consequence of Theorem 1.2. This means that the study of the countable Borel colorings is highly related to the study of countable unions of Borel rectangles. It is natural to ask for level by level versions of these two results, with respect to the Borel hierarchy. This work was initiated in [L-Z], where the authors prove the following.

Theorem 1.3 *(Lecomte, Zelený) Let $\xi\in\{1,2,3\}$. Then we can find a zero-dimensional Polish space $\mathbb{X}$, and an analytic relation $\mathbb{A}$ on $\mathbb{X}$ such that for any (zero-dimensional if $\xi=1$) Polish space X, and for any relation A on X, exactly one of the following holds:*

(a) there is a countable $\boldsymbol{\Delta}^0_\xi$-measurable coloring of A,

(b) there is $f\colon\mathbb{X}\to X$ continuous such that $\mathbb{A}\subseteq (f\times f)^{-1}(A)$.

In [L-Z], the authors note that the study of countable $\boldsymbol{\Delta}^0_\xi$-measurable colorings is highly related to the study of countable unions of $\boldsymbol{\Sigma}^0_\xi$ rectangles, since the existence of a countable $\boldsymbol{\Delta}^0_\xi$-measurable coloring of a relation A on a (zero-dimensional if $\xi=1$) Polish space X is equivalent to the fact that $\Delta(X)$ can be separated from A by a countable union of $\boldsymbol{\Sigma}^0_\xi$ rectangles, by the generalized reduction property for the class $\boldsymbol{\Sigma}^0_\xi$ (see 22.16 in [K]). In this direction, they prove the following.

Theorem 1.4 *(Lecomte, Zelený) Let $\xi \in \{1, 2\}$. Then we can find zero-dimensional Polish spaces $\mathbb{X}, \mathbb{Y}$, and disjoint analytic subsets $\mathbb{A}, \mathbb{B}$ of $\mathbb{X} \times \mathbb{Y}$ such that for any Polish spaces X, Y, and for any pair A, B of disjoint analytic subsets of $X \times Y$, exactly one of the following holds:*

(a) the set A can be separated from B by a $(\mathbf{\Sigma}^0_\xi \times \mathbf{\Sigma}^0_\xi)_\sigma$ set,

(b) $(\mathbb{X}, \mathbb{Y}, \mathbb{A}, \mathbb{B}) \leq (X, Y, A, B)$.

In fact, we can think of a number of related problems of this kind. We can study

- the finite or bounded finite Borel colorings,
- the separation of disjoint analytic sets by a finite or bounded finite union of Borel rectangles,
- the finite, bounded finite, or infinite Borel colorings of bounded complexity,
- the separation of disjoint analytic sets by a finite, bounded finite or infinite union of Borel rectangles of bounded complexity...

This last question has been studied in [Za] in the case of one rectangle. In [Za], the author characterizes when two disjoint analytic sets can be separated by a $\mathbf{\Sigma}^0_1$ (or $\mathbf{\Pi}^0_\xi$ when $\xi \leq 2$) rectangle. Louveau suggested that it could be very interesting to study the non-symmetric version of the problem to understand it better (we can also make this remark for countable unions of rectangles, which is another motivation for Theorem 1.5 to come). Zamora noticed that the problems of the separation of analytic sets by a $\mathbf{\Pi}^0_1 \times \mathbf{\Pi}^0_2$ set and by a $(\mathbf{\Sigma}^0_1 \times \mathbf{\Sigma}^0_2)_\sigma$ set are very much related (he derives a dichotomy for the rectangles from a dichotomy for the countable unions of rectangles). His technique cannot be extended to higher levels. However, the relation just mentioned is much stronger than in [Za], as we will see. The main results in this paper generalize these two Zamora results, and are, hopefully, steps towards the generalization of Theorem 1.4, and then Theorem 1.3. The first one is about countable unions of rectangles of the form $\mathbf{\Sigma}^0_1 \times \mathbf{\Sigma}^0_\xi$.

Theorem 1.5 *Let $\xi \geq 1$ be a countable ordinal. Then there are zero-dimensional Polish spaces $\mathbb{X}, \mathbb{Y}$, and disjoint analytic subsets $\mathbb{A}, \mathbb{B}$ of $\mathbb{X} \times \mathbb{Y}$ such that for any Polish spaces X, Y, and for any pair A, B of disjoint analytic subsets of $X \times Y$, exactly one of the following holds:*

(a) the set A can be separated from B by a $(\mathbf{\Sigma}^0_1 \times \mathbf{\Sigma}^0_\xi)_\sigma$ set,

(b) $(\mathbb{X}, \mathbb{Y}, \mathbb{A}, \mathbb{B}) \leq (X, Y, A, B)$.

The second one is about rectangles of the form $\mathbf{\Pi}^0_1 \times \mathbf{\Pi}^0_\xi$.

Theorem 1.6 *Let $\xi \geq 1$ be a countable ordinal. Then there are zero-dimensional Polish spaces $\mathbb{X}, \mathbb{Y}$, and disjoint analytic subsets $\mathbb{A}, \mathbb{B}$ of $\mathbb{X} \times \mathbb{Y}$ such that for any Polish spaces X, Y, and for any pair A, B of disjoint analytic subsets of $X \times Y$, exactly one of the following holds:*

(a) the set A can be separated from B by a $\mathbf{\Pi}^0_1 \times \mathbf{\Pi}^0_\xi$ set,

(b) $(\mathbb{X}, \mathbb{Y}, \mathbb{A}, \mathbb{B}) \leq (X, Y, A, B)$.

One of our key tools to prove these two results is the representation theorem for Borel sets by Debs and Saint Raymond. A classical result of Lusin-Souslin asserts that any Borel subset $\mathcal{B}$ of a Polish space is the bijective continuous image of a closed subset of the Baire space (see 13.7 in [K]). There is a level by level version of this result due to Kuratowski: the Baire class of the inverse map of the bijection is essentially equal to the Borel rank of $\mathcal{B}$ (see Theorem 1 in [Ku]).

The representation theorem for Borel sets by Debs and Saint Raymond refines this Kuratowski result (see Theorem I-6.6 in [D-SR]). We will state it and recall the material needed to state it in the next section. Initially, the representation theorem had three applications in [D-SR]: a theorem about continuous liftings, another one about compact covering maps, and a new proof (involving games as in the original paper) of the Louveau-Saint Raymond dichotomy characterizing when two disjoint analytic sets can be separated by a $\mathbf{\Sigma}^0_\xi$ (or $\mathbf{\Pi}^0_\xi$) set (see page 433 in [Lo-SR]). In [L3] and [L4], the representation theorem is used to prove a dichotomy about potential Wadge classes. Its proof provides another new proof of the Louveau-Saint Raymond theorem which does not involve games.

A very remarkable phenomenon happens in the present paper. In the applications just mentioned, the representation theorem was used only inside the proofs. Here, the representation theorem is used not only in the proofs of Theorems 1.5 and 1.6, but also to define the minimal objects $\mathbb{X}, \mathbb{Y}, \mathbb{A}, \mathbb{B}$. We believe that the minimal objects cannot be that simple for higher levels. Moreover, Theorem 1.4 provides an extension of Theorem 1.5 to countable unions of $\mathbf{\Sigma}^0_2$ rectangles. It is possible to prove such an extension using the representation theorem. However, we could not prove further extensions, leaving the general case of countable unions of rectangles of the form $\mathbf{\Sigma}^0_\eta \times \mathbf{\Sigma}^0_\xi$, or just $\mathbf{\Sigma}^0_\xi \times \mathbf{\Sigma}^0_\xi$, open for future work.

The organization of the paper is as follows. In Section 2, we recall the material about representation needed here, as well as some lemmas from [L3], and we give some effective facts needed to prove our main results. We prove Theorem 1.5 in Section 3, and Theorem 1.6 in Section 4.

2 Preliminaries

2.1 Representation of Borel sets

The following definition can be found in [D-SR].

Definition 2.1.1 *(Debs-Saint Raymond) A partial order relation R on $2^{<\omega}$ is a* **tree relation** *if, for $s \in 2^{<\omega}$,*

(a) $\emptyset\ R\ s$ (i.e., $\emptyset$ is the R-minimum element),

(b) the set $P_R(s) := \{t \in 2^{<\omega} \mid t\ R\ s\}$ is finite and linearly ordered by R ($h_R(s)$ will denote the number of strict R-predecessors of s, so that $h_R(s) = \text{Card}\big(P_R(s)\big) - 1$).

A basic exemple of a tree relation is given by the extension relation $\subseteq$.

• *Let R be a tree relation. An R-***branch** *is a $\subseteq$-maximal subset of $2^{<\omega}$ linearly ordered by R. We denote by $[R]$ the set of all* infinite *R-branches. For instance, if $\alpha \in 2^\omega$, then the strictly $\subseteq$-increasing sequence $h(\alpha)$ of all initial segments of α is an infinite $\subseteq$-branch.*

We equip $(2^{<\omega})^\omega$ with the product of the discrete topology on $2^{<\omega}$. If R is a tree relation, then the space $[R] \subseteq (2^{<\omega})^\omega$ is equipped with the topology induced by that of $(2^{<\omega})^\omega$, and is a Polish space. A basic clopen set is of the form $N^R_s := \big\{\gamma \in [R] \mid \gamma\big(h_R(s)\big) = s\big\}$, where $s \in 2^{<\omega}$. The map $h : 2^\omega \to [\subseteq]$ just defined in the last paragraph is a homeomorphism.

• *Let R, S be tree relations with $R \subseteq S$ (quite often, the big relation S will be the extension relation). The* **canonical map** *$\Pi : [R] \to [S]$ is defined by $\Pi(\gamma) :=$ the unique S-branch containing γ. The canonical map is continuous.*

• *Let S be a tree relation. We say that $R\subseteq S$ is* **distinguished** *in S if*

$$\forall s,t,u\in 2^{<\omega}\qquad \left.\begin{array}{c} s\ S\ t\ S\ u\\ \\ s\ R\ u\end{array}\right\}\ \Rightarrow\ s\ R\ t.$$

The idea is that if s is a good position when viewed from u, then s was already a good position when viewed from t.

• *We now iterate the notion of distinction.Let $\eta<\omega_1$. A family $(R^\rho)_{\rho\leq\eta}$ of tree relations is a* **resolution family** *if*

(a) $R^{\rho+1}$ is a distinguished subtree of R^ρ, for each $\rho<\eta$.

(b) $R^\lambda=\bigcap_{\rho<\lambda}\ R^\rho$, for each limit ordinal $\lambda\leq\eta$.

Before stating the representation theorem of Borel sets, we now give an idea of the objects it can provide. As mentioned in the introduction, it can be used to prove one of the initial cases of the Louveau-Saint Raymond dichotomy, characterizing when two disjoint analytic subsets A and B of a Polish space X can be separated by a $\mathbf{\Sigma}^0_2$ set, and due to Hurewicz. Let $\mathbb{P}_\infty$ be the $\mathbf{\Pi}^0_2$ set of the infinite binary sequences with infinitely many ones. The Hurewicz theorem says that the separation is not possible exactly when there is $f:2^\omega\rightarrow X$ continuous sending $\mathbb{P}_\infty$ into A and $\neg\mathbb{P}_\infty$ into B. We construct finite approximations of the map f. If we look at an initial segment of some $\alpha\in 2^\omega$, we have to guess where α will be, in $\mathbb{P}_\infty$ or not. If the initial segment seems to indicate that α will be in $\mathbb{P}_\infty$, then we will take the associated approximation of $f(\alpha)$ in A. Otherwise, we will take the associated approximation of $f(\alpha)$ in B. For instance, an initial segment finishing with a one seems to indicate that α will be in $\mathbb{P}_\infty$, and an initial segment finishing with a zero seems to indicate that α will not be in $\mathbb{P}_\infty$. A first approach is to consider the initial segments of α in $O:=\{s\in 2^{<\omega}\mid s\neq\emptyset\Rightarrow s(|s|-1)=1\}$. If there are infinitely many of them, then α is in $\mathbb{P}_\infty$, and we consider only them. Otherwise, we consider them (just in case we have to change our mind about the final position of α), and then all the initial segments after the last one. In other words, we extracted a subsequence of the sequence $h(\alpha)$ of the initial segments of α. One can check that the relation R defined by $s\ R\ u\Leftrightarrow s\subseteq u\ \wedge\ \big(s\in O\vee\forall s\subseteq t\subseteq u\ \ t\notin O\big)$ is a tree relation distinguished in $\subseteq$, the canonical map $\Pi:[R]\rightarrow[\subseteq]$ is a continuous bijection with $\mathbf{\Sigma}^0_2$-measurable inverse, and $\Pi^{-1}(h[\mathbb{P}_\infty])$ is the closed subset of $[R]$ made of the sequences with all members in O. The representation theorem of Borel sets can provide such objects, it extracts good subsequences.

The representation theorem of Borel sets is as follows in the successor case (see Theorems I-6.6 and I-3.8 in [D-SR]).

Theorem 2.1.2 *(Debs-Saint Raymond) Let η be a countable ordinal, and P be a $\mathbf{\Pi}^0_{\eta+1}$ subset of $[\subseteq]$. Then there is a resolution family $(R^\rho)_{\rho\leq\eta}$ such that*

(a) $R^0=\subseteq$,

(b) the canonical map $\Pi:[R^\eta]\rightarrow[R^0]$ is a continuous bijection with $\mathbf{\Sigma}^0_{\eta+1}$-measurable inverse,

(c) the set $\Pi^{-1}(P)$ is a closed subset of $[R^\eta]$.

For the limit case, in order to control the complexity of Π^{-1}, we need some more definition that can be found in [D-SR].

Definition 2.1.3 *(Debs-Saint Raymond) Let ξ be an infinite limit countable ordinal. We say that a resolution family $(R^\rho)_{\rho\leq\xi}$ with $R^0=\subseteq$ is* **uniform** *if*

$$\forall k\in\omega\ \exists\xi^k<\xi\ \forall s,t\in 2^{<\omega}\ \Big(min\big(h_{R^\xi}(s),h_{R^\xi}(t)\big)\leq k\ \wedge\ s\ R^{\xi^k}\ t\Big)\Rightarrow s\ R^\xi\ t.$$

Note that if ξ^k satisfies this formula, then so does any η with $\xi^k\leq\eta<\xi$ since $R^\eta\subseteq R^{\xi^k}$. In the sequel, ξ^k will be the least ξ' with $1\leq\xi'<\xi$ satisfying this formula with respect to k.

The representation theorem of Borel sets is as follows in the limit case (see Theorems I-6.6 and I-4.1 in [D-SR]).

Theorem 2.1.4 *(Debs-Saint Raymond) Let ξ be an infinite limit countable ordinal, and P be a $\mathbf{\Pi}^0_\xi$ subset of $[\subseteq]$. Then there is a uniform resolution family $(R^\rho)_{\rho\leq\xi}$ such that*

(a) $R^0=\subseteq$,

(b) the canonical map $\Pi:[R^\xi]\to[R^0]$ is a continuous bijection with $\mathbf{\Sigma}^0_\xi$-measurable inverse,

(c) the set $\Pi^{-1}(P)$ is a closed subset of $[R^\xi]$.

We will use the following extension of the property of distinction (see Lemma 2.6 in [L3]).

Lemma 2.1.5 *Let $\eta<\omega_1$, $(R^\rho)_{\rho\leq\eta}$ be a resolution family, and $\rho<\eta$. Assume that $s,t,u\in 2^{<\omega}$, $s\ R^0\ t\ R^\rho\ u$ and $s\ R^\rho\ u$. Then $s\ R^\rho\ t$. If moreover $s\ R^{\rho+1}\ u$, then $s\ R^{\rho+1}\ t$.*

Notation. Let $\eta<\omega_1$, $(R^\rho)_{\rho\leq\eta}$ be a resolution family with $R^0=\subseteq$, $s\in 2^{<\omega}$, and $\rho\leq\eta$. We define

$$s^\rho:=\begin{cases}\emptyset \text{ if } s=\emptyset,\\ s|\text{max}\{l<|s|\mid s|l\ R^\rho\ s\} \text{ if } s\neq\emptyset.\end{cases}$$

The sequence s^ρ is actually the immediate predecessor of s with respect to R^ρ. Lemmas 2.6 and 2.7 in [L3] allow us to define $\xi^s_1:=\text{sup}\{\rho\leq\eta\mid s^0\subseteq s^\rho\}$ and, inductively on $i\geq 1$ with $\xi^s_i<\eta$, $\xi^s_{i+1}:=\text{sup}\{\rho\leq\eta\mid s^{\xi^s_i+1}\subseteq s^\rho\}$. These lemmas imply the existence of a natural number $n\geq 1$ such that $\{s^{\xi^s_i}\mid 1\leq i\leq n\}$ is an enumeration of $\{s^\rho\mid\rho\leq\eta\}$. Moreover, $(\xi^s_i)_{1\leq i\leq n}$ is strictly increasing, $\xi^s_n=\eta$, $(s^{\xi^s_i})_{1\leq i\leq n}$ is strictly $\subseteq$-decreasing, and $s^{\xi^s_1}\subseteq s$. If $s\neq\emptyset$ and $1\leq i<n$, then $s^{\xi^s_{i+1}}\ R^{\xi^s_i+1}\ s^{\xi^s_i}$ and $s^{\xi^s_i+1}\subsetneqq s^{\xi^s_i}$. These last properties will be useful to establish the topological properties that we need (see Lemma 2.2.1).

2.2 Topologies

The reader should see [Mo] for the basic notions of effective descriptive set theory.

Notation. Let S be a recursively presented Polish space.

(1) The **Gandy-Harrington topology** on S is generated by the Σ^1_1 subsets of S, and is denoted Σ_S. Recall the following facts about Σ_S (see [L2]).

- Σ_S is finer than the initial topology of S.
- We set $\Omega_S:=\{s\in S\mid\omega^s_1=\omega^{\text{CK}}_1\}$. Then Ω_S is a Σ^1_1 subset of S and is dense in (S,Σ_S).
- $W\cap\Omega_S$ is a clopen subset of (Ω_S,Σ_S), for each Σ^1_1 subset W of S.
- (Ω_S,Σ_S) is a zero-dimensional Polish space. So we fix a complete compatible metric on (Ω_S,Σ_S).

(2) We call T_1 the usual topology on S, and T_η is the topology generated by the $\Sigma^1_1\cap\mathbf{\Pi}^0_{<\eta}$ subsets of S if $2\leq\eta<\omega_1^{\text{CK}}$ (see Definition 1.5 in [Lo]).

The next topological result is essentially Lemma 2.4 and the claim in the proof of Theorem 2.9 in [L3]. In our Cantor-like constructions of the functions desired in Theorems 1.5.(b) and 1.6.(b), it is enough to construct the finite approximations associated with the immediate predecessors of a given finite binary sequence, for all the relevant ordinals (in fact finitely many of them, those given after Lemma 2.1.5). The next lemma makes this possible.

Lemma 2.2.1 *Let S be a recursively presented Polish space, and $1\leq\eta<\omega_1^{CK}$.*

(a) (Louveau) Fix a Σ^1_1 subset A of S. Then $\overline{A}^{T_\eta}$ is $\mathbf{\Pi}^0_\eta$, Σ^1_1, and $T_{\eta+1}$-open.

(b) Let $p\geq 1$ be a natural number, $(\eta_i)_{1\leq i\leq p}$ be a strictly increasing sequence of ordinals between 1 and η, $(S_i)_{1\leq i\leq p}$ be a sequence of Σ^1_1 subsets of S, and O be a Σ^0_1 subset of S. Assume that $S_i\subseteq\overline{S_{i+1}}^{T_{\eta_i+1}}$ if $1\leq i<p$. Then $S_p\cap\bigcap_{1\leq i<p}\ \overline{S_i}^{T_{\eta_i}}\cap O$ is T_1-dense in $\overline{S_1}^{T_1}\cap O$.

(c) Let $(R^\rho)_{\rho\leq\eta}$ be a resolution family with $R^0=\subseteq$, s be a nonempty finite binary sequence, S_{s^ρ} be a Σ^1_1 subset of S (for $1\leq\rho\leq\eta$), E be a Σ^1_1 subset of S, and O be a Σ^0_1 subset of S. We assume that $S_{s^\eta}\subseteq\overline{E}^{T_{\eta+1}}$ and $S_t\subseteq\overline{S_u}^{T_\rho}$ if $u\ R^\rho\ t\subsetneqq s$ and $1\leq\rho\leq\eta$. Then $S_{s^\eta}\cap\bigcap_{1\leq\rho<\eta}\ \overline{S_{s^\rho}}^{T_\rho}\cap O$ and $E\cap\bigcap_{1\leq\rho\leq\eta}\ \overline{S_{s^\rho}}^{T_\rho}\cap O$ are T_1-dense in $\overline{S_{s^1}}^{T_1}\cap O$.

Proof. (a) See Lemma 1.7 in [Lo].

(b) Let D be a Σ^0_1 subset of S meeting $\overline{S_1}^{T_1}\cap O$. Then $S_1\cap D\cap O\neq\emptyset$, which proves the desired property for $p=1$. Then we argue inductively on p. So assume that the property is proved for p. Note that $S_p\subseteq\overline{S_{p+1}}^{T_{\eta_p+1}}$, and $S_p\cap\bigcap_{1\leq i<p}\ \overline{S_i}^{T_{\eta_i}}\cap D\cap O\neq\emptyset$, by induction assumption. Thus

$$\overline{S_{p+1}}^{T_{\eta_p+1}}\cap\bigcap_{1\leq i\leq p}\ \overline{S_i}^{T_{\eta_i}}\cap D\cap O\neq\emptyset.$$

As $\bigcap_{1\leq i\leq p}\ \overline{S_i}^{T_{\eta_i}}\cap D\cap O$ is T_{η_p+1}-open, $S_{p+1}\cap\bigcap_{1\leq i\leq p}\ \overline{S_i}^{T_{\eta_i}}\cap D\cap O\neq\emptyset$.

(c) We use the notation after Lemma 2.1.5. We enumerate $\{\xi^s_i\mid\xi^s_i\geq 1\}$ in an increasing way by $\{\eta^s_i\mid 1\leq i\leq p\}$, which means that we forget ξ^s_1 if it is 0. As $\eta\geq 1$, $p\geq 1$. Note that $s^{\eta^s_i+1}\subsetneqq s^{\eta^s_i}$ if $1\leq i<p$. We set $S_i:=S_{s^{\eta^s_i}}$, for $1\leq i\leq p$. Note that $S_i\subseteq\overline{S_{i+1}}^{T_{\eta^s_i+1}}$ if $1\leq i<p$ since $s^{\eta^s_{i+1}}\ R^{\eta^s_i+1}\ s^{\eta^s_i}$. Thus $S_{s^\eta}\cap\bigcap_{\eta^s_i<\eta}\ \overline{S_{s^{\eta^s_i}}}^{T_{\eta^s_i}}\cap O$ and $E\cap\bigcap_{\eta^s_i\leq\eta}\ \overline{S_{s^{\eta^s_i}}}^{T_{\eta^s_i}}\cap O$ are T_1-dense in $\overline{S_{s^1}}^{T_1}\cap O$, by (b) and since $s^{\eta^s_1}=s^1$. But if $1\leq\rho\leq\eta$, then there is $1\leq i\leq p$ with $s^\rho=s^{\eta^s_i}$. And $\rho\leq\eta^s_i$ since $s^{\eta^s_i+1}\subsetneqq s^{\eta^s_i}$ if $1\leq i<p$. Thus we are done since $S_{s^\eta}\cap\bigcap_{1\leq\rho<\eta}\ \overline{S_{s^\rho}}^{T_\rho}=S_{s^\eta}\cap\bigcap_{\eta^s_i<\eta}\ \overline{S_{s^{\eta^s_i}}}^{T_{\eta^s_i}}$ and $\bigcap_{1\leq\rho\leq\eta}\ \overline{S_{s^\rho}}^{T_\rho}=\bigcap_{\eta^s_i\leq\eta}\ \overline{S_{s^{\eta^s_i}}}^{T_{\eta^s_i}}$. □

2.3 Some general effective facts

Lemma 2.3.1 *Let $1\leq\eta,\xi<\omega_1^{CK}$, X,Y be recursively presented Polish spaces, A be a $\Sigma^1_1\cap\mathbf{\Sigma}^0_\eta$ subset of X, B be a $\Sigma^1_1\cap\mathbf{\Sigma}^0_\xi$ subset of Y, and C be a Σ^1_1 subset of $X\times Y$ disjoint from $A\times B$. Then we can find a $\Delta^1_1\cap\mathbf{\Sigma}^0_\eta$ set A' and a $\Delta^1_1\cap\mathbf{\Sigma}^0_\xi$ set B' such that $A'\times B'$ separates $A\times B$ from C. This also holds for the multiplicative classes.*

Proof. We argue as in the proof of Lemma 2.2 in [L-Z]. □

We now give a product version of Lemma 2.2.1.(a).

Notation. Let Y be a recursively presented Polish space. Recall the existence of Π^1_1 sets $W\subseteq\omega$ and $C\subseteq\omega\times Y$ such that $\Delta^1_1(Y)=\{C_n\mid n\in W\}$ and $\{(n,y)\in\omega\times Y\mid n\in W\ \wedge\ y\notin C_n\}$ is a Π^1_1 subset of $\omega\times Y$ (see Theorem 3.3.1 in [H-K-Lo]). Intuitively, W is the set of codes for the Δ^1_1 subsets of Y. We set, for $1\leq\xi<\omega_1^{\text{CK}}$,

$$W_\xi:=\{n\in W\mid C_n \text{ is a } \mathbf{\Pi}^0_\xi\cap\Delta^1_1 \text{ subset of } Y\}.$$

By Proposition 1.4 and Theorem 1.A in [Lo], the set $W_{<\xi}:=\bigcup_{1\leq\eta<\xi}W_\eta$ is Π^1_1. The sets W_ξ and $W_{<\xi}$ are the sets of codes for the $\mathbf{\Pi}^0_\xi\cap\Delta^1_1$ and $\mathbf{\Pi}^0_{<\xi}\cap\Delta^1_1$ subsets of Y respectively.

Lemma 2.3.2 *Let $1\leq\xi<\omega_1^{CK}$, X,Y be recursively presented Polish spaces, and B be a Σ^1_1 subset of $X\times Y$. Then $\overline{B}^{T_1\times T_\xi}$ is Σ^1_1.*

Proof. Assume first that $\xi=1$. Let $\big(N(X,m)\big)_{m\in\omega}$ and $\big(N(Y,n)\big)_{n\in\omega}$ be effective basis for the topology of X and Y, respectively. Then $(x,y)\notin\overline{B}^{T_1\times T_1}$ is equivalent to

$$\exists m,n\in\omega\ \ x\in N(X,m)\ \wedge\ y\in N(Y,n)\ \wedge$$
$$\forall(x',y')\in X\times Y\ \ \big(x'\notin N(X,m)\ \vee\ y'\notin N(Y,n)\ \vee\ (x',y')\notin B\big),$$

which shows that $\overline{B}^{T_1\times T_1}$ is Σ^1_1.

Assume now that $\xi\geq 2$. If $(x,y)\notin\overline{B}^{T_1\times T_\xi}$, then we can find $m\in\omega$ and a $\Sigma^1_1\cap\mathbf{\Pi}^0_{<\xi}$ subset S of Y such that $(x,y)\in N(X,m)\times S\subseteq\neg B$. In particular, S is contained in the Π^1_1 set

$$P:=\{y\in Y\mid\forall x\in N(X,m)\ (x,y)\notin B\}.$$

Theorems 1.A and 1.B in [Lo] provide a $\Delta^1_1\cap\mathbf{\Pi}^0_{<\xi}$ subset D of Y separating S from $\neg P$. This gives $n\in W_{<\xi}$ such that C_n separates S from $\neg P$. Thus $(x,y)\notin\overline{B}^{T_1\times T_\xi}$ is equivalent to

$$\exists m\in\omega\ \ \exists n\in W_{<\xi}\ \ x\in N(X,m)\ \wedge\ y\in C_n\ \wedge$$
$$\forall(x',y')\in X\times Y\ \big(x'\notin N(X,m)\ \vee\ (n\in W\wedge y'\notin C_n)\ \vee\ (x',y')\notin B\big),$$

which shows that $\overline{B}^{T_1\times T_\xi}$ is Σ^1_1. □

Theorem 2.3.3 *Let $1\leq\eta,\xi<\omega_1^{CK}$, X,Y be recursively presented Polish spaces, and A,B be disjoint Σ^1_1 subsets of $X\times Y$. We assume that A is separable from B by a $(\mathbf{\Sigma}^0_\eta\times\mathbf{\Sigma}^0_\xi)_\sigma$ set. Then A is separable from B by a $\Delta^1_1\cap\big((\Delta^1_1\cap\mathbf{\Sigma}^0_\eta)\times(\Delta^1_1\cap\mathbf{\Sigma}^0_\xi)\big)_\sigma$ set.*

Proof. We argue as in the proof of Theorem 2.3 in [L-Z]. □

The next result is similar to Theorem 2.5 in [L-Z].

Theorem 2.3.4 *Let $1 \leq \eta, \xi < \omega_1^{CK}$, X, Y be recursively presented Polish spaces, and A, B be disjoint Σ_1^1 subsets of $X \times Y$. The following are equivalent:*

(a) the set A cannot be separated from B by a $(\mathbf{\Sigma}_\eta^0 \times \mathbf{\Sigma}_\xi^0)_\sigma$ set.

(b) the set A cannot be separated from B by a $\Delta_1^1 \cap (\mathbf{\Sigma}_\eta^0 \times \mathbf{\Sigma}_\xi^0)_\sigma$ set.

(c) the set A cannot be separated from B by a $\mathbf{\Sigma}_1^0(T_\eta \times T_\xi)$ set.

(d) $A \cap \overline{B}^{T_\eta \times T_\xi} \neq \emptyset$.

Proof. Theorem 2.3.3 implies that (a) is indeed equivalent to (b), and actually to the fact that A cannot be separated from B by a $\Delta_1^1 \cap \big((\Delta_1^1 \cap \mathbf{\Sigma}_\eta^0) \times (\Delta_1^1 \cap \mathbf{\Sigma}_\xi^0)\big)_\sigma$ set. By Theorem 1.A in [Lo], a $\Delta_1^1 \cap \mathbf{\Sigma}_\xi^0$ set is a countable union of $\Delta_1^1 \cap \mathbf{\Pi}_{<\xi}^0$ sets, and thus T_ξ-open, if $\xi \geq 2$. Therefore (c) implies (a), and the converse is clear. It is also clear that (c) and (d) are equivalent. □

The following result is Lemma 3.3 in [Za], and is a consequence of Theorem 2.3.4.

Theorem 2.3.5 *Let $1 \leq \xi, \eta < \omega_1^{CK}$, X, Y be recursively presented Polish spaces, and A, B be disjoint Σ_1^1 subsets of $X \times Y$. The following are equivalent:*

(a) The set A cannot be separated from B by a $\mathbf{\Pi}_\eta^0 \times \mathbf{\Pi}_\xi^0$ set.

(b) $B \cap (\overline{proj_X[A]}^{T_\eta} \times \overline{proj_Y[A]}^{T_\xi}) \neq \emptyset$.

3 Countable unions of $\mathbf{\Sigma}_1^0 \times \mathbf{\Sigma}_\xi^0$ sets

Let Q be a $\mathbf{\Pi}_\xi^0$ subset of 2^ω which is not $\mathbf{\Sigma}_\xi^0$. Then $P := h[Q]$ is a $\mathbf{\Pi}_\xi^0$ subset of $[\subseteq]$ which is not $\mathbf{\Sigma}_\xi^0$ since h is a homeomorphism.

(A) The successor case

Assume that $\xi = \eta + 1$ is a countable ordinal. Theorem 2.1.2 gives a resolution family $(R^\rho)_{\rho \leq \eta}$. We set $\mathbb{X} := [R^\eta]$, $\mathbb{Y} := [\subseteq]$, $\mathbb{A} := \{(\beta, \alpha) \in \mathbb{X} \times \mathbb{Y} \mid \Pi(\beta) = \alpha \in P\}$ and

$$\mathbb{B} := \{(\beta, \alpha) \in \mathbb{X} \times \mathbb{Y} \mid \Pi(\beta) = \alpha \notin P\}.$$

The sets $\mathbb{A}$ and $\mathbb{B}$ are the diagonals of P and $\neg P$ respectively, viewed in $\mathbb{X} \times \mathbb{Y}$. Note that $\mathbb{X}$ and $\mathbb{Y}$ are zero-dimensional Polish spaces, $\mathbb{A}$ is a closed subset of $\mathbb{X} \times \mathbb{Y}$, and $\mathbb{B}$ is a difference of two closed subsets of $\mathbb{X} \times \mathbb{Y}$, disjoint from $\mathbb{A}$.

Lemma 3.1 *The set $\mathbb{A}$ is not separable from $\mathbb{B}$ by a $(\mathbf{\Sigma}_1^0 \times \mathbf{\Sigma}_\xi^0)_\sigma$ subset of $\mathbb{X} \times \mathbb{Y}$.*

Proof. We argue by contradiction, which gives a sequence $(O_n)_{n \in \omega}$ of open subsets of $[R^\eta]$ and a sequence $(S_n)_{n \in \omega}$ of $\mathbf{\Sigma}_\xi^0$ subsets of $[\subseteq]$ such that $\mathbb{A} \subseteq \bigcup_{n \in \omega} O_n \times S_n \subseteq \neg \mathbb{B}$. This implies that $P = \bigcup_{n \in \omega} \Pi[O_n] \cap S_n$. As Π^{-1} is $\mathbf{\Sigma}_\xi^0$-measurable, $\Pi[O_n]$ and P are $\mathbf{\Sigma}_\xi^0$ subsets of $[\subseteq]$, which is absurd. □

One of the key ideas of the proof of Theorem 1.5 is as follows. Since we work with rectangles, it is natural to consider projections. The representation theorem, applied on the left side to define our mimimum objects, will lead us to consider a lot of predecessors on the right side in the key Condition (5) to come (making the construction possible), and also closures of projections for some corresponding topologies. The projection associated with the smallest predecessor allows only to consider one topology on the left. This the the reason why we have the class of open sets in the left factor in the statement.

Proof of Theorem 1.5. The exactly part comes from Lemma 3.1. Assume that (a) does not hold. In order to simplify the notation, we will asume that $\xi < \omega_1^{\text{CK}}$, X and Y are recursively presented and A, B are Σ^1_1, so that $N := A \cap \overline{B}^{T_1 \times T_\xi}$ is a nonempty (by Theorem 2.3.4) Σ^1_1 (by Lemma 2.3.2) subset of $X \times Y$.

We set $\mathcal{I} := \{s \in 2^{<\omega} \mid N_s^{R^\eta} \cap \Pi^{-1}(P) \neq \emptyset\}$. The set $\mathcal{I}$ is essentially the tree associated with the closed set $\Pi^{-1}(P)$. It will determine if the finite approximation of $f \times g$ associated with s is contained in $N \subseteq A$ or B. As P is not empty, $\emptyset \in \mathcal{I}$. We construct, for $s \in 2^{<\omega}$,

- a point x_s of X and a Σ^0_1 subset X_s of X,
- a point y_s of Y and a Σ^0_1 subset Y_s of Y,
- a Σ^1_1 subset S_s of $X \times Y$.

We want these objects to satisfy the following conditions:

$$(1) \begin{cases} \overline{X_t} \subseteq X_s \text{ if } s \; R^\eta \; t \wedge s \neq t \\ \overline{Y_t} \subseteq Y_s \text{ if } s \; R^0 \; t \wedge s \neq t \\ S_t \subseteq S_s \text{ if } s \; R^\eta \; t \wedge (s, t \in \mathcal{I} \vee s, t \notin \mathcal{I}) \end{cases}$$

$$(2) \; x_s \in X_s \wedge y_s \in Y_s \wedge (x_s, y_s) \in S_s \subseteq (X_s \times Y_s) \cap \Omega_{X \times Y}$$

$$(3) \; \text{diam}(X_s), \text{diam}(Y_s), \text{diam}_{\text{GH}}(S_s) \leq 2^{-|s|}$$

$$(4) \; S_s \subseteq \begin{cases} N \text{ if } s \in \mathcal{I} \\ B \text{ if } s \notin \mathcal{I} \end{cases}$$

$$(5) \; \text{proj}_Y[S_t] \subseteq \overline{\text{proj}_Y[S_s]}^{T_\rho} \text{ if } s \; R^\rho \; t \wedge 1 \leq \rho \leq \eta$$

Assume that this is done. Let $\beta \in \mathbb{X}$. Note that $\beta(k) \; R^\eta \; \beta(k+1)$ for each $k \in \omega$. By (1),

$$\overline{X_{\beta(k+1)}} \subseteq X_{\beta(k)}.$$

Thus $(\overline{X_{\beta(k)}})_{k \in \omega}$ is a decreasing sequence of nonempty closed subsets of X with vanishing diameters. We define $\{f(\beta)\} := \bigcap_{k \in \omega} \overline{X_{\beta(k)}} = \bigcap_{k \in \omega} X_{\beta(k)}$, so that $f(\beta) = \lim_{k \to \infty} x_{\beta(k)}$ and f is continuous.

Now let $\alpha \in \mathbb{Y}$. By (1), $\overline{Y_{\alpha(k+1)}} \subseteq Y_{\alpha(k)}$. Thus $(\overline{Y_{\alpha(k)}})_{k \in \omega}$ is a decreasing sequence of nonempty closed subsets of Y with vanishing diameters. We define $\{g(\alpha)\} := \bigcap_{k \in \omega} \overline{Y_{\alpha(k)}} = \bigcap_{k \in \omega} Y_{\alpha(k)}$, so that $g(\alpha) = \lim_{k \to \infty} y_{\alpha(k)}$ and $g : \mathbb{Y} \to Y$ is continuous.

Let $(\beta,\alpha)\in\mathbb{A}$. Note that $\beta(k)\in\mathcal{I}$ for each $k\in\omega$. By (1)-(4), $(S_{\beta(k)})_{k\in\omega}$ is a decreasing sequence of nonempty clopen subsets of $N\cap\Omega_{X\times Y}$ with vanishing GH-diameters. We set $\{F(\beta)\}:=\bigcap_{k\in\omega}\ S_{\beta(k)}$. Note that $(x_{\beta(k)},y_{\beta(k)})_{k\in\omega}$ converges to $F(\beta)$ for $\varSigma_{X\times Y}$, and thus for the usual topology on $X\times Y$. As $\big(\beta(k)\big)_{k\in\omega}$ is a subsequence of $\big(\alpha(k)\big)_{k\in\omega}$, $\big(f(\beta),g(\alpha)\big)=F(\beta)$, which is therefore in $N\subseteq A$, showing that $\mathbb{A}\subseteq(f\times g)^{-1}(A)$.

Let $(\beta,\alpha)\in\mathbb{B}$. As $\Pi^{-1}(P)$ is a closed subset of $[R^\eta]$, there is $k_0\in\omega$ such that $\beta(k)\notin\mathcal{I}$ whenever $k\geq k_0$. By (1)-(4), $(S_{\beta(k)})_{k\geq k_0}$ is a decreasing sequence of nonempty clopen subsets of $B\cap\Omega_{X\times Y}$ with vanishing GH-diameters, and we define $\{G(\beta)\}:=\bigcap_{k\geq k_0}\ S_{\beta(k)}$. Note that $(x_{\beta(k)},y_{\beta(k)})_{k\in\omega}$ converges to $G(\beta)$. So $\big(f(\beta),g(\alpha)\big)=G(\beta)$, which is therefore in B, showing that $\mathbb{B}\subseteq(f\times g)^{-1}(B)$.

Let us prove that the construction is possible. Let $(x_\emptyset,y_\emptyset)\in N\cap\Omega_{X\times Y}$, $X_\emptyset,Y_\emptyset$ be $\varSigma^0_1$ sets with diameter at most 1 such that $(x_\emptyset,y_\emptyset)\in X_\emptyset\times Y_\emptyset$, and $S_\emptyset$ be a $\varSigma^1_1$ subset of $X\times Y$ with GH-diameter at most 1 and $(x_\emptyset,y_\emptyset)\in S_\emptyset\subseteq N\cap(X_\emptyset\times Y_\emptyset)\cap\Omega_{X\times Y}$. Assume that our objects satisfying (1)-(5) are constructed up to the length l, which is the case for $l=0$. So let $s\in 2^{l+1}$.

Claim *The set* $\text{proj}_Y[S_{s^\eta}]\cap\bigcap_{1\leq\rho<\eta}\ \overline{\text{proj}_Y[S_{s^\rho}]}^{T_\rho}\cap Y_{s^0}$ *is* T_1*-dense in* $\overline{\text{proj}_Y[S_{s^1}]}\cap Y_{s^0}$ *if* $\eta\geq 1$.

Indeed, we apply Lemma 2.2.1.(c) to $E:=Y$ and $O:=Y_{s^0}$. ⋄

Note that $s^1\subseteq s^0\subsetneqq s$ and $s^1\ R^1\ s^0$, so that $\text{proj}_Y[S_{s^0}]\subseteq\overline{\text{proj}_Y[S_{s^1}]}$. Thus $y_{s^0}\in\overline{\text{proj}_Y[S_{s^1}]}\cap Y_{s^0}$. This shows that $I:=\text{proj}_Y[S_{s^\eta}]\cap\bigcap_{1\leq\rho<\eta}\ \overline{\text{proj}_Y[S_{s^\rho}]}^{T_\rho}\cap Y_{s^0}$ is not empty, even if $\eta=0$.

Case 1 $s\notin\mathcal{I}$

1.1 If $s^\eta\notin\mathcal{I}$, then we choose $y_s\in I$, $x_s\in X_{s^\eta}$ with $(x_s,y_s)\in S_{s^\eta}$, $\varSigma^0_1$ sets X_s,Y_s with diameter at most 2^{-l-1} such that $(x_s,y_s)\in X_s\times Y_s\subseteq\overline{X_s}\times\overline{Y_s}\subseteq X_{s^\eta}\times Y_{s^0}$, and a $\varSigma^1_1$ subset S_s of $X\times Y$ with GH-diameter at most 2^{-l-1} such that $(x_s,y_s)\in S_s\subseteq S_{s^\eta}\cap\big(X_s\times(\bigcap_{1\leq\rho<\eta}\ \overline{\text{proj}_Y[S_{s^\rho}]}^{T_\rho}\cap Y_s)\big)$. If $t\ R^\eta\ s$ and $s\neq t$, then $t\ R^0\ s^\eta\ R^\eta\ s$, so that $t\ R^\eta\ s^\eta$, by Lemma 2.1.5. This implies that $\overline{X_s}\subseteq X_t$ and $\text{proj}_Y[S_{s^\eta}]\subseteq\overline{\text{proj}_Y[S_t]}^{T_\eta}$. Thus $\text{proj}_Y[S_s]\subseteq\overline{\text{proj}_Y[S_t]}^{T_\eta}$. If moreover $t\notin\mathcal{I}$, then $s^\eta\notin\mathcal{I}$ since $t\ R^\eta\ s^\eta$. Thus $S_{s^\eta}\subseteq S_t$ and $S_s\subseteq S_t$. Similarly, $\overline{Y_s}\subseteq Y_t$ if $t\ R^0\ s$ and $s\neq t$ (this is simpler). If $1\leq\rho<\eta$, $t\ R^\rho\ s$ and $s\neq t$, then $t\ R^\rho\ s^\rho$, $\text{proj}_Y[S_{s^\rho}]\subseteq\overline{\text{proj}_Y[S_t]}^{T_\rho}$ and $\text{proj}_Y[S_s]\subseteq\overline{\text{proj}_Y[S_t]}^{T_\rho}$.

1.2 If $s^\eta\in\mathcal{I}$, then we choose $y\in I$, and $x\in X_{s^\eta}$ with $(x,y)\in S_{s^\eta}$. Note that

$$(x,y)\in\overline{B}^{T_1\times T_\xi}\cap\big(X_{s^\eta}\times(\bigcap_{1\leq\rho\leq\eta}\overline{\text{proj}_Y[S_{s^\rho}]}^{T_\rho}\cap Y_{s^0})\big).$$

This gives $(x_s,y_s)\in B\cap\big(X_{s^\eta}\times(\bigcap_{1\leq\rho\leq\eta}\ \overline{\text{proj}_Y[S_{s^\rho}]}^{T_\rho}\cap Y_{s^0})\big)\cap\Omega_{X\times Y}$. We choose $\varSigma^0_1$ sets X_s,Y_s with diameter at most 2^{-l-1} such that $(x_s,y_s)\in X_s\times Y_s\subseteq\overline{X_s}\times\overline{Y_s}\subseteq X_{s^\eta}\times Y_{s^0}$, and a $\varSigma^1_1$ subset S_s of $X\times Y$ with GH-diameter at most 2^{-l-1} such that

$$(x_s,y_s)\in S_s\subseteq B\cap\big(X_s\times(\bigcap_{1\leq\rho\leq\eta}\overline{\text{proj}_Y[S_{s^\rho}]}^{T_\rho}\cap Y_s)\big)\cap\Omega_{X\times Y}.$$

As above, we check that these objects are as required.

Case 2 $s\in\mathcal{I}$

Note that $s^\eta\in\mathcal{I}$. We argue as in 1.1. $\square$

(B) The limit case

Assume that ξ is an infinite limit ordinal. We indicate the differences with the successor case. Theorem 2.1.4 gives a uniform resolution family $(R^\rho)_{\rho\leq\xi}$. We set $\mathbb{X}:=[R^\xi]$, $\mathbb{Y}:=[\subseteq]$,

$$\mathbb{A}:=\{(\gamma,\beta)\in\mathbb{X}\times\mathbb{Y}\mid\Pi(\gamma)=\beta\in P\},$$

and $\mathbb{B}:=\{(\gamma,\beta)\in\mathbb{X}\times\mathbb{Y}\mid\Pi(\gamma)=\beta\notin P\}$.

Proof of Theorem 1.5. This time, $\mathcal{I}:=\{s\in 2^{<\omega}\mid N_s^{R^\xi}\cap\Pi^{-1}(P)\neq\emptyset\}$. If $s\in 2^{<\omega}$, then we set $k_s:=\max\{h_{R^\xi}(t)\mid t\subseteq s\}$, and let $\xi(s)$ be the ordinal $\xi^{k(s)}$ given by the definition of a uniform resolution family. If $t,t'\subseteq s$ and $t\ R^{\xi(s)}\ t'$, then $t\ R^\xi\ t'$. Note that $1\leq\xi(t)\leq\xi(s)$ if $t\subseteq s$.

Conditions (1) and (5) become

$$(1')\ \begin{cases}\overline{X_t}\subseteq X_s\ \text{ if }\ s\ R^\xi\ t\ \wedge\ s\neq t\\ \overline{Y_t}\subseteq Y_s\ \text{ if }\ s\ R^0\ t\ \wedge\ s\neq t\\ S_t\subseteq S_s\ \text{ if }\ s\ R^\xi\ t\ \wedge\ (s,t\in\mathcal{I}\vee s,t\notin\mathcal{I})\end{cases}$$

$$(5')\ \text{proj}_Y[S_t]\subseteq\overline{\text{proj}_Y[S_s]}^{T_\rho}\ \text{ if }\ s\ R^\rho\ t\ \wedge\ 1\leq\rho\leq\xi(s)$$

As $s^{\xi(s)}\ R^{\xi(s)}\ s$, $s^{\xi(s)}\ R^\xi\ s$ and $s^{\xi(s)}\subseteq s^\xi$. As $s^\xi\subseteq s^{\xi^{k(s)}}=s^{\xi(s)}$, we get $s^{\xi(s)}=s^\xi$.

Claim The set $\text{proj}_Y[S_{s^\xi}]\cap\bigcap_{1\leq\rho<\xi(s)}\ \overline{\text{proj}_Y[S_{s^\rho}]}^{T_\rho}\cap Y_{s^0}$ is T_1-dense in $\overline{\text{proj}_Y[S_{s^1}]}^{T_1}\cap Y_{s^0}$.

We conclude as in the successor case, using the fact that $\xi(.)$ is increasing. $\square$

4 $\mathbf{\Pi}^0_1\times\mathbf{\Pi}^0_\xi$ sets

We consider P as in Section 3.

(A) The successor case

Assume that $\xi=\eta+1$ is a countable ordinal. Theorem 2.1.2 gives a resolution family $(R^\rho)_{\rho\leq\eta}$. We set $\mathbb{X}:=[R^\eta]\oplus\Pi^{-1}(\neg P)$, $\mathbb{Y}:=[\subseteq]\oplus\Pi^{-1}(\neg P)$,

$$\mathbb{A}:=\big\{\big((0,\beta),(1,\gamma)\big)\in\mathbb{X}\times\mathbb{Y}\mid\beta=\gamma\big\}\cup\big\{\big((1,\gamma),(0,\alpha)\big)\in\mathbb{X}\times\mathbb{Y}\mid\Pi(\gamma)=\alpha\big\}$$

and $\mathbb{B}:=\big\{\big((0,\beta),(0,\alpha)\big)\in\mathbb{X}\times\mathbb{Y}\mid\Pi(\beta)=\alpha\in P\big\}$. Note that $\mathbb{X}$ and $\mathbb{Y}$ are zero-dimensional Polish spaces, $\mathbb{A}$ is a closed subset of $\mathbb{X}\times\mathbb{Y}$, and $\mathbb{B}$ is a closed subset of $\mathbb{X}\times\mathbb{Y}$ disjoint from $\mathbb{A}$.

Lemma 4.1 *The set $\mathbb{A}$ is not separable from $\mathbb{B}$ by a $\mathbf{\Pi}^0_1\times\mathbf{\Pi}^0_\xi$ subset of $\mathbb{X}\times\mathbb{Y}$.*

Proof. Let C be a closed subset of $\mathbb{X}$ and S be a $\mathbf{\Pi}^0_\xi$ subset of $\mathbb{Y}$ with $\mathbb{A}\subseteq C\times S$. Note that $C\cap(\{0\}\times[R^\eta])=\{0\}\times C'$ for some closed subset C' of $[R^\eta]$. Similarly, $S\cap(\{0\}\times[\subseteq])=\{0\}\times S'$ for some $\mathbf{\Pi}^0_\xi$ subset S' of $[\subseteq]$. Let $\alpha\in[\subseteq]\setminus P$, and $\beta:=\gamma:=\Pi^{-1}(\alpha)$. Then $\big((0,\beta),(1,\gamma)\big)\in\mathbb{A}$, so that $\beta\in C'$ and $\alpha\in\Pi[C']$. Similarly, $\big((1,\gamma),(0,\alpha)\big)\in\mathbb{A}$, so that $\alpha\in S'$. This shows that $[\subseteq]\setminus P\subseteq\Pi[C']\cap S'$. As P is not $\mathbf{\Sigma}^0_\xi$, there is $\alpha\in\Pi[C']\cap S'\cap P$, and $\big((0,\beta),(0,\alpha)\big)\in\mathbb{B}\cap(C\times S)$ if $\beta:=\Pi^{-1}(\alpha)$. □

Proof of Theorem 1.6. The exactly part comes from Lemma 4.1. Assume that (a) does not hold. In order to simplify the notation, we will assume that $\xi<\omega_1^{\text{CK}}$, X and Y are recursively presented and A,B are Σ^1_1, so that $N:=B\cap(\overline{\text{proj}_X[A]}\times\overline{\text{proj}_Y[A]}^{T_\xi})$ is a nonempty Σ^1_1 subset of $X\times Y$, by Theorem 2.3.5.

We set $\mathcal{I}:=\{s\in 2^{<\omega}\mid N^{R^\eta}_s\cap\Pi^{-1}(P)\neq\emptyset\}$. As P is not empty, $\emptyset\in\mathcal{I}$. We define, for $t\in 2^{<\omega}$, $t_c\in 2$ by $t_c:=\chi_{\neg\mathcal{I}}(t)$. We construct

- a point $x_{\varepsilon,s}$ of X and a Σ^0_1 subset $X_{\varepsilon,s}$ of X, when $(\varepsilon,s)\in(\{0\}\times 2^{<\omega})\cup\big(\{1\}\times(\neg\mathcal{I})\big)$,
- a point $y_{\varepsilon,s}$ of Y and a Σ^0_1 subset $Y_{\varepsilon,s}$ of Y, when $(\varepsilon,s)\in(\{0\}\times 2^{<\omega})\cup\big(\{1\}\times(\neg\mathcal{I})\big)$,
- a Σ^1_1 subset $S_{\varepsilon,\varepsilon',s}$ of $X\times Y$, when $(\varepsilon,\varepsilon',s)\in 2^2\times 2^{<\omega}$, $(\varepsilon\neq\varepsilon'\wedge s\notin\mathcal{I})$ or $(\varepsilon=\varepsilon'=0\wedge s\in\mathcal{I})$.

We want these objects to satisfy the following conditions:

$$(1)\ \left\{\begin{array}{l}\overline{X_{\varepsilon,t}}\subseteq X_{\varepsilon,s}\ \text{ if }\ s\ R^\eta\ t\ \wedge\ s\neq t\\ \overline{Y_{0,t}}\subseteq Y_{0,s}\ \text{ if }\ s\ R^0\ t\ \wedge\ s\neq t\\ \overline{Y_{1,t}}\subseteq Y_{1,s}\ \text{ if }\ s\ R^\eta\ t\ \wedge\ s\neq t\\ S_{\varepsilon,\varepsilon',t}\subseteq S_{\varepsilon,\varepsilon',s}\ \text{ if }\ s\ R^\eta\ t\end{array}\right.$$

$$(2)\ x_{\varepsilon,s}\in X_{\varepsilon,s}\ \wedge\ y_{\varepsilon,s}\in Y_{\varepsilon,s}\ \wedge\ (x_{\varepsilon,s},y_{\varepsilon',s})\in S_{\varepsilon,\varepsilon',s}\subseteq(X_{\varepsilon,s}\times Y_{\varepsilon',s})\cap\Omega_{X\times Y}$$

$$(3)\ \text{diam}(X_{\varepsilon,s}),\text{diam}(Y_{\varepsilon,s}),\text{diam}_{\text{GH}}(S_{\varepsilon,\varepsilon',s})\leq 2^{-|s|}$$

$$(4)\ S_{\varepsilon,\varepsilon',s}\subseteq\left\{\begin{array}{l}N\ \text{ if }\ s\in\mathcal{I}\\ A\ \text{ if }\ s\notin\mathcal{I}\end{array}\right.$$

$$(5)\ \text{proj}_Y[S_{t_c,0,t}]\subseteq\overline{\text{proj}_Y[S_{s_c,0,s}]}^{T_\rho}\ \text{ if }\ s\ R^\rho\ t\ \wedge\ 1\leq\rho\leq\eta$$

Assume that this is done. Let $(0,\gamma)\in\mathbb{X}$. Note that $\gamma(k)\ R^\eta\ \gamma(k+1)$ for each $k\in\omega$. By (1), $\overline{X_{0,\gamma(k+1)}}\subseteq X_{0,\gamma(k)}$. Thus $(\overline{X_{0,\gamma(k)}})_{k\in\omega}$ is a decreasing sequence of nonempty closed subsets of X with vanishing diameters. We define $\{f(0,\gamma)\}:=\bigcap_{k\in\omega}\ \overline{X_{0,\gamma(k)}}=\bigcap_{k\in\omega}\ X_{0,\gamma(k)}$, so that

$$f(0,\gamma)=\lim_{k\to\infty}x_{0,\gamma(k)}$$

and f is continuous on $\{0\}\times[R^\eta]$. Now let $(1,\gamma)\in\mathbb{X}$. Note that moreover that there is $k_\gamma\in\omega$ minimal such that $\gamma(k)\notin\mathcal{I}$ if $k\geq k_\gamma$. We define $f(1,\gamma)$ similarly, using $(\overline{X_{1,\gamma(k)}})_{k\geq k_\gamma}$. Note that f is continuous on $\{1\}\times\Pi^{-1}(\neg\mathbb{P})$ since $k_{\gamma'}=k_\gamma$ if $\gamma'\in N^{R^\eta}_{\gamma(k_\gamma)}$.

Now let $(0,\alpha)\in\mathbb{Y}$. By (1), $\overline{Y_{0,\alpha(k+1)}}\subseteq Y_{0,\alpha(k)}$. Thus $(\overline{Y_{0,\alpha(k)}})_{k\in\omega}$ is a decreasing sequence of nonempty closed subsets of Y with vanishing diameters. We define

$$\{g(0,\alpha)\}:=\bigcap_{k\in\omega}\overline{Y_{0,\alpha(k)}}=\bigcap_{k\in\omega}Y_{0,\alpha(k)},$$

so that $g(0,\alpha)=\lim_{k\to\infty}y_{0,\alpha(k)}$. We define $g(1,\gamma)$ like $f(1,\gamma)$, so that $g:\mathbb{Y}\to Y$ is continuous.

Assume that $\big((0,\gamma),(1,\gamma)\big)\in\mathbb{A}$. As $\Pi^{-1}(P)$ is a closed subset of $[R^\eta]$, there is $k_0\in\omega$ such that $\gamma(k)\notin\mathcal{I}$ if $k\geq k_0$. By (1)-(4), $(S_{0,1,\gamma(k)})_{k\geq k_0}$ is a decreasing sequence of nonempty clopen subsets of $A\cap\Omega_{X\times Y}$ with vanishing GH-diameters. We set $\{F(\gamma)\}:=\bigcap_{k\geq k_0}S_{0,1,\gamma(k)}$. Note that $(x_{0,\gamma(k)},y_{1,\gamma(k)})_{k\in\omega}$ converges to $F(\gamma)$ for $\Sigma_{X\times Y}$, and thus for the usual topology on $X\times Y$. So $\big(f(0,\gamma),g(1,\gamma)\big)=F(\gamma)$, which is therefore in A. If now $\big((1,\gamma),(0,\alpha)\big)\in\mathbb{A}$, then we argue similarly, showing that $\mathbb{A}\subseteq(f\times g)^{-1}(A)$.

Let $\big((0,\gamma),(0,\alpha)\big)\in\mathbb{B}$. Note that $\gamma(k)\in\mathcal{I}$ for each $k\in\omega$. By (1)-(4), $(S_{0,0,\gamma(k)})_{k\in\omega}$ is a decreasing sequence of nonempty clopen subsets of $N\cap\Omega_{X\times Y}$ with vanishing GH-diameters, and we define $\{G(\gamma)\}:=\bigcap_{k\in\omega}S_{0,0,\gamma(k)}$. Note that $(x_{0,\gamma(k)},y_{0,\gamma(k)})_{k\in\omega}$ converges to $G(\gamma)$. So $\big(f(0,\gamma),g(0,\alpha)\big)=G(\gamma)$, which is therefore in $N\subseteq B$, showing that $\mathbb{B}\subseteq(f\times g)^{-1}(B)$.

Let us prove that the construction is possible. Let $(x_{0,\emptyset},y_{0,\emptyset})\in N\cap\Omega_{X\times Y}$, $X_{0,\emptyset},Y_{0,\emptyset}$ be Σ^0_1 sets with diameter at most 1 such that $(x_{0,\emptyset},y_{0,\emptyset})\in X_{0,\emptyset}\times Y_{0,\emptyset}$, and $S_{0,0,\emptyset}$ be a Σ^1_1 subset of $X\times Y$ with GH-diameter at most 1 and $(x_{0,\emptyset},y_{0,\emptyset})\in S_{0,0,\emptyset}\subseteq N\cap(X_{0,\emptyset}\times Y_{0,\emptyset})\cap\Omega_{X\times Y}$. Assume that our objects satisfying (1)-(5) are constructed up to the length l, which is the case for $l=0$. So let $s\in 2^{l+1}$.

Claim *The set* $\mathrm{proj}_Y[S_{s^\eta_c,0,s^\eta}]\cap\bigcap_{1\leq\rho<\eta}\overline{\mathrm{proj}_Y[S_{s^\rho_c,0,s^\rho}]}^{T_\rho}\cap Y_{0,s^0}$ *is* T_1*-dense in* $\overline{\mathrm{proj}_Y[S_{s^1_c,0,s^1}]}\cap Y_{0,s^0}$ *if* $\eta\geq 1$.

As in the proof of Theorem 1.5, we infer that

$$I:=\mathrm{proj}_Y[S_{s^\eta_c,0,s^\eta}]\cap\bigcap_{1\leq\rho<\eta}\overline{\mathrm{proj}_Y[S_{s^\rho_c,0,s^\rho}]}^{T_\rho}\cap Y_{0,s^0}$$

is not empty.

Case 1 $s\notin\mathcal{I}$

1.1 $s^\eta\notin\mathcal{I}$

Note that $s^\eta_c=1$. We choose $y_{0,s}\in I$, $x_{1,s}\in X_{1,s^\eta}$ with $(x_{1,s},y_{0,s})\in S_{1,0,s^\eta}$, Σ^0_1 sets $X_{1,s},Y_{0,s}$ with diameter at most 2^{-l-1} such that $(x_{1,s},y_{0,s})\in X_{1,s}\times Y_{0,s}\subseteq\overline{X_{1,s}}\times\overline{Y_{0,s}}\subseteq X_{1,s^\eta}\times Y_{0,s^0}$, and a Σ^1_1 subset $S_{1,0,s}$ of $X\times Y$ with GH-diameter at most 2^{-l-1} such that

$$(x_{1,s},y_{0,s})\in S_{1,0,s}\subseteq S_{1,0,s^\eta}\cap\Big(X_{1,s}\times\big(\bigcap_{1\leq\rho<\eta}\overline{\mathrm{proj}_Y[S_{s^\rho_c,0,s^\rho}]}^{T_\rho}\cap Y_{0,s}\big)\Big).$$

As in the proof of Theorem 1.5, we check that these objects are as required.

We also set $(x_{0,s},y_{1,s}):=(x_{0,s^\eta},y_{1,s^\eta})$, choose Σ^0_1 sets $X_{0,s},Y_{1,s}$ with diameter at most 2^{-l-1} such that $(x_{0,s},y_{1,s})\in X_{0,s}\times Y_{1,s}\subseteq\overline{X_{0,s}}\times\overline{Y_{1,s}}\subseteq X_{0,s^\eta}\times Y_{1,s^\eta}$, and a Σ^1_1 subset $S_{0,1,s}$ of $X\times Y$ with GH-diameter at most 2^{-l-1} such that $(x_{0,s},y_{1,s})\in S_{0,1,s}\subseteq S_{0,1,s^\eta}\cap(X_{0,s}\times Y_{1,s})$.

1.2 $s^\eta\in\mathcal{I}$

We choose $y\in I$, and $x\in X$ with $(x,y)\in S_{0,0,s^\eta}$. Note that

$$y\in\overline{\text{proj}_Y[A]}^{T_\xi}\cap\bigcap_{1\leq\rho\leq\eta}\overline{\text{proj}_Y[S_{s^\rho_c,0,s^\rho}]}^{T_\rho}\cap Y_{0,s^0}.$$

This gives $y'\in\text{proj}_Y[A]\cap\bigcap_{1\leq\rho\leq\eta}\overline{\text{proj}_Y[S_{s^\rho_c,0,s^\rho}]}^{T_\rho}\cap Y_{0,s^0}$, $x'\in X$ with

$$(x',y')\in A\cap\big(X\times(\bigcap_{1\leq\rho\leq\eta}\overline{\text{proj}_Y[S_{s^\rho_c,0,s^\rho}]}^{T_\rho}\cap Y_{0,s^0})\big),$$

and also $(x_{1,s},y_{0,s})\in A\cap\big(X\times(\bigcap_{1\leq\rho\leq\eta}\overline{\text{proj}_Y[S_{s^\rho_c,0,s^\rho}]}^{T_\rho}\cap Y_{0,s^0})\big)\cap\Omega_{X\times Y}$. We choose Σ^0_1 sets $X_{1,s},Y_{0,s}$ with diameter at most 2^{-l-1} such that

$$(x_{1,s},y_{0,s})\in X_{1,s}\times Y_{0,s}\subseteq\overline{X_{1,s}}\times\overline{Y_{0,s}}\subseteq X\times Y_{0,s^0},$$

and a Σ^1_1 subset $S_{1,0,s}$ of $X\times Y$ with GH-diameter at most 2^{-l-1} such that

$$(x_{1,s},y_{0,s})\in S_{1,0,s}\subseteq A\cap\big(X_{1,s}\times(\bigcap_{1\leq\rho\leq\eta}\overline{\text{proj}_Y[S_{s^\rho_c,0,s^\rho}]}^{T_\rho}\cap Y_{0,s})\big)\cap\Omega_{X\times Y}.$$

If $t\ R^\eta\ s$ and $s\neq t$, then $t\subseteq s^\eta\ R^\eta\ s$, $t\ R^\eta\ s^\eta$ and $t\in\mathcal{I}$. Thus $X_{1,t}$ and $S_{1,0,t}$ do not have to be considered. As in the proof of Theorem 1.5, $\overline{Y_{0,s}}\subseteq Y_{0,t}$ if $t\ R^0\ s$ and $s\neq t$, and Condition (5) holds.

Note also that $(x_{0,s^\eta},y_{0,s^\eta})\in S_{0,0,s^\eta}$, so that $x_{0,s^\eta}\in\overline{\text{proj}_X[A]}\cap X_{0,s^\eta}$. This gives a point x' of $\text{proj}_X[A]\cap X_{0,s^\eta}$, and $y'\in Y$ with $(x',y')\in A\cap(X_{0,s^\eta}\times Y)$, and $(x_{0,s},y_{1,s})\in A\cap(X_{0,s^\eta}\times Y)\cap\Omega_{X\times Y}$. We choose Σ^0_1 sets $X_{0,s},Y_{1,s}$ with diameter at most 2^{-l-1} such that

$$(x_{0,s},y_{1,s})\in X_{0,s}\times Y_{1,s}\subseteq\overline{X_{0,s}}\times\overline{Y_{1,s}}\subseteq X_{0,s^\eta}\times Y,$$

and a Σ^1_1 subset $S_{0,1,s}$ of $X\times Y$ with GH-diameter at most 2^{-l-1} such that

$$(x_{0,s},y_{1,s})\in S_{0,1,s}\subseteq A\cap(X_{0,s}\times Y_{0,s})\cap\Omega_{X\times Y}.$$

Here also, $Y_{1,t}$ and $S_{0,1,t}$ do not have to be considered. As in the proof of Theorem 1.5, $\overline{X_{0,s}}\subseteq X_{0,t}$ if $t\ R^\eta\ s$ and $s\neq t$, and Condition (5) is without object.

Case 2 $s\in\mathcal{I}$

Note that $s^\eta\in\mathcal{I}$. We argue as in the first part of 1.1 to construct $x_{0,s}$, $y_{0,s}$, $X_{0,s}$, $Y_{0,s}$ and $S_{0,0,s}$.□

(B) The limit case

Assume that ξ is an infinite limit ordinal. We indicate the differences with the successor case. Theorem 2.1.4 gives a uniform resolution family $(R^\rho)_{\rho\leq\xi}$. We set $\mathbb{X}:=[R^\xi]\oplus\Pi^{-1}(\neg P)$,

$$\mathbb{Y}:=[\subseteq]\oplus\Pi^{-1}(\neg P),$$

$\mathbb{A}:=\big\{\big((0,\beta),(1,\gamma)\big)\in\mathbb{X}\times\mathbb{Y}\mid\beta=\gamma\big\}\cup\big\{\big((1,\gamma),(0,\alpha)\big)\in\mathbb{X}\times\mathbb{Y}\mid\Pi(\gamma)=\alpha\big\}$ and

$$\mathbb{B}:=\big\{\big((0,\beta),(0,\alpha)\big)\in\mathbb{X}\times\mathbb{Y}\mid\Pi(\beta)=\alpha\in P\big\}.$$

Proof of Theorem 1.6. Conditions (1) and (5) become

$$(1')\ \begin{cases}\overline{X_{\varepsilon,t}}\subseteq X_{\varepsilon,s}\ \text{ if }\ s\ R^\xi\ t\ \wedge\ s\neq t\\ \overline{Y_{0,t}}\subseteq Y_{0,s}\ \text{ if }\ s\ R^0\ t\ \wedge\ s\neq t\\ \overline{Y_{1,t}}\subseteq Y_{1,s}\ \text{ if }\ s\ R^\xi\ t\ \wedge\ s\neq t\\ S_{\varepsilon,\varepsilon',t}\subseteq S_{\varepsilon,\varepsilon',s}\ \text{ if }\ s\ R^\xi\ t\end{cases}$$

$$(5')\ \text{proj}_Y[S_{t_c,0,t}]\subseteq\overline{\text{proj}_Y[S_{s_c,0,s}]}^{T_\rho}\ \text{ if }\ s\ R^\rho\ t\ \wedge\ 1\leq\rho\leq\xi(s)$$

Claim The set $\text{proj}_Y[S_{s_c^\xi,0,s^\xi}]\cap\bigcap_{1\leq\rho<\xi(s)}\ \overline{\text{proj}_Y[S_{s_c^\rho,0,s^\rho}]}^{T_\rho}\cap Y_{0,s^0}$ is T_1-dense in

$$\overline{\text{proj}_Y[S_{s_c^1,0,s^1}]}^{T_1}\cap Y_{0,s^0}.$$

We conclude as in the successor case. □

5 References

[D-SR] G. Debs and J. Saint Raymond, Borel liftings of Borel sets: some decidable and undecidable statements, *Mem. Amer. Math. Soc.* 187, 876 (2007)

[H-K-Lo] L. A. Harrington, A. S. Kechris and A. Louveau, A Glimm-Effros dichotomy for Borel equivalence relations, *J. Amer. Math. Soc.* 3 (1990), 903-928

[K] A. S. Kechris, *Classical Descriptive Set Theory,* Springer-Verlag, 1995

[K-S-T] A. S. Kechris, S. Solecki and S. Todorčević, Borel chromatic numbers, *Adv. Math.* 141 (1999), 1-44

[Ku] C. Kuratowski, Sur une généralisation de la notion d'homéomorphie, *Fund. Math.* 22, 1 (1934), 206-220

[L1] D. Lecomte, On minimal non potentially closed subsets of the plane, *Topology Appl.* 154, 1 (2007), 241-262

[L2] D. Lecomte, A dichotomy characterizing analytic graphs of uncountable Borel chromatic number in any dimension, *Trans. Amer. Math. Soc.* 361 (2009), 4181-4193

[L3] D. Lecomte, How can we recognize potentially $\mathbf{\Pi}^0_\xi$ subsets of the plane?, *J. Math. Log.* 9, 1 (2009), 39-62

[L4] D. Lecomte, Potential Wadge classes, *Mem. Amer. Math. Soc.,* 221, 1038 (2013)

[L-Z] D. Lecomte and M. Zelený, Baire-class ξ colorings: the first three levels, *Trans. Amer. Math. Soc.* 366, 5 (2014), 2345-2373

[Lo] A. Louveau, A separation theorem for Σ^1_1 sets, *Trans. Amer. Math. Soc.* 260 (1980), 363-378

[Lo-SR] A. Louveau and J. Saint Raymond, Borel classes and closed games: Wadge-type and Hurewicz-type results, *Trans. Amer. Math. Soc.* 304 (1987), 431-467

[M] B. D. Miller, The graph-theoretic approach to descriptive set theory, *Bull. Symbolic Logic* 18, 4 (2012), 554-575

[Mo] Y. N. Moschovakis, *Descriptive set theory,* North-Holland, 1980

[Za] R. Zamora, Separation of analytic sets by rectangles of low complexity, *Topology Appl.* 182 (2015), 77-97