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BASIS THEOREMS FOR NON-POTENTIALLY CLOSED SETS AND GRAPHS OF UNCOUNTABLE BOREL CHROMATIC NUMBER

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We show that there is an antichain basis for neither (1) the class of non-potentially closed Borel subsets of the plane under Borel rectangular reducibility nor (2) the class of analytic graphs of uncountable Borel chromatic number under Borel reducibility.

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Introduction

This paper is a contribution to the descriptive set-theoretic study of the Borel structure of the plane. We will employ standard notation as well as various classical theorems, for which we suggest Kechris [3] as a reference.

Suppose that Γ is a class of subsets of Polish spaces, X and Y are Polish spaces, and $R \subseteq X \times Y$ is Borel. We say that R *is potentially* Γ if there are Polish topologies τ_X and τ_Y, compatible with the underlying Borel structure of X and Y, such that $R \in \Gamma(X \times Y, \tau_X \times \tau_Y)$. One of the main questions underlying our work here is:

Question 1. Under what circumstances is R potentially closed?

Various answers to questions of this sort have appeared in the descriptive set-theoretic literature over the last few years. Perhaps the best known example comes

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from the study of Borel equivalence relations. Recall that E_0 is the equivalence relation on $2^{\mathbb{N}}$ given by

$$\alpha E_0 \beta \Leftrightarrow \exists n \in \mathbb{N}\, \forall m \geq n\ (\alpha(m) = \beta(m)).$$

Recall also that if E and F are Borel equivalence relations on X and Y, then a *reduction of E to F* is a function $\pi : X \to Y$ such that

$$\forall x_1, x_2 \in X\ (x_1 E x_2 \Leftrightarrow \pi(x_1) F \pi(x_2)),$$

and an *embedding of E into F* is an injective reduction of E to F. We write $E \sqsubseteq_c F$ to indicate the existence of a continuous embedding of E into F.

Theorem 2 (Harrington-Kechris-Louveau [2]). *Suppose that E is a Borel equivalence relation on a Polish space. Then exactly one of the following holds:*

1. *E is potentially closed;*
2. *$E_0 \sqsubseteq_c E$.*

A *quasi-order on Q* is a reflexive, transitive binary relation $\leq$ on Q. Suppose $D \subseteq Q$. We say *D forms a basis for Q under $\leq$*, or *D is dense in $(Q, \leq)$*, if $\forall q \in Q\, \exists d \in D\ (d \leq q)$. Theorem 2 provides an ideal solution to the special case of Question 1 for Borel equivalence relations, as it implies that $\{E_0\}$ is a one-element basis for the class of non-potentially closed Borel equivalence relations under Borel reducibility, the de facto standard among quasi-orders on Borel equivalence relations.

A *directed graph on X* is an irreflexive set $\mathcal{G} \subseteq X \times X$. A *coloring of $\mathcal{G}$* is a function $c : X \to Z$ such that

$$\forall x_1, x_2 \in X\ ((x_1, x_2) \in \mathcal{G} \Rightarrow c(x_1) \neq c(x_2)).$$

Recall that if $A \subseteq X$, then a function $\pi : A \to Y$ *is Γ-measurable* if the pre-image of every open subset of Y under π is in Γ. The *Γ-chromatic number of $\mathcal{G}$*, or $\chi_\Gamma(\mathcal{G})$, is the least cardinal κ for which there is a Polish space Z and a Γ-measurable coloring $c : X \to Z$ of $\mathcal{G}$ such that $\kappa = |c[X]|$. Let $\chi_B(\mathcal{G})$ denote the *Borel chromatic number of $\mathcal{G}$*. The other main question underlying our work here is:

Question 3. Under what circumstances is $\chi_B(\mathcal{G}) \leq \aleph_0$?

Questions 1 and 3 are very much connected. For example, in §5 we note that R is potentially closed if and only if the directed graph $\mathcal{G}_R$ on $(X \times Y) \setminus R$ given by

$$\mathcal{G}_R = \{((x_1, y_1), (x_2, y_2)) : (x_1, y_2) \in R\}$$

has countable Borel chromatic number (this is essentially due to Lecomte [6].) Moreover, if $\mathcal{G}$ is potentially closed, then $\chi_B(\mathcal{G}) \leq \aleph_0$, and if $\overline{\mathcal{G}} \setminus \mathcal{G}$ is contained in the set $\Delta(X) = \{(x, x) : x \in X\}$, then $\mathcal{G}$ is potentially closed if and only if $\chi_B(\mathcal{G}) \leq \aleph_0$. Along similar lines, it is not difficult to see that if E is a countable Borel equivalence relation, then E is potentially closed if and only if $\chi_B(E \setminus \Delta(X)) \leq \aleph_0$.

Suppose that $\mathcal{G}$ is a directed graph on X and $\mathcal{H}$ is a directed graph on Y. A *homomorphism from $\mathcal{G}$ to $\mathcal{H}$* is a function $\pi : X \to Y$ such that

$$\forall x_1, x_2 \in X \ ((x_1, x_2) \in \mathcal{G} \Rightarrow (\pi(x_1), \pi(x_2)) \in \mathcal{H}).$$

A *graph* is a symmetric directed graph. Fix, from this point forward, sequences $s_n \in 2^n$ such that $\forall s \in 2^{<\mathbb{N}} \exists n \in \mathbb{N} \ (s \subseteq s_n)$. For each $i \in \{0,1\}$, set $\bar{\imath} = 1 - i$, and let $\mathcal{G}_0$ denote the graph on $2^{\mathbb{N}}$ given by

$$\mathcal{G}_0 = \{(s_n i \alpha, s_n \bar{\imath} \alpha) : i \in \{0,1\} \text{ and } n \in \mathbb{N} \text{ and } \alpha \in 2^{\mathbb{N}}\}.$$

Theorem 4 (Kechris-Solecki-Todorćević [4]). *Suppose $\mathcal{G}$ is an analytic graph on a Polish space. Then exactly one of the following holds:*

1. $\chi_B(\mathcal{G}) \leq \aleph_0$;
2. *There is a continuous homomorphism from $\mathcal{G}_0$ to $\mathcal{G}$.*

Theorem 4 provides an ideal solution to Question 3, as it implies that $\{\mathcal{G}_0\}$ is a one-element basis for the class of analytic graphs of uncountable Borel chromatic number under Borel homomorphism, the Borel analog of the de facto standard among quasi-orders on graphs.

We say that a set $B \subseteq X$ *is globally Baire* if for every Polish space Z and every Borel function $\pi : Z \to X$, the set $\pi^{-1}(B)$ has the property of Baire. By a result of Lusin-Sierpiński, every $\sigma(\mathbf{\Sigma}^1_1)$ set is globally Baire (see, for example, Theorem 21.6 of Kechris [3]). Moreover, under appropriate strong set-theoretic hypotheses, the class of globally Baire measurable sets is quite rich. For instance, under projective determinacy, every projective set is globally Baire (see, for example, Theorem 38.17 of Kechris [3]).

As noted in §6.C of Kechris-Solecki-Todorćević [4], Theorem 4 implies that condition (1) is equivalent to the existence of a globally Baire measurable coloring with countable range, and condition (2) is equivalent to the existence of a Baire measurable homomorphism from $\mathcal{G}_0$ to $\mathcal{G}$. It should be noted, however, that homomorphisms cannot be replaced with substantially stronger notions in the statement of Theorem 4. Recall that a set is $D_2(\mathbf{\Sigma}^0_1)$ if it is the difference of two open sets.

Theorem 5 (Lecomte [6]). *There is a $D_2(\mathbf{\Sigma}^0_1)$ graph $\mathcal{G}$ on $2^{\mathbb{N}}$ of uncountable Borel chromatic number such that there is no injective Baire measurable homomorphism from $\mathcal{G}_0$ to $\mathcal{G}$.*

This leads us to a natural revision of Question 3. A *reduction of $\mathcal{G}$ to $\mathcal{H}$* is a function $\pi : X \to Y$ such that

$$\forall x_1, x_2 \in X \ ((x_1, x_2) \in \mathcal{G} \Leftrightarrow (\pi(x_1), \pi(x_2)) \in \mathcal{H}).$$

An *embedding of $\mathcal{G}$ into $\mathcal{H}$* is an injective reduction of $\mathcal{G}$ to $\mathcal{H}$. We use $\leq_{\mathrm{GB}}$ to denote the quasi-order of globally Baire measurable reducibility on the class of analytic directed graphs of uncountable Borel chromatic number. (To see that this is actually a quasi-order, it is enough to show that the class of globally Baire measurable functions is closed under composition, which we do in §4.) As it should cause no confusion, we also use $\sqsubseteq_c$ to denote the quasi-order of continuous embeddability on the class of analytic directed graphs of uncountable Borel chromatic number.

Question 6. Is there a simple basis for the class of analytic graphs of uncountable Borel chromatic number under some quasi-order which lies between $\sqsubseteq_c$ and $\leq_{\mathrm{GB}}$?

There is a similar revision of Question 1. Suppose that X_1, Y_1, X_2, and Y_2 are Polish spaces, $R_1 \subseteq X_1 \times Y_1$, and $R_2 \subseteq X_2 \times Y_2$. A *rectangular reduction of* R_1 *to* R_2 is a pair of functions (π_X, π_Y), where $\pi_X : X_1 \to X_2$, $\pi_Y : Y_1 \to Y_2$, and

$$\forall x_1 \in X_1 \, \forall y_1 \in Y_1 \; ((x_1, y_1) \in R_1 \Leftrightarrow (\pi_X(x_1), \pi_Y(y_1)) \in R_2).$$

A *rectangular embedding of* R_1 *into* R_2 is a rectangular reduction (π_X, π_Y) of R_1 to R_2 such that both π_X and π_Y are injective. We use $\leq^r_{\mathrm{GB}}$ to denote the quasi-order of globally Baire measurable rectangular reducibility on the class of non-potentially closed Borel sets, and we use $\sqsubseteq^r_c$ to denote the quasi-order of continuous rectangular embeddability on the class of non-potentially closed Borel sets.

Question 7. Is there a simple basis for the class of non-potentially closed Borel sets under some quasi-order which lies between $\sqsubseteq^r_c$ and $\leq^r_{\mathrm{GB}}$?

Among sets of sufficiently low potential complexity, this question has a positive answer. Recall that a set is $\check{D}_2(\mathbf{\Sigma^0_1})$ if it is the complement of a $D_2(\mathbf{\Sigma^0_1})$ set, and let $\leq_{\mathrm{lex}}$ denote the lexicographic ordering of $2^{\mathbb{N}}$.

Theorem 8 (Lecomte [5]). *Suppose that X and Y are Polish spaces and $R \subseteq X \times Y$ is potentially $\check{D}_2(\mathbf{\Sigma^0_1})$. Then exactly one of the following holds:*

1. *R is potentially closed;*
2. *$(2^{\mathbb{N}} \times 2^{\mathbb{N}}) \setminus \Delta(2^{\mathbb{N}}) \sqsubseteq^r_c R$ or $<_{\mathrm{lex}} \sqsubseteq^r_c R$.*

Recall that an *oriented graph* is an antisymmetric directed graph. Louveau has pointed out that the analog of Theorem 4 for directed graphs holds when $\mathcal{G}_0$ is replaced with the oriented graph $\mathcal{G}_0^{\rightarrow}$ on $2^{\mathbb{N}}$ given by

$$\mathcal{G}_0^{\rightarrow} = \{(s_n 0\alpha, s_n 1\alpha) : n \in \mathbb{N} \text{ and } \alpha \in 2^{\mathbb{N}}\}.$$

As a corollary of this, one can obtain a positive answer to the analog of Question 1 for a natural weakening of continuous rectangular reducibility:

Theorem 9 (Lecomte [6]). *Suppose that X and Y are Polish spaces and $R \subseteq X \times Y$ is Borel. Then exactly one of the following holds:*

1. *R is potentially closed;*
2. *There are continuous maps $\pi_X : 2^{\mathbb{N}} \to X$ and $\pi_Y : 2^{\mathbb{N}} \to Y$ such that*

$$\forall (\alpha, \beta) \in \overline{\mathcal{G}_0^{\rightarrow}} \; ((\alpha, \beta) \in \mathcal{G}_0^{\rightarrow} \Leftrightarrow (\pi_X(\alpha), \pi_Y(\beta)) \in R).$$

However, the analog of this result for rectangular reducibility is false:

Theorem 10 (Lecomte [6]). *There is a set of continuum-many non-potentially closed $D_2(\mathbf{\Sigma^0_1})$ subsets of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ which are pairwise incomparable and minimal with respect to every quasi-order $\preceq$ which lies between $\sqsubseteq^r_c$ and $\leq^r_{\mathrm{GB}}$. In particular, every basis for the class of non-potentially closed Borel sets under $\preceq$ has cardinality at least $\mathfrak{c}$.*

Along similar lines, we have the following:

Theorem 11 (Lecomte [6]). *There is a set of continuum-many $D_2(\mathbf{\Sigma^0_1})$ graphs on $2^{\mathbb{N}}$ of uncountable Borel chromatic number which are pairwise incomparable and minimal with respect to every quasi-order $\preceq$ which lies between $\sqsubseteq_c$ and $\leq_{\mathrm{GB}}$. In particular, every basis for the class of analytic graphs of uncountable Borel chromatic number under $\preceq$ has cardinality at least $\mathfrak{c}$.*

While Theorems 10 and 11 certainly force us to think carefully about what we mean by a "simple basis" in Questions 6 and 7, they do not rule out positive answers to these questions, and this is really the starting point of this paper. In particular, Theorems 10 and 11 leave open:

Question 12. Does the class of $\preceq$-minimal non-potentially closed Borel sets form a basis for the class of non-potentially closed Borel sets under $\preceq$?

Question 13. Does the class of $\preceq$-minimal analytic graphs of uncountable Borel chromatic number form a basis for the class of analytic graphs of uncountable Borel chromatic number under $\preceq$?

Associated with each pair $S \in \mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n) \times \mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n)$ is the K_σ directed graph $\mathcal{G}^S$ on $2^{\mathbb{N}}$ given by

$$\mathcal{G}^S = \{(si\alpha, t\bar{\imath}\alpha) : i \in \{0,1\} \text{ and } (s,t) \in S^i \text{ and } \alpha \in 2^{\mathbb{N}}\},$$

where $S = (S^0, S^1)$. We say that S *is dense* if $\forall r \in 2^{<\mathbb{N}}\, \exists (s,t) \in S^0\ (r \subseteq s,t)$. As noted in §1, if S is dense, then $\chi_B(\mathcal{G}^S) > \aleph_0$. Our first theorem is a weak positive answer to a special case of Question 6:

Theorem 14. *Suppose that $\mathcal{G}$ is an analytic directed graph on a Polish space which admits a globally Baire measurable reduction to a locally countable analytic directed graph on a Polish space. Then exactly one of the following holds:*

1. $\chi_B(\mathcal{G}) \leq \aleph_0$;
2. *There is a dense pair S such that $\mathcal{G}^S \sqsubseteq_c \mathcal{G}$.*

It is worth noting that our proof of Theorem 14 can be modified so as to give a new proof of Theorem 4 which avoids the need for the techniques of effective descriptive set theory.

While the directed graphs of the form $\mathcal{G}^S$ are certainly simple when compared to arbitrary analytic directed graphs, Theorem 14 still leaves much to be desired, as continuous embeddability of such directed graphs is already quite complicated. One response to this criticism is that Theorem 14 easily implies much nicer results in natural special cases. Recall that a graph $\mathcal{G}$ *is acyclic* if there is at most one injective $\mathcal{G}$-path between any two points.

Theorem 15. *Suppose that $\mathcal{G}$ is an analytic graph on a Polish space which admits a globally Baire measurable reduction to a locally countable analytic graph on a Polish space, as well as a globally Baire measurable reduction to an acyclic analytic graph on a Polish space. Then exactly one of the following holds:*

1. $\chi_B(\mathcal{G}) \leq \aleph_0$;
2. $\mathcal{G}_0 \sqsubseteq_c \mathcal{G}$.

This strengthens a special case of Theorem 6.6 of Kechris-Solecki-Todorćevič [4]. A similar fact holds for oriented graphs when $\mathcal{G}_0$ is replaced with $\mathcal{G}_0^{\rightarrow}$, which can be used to give a new proof of Theorem 28 of Lecomte [6].

We can also use Theorem 14 to extend a well known dichotomy theorem from countable Borel equivalence relations to countable analytic equivalence relations. Recall that a *partial transversal of E* is a set which intersects every equivalence class of E in at most one point.

Theorem 16. *Suppose that X is a Polish space and E is a countable analytic equivalence relation on X. Then exactly one of the following holds:*

1. *X is the union of countably many Borel partial transversals of E;*
2. $E_0 \sqsubseteq_c E$.

A better response to the criticism of Theorem 14 mentioned earlier, however, is that the real intent behind the theorem is not to provide a positive answer to Question 6 at all, but instead to provide a means of obtaining negative answers!

In order to describe how our basis theorem can give rise to anti-basis results, we must step back and examine a natural quasi-order that lives on $\mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n) \times \mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n)$. Note first that the directed graphs $\mathcal{G}^S$ have natural finite approximations $\mathcal{G}_n^S$ on 2^n, given by $\mathcal{G}_0^S = \emptyset$ and

$$\begin{aligned}\mathcal{G}_{n+1}^S = \{&(si, ti) : i \in \{0,1\} \text{ and } (s,t) \in \mathcal{G}_n^S\} \cup \\ &\{(si, t\bar{\imath}) : i \in \{0,1\} \text{ and } (s,t) \in S^i\}.\end{aligned}$$

An *aligned function on* $2^{<\mathbb{N}}$ is a function $\varphi : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ for which there are natural numbers k_i^φ and pairs $u_i^\varphi \in 2^{k_i^\varphi} \times 2^{k_i^\varphi}$, for each $i \in \mathbb{N}$, such that

$$\varphi(\emptyset) = \emptyset \text{ and } \forall n \in \mathbb{N}\, \forall s \in 2^{n+1}\ (\varphi(s) = (u_0^\varphi)_{s(0)} \ldots (u_n^\varphi)_{s(n)}).$$

As it should cause no confusion, let $\leq_{\text{lex}}$ also denote the lexicographic ordering of $2^{<\mathbb{N}}$. We say that φ *is order-preserving* if $\forall n \in \mathbb{N}\ ((u_n^\varphi)_0 \leq_{\text{lex}} (u_n^\varphi)_1)$, and we say that φ *is order-reversing* if $\forall n \in \mathbb{N}\ ((u_n^\varphi)_1 \leq_{\text{lex}} (u_n^\varphi)_0)$. We say that φ *is monotonic* if it is order-preserving or order-reversing. An *aligned embedding of S into T* is an aligned function $\varphi : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ such that

$$\forall n \in \mathbb{N}\ (\varphi|2^{n+1} \text{ is an embedding of } \mathcal{G}^S_{n+1} \text{ into } \mathcal{G}^T_{k^\varphi_0+\cdots+k^\varphi_n}).$$

Suppose now that $\leq_P$ and $\leq_Q$ are quasi-orders on P and Q. A *homomorphism from $\leq_P$ to $\leq_Q$* is a function $\pi : P \to Q$ such that

$$\forall p_1, p_2 \in P\ (p_1 \leq_P p_2 \Rightarrow \pi(p_1) \leq_Q \pi(p_2)).$$

In the special case that $\pi[P]$ is dense in $(Q, \leq_Q)$, it follows that π sends bases for P under $\leq_P$ to bases for Q under $\leq_Q$. In §1, we note that the map $S \mapsto \mathcal{G}^S$ is a homomorphism from monotonic aligned embeddability to continuous embeddability. Theorem 14 therefore amounts to the fact that the image of the set of dense pairs under this homomorphism is dense in the appropriate class of analytic directed graphs under $\sqsubseteq_c$. We actually obtain much more information from this point of view, however, as it gives explicit means of transforming bases for the set of dense pairs under monotonic aligned embeddability into bases for the appropriate class of analytic directed graphs under $\sqsubseteq_c$.

Unfortunately, homomorphisms are seldom sufficient to transfer anti-basis theorems from $(P, \leq_P)$ to $(Q, \leq_Q)$. Suppose that $\sqsubseteq_Q$ is a quasi-order on Q which is contained in $\leq_Q$. A *basis embedding of $\leq_P$ into* $(\sqsubseteq_Q, \leq_Q)$ is a homomorphism from $\leq_P$ to $\sqsubseteq_Q$ with the additional property that

$$\forall p_1, p_2 \in P\, \forall q \leq_Q \pi(p_1), \pi(p_2)\, \exists p \in P\ (p \leq_P p_1, p_2 \text{ and } \pi(p) \sqsubseteq_Q q).$$

We say that a set $A \subseteq P$ *is a weak antichain* if

$$\forall p_1, p_2 \in A\ (p_1 \neq p_2 \Rightarrow p_1 \not\leq_P p_2),$$

and we say that a set $A \subseteq P$ *is a strong antichain* if

$$\forall p_1, p_2 \in A\ (p_1 \neq p_2 \Rightarrow \forall p \in A\ (p \not\leq_P p_1 \text{ or } p \not\leq_P p_2)).$$

As noted in §5, basis embeddings send strong antichains to sets which are strong antichains with respect to every quasi-order which lies between $\sqsubseteq_Q$ and $\leq_Q$. As the existence of strong antichains of cardinality κ rules out the existence of bases of cardinality strictly less than κ, this is one example of the fashion in which basis embeddings can be used to transfer anti-basis results from $\leq_P$ to all quasi-orders which lie between $\sqsubseteq_Q$ and $\leq_Q$.

Associated with each set $A \in \mathcal{P}(2^{<\mathbb{N}}) \times \mathcal{P}(2^{<\mathbb{N}})$ is the directed graph $\mathcal{G}^A = \mathcal{G}^{S^A}$, where $S^A \in \mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n) \times \mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n)$ is given by

$$(S^A)^i = \{(s, s) : s \in A^i\}.$$

We say that *A is dense* if $\forall r \in 2^{<\mathbb{N}}\, \exists s \in A^0\ (r \subseteq s)$. If A is dense, then so too is S^A, so $\chi_B(\mathcal{G}^A) > \aleph_0$, thus $\mathcal{G}^A$ is not potentially closed.

There are two main advantages in restricting our attention to the sets of the form S^A. First, it is easy to see that $\overline{\mathcal{G}^A} \setminus \mathcal{G}^A \subseteq \Delta(2^{\mathbb{N}})$, thus the directed graphs of this form are $D_2(\mathbf{\Sigma^0_1})$. Second, the restriction of aligned embeddability to the sets of the form S^A takes a significantly simpler form on $\mathcal{P}(2^{<\mathbb{N}}) \times \mathcal{P}(2^{<\mathbb{N}})$. A *nicely aligned function on* $2^{<\mathbb{N}}$ is a function $\psi : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ for which there are natural numbers k_i^ψ and sequences $u_i^\psi \in 2^{k_i^\psi}$, for each $i \in \mathbb{N}$, as well as a sequence $\alpha^\psi \in 2^{\mathbb{N}}$ such that for all $n \in \mathbb{N}$ and $s \in 2^n$,

$$\psi(s) = u_0^\psi |s(0) - \alpha^\psi(0)| \dots u_{n-1}^\psi |s(n-1) - \alpha^\psi(n-1)| u_n^\psi.$$

We say that ψ *is order-preserving* if $\alpha^\psi = 0^\infty$, and ψ *is order-reversing* if $\alpha^\psi = 1^\infty$. We say that ψ *is monotonic* if it is order-preserving or order-reversing. An *aligned embedding of* A *into* B is a nicely aligned function $\psi : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ such that

$$\forall i \in \{0,1\}\, \forall s \in 2^{<\mathbb{N}}\ (s \in A^i \Leftrightarrow \psi(s) \in B^i).$$

Theorem 17. *The map* $A \mapsto \mathcal{G}^A$ *is a basis embedding of monotonic aligned embeddability on the set of dense pairs into* $(\sqsubseteq_c^r, \leq_{\mathrm{GB}}^r)$.

Theorem 18. *The map* $A \mapsto \mathcal{G}^A$ *is a basis embedding of monotonic aligned embeddability on the set of dense pairs into* $(\sqsubseteq_c, \leq_{\mathrm{GB}})$.

The upshot of Theorems 17 and 18 is that we can obtain negative solutions to Questions 6 and 7 by studying monotonic aligned embeddability on $\mathcal{P}(2^{<\mathbb{N}}) \times \mathcal{P}(2^{<\mathbb{N}})$. By combining this observation with purely combinatorial arguments, we obtain strong negative answers to Questions 12 and 13:

Theorem 19. *There is a non-potentially closed* $D_2(\mathbf{\Sigma^0_1})$ *set* $S \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ *such that for all Borel sets* $R \leq_{\mathrm{GB}}^r S$*, exactly one of the following holds:*

1. *R is potentially closed;*
2. *There is a strong* $\leq_{\mathrm{GB}}^r$*-antichain of continuum-many* $D_2(\mathbf{\Sigma^0_1})$ *subsets of* $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ *which are* $\sqsubseteq_c^r$*-below* R*.*

In particular, if $\preceq$ *is a quasi-order which lies between* $\sqsubseteq_c^r$ *and* $\leq_{\mathrm{GB}}^r$*, then no weak* $\preceq$*-antichain is a basis for the class of non-potentially closed Borel sets under* $\preceq$*, thus the class of* $\preceq$*-minimal sets is not a basis for the class of non-potentially closed Borel sets under* $\preceq$*.*

Theorem 20. *There is a* $D_2(\mathbf{\Sigma^0_1})$ *graph* $\mathcal{H}$ *on* $2^{\mathbb{N}}$ *of uncountable Borel chromatic number such that for all analytic* $\mathcal{G} \leq_{\mathrm{GB}} \mathcal{H}$*, exactly one of the following holds:*

1. $\chi_B(\mathcal{G}) \leq \aleph_0$*;*
2. *There is a strong* $\leq_{\mathrm{GB}}$*-antichain of continuum-many* $D_2(\mathbf{\Sigma^0_1})$ *graphs on* $2^{\mathbb{N}}$ *which are* $\sqsubseteq_c$*-below* $\mathcal{G}$*.*

In particular, if $\preceq$ *is a quasi-order which lies between* $\sqsubseteq_c$ *and* $\leq_{\mathrm{GB}}$*, then no weak* $\preceq$*-antichain is a basis for the class of analytic graphs of uncountable Borel chromatic number under* $\preceq$*, thus the class of* $\preceq$*-minimal graphs is not a basis for the class of analytic graphs of uncountable Borel chromatic number under* $\preceq$*.*

The organization of the paper is as follows. In §1, we establish various basic properties of the directed graphs of the form $\mathcal{G}^S$. In §2, we prove a technical dichotomy theorem for locally countable Borel directed graphs which lies at the heart of all of our results. In §3, we establish some straightforward generalizations of classical descriptive set-theoretic facts which can be used to generalize our technical dichotomy theorem, first to locally countable analytic directed graphs, and then to analytic directed graphs which admit globally Baire measurable reductions into locally countable analytic directed graphs. In §4, we obtain Theorem 14 as a corollary, and use this to prove Theorem 15 and its oriented analog. Here we also give the new proof of Theorem 28 of Lecomte [6] mentioned earlier, as well as the proof of Theorem 16. In §5, we establish Theorems 17 and 18. In §6, we turn our attention to the family of locally countable directed graphs which were considered in Lecomte [6]. These can be described as the directed graphs of the form $\mathcal{G}^A$, where $A \in \mathcal{P}(2^{<\mathbb{N}}) \times \mathcal{P}(2^{<\mathbb{N}})$ satisfies a natural homogeneity condition. Using these, we obtain new proofs of Theorems 10 and 11. We also show that if we restrict our attention below the directed graphs of this form, then the minimal sets do form a basis, thus we must look elsewhere in order to obtain Theorems 19 and 20. We do this in §7.

1. Basic properties of the directed graphs of the form $\mathcal{G}^S$

We say that a set $B \subseteq X$ *is $\mathcal{G}$-discrete* if $\mathcal{G} \cap (B \times B) = \emptyset$. Note that $\chi_\Gamma(\mathcal{G}) \leq \aleph_0$ if and only if X can be partitioned into countably many $\mathcal{G}$-discrete sets in Γ. It follows that if Γ is closed under complements and finite intersections, then $\chi_\Gamma(\mathcal{G}) \leq \aleph_0$ if and only if X is the union of countably many $\mathcal{G}$-discrete sets in Γ. Let BP denote the class of all subsets of Polish spaces which have the property of Baire.

Proposition 1.1. *Suppose that S is a dense pair and $C \subseteq 2^{\mathbb{N}}$ is non-meager and has the property of Baire. Then $\chi_{\mathrm{BP}}(\mathcal{G}^S \cap (C \times C)) > \aleph_0$.*

Proof. It is enough to show that no non-meager set $B \subseteq 2^{\mathbb{N}}$ with the property of Baire is $\mathcal{G}^S$-discrete. Towards this end, fix $r \in 2^{<\mathbb{N}}$ such that B is comeager in the basic clopen set $\mathcal{N}_r = \{\alpha \in 2^{\mathbb{N}} : r \subseteq \alpha\}$, and fix $(s,t) \in S^0$ such that $r \subseteq s, t$. Then there are comeagerly many $\alpha \in 2^{\mathbb{N}}$ such that $s0\alpha, t1\alpha \in B$. As $(s0\alpha, t1\alpha) \in \mathcal{G}^S$, it follows that B is not $\mathcal{G}^S$-discrete. □

The *symmetrization of* $\mathcal{G}$ is $\mathcal{G}^{\pm 1} = \{(x,y) : (x,y) \in \mathcal{G} \text{ or } (y,x) \in \mathcal{G}\}$. We say that $\mathcal{G}$ *is acyclic* if $\mathcal{G}^{\pm 1}$ is acyclic. Let $E_{\mathcal{G}}$ denote the smallest equivalence relation containing $\mathcal{G}$. We say that a (directed) graph $\mathcal{G}$ *is a (directed) graphing of* E if $E = E_{\mathcal{G}}$. In case $\mathcal{G}$ is acyclic, we say that $\mathcal{G}$ *is a (directed) treeing of* E.

Definition 1.2. Let S_0 denote the element of $\mathcal{P}(\bigcup_{n \in \mathbb{N}} 2^n \times 2^n) \times \mathcal{P}(\bigcup_{n \in \mathbb{N}} 2^n \times 2^n)$ given by $(S_0)^0 = (S_0)^1 = \{(s_n, s_n) : n \in \mathbb{N}\}$.

Definition 1.3. Let $S_0^{\rightarrow}$ denote the element of $\mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n) \times \mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n)$ given by $(S_0^{\rightarrow})^0 = \{(s_n, s_n) : n \in \mathbb{N}\}$ and $(S_0^{\rightarrow})^1 = \emptyset$.

Note that $\mathcal{G}_0 = \mathcal{G}_0^{S_0}$ and $\mathcal{G}_0^{\rightarrow} = \mathcal{G}^{S_0^{\rightarrow}}$.

Proposition 1.4. *$\mathcal{G}_0^{\rightarrow}$ is a directed treeing of E_0.*

Proof. As $\mathcal{G}_0$ is the symmetrization of $\mathcal{G}_0^{\rightarrow}$, it is enough to show that $\mathcal{G}_0$ is a treeing of E_0. It is clear that $\mathcal{G}_0 \subseteq E_0$, thus $E_{\mathcal{G}_0} \subseteq E_0$. To see that $E_0 \subseteq E_{\mathcal{G}_0}$, note first that each of the graphs $\mathcal{G}_n^{S_0}$ is a tree, by a straightforward inductive argument. Suppose now that $\alpha E_0 \beta$, and fix $n \in \mathbb{N}$ such that $\forall m \geq n\ (\alpha(m) = \beta(m))$. As $\alpha|n$ and $\beta|n$ are $\mathcal{G}_n^{S_0}$-connected, it follows that $\alpha E_{\mathcal{G}_0} \beta$.

It only remains to verify that $\mathcal{G}_0$ is acyclic, or equivalently, that if $k \geq 2$ and $\langle \alpha_0, \alpha_1, \ldots, \alpha_k \rangle$ is an injective path through $\mathcal{G}_0$, then $(\alpha_0, \alpha_k) \notin \mathcal{G}_0$. Towards this end, fix $n \in \mathbb{N}$ sufficiently large that $\forall i, j \leq k\, \forall m \geq n\ (\alpha_i(m) = \alpha_j(m))$. Then $\langle \alpha_0|n, \alpha_1|n, \ldots, \alpha_k|n \rangle$ is an injective path through $\mathcal{G}_n^{S_0}$. As $\mathcal{G}_n^{S_0}$ is a tree, it follows that $(\alpha_0|n, \alpha_k|n) \notin \mathcal{G}_n^{S_0}$, thus $(\alpha_0, \alpha_k) \notin \mathcal{G}_0$. $\square$

We say that S *is strongly dense* if $\forall n \in \mathbb{N}\ ((s_n, s_n) \in S^0)$.

Proposition 1.5. *Suppose that S is strongly dense and $\mathcal{G}^S$ is an acyclic graph. Then $S = S_0$.*

Proof. The fact that S is strongly dense ensures that $(S_0)^0 \subseteq S^0$, and the fact that $\mathcal{G}^S$ is symmetric ensures that $S^0 = S^1$, so $(S_0)^1 \subseteq S^1$, thus $\mathcal{G}_0 \subseteq \mathcal{G}^S$. Observe now that if $i \in \{0,1\}$ and $(s,t) \in S^i \setminus (S_0)^i$, then $\langle si0^\infty, t\bar{\imath}0^\infty \rangle$ is a path through $\mathcal{G}^S \setminus \mathcal{G}_0$. As Proposition 1.4 ensures that there is a path from $si0^\infty$ to $t\bar{\imath}0^\infty$ through $\mathcal{G}_0$, it follows that $\mathcal{G}^S$ is not acyclic. $\square$

Proposition 1.6. *Suppose that S is strongly dense and $\mathcal{G}^S$ is an acyclic oriented graph. Then $S = S_0^{\rightarrow}$.*

Proof. The fact that S is strongly dense ensures that $(S_0^{\rightarrow})^0 \subseteq S^0$, thus $\mathcal{G}_0^{\rightarrow} \subseteq \mathcal{G}^S$, and the fact that $\mathcal{G}^S$ is antisymmetric ensures that $S^0 \cap S^1 = \emptyset$, thus $(S_0^{\rightarrow})^0 \cap S^1 = \emptyset$. Observe now that if $i \in \{0,1\}$ and $(s,t) \in S^i \setminus (S_0^{\rightarrow})^i$, then $\langle si0^\infty, t\bar{\imath}0^\infty \rangle$ is a path through $\mathcal{G}^S \setminus \mathcal{G}_0$. As Proposition 1.4 ensures that there is a path from $si0^\infty$ to $t\bar{\imath}0^\infty$ through $\mathcal{G}_0$, it follows that $\mathcal{G}^S$ is not acyclic. $\square$

Along similar lines, we have the following:

Proposition 1.7. *Suppose that S is strongly dense and $\mathcal{G}^S \cup \Delta(2^{\mathbb{N}})$ is an equivalence relation. Then $S^0 = S^1 = \bigcup_{n\in\mathbb{N}} 2^n \times 2^n$.*

Proof. As S is strongly dense, it follows from Proposition 1.4 that $E_0 = \mathcal{G}^S \cup \Delta(2^{\mathbb{N}})$. Given $s, t \in 2^n$ and $i \in \{0,1\}$, the fact that $si0^\infty E_0 t\bar{\imath}0^\infty$ ensures that $(s,t) \in S^i$, and the proposition follows. $\square$

Given an aligned function $\varphi : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$, define $\overline{\varphi} : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by

$$\overline{\varphi}(\alpha) = (u_0^{\varphi})_{\alpha(0)}(u_1^{\varphi})_{\alpha(1)} \dots .$$

Proposition 1.8. *Suppose that φ is an aligned embedding of S into T. Then $\overline{\varphi}$ is a continuous embedding of $\mathcal{G}^S$ into $\mathcal{G}^T$.*

Proof. It is clear that $\overline{\varphi}$ is an aligned embedding of E_0 into E_0. To see that $\overline{\varphi}$ is an embedding of $\mathcal{G}^S$ into $\mathcal{G}^T$, it only remains to verify that if $\alpha E_0 \beta$, then $(\alpha, \beta) \in \mathcal{G}^S \Leftrightarrow (\overline{\varphi}(\alpha), \overline{\varphi}(\beta)) \in \mathcal{G}^T$. Towards this end, fix a positive natural number n with the property that $\forall m \geq n \ (\alpha(m) = \beta(m))$, and observe that

$$\begin{aligned} (\alpha, \beta) \in \mathcal{G}^S &\Leftrightarrow (\alpha|n, \beta|n) \in \mathcal{G}_n^S \\ &\Leftrightarrow (\varphi(\alpha|n), \varphi(\beta|n)) \in \mathcal{G}^T_{k_0^{\varphi} + \dots + k_{n-1}^{\varphi}} \\ &\Leftrightarrow (\overline{\varphi}(\alpha), \overline{\varphi}(\beta)) \in \mathcal{G}^T. \end{aligned}$$

□

As a consequence, we say that $\pi : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ *is an (order-preserving, order-reversing, or monotonic) aligned embedding of $\mathcal{G}^S$ into $\mathcal{G}^T$* if it is of the form $\overline{\varphi}$, for some (order-preserving, order-reversing, or monotonic) aligned embedding φ of S into T.

Proposition 1.9. *The map $S \mapsto \mathcal{G}^S$ is a homomorphism from aligned embeddability into continuous embeddability.*

Proof. This is a direct consequence of Proposition 1.8. □

2. A dichotomy for locally countable Borel directed graphs

We begin this section by introducing a natural generalization of the notion of chromatic number. Suppose that $\boldsymbol{\mathcal{G}} = \langle \mathcal{G}_i \rangle_{i \in I}$ is a sequence of directed graphs on X. A *coloring of* $\boldsymbol{\mathcal{G}}$ is a function $c : X \to Z$ such that

$$\forall z \in Z \, \exists i \in I \ \ (c^{-1}(z) \text{ is } \mathcal{G}_i\text{-discrete}).$$

The *Borel chromatic number of* $\boldsymbol{\mathcal{G}}$, or $\chi_B(\boldsymbol{\mathcal{G}})$, is the least cardinal κ for which there is a Polish space Z and a Borel coloring $c : X \to Z$ of $\boldsymbol{\mathcal{G}}$ such that $|c[X]| = \kappa$. Observe that if $\boldsymbol{\mathcal{G}} = \langle \mathcal{G} \rangle$, then $\chi_B(\boldsymbol{\mathcal{G}}) = \chi_B(\mathcal{G})$.

We say that a set $B \subseteq X$ *is $\boldsymbol{\mathcal{G}}$-discrete* if there exists $i \in I$ such that B is $\mathcal{G}_i$-discrete. Note that $\chi_B(\boldsymbol{\mathcal{G}}) \leq \aleph_0$ if and only if X is the union of countably many $\boldsymbol{\mathcal{G}}$-discrete Borel sets.

Proposition 2.1. *Suppose that X is a Polish space and $\boldsymbol{\mathcal{G}} = \langle \mathcal{G}_i \rangle_{i \in I}$ is a countable sequence of analytic directed graphs on X. Then every $\boldsymbol{\mathcal{G}}$-discrete analytic set is contained in a $\boldsymbol{\mathcal{G}}$-discrete Borel set.*

Proof. Simply note that if Z is a Polish space and $A \subseteq Z \times X$ is $\mathbf{\Sigma^1_1}$, then for each $z \in Z$, the set A_z is $\boldsymbol{\mathcal{G}}$-discrete if and only if

$$\exists i \in I \, \forall x, y \in X \ (x \notin A_z \text{ or } y \notin A_z \text{ or } (x, y) \notin \mathcal{G}_i),$$

so the set $\{z \in Z : A_z \text{ is } \boldsymbol{\mathcal{G}}\text{-discrete}\}$ is $\mathbf{\Pi^1_1}$, thus the proposition follows from the first reflection theorem (see, for example, Theorem 35.10 of Kechris [3]). □

Proposition 2.2. *Suppose that X and Y are Polish spaces, $\mathcal{G}$ is a locally countable Borel directed graph on X of uncountable Borel chromatic number, $\mathcal{H}$ is a directed graph on Y, and $\pi : X \to Y$ is a Borel reduction of $\mathcal{G}$ to $\mathcal{H}$. Then there is a Borel set $B \subseteq X$ such that $\chi_B(\mathcal{G}|B) > \aleph_0$ and $\pi|B$ is injective.*

Proof. Set $A = \{x \in X : \mathcal{G}_x \cup \mathcal{G}^x \neq \emptyset\}$. Since $\mathcal{G}$ is locally countable, the Lusin-Novikov uniformization theorem (see, for example, Theorem 18.10 of Kechris [3]) implies that A is Borel. It then follows that $\chi_B(\mathcal{G}|A) > \aleph_0$.

Lemma 2.3. *$\pi|A$ is countable-to-one.*

Proof. As $\mathcal{G}$ is locally countable, it is enough to show that

$$\forall x_1, x_2 \in A \ (\pi(x_1) = \pi(x_2) \Rightarrow x_1 E_{\mathcal{G}} x_2).$$

Towards this end, suppose that $x_1, x_2 \in A$ and $\pi(x_1) = \pi(x_2)$, fix $x'_i \in \mathcal{G}_{x_i} \cup \mathcal{G}^{x_i}$, and observe that $(\pi(x_i), \pi(x'_i)) \in \mathcal{H}^{\pm 1}$, so $(\pi(x_1), \pi(x'_2)) \in \mathcal{H}^{\pm 1}$, thus $(x_1, x'_2) \in \mathcal{G}^{\pm 1}$. As $(x_2, x'_2) \in \mathcal{G}^{\pm 1}$, it follows that $x_1 E_{\mathcal{G}} x_2$. □

Lemma 2.3 and the Lusin-Novikov uniformization theorem imply that there are Borel sets $A_n \subseteq A$ such that $A = \bigcup_{n \in \mathbb{N}} A_n$ and $\forall n \in \mathbb{N}$ ($\pi|A_n$ is injective). Fix $n \in \mathbb{N}$ such that $\chi_B(\mathcal{G}|A_n) > \aleph_0$, and set $B = A_n$. □

We are now ready for the main result of this section:

Theorem 2.4. *Suppose that X is a Polish space, $\mathcal{G}$ is a locally countable Borel directed graph on X, $\mathcal{F}$ is a finite subset of $\mathcal{P}(\bigcup_{n \in \mathbb{N}} 2^n \times 2^n) \times \mathcal{P}(\bigcup_{n \in \mathbb{N}} 2^n \times 2^n)$, and $\pi^T : X \to 2^{\mathbb{N}}$ is a Borel reduction of $\mathcal{G}$ to $\mathcal{G}^T$, for each $T \in \mathcal{F}$. Then exactly one of the following holds:*

1. $\chi_B(\mathcal{G}) \leq \aleph_0$;
2. *There is a strongly dense pair S and a continuous embedding $\pi : 2^{\mathbb{N}} \to X$ of $\mathcal{G}^S$ into $\mathcal{G}$ such that $\pi^T \circ \pi$ is a monotonic aligned embedding of $\mathcal{G}^S$ into $\mathcal{G}^T$, for each $T \in \mathcal{F}$.*

Proof. Proposition 1.1 implies that (1) and (2) are mutually exclusive, since colorings can be pulled back through homomorphisms. It is therefore sufficient to prove $\neg(1) \Rightarrow (2)$. Towards this end, suppose that $\chi_B(\mathcal{G}) > \aleph_0$. By repeatedly applying Proposition 2.2, we can assume that each of the functions π^T is injective. Fix a Borel linear order $\leq$ on X. Since the union of finitely many directed graphs of countable Borel chromatic number has countable Borel chromatic number, it follows that after replacing $\leq$ with $\geq$ if necessary, there is a sequence $\sigma \in 2^{\mathcal{F}}$ such that the Borel oriented graph

$$\mathcal{H} = \{(x, y) \in \mathcal{G} : x \leq y \text{ and } \forall T \in \mathcal{F}\ (\sigma(T) = 0 \Leftrightarrow \pi^T(x) \leq_{\text{lex}} \pi^T(y))\}$$

has uncountable Borel chromatic number.

By Theorem 1 of Feldman-Moore [1], there is a countable group G of Borel automorphisms of X such that $E_{\mathcal{G}} = \bigcup_{g \in G} \text{graph}(g)$. Fix an increasing sequence of finite symmetric neighborhoods $H_0 \subseteq H_1 \subseteq \cdots \subseteq G$ of 1_G such that $G = \bigcup_{n \in \mathbb{N}} H_n$. Let F_n denote the equivalence relation on $2^{\mathbb{N}}$ given by

$$\alpha F_n \beta \Leftrightarrow \forall m \geq n\ (\alpha(m) = \beta(m)).$$

By standard change of topology results (see, for example, §13 of Kechris [3]), we can assume that X is a zero-dimensional Polish space, G acts on X by homeomorphisms, and for all $g, h \in G$, $T \in \mathcal{F}$, $s \in 2^{<\mathbb{N}}$, and $n \in \mathbb{N}$, the sets $\{x \in X : (g \cdot x, h \cdot x) \in \mathcal{G}\}$, $\{x \in X : \pi^T(x) F_n \pi^T(g \cdot x)\}$, and $\{x \in X : s \subseteq \pi^T(x)\}$ are clopen.

We will recursively find clopen sets $A_n \subseteq X$, $g_n \in G$, $S_n \in \mathcal{P}(2^n \times 2^n) \times \mathcal{P}(2^n \times 2^n)$, $k_n \in \mathbb{N}$, and $\pi_n^T : 2^n \to 2^{k_0 + \cdots + k_{n-1}}$, for each $T \in \mathcal{F}$. From these, we define Borel automorphisms $h_s : X \to X$ by

$$h_s \cdot x = \begin{cases} x & \text{if } s = \emptyset, \\ g_0^{s(0)} \ldots g_n^{s(n)} \cdot x & \text{if } s \in 2^{n+1}. \end{cases}$$

From these, we define sequences $\boldsymbol{\mathcal{H}}^n = \langle \mathcal{H}_s \rangle_{s \in 2^n}$, where $\mathcal{H}_s$ is the Borel oriented graph on A_n given by

$$\mathcal{H}_s = \{(x, y) \in A_n \times A_n : (h_s \cdot x, h_s \cdot y) \in \mathcal{H}\}.$$

We begin by setting $A_0 = X$ and $\pi_0^T(\emptyset) = \emptyset$, for each $T \in \mathcal{F}$. Suppose now that we have found $\langle (A_m, \langle \pi_m^T \rangle_{T \in \mathcal{F}}) \rangle_{m \leq n}$ and $\langle (g_m, S_m, k_m) \rangle_{m < n}$ which satisfy the following conditions:

(a) $\forall m \leq n\ (\chi_B(\boldsymbol{\mathcal{H}}^m) > \aleph_0)$;
(b) $\forall m < n\ ((s_m, s_m) \in (S_m)^0)$;
(c) $\forall m < n\ (A_{m+1} \subseteq A_m \cap g_m^{-1}(A_m))$;
(d) $\forall m < n\, \forall x \in A_{m+1}\, \forall s, t \in 2^m\, \forall i \in \{0, 1\}$
$((s, t) \in (S_m)^i \Leftrightarrow (h_s g_m^i \cdot x, h_t g_m^{\bar{i}} \cdot x) \in \mathcal{G})$;
(e) $\forall m < n\, \forall x \in A_{m+1}\, \forall T \in \mathcal{F}\ (\pi^T(x) F_{k_0 + \cdots + k_m} \pi^T(g_m \cdot x))$;
(f) $\forall m \leq n\, \forall x \in A_m\, \forall s \in 2^m\, \forall T \in \mathcal{F}\ (\pi_m^T(s) \subseteq \pi^T(h_s \cdot x))$;
(g) $\forall m < n\, \forall T \in \mathcal{F}\ (\sigma(T) = 0 \Leftrightarrow \pi_{m+1}^T(s_m 0) \leq_{\text{lex}} \pi_{m+1}^T(s_m 1))$;
(h) $\forall m < n\, \forall s, t \in 2^m\, \forall h \in H_m\ (h h_s[A_{m+1}] \cap h_t g_m[A_{m+1}] = \emptyset)$;
(i) $\forall m < n\, \forall s \in 2^{m+1}\ (\text{diam}(h_s[A_{m+1}]) \leq 1/(m+1))$.

Let $\mathbb{P}$ denote the set of tuples $p = (g_p, S_p, k_p, \langle \pi_p^T \rangle_{T \in \mathcal{F}})$ in the set

$$G \times (\mathcal{P}(2^n \times 2^n) \times \mathcal{P}(2^n \times 2^n)) \times \mathbb{N} \times ((2^{k_0 + \cdots + k_{n-1} + k_p})^{2^{n+1}})^{\mathcal{F}},$$

where $(s_n, s_n) \in (S_p)^0$. For each $p \in \mathbb{P}$, let A_p denote the set of x such that:

(c′) $x \in A_n \cap g_p^{-1}(A_n)$;
(d′) $\forall s, t \in 2^n \, \forall i \in \{0,1\} \, ((s,t) \in (S_p)^i \Leftrightarrow (h_s g_p^i \cdot x, h_t g_p^{\bar{i}} \cdot x) \in \mathcal{G})$;
(e′) $\forall T \in \mathcal{F} \, (\pi^T(x) F_{k_0 + \cdots + k_{n-1} + k_p} \pi^T(g_p \cdot x))$;
(f′) $\forall s \in 2^n \, \forall i \in \{0,1\} \, \forall T \in \mathcal{F} \, (\pi_p^T(si) \subseteq \pi^T(h_s g_p^i \cdot x))$;
(g′) $\forall T \in \mathcal{F} \, (\sigma(T) = 0 \Leftrightarrow \pi_p^T(s_n 0) \leq_{\mathrm{lex}} \pi_p^T(s_n 1))$;
(h′) $\forall s, t \in 2^n \, \forall h \in H_n \, (g_p \cdot x \neq h_t^{-1} h h_s \cdot x)$.

Note that the first five conditions are clopen and the sixth is open, so that each of the sets A_p is open. For each $p \in \mathbb{P}$, set $\boldsymbol{\mathcal{H}}^p = \langle \mathcal{H}_s^p \rangle_{s \in 2^{n+1}}$, where

$$\mathcal{H}_{si}^p = \{(x,y) \in A_p \times A_p : (h_s g_p^i \cdot x, h_s g_p^i \cdot y) \in \mathcal{H}\},$$

for each $s \in 2^n$ and $i \in \{0,1\}$.

Lemma 2.5. *There exists $p \in \mathbb{P}$ such that $\chi_B(\boldsymbol{\mathcal{H}}^p) > \aleph_0$.*

Proof. Suppose, towards a contradiction, that for each $p \in \mathbb{P}$ there are $\boldsymbol{\mathcal{H}}^p$-discrete Borel sets $B_{p,k} \subseteq X$, for $k \in \mathbb{N}$, such that $A_p = \bigcup_{k \in \mathbb{N}} B_{p,k}$. For each $p \in \mathbb{P}$ and $k \in \mathbb{N}$, fix $i_{p,k} \in \{0,1\}$ such that $g_p^{i_{p,k}}[B_{p,k}]$ is $\boldsymbol{\mathcal{H}}^n$-discrete. Define

$$A = A_n \setminus \bigcup_{p \in \mathbb{P}, k \in \mathbb{N}} g_p^{i_{p,k}}[B_{p,k}].$$

Then $\chi_B(\boldsymbol{\mathcal{H}}^n | A) > \aleph_0$. Let $\mathcal{K}_n$ denote the graph on X consisting of pairs (x,y) of distinct elements of X such that

$$\exists s, t \in 2^n \, \exists h \in H_n \, (h_t^{-1} h h_s \cdot x = y).$$

Then $\mathcal{K}_n$ has bounded vertex degree, and therefore finite Borel chromatic number (by Proposition 4.5 of Kechris-Solecki-Todorćevič [4]), so there is a $\mathcal{K}_n$-discrete Borel set $A' \subseteq A$ such that $\chi_B(\boldsymbol{\mathcal{H}}^n | A') > \aleph_0$.

Fix $x, y \in A'$ such that $(h_{s_n} \cdot x, h_{s_n} \cdot y) \in \mathcal{H}$. Then there exists $g_p \in G$ such that $g_p \cdot x = y$, so that condition (c′) holds. Let

$$(S_p)^i = \{(s,t) \in 2^n \times 2^n : (h_s g_p^i \cdot x, h_t g_p^{\bar{i}} \cdot x) \in \mathcal{G}\},$$

so that $(s_n, s_n) \in (S_p)^0$ and condition (d′) holds. Fix $k_p \in \mathbb{N}$ such that

$$\forall T \in \mathcal{F} \, (\pi^T(x) F_{k_0 + \cdots + k_{n-1} + k_p} \pi^T(y)),$$

so that (e′) holds. For each $T \in \mathcal{F}$, define $\pi_p^T : 2^{n+1} \to 2^{k_0 + \cdots + k_{n-1} + k_p}$ by

$$\pi_p^T(si) = \pi^T(h_s g_p^i \cdot x) | (k_0 + \cdots + k_{n-1} + k_p),$$

so that condition (f′) holds. The fact that $(h_{s_n} \cdot x, h_{s_n} \cdot y) \in \mathcal{H}$ ensures that condition (g′) holds, and the fact that A' is $\mathcal{K}_n$-discrete implies that condition (h′) holds. Then

$p = (g_p, S_p, k_p, \langle \pi_p^T \rangle_{T \in \mathcal{F}})$ is in $\mathbb{P}$, so there exists $k \in \mathbb{N}$ such that at least one of x, y is in $g_p^{i_{p,k}}[B_{p,k}]$, which contradicts the definition of A. □

By Lemma 2.5, there exists $p \in \mathbb{P}$ such that $\chi_B(\boldsymbol{\mathcal{H}}^p) > \aleph_0$. Set

$$g_n = g_p,\ S_n = S_p,\ k_n = k_p \text{ and } \pi_{n+1}^T = \pi_p^T.$$

Fix a cover of A_p by countably many clopen sets $U_i \subseteq A_p$ such that:

(h″) $\forall i \in \mathbb{N} \, \forall s, t \in 2^n \, \forall h \in H_n \ (hh_s[U_i] \cap h_t g_n[U_i] = \emptyset)$;
(i″) $\forall i \in \mathbb{N} \, \forall s \in 2^{n+1} \ (\mathrm{diam}(h_s[U_i]) \leq 1/(n+1))$.

Fix $i \in \mathbb{N}$ such that $\chi_B(\boldsymbol{\mathcal{H}}^p | U_i) > \aleph_0$, and set $A_{n+1} = U_i$. As $\chi_B(\boldsymbol{\mathcal{H}}^{n+1}) = \chi_B(\boldsymbol{\mathcal{H}}^p | U_i)$, it follows that the analogs of conditions (a) – (i) hold at $n+1$.

This completes the recursive construction. Set $S^i = \bigcup_{n \in \mathbb{N}} (S_n)^i$. Conditions (c) and (i) ensure that, for each $\alpha \in 2^{\mathbb{N}}$, the sets $h_{\alpha|0}[A_0], h_{\alpha|1}[A_1], \ldots$ are decreasing and of vanishing diameter, and since they are clopen, they therefore have singleton intersection. Define $\pi : 2^{\mathbb{N}} \to X$ by

$$\pi(\alpha) = \text{ the unique element of } \bigcap_{n \in \mathbb{N}} h_{\alpha|n}[A_n].$$

It follows from conditions (h) and (i) that π is a continuous injection.

Lemma 2.6. *Suppose that $n \in \mathbb{N}$, $s \in 2^n$, and $\alpha \in 2^{\mathbb{N}}$. Then $\pi(s\alpha) = h_s \cdot \pi(0^n\alpha)$.*

Proof. Simply observe that

$$\begin{aligned}
\{\pi(s\alpha)\} &= \bigcap_{i \geq n} h_{(s\alpha)|i}[A_i] \\
&= \bigcap_{i \geq 0} h_s h_{0^n(\alpha|i)}[A_{i+n}] \\
&= h_s \left[\bigcap_{i \geq 0} h_{0^n(\alpha|i)}[A_{i+n}] \right] \\
&= h_s \left[\bigcap_{i \geq n} h_{(0^n\alpha)|i}[A_i] \right] \\
&= \{h_s \cdot \pi(0^n\alpha)\},
\end{aligned}$$

thus $\pi(s\alpha) = h_s \cdot \pi(0^n\alpha)$. □

Lemma 2.7. *Suppose that $\alpha E_0 \beta$. Then $(\alpha, \beta) \in \mathcal{G}^S \Leftrightarrow (\pi(\alpha), \pi(\beta)) \in \mathcal{G}$.*

Proof. Fix $n \in \mathbb{N}$ maximal such that $\alpha(n) \neq \beta(n)$, let $s = \alpha|n$ and $t = \beta|n$, set $i = \alpha(n)$, and fix $\gamma \in 2^{\mathbb{N}}$ such that $\alpha = si\gamma$ and $\beta = t\bar{\imath}\gamma$. Lemma 2.6 ensures that $\pi(\alpha) = \pi(si\gamma) = h_{si} \cdot \pi(0^{n+1}\gamma)$ and $\pi(\beta) = \pi(t\bar{\imath}\gamma) = h_{t\bar{\imath}} \cdot \pi(0^{n+1}\gamma)$, and since

$\pi(0^{n+1}\gamma) \in A_{n+1}$, condition (d) then ensures that $(\alpha, \beta) \in \mathcal{G}^S \Leftrightarrow (s,t) \in (S_n)^i \Leftrightarrow (\pi(\alpha), \pi(\beta)) \in \mathcal{G}$. $\square$

Lemma 2.8. *Suppose that* $(\alpha, \beta) \notin E_0$. *Then* $(\pi(\alpha), \pi(\beta)) \notin E_{\mathcal{G}}$.

Proof. It is enough to check that if $\alpha, \beta \in 2^{\mathbb{N}}$ and $\alpha(n) \neq \beta(n)$, then there is no $h \in H_n$ such that $h \cdot \pi(\alpha) = \pi(\beta)$. Suppose, towards a contradiction, that there is such an h. As H_n is symmetric, we can assume that $\alpha(n) = 0$ and $\beta(n) = 1$. Set $s = \alpha|n$ and $t = \beta|n$, and fix $\gamma, \delta \in 2^{\mathbb{N}}$ such that $\alpha = s0\gamma$ and $\beta = t1\delta$. Lemma 2.6 ensures that $\pi(\alpha) = h_s \cdot \pi(0^{n+1}\gamma)$ and $\pi(\beta) = h_t g_n \cdot \pi(0^{n+1}\delta)$. As $\pi(0^{n+1}\gamma), \pi(0^{n+1}\delta) \in A_{n+1}$, it follows that $\pi(\beta) \in hh_s[A_{n+1}] \cap h_t g_n[A_{n+1}]$, which contradicts condition (h). $\square$

Lemma 2.9. $\forall \alpha \in 2^{\mathbb{N}}$ $(\pi_n^T(\alpha|n) \subseteq \pi^T \circ \pi(\alpha))$.

Proof. Suppose that $\alpha \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$, fix $x \in A_n$ such that $\pi(\alpha) = h_{\alpha|n} \cdot x$, and observe that $\pi_n^T(\alpha|n) \subseteq \pi^T(h_{\alpha|n} \cdot x)$, by condition (f). $\square$

Lemmas 2.7 and 2.8 easily imply that π is an embedding of $\mathcal{G}^S$ into $\mathcal{G}$, so it only remains to show that if $T \in \mathcal{F}$, then $\pi^T \circ \pi$ is a monotonic aligned embedding of $\mathcal{G}^S$ into $\mathcal{G}^T$. Towards this end, note that if $n \in \mathbb{N}$ and $i \in \{0,1\}$, then $\pi_n^T(s_n) \subseteq \pi_{n+1}^T(s_n i)$, by Lemma 2.9. It follows that there is a unique pair $u_n \in 2^{k_n} \times 2^{k_n}$ such that $\pi_{n+1}^T(s_n i) = \pi_n^T(s_n)(u_n)_i$, for each $i \in \{0,1\}$.

Lemma 2.10. $\forall n \in \mathbb{N}$ $((u_n)_0 \neq (u_n)_1)$.

Proof. Fix $n \in \mathbb{N}$ and set $x = \pi(s_n 00^\infty)$ and $y = \pi(s_n 10^\infty)$. As π is injective, it follows that $x \neq y$. As π^T is injective, it follows that $\pi^T(x) \neq \pi^T(y)$. Lemma 2.9 and condition (e) then imply that there exists $\alpha \in 2^{\mathbb{N}}$ such that $\pi^T(x) = \pi_n^T(s_n)(u_n)_0\alpha$ and $\pi^T(y) = \pi_n^T(s_n)(u_n)_1\alpha$, thus $(u_n)_0 \neq (u_n)_1$. $\square$

Lemma 2.11. $\forall n \in \mathbb{N}\, \forall s \in 2^n\, \forall i \in \{0,1\}$ $(\pi_{n+1}^T(si) = \pi_n^T(s)(u_n)_i)$.

Proof. Since $\pi_{n+1}^T(s_n i) = \pi_n^T(s_n)(u_n)_i$, Lemmas 2.6 and 2.9 imply that

$$\pi_n^T(s) \subseteq \pi_{n+1}^T(si) \subseteq \pi^T \circ \pi(si0^\infty) = \pi^T(h_s g_n^i \cdot \pi(0^\infty)),$$

and condition (e) ensures that $\pi^T(h_s g_n^i \cdot \pi(0^\infty)) F_{k_0+\cdots+k_{n-1}} \pi^T(h_{s_n} g_n^i \cdot \pi(0^\infty))$, it follows that $\pi_{n+1}^T(si) = \pi_n^T(s)(u_n)_i$. $\square$

Suppose now that $T \in \mathcal{F}$, and observe that if $n \in \mathbb{N}$ and $s, t \in 2^n$, then

$$\begin{aligned}(s,t) \in \mathcal{G}_n^S &\Leftrightarrow (s0^\infty, t0^\infty) \in \mathcal{G}^S \\ &\Leftrightarrow (\pi(s0^\infty), \pi(t0^\infty)) \in \mathcal{G} \\ &\Leftrightarrow (\pi^T \circ \pi(s0^\infty), \pi^T \circ \pi(t0^\infty)) \in \mathcal{G}^T \\ &\Leftrightarrow (\pi_n^T(s), \pi_n^T(t)) \in \mathcal{G}^T_{k_0+\cdots+k_{n-1}},\end{aligned}$$

thus $\pi^T \circ \pi$ is an aligned embedding of $\mathcal{G}^S$ into $\mathcal{G}^T$. Condition (g) ensures that if $\sigma(T) = 0$, then $\pi^T \circ \pi$ is order-preserving, and if $\sigma(T) = 1$, then $\pi^T \circ \pi$ is order-reversing. In either case, it follows that $\pi^T \circ \pi$ is monotonic, and this completes the proof of the theorem. $\square$

3. Generalizations

In this section, we will establish a generalization of Theorem 2.4. First, however, we must generalize some basic descriptive set-theoretic facts.

Proposition 3.1. *Suppose that X and Y are Polish spaces, $\varphi : X \to Y$ is a Borel function, and $A \subseteq X$ has the property of Baire. Then there is a Borel set $B \subseteq X$ such that $B \subseteq A$, $A \setminus B$ is meager, and $\varphi[B]$ is Borel.*

Proof. Define an equivalence relation E on X by $x_1 E x_2 \Leftrightarrow \varphi(x_1) = \varphi(x_2)$. A *selector for E* is a function $s : X \to X$ such that

$$\forall x \in X\ (xEs(x)) \text{ and } \forall x_1, x_2 \in X\ (x_1 E x_2 \Rightarrow s(x_1) = s(x_2)).$$

A *partial transversal of E* is a set which intersects every E-class in at most one point. The *E-saturation of $B \subseteq X$* is given by

$$[B]_E = \{x \in X : \exists y \in B\ (xEy)\},$$

and we say that $B \subseteq X$ *is E-invariant* if $B = [B]_E$.

As A has the property of Baire, there is a Borel set $C \subseteq A$ such that $A \setminus C$ is meager. By the Jankov-von Neumann uniformization theorem (see, for example, Theorem 18.1 of Kechris [3]), there is a $\sigma(\mathbf{\Sigma}^1_1)$-measurable function $\psi : \varphi[C] \to C$ such that $\forall y \in \varphi[C]\ (y = \varphi \circ \psi(y))$. Then $s = \psi \circ \varphi | C$ is a $\sigma(\mathbf{\Sigma}^1_1)$-measurable selector for $E|C$. Fix a Borel set $D \subseteq C$ such that $C \setminus D$ is meager and $s|D$ is Borel. Then the set $s[D]$ is an analytic partial transversal of $E|C$. As the property of being a partial transversal of $E|C$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, the first reflection theorem implies that there is a Borel partial transversal $B' \supseteq s[D]$ of $E|C$. By Souslin's Theorem (see, for example, Theorem 14.11 of Kechris [3]), the set

$$\begin{aligned} B &= [B']_{E|C} \\ &= \{x \in C : \exists y \in B'\ (xEy)\} \\ &= \{x \in C : \exists! y \in B'\ (xEy)\} \end{aligned}$$

is Borel. Observe that $B \subseteq C \subseteq A$. As $D \subseteq B$, it follows that $A \setminus B \subseteq (A \setminus C) \cup (C \setminus D)$ is meager, and the Lusin-Souslin theorem (see, for example, Theorem 15.1 of Kechris [3]) ensures that $\varphi[B] = \varphi[B']$ is Borel. $\square$

Proposition 3.2. *Suppose that X and Y are Polish spaces, $C \subseteq X$ is a comeager Borel set, $\pi : C \to Y$ is a Borel function, and $A \subseteq Y$ is globally Baire. Then there is a comeager Borel set $D \subseteq C$ such that $A \cap \pi[D]$ and $\pi[D]$ are Borel.*

Proof. Fix $y_0 \in Y$ and define $\hat{\pi} : X \to Y$ by

$$\hat{\pi}(x) = \begin{cases} \pi(x) & \text{if } x \in C, \\ y_0 & \text{otherwise.} \end{cases}$$

Set $A' = Y \setminus A$, and observe that both $\hat{\pi}^{-1}(A)$ and $\hat{\pi}^{-1}(A')$ have the property of Baire. By Proposition 3.1, there are Borel sets $B \subseteq \hat{\pi}^{-1}(A)$ and $B' \subseteq \hat{\pi}^{-1}(A')$ such that $\hat{\pi}^{-1}(A) \setminus B, \hat{\pi}^{-1}(A') \setminus B'$ are meager and $\hat{\pi}[B], \hat{\pi}[B']$ are Borel. Then the set $D' = B \cup B'$ is comeager, and both $A \cap \hat{\pi}[D'] = \hat{\pi}[B]$ and $\hat{\pi}[D'] = \hat{\pi}[B] \cup \hat{\pi}[B']$ are Borel. It follows that the set $D = C \cap D'$ is comeager, and since $A \cap \pi[D] \setminus \{y_0\} = A \cap \hat{\pi}[D'] \setminus \{y_0\}$ and $\pi[D] \setminus \{y_0\} = \hat{\pi}[D'] \setminus \{y_0\}$, it follows that $A \cap \pi[D]$ and $\pi[D]$ are Borel. □

Proposition 3.3. *Suppose that X, Y, and Z are Polish spaces, $C \subseteq X$ is a comeager Borel set, $\pi : C \to Y$ is Borel, $A \subseteq Y$, $\varphi : A \to Z$ is globally Baire measurable, and $\pi[C] \subseteq A$. Then there is a Borel set $B \subseteq Y$ such that $B \subseteq A$, $\pi^{-1}(B)$ is comeager, and $\varphi|B$ is Borel.*

Proof. Fix an open basis $\langle U_n \rangle_{n \in \mathbb{N}}$ for Z. For each $n \in \mathbb{N}$, the set $\varphi^{-1}(U_n)$ is globally Baire, so Proposition 3.2 ensures that there is a comeager Borel set $C_n \subseteq C$ such that $\varphi^{-1}(U_n) \cap \pi[C_n]$ and $\pi[C_n]$ are Borel. Then the set $B = \bigcap_{n \in \mathbb{N}} \pi[C_n]$ is Borel, as is each set of the form $\varphi^{-1}(U_n) \cap B$, thus $\varphi|B$ is Borel. As $\bigcap_{n \in \mathbb{N}} C_n \subseteq \pi^{-1}(B)$, it follows that $\pi^{-1}(B)$ is comeager. □

Proposition 3.4. *Suppose that X, Y, and Z are Polish spaces and $\psi : X \to Y$ and $\varphi : Y \to Z$ are globally Baire measurable. Then $\varphi \circ \psi$ is globally Baire measurable.*

Proof. We must show that if W is a Polish space, $\pi : W \to X$ is Borel, and $U \subseteq Z$ is open, then the set $(\varphi \circ \psi \circ \pi)^{-1}(U)$ has the property of Baire. Towards this end, note first that by Proposition 3.3, there is a Borel set $A \subseteq X$ such that $\pi^{-1}(A)$ is comeager and $\psi|A$ is Borel. By another application of Proposition 3.3, there is a Borel set $B \subseteq Y$ such that the set $C = (\psi \circ \pi|\pi^{-1}(A))^{-1}(B)$ is comeager and $\varphi|B$ is Borel, thus $\varphi \circ \psi \circ \pi|C$ is Borel. It follows that $(\varphi \circ \psi \circ \pi)^{-1}(U)$ is the union of a Borel set with a meager set, and therefore has the property of Baire. □

Proposition 3.5. *Suppose that X, Y, and Z are Polish spaces, $\pi : Z \to X$ is Borel, and $R \subseteq X \times Y$ is an analytic set with countable vertical sections. Then there is a Borel set $B \subseteq X$ such that $\pi^{-1}(B)$ is comeager and $R \cap (B \times Y)$ is Borel.*

Proof. By a result of Lusin (see, for example, Exercise 35.13 of Kechris [3]), there are Borel functions $f_n : X \to Y$ such that $R \subseteq \bigcup_{n \in \mathbb{N}} \text{graph}(f_n)$. For each $n \in \mathbb{N}$, define $A_n \subseteq X$ by

$$A_n = \{x \in X : (x, f_n(x)) \in R\}.$$

By Proposition 3.2, there are comeager Borel sets $C_n \subseteq Z$ such that $A_n \cap \pi[C_n]$ and $\pi[C_n]$ are Borel. Then the set $B = \bigcap_{n\in\mathbb{N}} \pi[C_n]$ is Borel, as is $R \cap (B \times Y) = \bigcup_{n\in\mathbb{N}} \mathrm{graph}(f_n|(A_n \cap B))$. As $\bigcap_{n\in\mathbb{N}} C_n \subseteq \pi^{-1}(B)$, it follows that $\pi^{-1}(B)$ is comeager, which completes the proof of the proposition. $\square$

Remark 3.6. Our assumption that R has countable vertical sections cannot be entirely removed. To see this, set $X = Y = Z = \mathbb{R}$ and $\pi = \mathrm{id}$, fix a co-analytic set $A \subseteq \mathbb{R}$ which is not analytic, define $R' = \{(x, y) \in X \times Y : x + y \in A\}$, and observe that the set $R = (X \times Y) \setminus R'$ is analytic. If $C \subseteq X$ and $D \subseteq Y$ are comeager, then for each $z \in A$, there exists $(x, y) \in C \times D$ such that $z = x + y$, thus $A = \{x + y : (x, y) \in R' \cap (C \times D)\}$. It follows that if $R \cap (C \times D)$ is co-analytic, then A is analytic, a contradiction.

We are now ready to return to Borel chromatic numbers.

Proposition 3.7. *Suppose that X is a Polish space and $\mathcal{G}$ is a locally countable analytic directed graph on X of uncountable Borel chromatic number. Then there is a Borel set $B \subseteq X$ such that $\chi_B(\mathcal{G}|B) > \aleph_0$ and $\mathcal{G}|B$ is Borel.*

Proof. By the directed analog of Theorem 6.3 of Kechris-Solecki-Todorčević [4], there is a continuous homomorphism $\pi : 2^{\mathbb{N}} \to X$ from $\mathcal{G}_0^{\rightarrow}$ to $\mathcal{G}$. By Proposition 3.5, there is a Borel set $B \subseteq X$ such that $\pi^{-1}(B)$ is comeager and $\mathcal{G} \cap (B \times X)$ is Borel. It follows that $\mathcal{G}|B$ is Borel, and Proposition 1.1 ensures that $\chi_B(\mathcal{G}_0^{\rightarrow}|\pi^{-1}(B)) > \aleph_0$, thus $\chi_B(\mathcal{G}|B) > \aleph_0$. $\square$

Proposition 3.8. *Suppose that X and Y are Polish spaces, $\mathcal{G}$ is an analytic directed graph on X, $A \subseteq X$ is analytic, $\chi_B(\mathcal{G} \cap (A \times A)) > \aleph_0$, and $\varphi : A \to Y$ is globally Baire measurable. Then there is a Borel set $B \subseteq X$ such that $B \subseteq A$, $\chi_B(\mathcal{G}|B) > \aleph_0$, and $\varphi|B$ is Borel.*

Proof. By the directed analog of Theorem 6.3 of Kechris-Solecki-Todorčević [4], there is a continuous homomorphism $\pi : 2^{\mathbb{N}} \to X$ from $\mathcal{G}_0^{\rightarrow}$ to $\mathcal{G} \cap (A \times A)$. Then $\pi[2^{\mathbb{N}}] \subseteq A$, so Proposition 3.3 implies that there is a Borel set $B \subseteq X$ such that $\pi^{-1}(B)$ is comeager and $\varphi|B$ is Borel, and Proposition 1.1 ensures that $\chi_B(\mathcal{G}_0^{\rightarrow}|\pi^{-1}(B)) > \aleph_0$, thus $\chi_B(\mathcal{G}|B) > \aleph_0$. $\square$

Proposition 3.9. *Suppose that X and Y are Polish spaces, $\mathcal{G}$ and $\mathcal{H}$ are analytic directed graphs on X and Y of uncountable Borel chromatic number, and $\pi : X \to Y$ is a globally Baire measurable reduction of $\mathcal{G}$ to $\mathcal{H}$. Then there is a Borel set $B \subseteq Y$ such that $\mathcal{H}|B$ is a directed graph of uncountable Borel chromatic number which admits a Borel embedding into $\mathcal{G}$.*

Proof. By Proposition 3.8, there is a Borel set $A \subseteq X$ such that $\chi_B(\mathcal{G}|A) > \aleph_0$ and $\pi|A$ is Borel. By the Jankov-von Neumann uniformization theorem, there is a

$\sigma(\mathbf{\Sigma}^1_1)$-measurable function $\varphi : \pi[A] \to A$ such that

$$\forall y \in \pi[A]\ (\pi \circ \varphi(y) = y).$$

By Proposition 3.8, there is a Borel set $B \subseteq \pi[A]$ such that $\chi_B(\mathcal{H}|B) > \aleph_0$ and $\varphi|B$ is Borel, and it is clear that $\varphi|B$ is an embedding of $\mathcal{H}|B$ into $\mathcal{G}$. $\square$

We are now ready to prove the promised generalization of Theorem 2.4:

Theorem 3.10. *Suppose that X is a Polish space, $\mathcal{G}$ is an analytic directed graph on X which admits a globally Baire measurable reduction into a locally countable analytic directed graph on a Polish space, $\mathcal{F}$ is a finite subset of $\mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n) \times \mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n)$, and $\pi^T : X \to 2^{\mathbb{N}}$ is a globally Baire measurable reduction of $\mathcal{G}$ to $\mathcal{G}^T$, for each $T \in \mathcal{F}$. Then exactly one of the following holds:*

1. $\chi_B(\mathcal{G}) \leq \aleph_0$;
2. *There is a strongly dense pair S and a continuous embedding $\pi : 2^{\mathbb{N}} \to X$ of $\mathcal{G}^S$ into $\mathcal{G}$ such that $\pi^T \circ \pi$ is a monotonic aligned embedding of $\mathcal{G}^S$ into $\mathcal{G}^T$, for each $T \in \mathcal{F}$.*

Proof. It is again sufficient to show $\neg(1) \Rightarrow (2)$. By Proposition 3.9, there is a locally countable analytic directed graph $\mathcal{H}$ on a Polish space Y such that $\chi_B(\mathcal{H}) > \aleph_0$, as well as a Borel embedding φ of $\mathcal{H}$ into $\mathcal{G}$. By Proposition 3.7, there is a Borel set $B \subseteq Y$ such that $\chi_B(\mathcal{H}|B) > \aleph_0$ and $\mathcal{H}|B$ is Borel. As each function of the form $\pi^T \circ \varphi|B$ is globally Baire measurable, it follows from $|\mathcal{F}|$ many applications of Proposition 3.8 that there is a Borel set $C \subseteq B$ such that $\chi_B(\mathcal{H}|C) > \aleph_0$ and the functions of the form $\pi^T \circ \varphi|C$, for $T \in \mathcal{F}$, are Borel. Fix a Polish topology on C, compatible with its underlying Borel structure, with respect to which $\varphi|C$ and the functions of the form $\pi^T \circ \varphi|C$, for $T \in \mathcal{F}$, are continuous. By Theorem 2.4, there is a strongly dense pair S and a continuous embedding $\psi : 2^{\mathbb{N}} \to C$ of $\mathcal{G}^S$ into $\mathcal{H}|C$ such that $(\pi^T \circ \varphi|C) \circ \psi$ is a monotonic aligned embedding of $\mathcal{G}^S$ into $\mathcal{G}^T$, for each $T \in \mathcal{F}$, and it follows that the function $\pi = (\varphi|C) \circ \psi$ is as desired. $\square$

4. Basis theorems

In this section, we derive several basis results from Theorem 3.10.

Theorem 4.1. *Suppose that X is a Polish space and $\mathcal{G}$ is an analytic directed graph on X which admits a globally Baire measurable reduction to a locally countable analytic directed graph on a Polish space. Then exactly one of the following holds:*

1. $\chi_B(\mathcal{G}) \leq \aleph_0$;
2. *There is a strongly dense pair S such that $\mathcal{G}^S \sqsubseteq_c \mathcal{G}$.*

Proof. This is a direct consequence of Theorem 3.10. $\square$

Theorem 4.2. *Suppose that X is a Polish space and $\mathcal{G}$ is an analytic graph on X which admits a globally Baire measurable reduction to a locally countable analytic graph on a Polish space, as well as a globally Baire measurable reduction to an acyclic analytic graph on a Polish space. Then exactly one of the following holds:*

1. $\chi_B(\mathcal{G}) \leq \aleph_0$;
2. $\mathcal{G}_0 \sqsubseteq_c \mathcal{G}$.

Proof. It is again enough to show that $\neg(1) \Rightarrow (2)$. By Proposition 3.9 and standard change of topology results, there is an acyclic analytic graph $\mathcal{H} \sqsubseteq_c \mathcal{G}$ of uncountable Borel chromatic number. Theorem 4.1 then implies that there is a strongly dense pair S such that $\mathcal{G}^S \sqsubseteq_c \mathcal{H}$. Then $\mathcal{G}^S$ is an acyclic graph, so Proposition 1.5 implies that $S = S_0$, and the theorem follows. □

Theorem 4.3. *Suppose that X is a Polish space and $\mathcal{G}$ is an analytic oriented graph on X which admits a globally Baire measurable reduction to a locally countable analytic oriented graph on a Polish space, as well as a globally Baire measurable reduction to an acyclic analytic oriented graph on a Polish space. Then exactly one of the following holds:*

1. $\chi_B(\mathcal{G}) \leq \aleph_0$;
2. $\mathcal{G}_0^{\rightarrow} \sqsubseteq_c \mathcal{G}$.

Proof. It is once more enough to show that $\neg(1) \Rightarrow (2)$. By Proposition 3.9 and standard change of topology results, there is an acyclic analytic oriented graph $\mathcal{H} \sqsubseteq_c \mathcal{G}$ of uncountable Borel chromatic number. Theorem 4.1 then implies that there is a strongly dense pair S such that $\mathcal{G}^S \sqsubseteq_c \mathcal{H}$. Then $\mathcal{G}^S$ is an acyclic oriented graph, so Proposition 1.6 implies that $S = S_0^{\rightarrow}$, and the theorem follows. □

Next we give a new proof of Theorem 28 of Lecomte [6]:

Theorem 4.4. *Suppose that X is a non-empty Polish space and $\mathcal{G}$ is a directed graph on X of the form $\bigcup_{n \in \mathbb{N}} \mathrm{graph}(f_n)$, where:*

1. *$f_0, f_1, \ldots$ are partial homeomorphisms with open domains and ranges;*
2. *$\Delta(X) \subseteq \overline{\mathcal{G}}$;*
3. *For each $n \in \mathbb{N}$, $s \in \mathbb{N}^{n+1}$ such that $\forall i < n\ (s(i) \neq s(i+1))$, $t \in \mathbb{Z}^{n+1}$, and non-empty open set $U \subseteq X$, there is a non-empty open set $V \subseteq U$ such that $f_s^t | V$ is fixed-point free, where $f_s^t = f_{s(0)}^{t(0)} \cdots f_{s(n)}^{t(n)}$.*

Then $\mathcal{G}_0^{\rightarrow} \sqsubseteq_c \mathcal{G}$.

Proof. Let $S_n = \{(s,t) \in \mathbb{N}^{n+1} \times \mathbb{Z}^{n+1} : \forall i < n\ (s(i) \neq s(i+1))\}$, and set $S = \bigcup_{n \in \mathbb{N}} S_n$. Condition (3) ensures that for each $(s,t) \in S$, the set

$$M_{(s,t)} = \{x \in X : x = f_s^t(x)\}$$

is meager, thus so too is the set $M = \bigcup_{(s,t)\in S} M_{(s,t)}$. Let $E = E_{\mathcal{G}}$. As each f_n sends meager sets to meager sets, it follows that the E-saturation of M is also meager, thus $C = X \setminus [M]_E$ is an E-invariant comeager Borel set.

Lemma 4.5. *$\mathcal{G}|C$ is oriented.*

Proof. Suppose, towards a contradiction, that there exist $x, y \in C$ such that $(x, y), (y, x) \in \mathcal{G}|C$, and fix $m, n \in \mathbb{N}$ such that $x = f_m(y)$ and $y = f_n(x)$. If $m = n$, then set $s = \langle m \rangle = \langle n \rangle$ and $t = \langle 2 \rangle$. Otherwise, set $s = \langle m, n \rangle$ and $t = \langle 1, 1 \rangle$. In either case, it follows that $(s, t) \in S$ and $x = f_s^t(x)$, a contradiction. □

Lemma 4.6. *$\mathcal{G}|C$ is acyclic.*

Proof. Suppose, towards a contradiction, that $\mathcal{G}|C$ is not acyclic, and fix $n \in \mathbb{N}$ least for which there exists $s \in \mathbb{N}^{n+1}$, $t \in \mathbb{Z}^{n+1}$, and $x \in C$ such that $x = f_s^t(x)$. The minimality of n ensures that $(s, t) \in S$, a contradiction. □

Lemma 4.7. $\chi_B(\mathcal{G}|C) > \aleph_0$.

Proof. It is enough to check that no non-meager Borel set $B \subseteq X$ is $\mathcal{G}$-discrete. Towards this end, fix a non-empty open set $U \subseteq X$ such that $B \cap U$ is comeager in U, and fix $x \in U$. As $(x, x) \in \overline{\mathcal{G}}$, there exists $n \in \mathbb{N}$ and $y \in \mathrm{dom}(f_n) \cap U$ such that $f_n(y) \in U$, so that the set $V = \mathrm{dom}(f_n) \cap U \cap f_n^{-1}(U)$ is non-empty. As B is comeager in both V and $f_n(V)$, it follows that $B \cap f_n^{-1}(B)$ is non-empty, so B is not $\mathcal{G}$-discrete. □

Theorem 4.3 implies that there is a continuous embedding π of $\mathcal{G}_0^{\rightarrow}$ into $\mathcal{G} \cap (C \times C)$. As $\pi[2^{\mathbb{N}}] \subseteq C$, it follows that π is a continuous embedding of $\mathcal{G}_0^{\rightarrow}$ into $\mathcal{G}$. □

We close this section with a basis result for equivalence relations:

Theorem 4.8. *Suppose that X is a Polish space and E is a countable analytic equivalence relation on X. Then exactly one of the following holds:*

1. *X is the union of countably many Borel partial transversals of E;*
2. *$E_0 \sqsubseteq_c E$.*

Proof. It is clear that condition (1) is equivalent to

1′. $\chi_B(E \setminus \Delta(X)) \leq \aleph_0$,

and condition (2) is equivalent to

2′. $E_0 \setminus \Delta(2^{\mathbb{N}}) \sqsubseteq_c E \setminus \Delta(X)$.

Define S by $S^0 = S^1 = \bigcup_{n \in \mathbb{N}} 2^n \times 2^n$. Then $\mathcal{G}^S = E_0 \setminus \Delta(2^{\mathbb{N}})$, thus $\chi_B(E_0 \setminus \Delta(2^{\mathbb{N}})) = \chi_B(\mathcal{G}^S) > \aleph_0$, by Proposition 1.1. As colorings can be pulled-back through homomorphisms, it follows that conditions (1′) and (2′) are mutually exclusive. To see that $\neg(1') \Rightarrow (2')$, observe that if (1′) fails, then Theorem 4.1 implies that there is a strongly dense pair T such that $\mathcal{G}^T \sqsubseteq_c E \setminus \Delta(X)$. Then $\mathcal{G}^T \cup \Delta(2^{\mathbb{N}})$ is an equivalence relation, so Proposition 1.7 ensures that $S = T$, thus $E_0 \setminus \Delta(2^{\mathbb{N}}) \sqsubseteq_c E \setminus \Delta(X)$. □

5. Basis embeddings

In this section, we will show that the map $A \mapsto \mathcal{G}^A$ is a basis embedding. We begin with some connections between our notions of aligned embeddability:

Proposition 5.1. *Suppose that there is an aligned embedding of S into S^A. Then there exists B such that $S = S^B$.*

Proof. It is enough to show that $\forall i \in \{0,1\} \forall (s,t) \in S^i \ (s = t)$. Towards this end, suppose that φ is an aligned embedding of S into S^A, and for each $n \in \mathbb{N}$, let $\mathcal{G}_n^A = \mathcal{G}_n^{S^A}$. Observe that if $(s,t) \in S^i \cap (2^n \times 2^n)$, then $(si, t\bar{\imath}) \in \mathcal{G}_{n+1}^S$, thus

$$(\varphi(si), \varphi(t\bar{\imath})) \in \mathcal{G}^A_{k_0^\varphi + \cdots + k_n^\varphi}.$$

Fix $j < k_0^\varphi + \cdots + k_n^\varphi$ with $(\varphi(s)(u_n^\varphi)_i | j, \varphi(t)(u_n^\varphi)_{\bar{\imath}} | j) \in (S^A)^0 \cup (S^A)^1$ and

$$\forall k > j \ ([\varphi(s)(u_n^\varphi)_i](k) = [\varphi(t)(u_n^\varphi)_{\bar{\imath}}](k)).$$

As $(u_n^\varphi)_i \neq (u_n^\varphi)_{\bar{\imath}}$, it follows that $j \geq k_0^\varphi + \cdots + k_{n-1}^\varphi$, and since $(S^A)^0 \cup (S^A)^1 \subseteq \Delta(2^{<\mathbb{N}})$, this implies that $\varphi(s) = \varphi(t)$, thus $s = t$. □

Proposition 5.2. *Suppose that there is a monotonic aligned embedding of A into B. Then there is a monotonic aligned embedding of S^A into S^B.*

Proof. Suppose that $\psi : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ is an order-preserving aligned embedding of A into B, and let $\varphi : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ be the order-preserving aligned function given by $(u_n^\varphi)_i = u_n^\psi i$. Given distinct sequences $s, t \in 2^{n+1}$, fix $m < n+1$ largest such that $s(m) \neq t(m)$, set $i = s(m)$, and observe that

$$\begin{aligned}
(s,t) \in \mathcal{G}_{n+1}^A &\Leftrightarrow (s|m, t|m) \in (S^A)^i \\
&\Leftrightarrow s|m = t|m \text{ and } s|m \in A^i \\
&\Leftrightarrow \psi(s|m) = \psi(t|m) \text{ and } \psi(s|m) \in B^i \\
&\Leftrightarrow (\psi(s|m), \psi(t|m)) \in (S^B)^i \\
&\Leftrightarrow (\varphi(s), \varphi(t)) \in \mathcal{G}^B_{k_0^\varphi + \cdots + k_n^\varphi},
\end{aligned}$$

thus φ is an order-preserving aligned embedding of S^A into S^B. The case that ψ is order-reversing is handled similarly, with $(u_n^\varphi)_i = u_n^\psi \bar{\imath}$. □

Proposition 5.3. *Suppose that A^0 contains sequences of every length and there is a monotonic aligned embedding of S^A into S^B. Then there is a monotonic aligned embedding of A into B.*

Proof. Fix an order-preserving aligned embedding φ of S^A into S^B.

Lemma 5.4. *For all n, there is a unique k_n with $[(u_n^\varphi)_0](k_n) \neq [(u_n^\varphi)_1](k_n)$.*

Proof. Fix $s \in A^0 \cap 2^n$. Then $(s0, s1) \in \mathcal{G}^A_{n+1}$, thus

$$(\varphi(s0), \varphi(s1)) \in \mathcal{G}^B_{k_0^\varphi + \cdots + k_n^\varphi}.$$

Fix a positive integer $j < k_0^\varphi + \cdots + k_n^\varphi$ such that $(\varphi(s)(u_n^\varphi)_0|j, \varphi(s)(u_n^\varphi)_1|j) \in S^B$ and $\forall k > j\ ([\varphi(s)(u_n^\varphi)_0](k) = [\varphi(s)(u_n^\varphi)_1](k))$, and observe that $k_n = j - (k_0^\varphi + \cdots + k_{n-1}^\varphi)$ is as desired. ◻

Let ψ be the order-preserving nicely aligned function obtained by setting $u_0^\psi = (u_0^\varphi)_0|k_0 = (u_0^\varphi)_1|k_0$ and letting u_{n+1}^ψ be the concatenation of $(u_n^\varphi)_0|(k_n, k_n^\varphi) = (u_n^\varphi)_1|(k_n, k_n^\varphi)$ with $(u_{n+1}^\varphi)_0|k_{n+1} = (u_{n+1}^\varphi)_1|k_{n+1}$. If $s \in 2^n$, then

$$\begin{aligned} s \in A^i &\Leftrightarrow (si, s\bar{\imath}) \in \mathcal{G}^A_{n+1} \\ &\Leftrightarrow (\varphi(si), \varphi(s\bar{\imath})) \in \mathcal{G}^B_{k_0^\varphi + \cdots + k_n^\varphi} \\ &\Leftrightarrow \psi(s) \in B^i, \end{aligned}$$

thus ψ is an order-preserving aligned embedding of A into B. The case that φ is order-reversing is handled similarly. □

We are now prepared for our first basis embeddability result:

Theorem 5.5. *The map $A \mapsto \mathcal{G}^A$ is a basis embedding of monotonic aligned embeddability on the set of dense pairs into $(\sqsubseteq_c, \leq_{\mathrm{GB}})$.*

Proof. By Proposition 1.9, the map $S \mapsto \mathcal{G}^S$ is a homomorphism from monotonic aligned embeddability into $\sqsubseteq_c$, so it only remains to show that if A and B are dense and $\mathcal{G} \leq_{\mathrm{GB}} \mathcal{G}^A, \mathcal{G}^B$ is an analytic graph of uncountable Borel chromatic number, then there is a dense pair C which admits monotonic aligned embeddings into A and B and for which $\mathcal{G}^C \sqsubseteq_c \mathcal{G}$. Towards this end, observe that by Theorem 3.10, there is a strongly dense pair S which admits monotonic aligned embeddings into S^A and S^B and for which $\mathcal{G}^S \sqsubseteq_c \mathcal{G}$. By Proposition 5.1, there exists C such that $S = S^C$. As S is strongly dense, it follows that C is dense and contains sequences of every length, thus Proposition 5.3 implies that there are monotonically aligned embeddings of C into A and B. □

We next note some connections between non-potentially closed sets and graphs of uncountable Borel chromatic number:

Proposition 5.6. *Suppose that X is a Polish space and $\mathcal{G}$ is a directed graph on X which is potentially closed. Then $\chi_B(\mathcal{G}) \leq \aleph_0$.*

Proof. Fix sequences $\langle A_n \rangle_{n \in \mathbb{N}}$ and $\langle B_n \rangle_{n \in \mathbb{N}}$ of Borel subsets of X such that

$$(X \times X) \setminus \mathcal{G} = \bigcup_{n \in \mathbb{N}} A_n \times B_n.$$

Set $C_n = A_n \cap B_n$, and observe that $\Delta(X) \subseteq \bigcup_{n \in \mathbb{N}} C_n \times C_n$, thus $X = \bigcup_{n \in \mathbb{N}} C_n$. As each C_n is $\mathcal{G}$-discrete, it follows that $\chi_B(\mathcal{G}) \leq \aleph_0$. □

Proposition 5.7. *Suppose that X is a Polish space and $\mathcal{G}$ is a directed graph on X such that $\overline{\mathcal{G}} \setminus \mathcal{G} \subseteq \Delta(X)$. Then the following are equivalent:*

1. *$\mathcal{G}$ is potentially closed;*
2. *$\chi_B(\mathcal{G}) \leq \aleph_0$.*

Proof. By Proposition 5.6, it is enough to show (2) $\Rightarrow$ (1). Towards this end, suppose that $\chi_B(\mathcal{G}) \leq \aleph_0$, and fix a sequence $\langle B_n \rangle_{n \in \mathbb{N}}$ of $\mathcal{G}$-discrete Borel sets which cover X, as well as an open basis $\langle U_n \rangle_{n \in \mathbb{N}}$ for X, and set

$$S = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \mathcal{G} \cap (U_m \times U_n) = \emptyset\}.$$

Our assumption that $\overline{\mathcal{G}} \setminus \mathcal{G} \subseteq \Delta(X)$ ensures that if $(x, y) \notin \mathcal{G} \cup \Delta(X)$, then there exists $(m, n) \in S$ such that $(x, y) \in U_m \times U_n$, and it follows that

$$(X \times X) \setminus \mathcal{G} = \bigcup_{n \in \mathbb{N}} B_n \times B_n \cup \bigcup_{(m,n) \in S} U_m \times U_n,$$

thus $\mathcal{G}$ is potentially closed. □

Proposition 5.8 (Lecomte [6]). *Suppose that X and Y are Polish spaces and $R \subseteq X \times Y$ is Borel. Then the following are equivalent:*

1. *R is potentially closed;*
2. *$\chi_B(\mathcal{G}_R) \leq \aleph_0$.*

Proof. To see (1) $\Rightarrow$ (2), suppose that there are Borel sets $A_n \subseteq X$ and $B_n \subseteq Y$ such that $(X \times Y) \setminus R = \bigcup_{n \in \mathbb{N}} A_n \times B_n$, and observe that the function which assigns to each pair $(x, y) \in (X \times Y) \setminus R$ the least n such that $(x, y) \in A_n \times B_n$ is a Borel coloring of $\mathcal{G}_R$.

To see (2) $\Rightarrow$ (1), suppose that $c : (X \times Y) \setminus R \to \mathbb{N}$ is a Borel coloring of $\mathcal{G}_R$, and observe that if $(x, y) \in \mathrm{proj}_X[c^{-1}(n)] \times \mathrm{proj}_Y[c^{-1}(n)]$, then there exists $(x', y') \in X \times Y$ such that $c(x, y') = c(x', y) = n$, thus $(x, y) \notin R$. The first reflection theorem then implies that there are Borel sets $A_n \supseteq \mathrm{proj}_X[c^{-1}(n)]$ and $B_n \supseteq \mathrm{proj}_Y[c^{-1}(n)]$ such that $R \cap (A_n \times B_n) = \emptyset$, and it follows that $(X \times Y) \setminus R = \bigcup_{n \in \mathbb{N}} A_n \times B_n$, thus R is potentially closed. □

Next we have an analog of Proposition 3.8 for rectangular reductions:

Proposition 5.9. *Suppose that X_1, Y_1, X_2, and Y_2 are Polish spaces, $R \subseteq X_1 \times Y_1$ is a non-potentially closed Borel set, and $\varphi_X : X_1 \to X_2$ and $\varphi_Y : Y_1 \to Y_2$ are globally Baire measurable. Then there are Borel sets $A \subseteq X_1$ and $B \subseteq Y_1$ such that $R|(A \times B)$ is not potentially closed and both $\varphi_X|A$ and $\varphi_Y|B$ are Borel.*

Proof. Proposition 5.8 implies that $\chi_B(\mathcal{G}_R) > \aleph_0$. By the directed analog of Theorem 6.3 of Kechris-Solecki-Todorčević [4], there is a Borel homomorphism (π_X, π_Y) from $\mathcal{G}_0^{\rightarrow}$ to $\mathcal{G}_R$. By Proposition 3.3, there are Borel sets $A \subseteq X_1$ and $B \subseteq Y_1$ such that $\pi_X^{-1}(A)$ and $\pi_Y^{-1}(B)$ are comeager and both $\varphi_X|A$ and $\varphi_Y|B$ are Borel. As the set $C = \pi_X^{-1}(A) \cap \pi_Y^{-1}(B)$ is comeager, Proposition 1.1 ensures that $\chi_B(\mathcal{G}_0^{\rightarrow}|C) > \aleph_0$, so $\chi_B(\mathcal{G}_{R|(A\times B)}) > \aleph_0$, thus $R|(A \times B)$ is not potentially closed. $\square$

We also need the following two technical observations:

Proposition 5.10. *Suppose that X and Y are Polish spaces, R is a subset of $X \times X$ such that $\Delta(X) \subseteq \overline{R} \setminus R$, S is a subset of $Y \times Y$ such that $\overline{S} \setminus S \subseteq \Delta(Y)$, and (π_1, π_2) is a continuous rectangular reduction of R to S. Then $\pi_1 = \pi_2$.*

Proof. Simply observe that for each $x \in X$, the point (x, x) is in $\overline{R} \setminus R$, so the point $(\pi_1(x), \pi_2(x))$ is in $\overline{S} \setminus S$, thus $\pi_1(x) = \pi_2(x)$. $\square$

Proposition 5.11. *Suppose that X is a Polish space, $\mathcal{G}$ is an analytic directed graph on X, $A, B \subseteq X$ are analytic sets, $C = A \cap B$, and $\chi_B(\mathcal{G} \cap (C \times C)) \leq \aleph_0$. Then $\chi_B(\mathcal{G} \cap (A \times B)) \leq \aleph_0$.*

Proof. Fix a Borel coloring $c : X \to \mathbb{N}$ of $\mathcal{G} \cap (C \times C)$, and define $d : X \to \mathbb{N} \times \{0, 1\} \times \{0, 1\}$ by

$$d(x) = (c(x), \chi_A(x), \chi_B(x)).$$

Then d is a $\sigma(\mathbf{\Sigma}_1^1)$-measurable coloring of $\mathcal{G} \cap (A \times B)$, thus the directed analogs of the remarks from the first paragraph of §6.C of Kechris-Solecki-Todorčević [4] imply that $\chi_B(\mathcal{G} \cap (A \times B)) \leq \aleph_0$. $\square$

We are now ready to prove a basis theorem for non-potentially closed sets which lie below directed graphs of the form $\mathcal{G}^A$:

Theorem 5.12. *Suppose that X and Y are Polish spaces, $R \subseteq X \times Y$ is Borel, and there is a pair A such that $R \leq_{\mathrm{GB}}^r \mathcal{G}^A$. Then exactly one of the following holds:*

1. *R is potentially closed;*
2. *There is a dense pair B such that $\mathcal{G}^B \sqsubseteq_c^r R$.*

Proof. To see that (1) and (2) are mutually exclusive, observe that if both conditions (1) and (2) hold, then $\mathcal{G}^B$ is potentially closed, which is ruled out by Propositions 1.1 and 5.6. To see $\neg(1) \Rightarrow (2)$, suppose that R is not potentially closed, and fix a globally Baire measurable rectangular reduction (π_X, π_Y) from R to $\mathcal{G}^A$. By Proposition 5.9 and standard change of topology results, we can assume that (π_X, π_Y) is continuous.

By Theorem 9 of Lecomte [6], there are continuous functions $\varphi_X : 2^{\mathbb{N}} \to X$ and $\varphi_Y : 2^{\mathbb{N}} \to Y$ such that

$$\forall (\alpha, \beta) \in \overline{\mathcal{G}_0^{\rightarrow}} \ ((\alpha, \beta) \in \mathcal{G}_0^{\rightarrow} \Leftrightarrow (\varphi_X(\alpha), \varphi_Y(\beta)) \in R).$$

We can actually give a direct proof of this: Proposition 5.8 implies that $\chi_B(\mathcal{G}_R) > \aleph_0$, so the directed analog of Theorem 6.3 of Kechris-Solecki-Todorčević [4] ensures that there is a continuous homomorphism $\varphi = (\varphi_X, \varphi_Y)$ from $\mathcal{G}_0^{\rightarrow}$ to $\mathcal{G}_R$, thus φ_X and φ_Y are as desired.

It follows that $\mathcal{G}^A \cap (\pi_X \circ \varphi_X[2^{\mathbb{N}}] \times \pi_Y \circ \varphi_Y[2^{\mathbb{N}}])$ is not potentially closed. As the sets $\pi_X \circ \varphi_X[2^{\mathbb{N}}]$ and $\pi_Y \circ \varphi_Y[2^{\mathbb{N}}]$ are compact, Proposition 5.7 implies that $\chi_B(\mathcal{G}^A \cap (\pi_X \circ \varphi_X[2^{\mathbb{N}}] \times \pi_Y \circ \varphi_Y[2^{\mathbb{N}}])) > \aleph_0$. Define $C \subseteq 2^{\mathbb{N}}$ by

$$C = \pi_X \circ \varphi_X[2^{\mathbb{N}}] \cap \pi_Y \circ \varphi_Y[2^{\mathbb{N}}],$$

and observe that $\chi_B(\mathcal{G}^A | C) > \aleph_0$, by Proposition 5.11. The Jankov-von Neumann uniformization theorem implies that there are $\sigma(\mathbf{\Sigma}^1_1)$-measurable functions $\psi_X : C \to X$ and $\psi_Y : C \to Y$ such that

$$\forall \alpha \in C \ (\alpha = \pi_X \circ \psi_X(\alpha) = \pi_Y \circ \psi_Y(\alpha)).$$

By Proposition 3.8, there is a Borel set $D \subseteq C$ such that $\chi_B(\mathcal{G}^A | D) > \aleph_0$ and both $\psi_X | D$ and $\psi_Y | D$ are Borel. Fix a Polish topology on D, compatible with the Borel structure that it inherits from $2^{\mathbb{N}}$, in which $\psi_X | D$ and $\psi_Y | D$ are continuous. By Theorem 3.10, there is a strongly dense pair S which admits a monotonic aligned embedding into S^A and for which there is a continuous embedding π of $\mathcal{G}^S$ into $\mathcal{G}^A | D$. By Proposition 5.1, there is a pair B such that $S = S^B$. As S is dense, it follows that so too is B, and $(\psi_X \circ \pi, \psi_Y \circ \pi)$ is a continuous rectangular embedding of $\mathcal{G}^B$ into R. □

As a consequence, we obtain our second basis embedding result:

Theorem 5.13. *The map $A \mapsto \mathcal{G}^A$ is a basis embedding of monotonic aligned embeddability on the set of dense pairs into $(\sqsubseteq_c^r, \leq_{\mathrm{GB}}^r)$.*

Proof. By Proposition 1.9, the map $S \mapsto \mathcal{G}^S$ is a homomorphism from monotonic aligned embeddability into $\sqsubseteq_c^r$, so it only remains to show that if A and B are dense, $R \subseteq X \times Y$ is a non-potentially closed Borel set, and $R \leq_{\mathrm{GB}}^r \mathcal{G}^A, \mathcal{G}^B$, then there is a dense pair D which admits monotonic aligned embeddings into A and B and for which $\mathcal{G}^D \sqsubseteq_c^r R$. Towards this end, fix globally Baire measurable rectangular reductions (π_X^A, π_Y^A) and (π_X^B, π_Y^B) of R into $\mathcal{G}^A$ and $\mathcal{G}^B$. By Proposition 5.9 and standard change of topology results, we can assume that these maps are continuous. By Theorem 5.12, there is a continuous rectangular embedding (φ_X, φ_Y) of $\mathcal{G}^C$ into R, for some dense pair C. Proposition 5.10 then implies that $\pi_X^A \circ \varphi_X = \pi_Y^A \circ \varphi_Y$ and $\pi_X^B \circ \varphi_X = \pi_Y^B \circ \varphi_Y$, thus the maps $\psi^A = \pi_X^A \circ \varphi_X$ and $\psi^B = \pi_X^B \circ \varphi_X$ are continuous reductions of $\mathcal{G}^C$ to $\mathcal{G}^A$ and $\mathcal{G}^B$. Theorem 3.10 now implies that there is a continuous embedding ψ of $\mathcal{G}^S$ into $\mathcal{G}^C$ and monotonic aligned embeddings of S into S^A and S^B, for some strongly dense pair S. Then $(\varphi_X \circ \psi, \varphi_Y \circ \psi)$ is a continuous rectangular embedding of $\mathcal{G}^S$ into R. By Proposition 5.1, there is a pair D such that $S = S^D$. As S is strongly dense, it follows that D is dense and contains sequences of every length, thus Proposition 5.3 implies that there are monotonic aligned embeddings of D into A and B. $\square$

We close this section with some simple properties of basis embeddings:

Proposition 5.14. *Suppose that $\pi : P \to Q$ is a basis embedding of $\leq_P$ into $(\sqsubseteq_Q, \leq_Q)$ and $\preceq$ is a quasi-order which lies between $\sqsubseteq_Q$ and $\leq_Q$.*

1. *If p is $\leq_P$-minimal, then $\pi(p)$ is $\preceq$-minimal.*
2. *If A is a strong $\leq_P$-antichain, then $\pi[A]$ is a strong $\preceq$-antichain.*

Proof. To see (1), suppose that p is $\leq_P$-minimal and $q \preceq \pi(p)$. Then there exists $r \leq_P p$ such that $\pi(r) \sqsubseteq_Q q$. The $\preceq$-minimality of p implies that $p \leq_P r$, and it follows that $\pi(p) \sqsubseteq_Q \pi(r) \sqsubseteq_Q q$, thus $\pi(p) \preceq q$.

To see (2), observe that if A is a strong $\leq_P$-antichain, $p_1, p_2 \in A$, $\pi(p_1) \neq \pi(p_2)$, and $q \leq_Q \pi(p_1), \pi(p_2)$, then there exists $p \leq_P p_1, p_2$, a contradiction. $\square$

6. Homogeneous pairs

For each $x \in \prod_{n \in \mathbb{N}} 2^{n+1} \times \prod_{n \in \mathbb{N}} 2^{n+1}$, define $A^x \in \mathcal{P}(2^{<\mathbb{N}}) \times \mathcal{P}(2^{<\mathbb{N}})$ by

$$(A^x)^i = \{s \in 2^{<\mathbb{N}} : |\text{supp}(s)| \in \text{supp}(x^i(|s|))\}.$$

Proposition 6.1. *Suppose that there is a monotonic aligned embedding of A into A^x. Then there exists y such that $A = A^y$.*

Proof. Suppose that ψ is an order-preserving aligned embedding of A into A^x, define $y \in \prod_{n \in \mathbb{N}} 2^{n+1} \times \prod_{n \in \mathbb{N}} 2^{n+1}$ by

$$[y^i(n)](k) = [x^i(n + k_0^\psi + \cdots + k_n^\psi)](k + |\text{supp}(u_0^\psi)| + \cdots + |\text{supp}(u_n^\psi)|),$$

and observe that if $s \in 2^n$ and $i \in \{0,1\}$, then

$$\begin{aligned} s \in A^i &\Leftrightarrow \psi(s) \in (A^x)^i \\ &\Leftrightarrow |\mathrm{supp}(\psi(s))| \in \mathrm{supp}(x^i(|\psi(s)|)) \\ &\Leftrightarrow |\mathrm{supp}(s)| + |\mathrm{supp}(u_0^\psi)| + \cdots + |\mathrm{supp}(u_n^\psi)| \in \\ &\quad\ \mathrm{supp}(x^i(n + k_0^\psi + \cdots + k_n^\psi)) \\ &\Leftrightarrow |\mathrm{supp}(s)| \in \mathrm{supp}(y^i(|s|)) \\ &\Leftrightarrow s \in (A^y)^i, \end{aligned}$$

thus $A = A^y$. Similarly, if ψ is order-reversing, define

$$[y^i(n)](k) = [x^i(n + k_0^\psi + \cdots + k_n^\psi)]((n-k) + |\mathrm{supp}(u_0^\psi)| + \cdots + |\mathrm{supp}(u_n^\psi)|),$$

and observe that if $s \in 2^n$ and $i \in \{0,1\}$, then

$$\begin{aligned} s \in A^i &\Leftrightarrow \psi(s) \in (A^x)^i \\ &\Leftrightarrow |\mathrm{supp}(\psi(s))| \in \mathrm{supp}(x^i(|\psi(s)|)) \\ &\Leftrightarrow (n - |\mathrm{supp}(s)|) + |\mathrm{supp}(u_0^\psi)| + \cdots + |\mathrm{supp}(u_n^\psi)| \in \\ &\quad\ \mathrm{supp}(x^i(n + k_0^\psi + \cdots + k_n^\psi)) \\ &\Leftrightarrow |\mathrm{supp}(s)| \in \mathrm{supp}(y^i(|s|)) \\ &\Leftrightarrow s \in (A^y)^i, \end{aligned}$$

thus $A = A^y$. $\square$

We say that x is *dense* if $\forall j \in \mathbb{N}\, \exists n \in \mathbb{N}\ (\mathrm{supp}(x^0(n)) \cap [j, n-j]) \neq \emptyset)$. This holds, for example, if $[x^0(n)](\lfloor n/2 \rfloor) = 1$, for infinitely many $n \in \mathbb{N}$.

Proposition 6.2. *x is dense if and only if A^x is dense.*

Proof. To see $(\Rightarrow)$, we must show that if x is dense and $r \in 2^{<\mathbb{N}}$, then there exists $s \in (A^x)^0$ such that $r \subseteq s$. Towards this end, note that the density of x ensures that there exists $n \in \mathbb{N}$ and $k \in \mathrm{supp}(x^0(n)) \cap [|r|, n - |r|]$. Then $k - |\mathrm{supp}(r)| \geq 0$ and $n - |r| - (k - |\mathrm{supp}(r)|) \geq |\mathrm{supp}(r)| \geq 0$, and $k = |\mathrm{supp}(r 1^{k - |\mathrm{supp}(r)|} 0^{n - |r| - (k - |\mathrm{supp}(r)|)})|$ is in $\mathrm{supp}(x^0(n))$, thus $s = r 1^{k - |\mathrm{supp}(r)|} 0^{n - |r| - (k - |\mathrm{supp}(r)|)}$ is in $(A^x)^0$.

To see $(\Leftarrow)$, we must show that if A^x is dense and $j \in \mathbb{N}$, then there exists $n \in \mathbb{N}$ such that $\mathrm{supp}(x^0(n)) \cap [j, n-j] \neq \emptyset$. Fix $s \in (A^x)^0$ such that $0^j 1^j \subseteq s$, put $n = |s|$ and $k = |\mathrm{supp}(s)|$, and observe that $k \in \mathrm{supp}(x^0(n)) \cap [j, n-j]$. $\square$

We say that a function $\xi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ *is aligned* if $\xi_0(n) \leq \xi_1(n)$, $\xi_0(n) \leq \xi_0(n+1)$, and $\xi_0(n+1) - \xi_0(n) < \xi_1(n+1) - \xi_1(n)$, for all $n \in \mathbb{N}$. Such a map is an *order-preserving aligned embedding of x into y* if

$$\forall n \in \mathbb{N}\, \forall i \in \{0,1\}\, \forall k \leq n\ ([x^i(n)](k) = [y^i(\xi_1(n))](\xi_0(n) + k)),$$

and such a map is an *order-reversing aligned embedding of x into y* if

$$\forall n \in \mathbb{N}\, \forall i \in \{0,1\}\, \forall k \leq n\ ([x^i(n)](k) = [y^i(\xi_1(n))](\xi_0(n) + (n-k))).$$

A *monotonic aligned embedding of x into y* is a function $\xi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ which is either an order-preserving aligned embedding of x into y or an order-reversing aligned embedding of x into y.

Proposition 6.3. *There is a monotonic aligned embedding of x into y if and only if there is a monotonic aligned embedding of A^x into A^y.*

Proof. To see ($\Rightarrow$), suppose that $\xi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is an order-preserving aligned embedding of x into y. Set $u_0^\psi = 1^{\xi_0(0)} 0^{\xi_1(0) - \xi_0(0)}$ and

$$u_{n+1}^\psi = 1^{\xi_0(n+1) - \xi_0(n)} 0^{(\xi_1(n+1) - \xi_1(n)) - (\xi_0(n+1) - \xi_0(n)) - 1}.$$

Let ψ denote the corresponding order-preserving aligned function, and note that if $s \in 2^n$ and $i \in \{0, 1\}$, then $|\mathrm{supp}(\psi(s))| = \xi_0(n) + |\mathrm{supp}(s)|$ and $|\psi(s)| = \xi_1(n)$, so

$$\begin{aligned} s \in (A^x)^i &\Leftrightarrow |\mathrm{supp}(s)| \in \mathrm{supp}(x^i(n)) \\ &\Leftrightarrow \xi_0(n) + |\mathrm{supp}(s)| \in \mathrm{supp}(y^i(\xi_1(n))) \\ &\Leftrightarrow |\mathrm{supp}(\psi(s))| \in \mathrm{supp}(y^i(|\psi(s)|)) \\ &\Leftrightarrow \psi(s) \in (A^y)^i, \end{aligned}$$

thus ψ is an aligned embedding of A^x into A^y.

Similarly, if ξ is order-reversing, define u_n^ψ as before, let ψ denote the corresponding order-reversing aligned function, and note that if $s \in 2^n$ and $i \in \{0, 1\}$, then $|\mathrm{supp}(\psi(s))| = \xi_0(n) + (n - |\mathrm{supp}(s)|)$ and $|\psi(s)| = \xi_1(n)$, so

$$\begin{aligned} s \in (A^x)^i &\Leftrightarrow |\mathrm{supp}(s)| \in \mathrm{supp}(x^i(n)) \\ &\Leftrightarrow \xi_0(n) + (n - |\mathrm{supp}(s)|) \in \mathrm{supp}(y^i(\xi_1(n))) \\ &\Leftrightarrow |\mathrm{supp}(\psi(s))| \in \mathrm{supp}(y^i(|\psi(s)|)) \\ &\Leftrightarrow \psi(s) \in (A^y)^i, \end{aligned}$$

thus ψ is an aligned embedding of A^x into A^y.

To see ($\Leftarrow$), suppose that ψ is a monotonic aligned embedding of A^x into A^y, and define an aligned function $\xi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by setting

$$\xi(n) = (|\mathrm{supp}(u_0^\psi)| + \cdots + |\mathrm{supp}(u_n^\psi)|, n + k_0^\psi + \cdots + k_n^\psi).$$

If ψ is order-preserving and $k \leq n$, then

$$\begin{aligned} k \in \mathrm{supp}(x^i(n)) &\Leftrightarrow 1^k 0^{n-k} \in (A^x)^i \\ &\Leftrightarrow \psi(1^k 0^{n-k}) \in (A^y)^i \\ &\Leftrightarrow |\mathrm{supp}(\psi(1^k 0^{n-k}))| \in \mathrm{supp}(y^i(|\psi(1^k 0^{n-k})|)) \\ &\Leftrightarrow \xi_0(n) + k \in \mathrm{supp}(y^i(\xi_1(n))), \end{aligned}$$

thus ξ is an order-preserving aligned embedding of x into y.

Similarly, if ψ is order-reversing and $k \leq n$, then

$$\begin{aligned} k \in \operatorname{supp}(x^i(n)) &\Leftrightarrow 1^k 0^{n-k} \in (A^x)^i \\ &\Leftrightarrow \psi(1^k 0^{n-k}) \in (A^y)^i \\ &\Leftrightarrow |\operatorname{supp}(\psi(1^k 0^{n-k}))| \in \operatorname{supp}(y^i(|\psi(1^k 0^{n-k})|)) \\ &\Leftrightarrow \xi_0(n) + (n-k) \in \operatorname{supp}(y^i(\xi_1(n))), \end{aligned}$$

thus ξ is an order-reversing aligned embedding of x into y. □

Given $s \in 2^{m+1} \times 2^{m+1}$ and $j \leq k \leq m$, we use $s|[j,k]$ to denote the pair in $2^{(k-j)+1} \times 2^{(k-j)+1}$ given by

$$(s|[j,k])^i(l) = s^i(j+l).$$

We say that t *occurs in* s (*at position* j), or $t \sqsubseteq s$, if $t = s|[j,k]$, for some $k \in \mathbb{N}$. We use $x(n)$ as shorthand for the pair $(x^0(n), x^1(n))$. We say that s *occurs in* x, or $s \sqsubseteq x$, if $s \sqsubseteq x(n)$, for some $n \in \mathbb{N}$. We say that s *occurs recurrently in* x if $\forall j \in \mathbb{N}\, \exists n \in \mathbb{N}\ (s \sqsubseteq x(n)|[j, n-j])$.

Proposition 6.4. *Suppose that $x(n)$ occurs recurrently in y, for all $n \in \mathbb{N}$. Then there is an order-preserving aligned embedding of x into y.*

Proof. It is enough to find $\xi(n) \in \mathbb{N} \times \mathbb{N}$, for all $n \in \mathbb{N}$, such that:

1. $\forall m < n\ (\xi_0(m) \leq \xi_0(m+1))$;
2. $\forall m < n\ (\xi_0(m+1) - \xi_0(m) < \xi_1(m+1) - \xi_1(m))$;
3. $x(n)$ occurs in $y(\xi_1(n))$ at position $\xi_0(n)$.

Suppose that we have found $\xi(m)$, for all $m < n$. Fix $j > \max_{m<n} \xi_1(m)$. As $x(n)$ occurs recurrently in y, there exists $\xi_1(n) \in \mathbb{N}$ such that $x(n) \sqsubseteq y(\xi_1(n))|[j, \xi_1(n)-j]$. Fix $\xi_0(n) \in [j, \xi_1(n) - j]$ such that $x(n)$ occurs in $y(\xi_1(n))$ at position $\xi_0(n)$, so that condition (3) holds. To see that conditions (1) and (2) hold, suppose that $m = n-1$ is non-negative, and observe that $\xi_0(m) \leq \xi_1(m) < j \leq \xi_0(m+1)$ and $\xi_0(m+1) - \xi_0(m) \leq \xi_0(m+1) \leq \xi_1(m+1) - j < \xi_1(m+1) - \xi_1(m)$. □

We say that s *is avoidable in* x if for all $j \in \mathbb{N}$ there exist $k, l, n \in \mathbb{N}$ such that $s \not\sqsubseteq x(n)|[k,l]$ and $\operatorname{supp}(x^0(n)) \cap [k+j, l-j] \neq \emptyset$.

Proposition 6.5. *Suppose that s is avoidable in x. Then there is a dense sequence y such that $s \not\sqsubseteq y$ and there is an order-preserving aligned embedding of y into x.*

Proof. It is clearly sufficient to recursively construct $y(n) \in 2^{n+1} \times 2^{n+1}$ and $\xi(n) \in \mathbb{N} \times \mathbb{N}$, for all $n \in \mathbb{N}$, such that:

1. $\forall m < n\ (\xi_0(m) \leq \xi_0(m+1))$;
2. $\forall m < n\ (\xi_0(m+1) - \xi_0(m) < \xi_1(m+1) - \xi_1(m))$;
3. $y(n)$ occurs in $x(\xi_1(n))$ at position $\xi_0(n)$;

4. $[y^0(n)](\lfloor n/2 \rfloor) = 1$;
5. $s \not\sqsubseteq y(n)$.

Suppose that we have found $y(m)$ and $\xi(m)$, for all $m < n$. Fix $j > n + \max_{m<n} \xi_1(m)$. As s is avoidable in x, there exist $k, l, \xi_1(n) \in \mathbb{N}$ such that $s \not\sqsubseteq x(\xi_1(n))|[k, l]$ and $\mathrm{supp}(x^0(\xi_1(n))) \cap [k + j, l - j] \neq \emptyset$. Then there exists $\xi_0(n) \in [k + j - n, l - j]$ such that

$$[x^0(\xi_1(n))](\xi_0(n) + \lfloor n/2 \rfloor) = 1.$$

Set $y(n) = x(\xi_1(n))|[\xi_0(n), \xi_0(n) + n]$, so that conditions (3), (4), and (5) hold. To see that conditions (1) and (2) hold, suppose that $m = n - 1$ is non-negative, and observe that $\xi_0(m) \leq \xi_1(m) < j - n \leq \xi_0(m + 1)$ and $\xi_0(m + 1) - \xi_0(m) \leq \xi_0(m + 1) \leq l - j < \xi_1(m + 1) - \xi_1(m)$. □

Next we connect the notions of recurrent occurrence and unavoidability:

Proposition 6.6. *Suppose that x is dense and s is unavoidable in x. Then s occurs recurrently in x.*

Proof. As s is unavoidable in x, there exists $j' \in \mathbb{N}$ such that

$$\forall k, l, n \in \mathbb{N}\ (\mathrm{supp}(x^0(n)) \cap [k + j', l - j'] \neq \emptyset \Rightarrow s \sqsubseteq x(n)|[k, l]).$$

Given $j \in \mathbb{N}$, set $j'' = j + j'$. The density of x ensures the existence of $n \in \mathbb{N}$ such that $\mathrm{supp}(x^0(n)) \cap [j'', n - j''] \neq \emptyset$. Set $k = j$ and $l = n - j$. Then $j'' = k + j'$ and $n - j'' = l - j'$. As $\mathrm{supp}(x^0(n)) \cap [k + j', l - j'] \neq \emptyset$, it follows that $s \sqsubseteq x(n)|[k, l] = x(n)|[j, n - j]$, thus s occurs recurrently in x. □

These notions are essentially preserved under monotonic aligned embeddings:

Proposition 6.7. *Suppose that there is an order-preserving aligned embedding of y into x.*

1. *If s occurs recurrently in y, then s occurs recurrently in x.*
2. *If s is avoidable in y, then s is avoidable in x.*

Proof. To see (1), suppose that s occurs recurrently in y, and observe that if $j \in \mathbb{N}$, then there exists $n' \in \mathbb{N}$ such that $s \sqsubseteq y(n')|[j, n' - j]$. Then there exists $n \in \mathbb{N}$ such that $y(n') \sqsubseteq x(n)$, so $s \sqsubseteq y(n')|[j, n' - j] \sqsubseteq x(n)|[j, n - j]$, thus s occurs recurrently in x.

To see (2), suppose that s is avoidable in y, and observe that if $j \in \mathbb{N}$, then there exist $k', l', n' \in \mathbb{N}$ such that $s \not\sqsubseteq y(n')|[k', l']$ and $\mathrm{supp}(y^0(n')) \cap [k' + j, l' - j] \neq \emptyset$. Then there exist $k, l, n \in \mathbb{N}$ such that $y(n')|[k', l'] = x(n)|[k, l]$, so $s \not\sqsubseteq x(n)|[k, l]$ and $\mathrm{supp}(x^0(n)) \cap [k + j, l - j] \neq \emptyset$, thus s is avoidable in x. □

For each $s \in 2^n \times 2^n$, define $\tilde{s} \in 2^n \times 2^n$ by $\tilde{s}^i(k) = s^i(n - k - 1)$.

Proposition 6.8. *Suppose that there is an order-reversing aligned embedding of y into x.*

1. *If s occurs recurrently in y, then $\tilde{s}$ occurs recurrently in x.*
2. *If s is avoidable in y, then $\tilde{s}$ is avoidable in x.*

Proof. This follows exactly as in the proof of Proposition 6.7. □

Define $\tilde{x}$ by $[\tilde{x}^i(n)](k) = [x^i(n)](n-k)$.

Proposition 6.9. *Suppose that x is dense. Then the following are equivalent:*

1. *$\forall n \in \mathbb{N}$ ($x(n)$ is unavoidable in x);*
2. *x is a minimal dense pair under monotonic aligned embeddability.*

Proof. To see (1) $\Rightarrow$ (2), suppose that y is a dense pair. If there is an order-preserving aligned embedding of y into x, then Proposition 6.7 implies that each $x(n)$ is unavoidable in y, Proposition 6.6 ensures that each $x(n)$ occurs recurrently in y, and Proposition 6.4 ensures that there is an order-preserving aligned embedding of x into y. Similarly, if there is an order-reversing aligned embedding of y into x, then Proposition 6.8 implies that each $\tilde{x}(n)$ is unavoidable in y, Proposition 6.6 ensures that each $\tilde{x}(n)$ occurs recurrently in y, and Proposition 6.4 ensures that there is an order-preserving aligned embedding of $\tilde{x}$ into y, thus there is an order-reversing aligned embedding of x into y.

To see (2) $\Rightarrow$ (1) suppose, towards a contradiction, that x is a minimal dense pair under monotonic aligned embeddability, and there exists $n \in \mathbb{N}$ such that $x(n)$ is avoidable in x. By Proposition 6.5, there is a dense pair y such that $x(n) \not\sqsubseteq y$ and there is an order-preserving aligned embedding of y into x. As the minimality of x implies that there is a monotonic aligned embedding of x into y, it follows that there is an order-reversing aligned embedding of x into y, since otherwise we would have that $x(n) \sqsubseteq y$, a contradiction. Proposition 6.8 then implies that that $\tilde{x}(n)$ is avoidable in y, thus Proposition 6.5 implies that there is a dense sequence z such that $\tilde{x}(n) \not\sqsubseteq z$ and there is an order-preserving aligned embedding of z into y. The minimality of x implies that there is a monotonic aligned embedding of x into z, and it follows that either $x(n) \sqsubseteq z$ or $\tilde{x}(n) \sqsubseteq z$, the desired contradiction. □

Associated with each pair x is the graph $\mathcal{G}^x = \mathcal{G}^{A^x}$. For each $\alpha \in 2^{\mathbb{N}}$, define $x_\alpha \in \prod_{n\in\mathbb{N}} 2^{n+1} \times \prod_{n\in\mathbb{N}} 2^{n+1}$ by

$$[x^0_\alpha(n)](l) = [x^1_\alpha(n)](l) = \begin{cases} 1 \text{ if } \exists i \in \operatorname{supp}(\alpha)\ (l \equiv 2^{2i} \pmod{2^{2i+1}}), \\ 0 \text{ otherwise.} \end{cases}$$

The graphs of the form $\mathcal{G}^{x_\alpha}$ are essentially the same as those in the pairwise incompatible family of minimal non-potentially closed analytic sets in Lecomte [6]. We will now establish a few simple combinatorial facts which culminate in a new proof of Theorem 6 of Lecomte [6].

Proposition 6.10. *Suppose that $\alpha \neq 0^\infty$. Then x_α is dense.*

Proof. Fix $i \in \mathrm{supp}(\alpha)$. Given $j \in \mathbb{N}$, set $n = 2j + 2^{2i+1}$ and fix $l \in [j, n-j]$ such that $l \equiv 2^{2i} \pmod{2^{2i+1}}$. Then $[x^0_\alpha(n)](l) = 1$, thus $\mathrm{supp}(x^0_\alpha(n)) \cap [j, n-j] \neq \emptyset$. □

Proposition 6.11. *$\forall n \in \mathbb{N}$ ($x_\alpha(n)$ is unavoidable in x_α).*

Proof. Given $n \in \mathbb{N}$, fix $m \in \mathbb{N}$ sufficiently large that $n < 2^{2m}$. Put $j' = 2^{2m}$ and suppose that $k', l', n' \in \mathbb{N}$ and $\mathrm{supp}(x^0(n')) \cap [k'+j', l'-j'] \neq \emptyset$. Then $|l'-k'| \geq 2^{2m+1}$. Fix $j \in [k', k'+2^{2m}]$ such that $j \equiv 0 \pmod{2^{2m}}$. If $l \leq n$, then

$$\begin{aligned}
[x^0_\alpha(n)](l) = 1 &\Leftrightarrow \exists i \in \mathrm{supp}(\alpha)\ (l \equiv 2^{2i} \pmod{2^{2i+1}}) \\
&\Leftrightarrow \exists i \in \mathrm{supp}(\alpha|m)\ (l \equiv 2^{2i} \pmod{2^{2i+1}}) \\
&\Leftrightarrow \exists i \in \mathrm{supp}(\alpha|m)\ (j + l \equiv 2^{2i} \pmod{2^{2i+1}}) \\
&\Leftrightarrow \exists i \in \mathrm{supp}(\alpha)\ (j + l \equiv 2^{2i} \pmod{2^{2i+1}}) \\
&\Leftrightarrow [x^0_\alpha(n')](j+l) = 1,
\end{aligned}$$

so $x_\alpha(n)$ occurs in $x_\alpha(n')$ at position j, thus $x_\alpha(n) \sqsubseteq x_\alpha(n')|[k', l']$. □

As a corollary, we obtain:

Proposition 6.12. *Suppose that $\alpha \neq 0^\infty$. Then x_α is a minimal dense pair under monotonic aligned embeddability.*

Proof. This follows from Propositions 6.9, 6.10, and 6.11. □

By employing a variant of the above arguments, one can show that each x_α is minimal among all pairs (not just the dense ones). We will have no need for this strengthening, however.

Proposition 6.13. *Suppose that $J \subseteq \{0, \ldots, n\}$ is an interval of cardinality 2^{2m}. Then $0 \leq |\mathrm{supp}(x^0_\alpha(n)) \cap J| - \sum_{i \in \mathrm{supp}(\alpha|m)} 2^{2m-2i-1} \leq 1$.*

Proof. Fix $j_0 \in J$ such that $j_0 \equiv 0 \pmod{2^{2m}}$. If $j \in J \setminus \{j_0\}$, then

$$\begin{aligned}
[x^0_\alpha(n)](j) = 1 &\Leftrightarrow \exists i \in \mathrm{supp}(\alpha)\ (j \equiv 2^{2i} \pmod{2^{2i+1}}) \\
&\Leftrightarrow \exists i \in \mathrm{supp}(\alpha|m)\ (j \equiv 2^{2i} \pmod{2^{2i+1}}),
\end{aligned}$$

thus $|\mathrm{supp}(x^0_\alpha(n)) \cap (J \setminus \{j_0\})| = \sum_{i \in \mathrm{supp}(\alpha|m)} 2^{2m-2i-1}$. □

As a corollary, we obtain:

Proposition 6.14. *Suppose that $\alpha \neq \beta$. Then there is no monotonic aligned embedding of x_α into x_β.*

Proof. Fix $m \in \mathbb{N}$ such that $\alpha|m \neq \beta|m$. Then

$$\left| \sum_{i \in \mathrm{supp}(\alpha|m)} 2^{2m-2i-1} - \sum_{i \in \mathrm{supp}(\beta|m)} 2^{2m-2i-1} \right| \geq 2,$$

so Proposition 6.13 implies that neither $x_\alpha(2^{2m})$ nor $\tilde{x}_\alpha(2^{2m})$ occurs in x_β, thus there is no monotonic aligned embedding of x_α into x_β. $\square$

We say that x *is symmetric* if $x^0 = x^1$.

Proposition 6.15. *There is an antichain of continuum-many symmetric minimal dense pairs under monotonic aligned embeddability.*

Proof. Propositions 6.12 and 6.14 imply that $\{x_\alpha\}_{\alpha \neq 0^\infty}$ is as desired. $\square$

Using this, we can now give our new proof of Theorem 6 of Lecomte [6]:

Theorem 6.16. *There is a set of continuum-many non-potentially closed $D_2(\mathbf{\Sigma^0_1})$ subsets of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ which are pairwise incomparable and minimal with respect to every quasi-order $\preceq$ which lies between $\sqsubseteq_c^r$ and $\leq_{\mathrm{GB}}^r$. In particular, every basis for the class of non-potentially closed Borel sets under $\preceq$ has cardinality at least $\mathfrak{c}$.*

Proof. There is an antichain $\mathcal{F} \subseteq \prod_{n \in \mathbb{N}} 2^{n+1} \times \prod_{n \in \mathbb{N}} 2^{n+1}$ of continuum-many minimal dense pairs under monotonic aligned embeddability, by Proposition 6.15. Propositions 6.1, 6.2, and 6.3 imply that $\{A^x : x \in \mathcal{F}\}$ is a pairwise incomparable family of continuum-many minimal dense pairs under monotonic aligned embeddability, and Theorem 5.13 and Proposition 5.14 then imply that the family $\{\mathcal{G}^x : x \in \mathcal{F}\}$ is as desired. $\square$

Along similar lines, we have the following:

Theorem 6.17 (Lecomte [6]). *There is a set of continuum-many $D_2(\mathbf{\Sigma^0_1})$ graphs on $2^{\mathbb{N}}$ of uncountable Borel chromatic number which are pairwise incomparable and minimal with respect to every quasi-order $\preceq$ which lies between $\sqsubseteq_c$ and $\leq_{\mathrm{GB}}$. In particular, every basis for the class of analytic graphs of uncountable Borel chromatic number under $\preceq$ has cardinality at least $\mathfrak{c}$.*

Proof. There is an antichain $\mathcal{F} \subseteq \prod_{n \in \mathbb{N}} 2^{n+1} \times \prod_{n \in \mathbb{N}} 2^{n+1}$ of continuum-many symmetric minimal dense pairs under monotonic aligned embeddability, by Proposition 6.15. Propositions 6.1, 6.2, and 6.3 imply that $\{A^x : x \in \mathcal{F}\}$ is a pairwise incomparable family of continuum-many minimal dense pairs under monotonic aligned embeddability, and Theorem 5.5 and Proposition 5.14 then imply that the family $\{\mathcal{G}^x : x \in \mathcal{F}\}$ is as desired. $\square$

Remark 6.18. Define $x^{\rightarrow}$ by $(x^{\rightarrow})^0 = x^0$ and $(x^{\rightarrow})^1 = \langle 0^{n+1} \rangle_{n \in \mathbb{N}}$. The analog of Theorem 6.17 for oriented graphs can be easily obtained by using $(x_\alpha)^{\rightarrow}$ in place of x_α.

We next turn our attention to a basis theorem:

Proposition 6.19. *Suppose that x is dense. Then there is a monotonic aligned embedding of y into x, for some y which is a minimal dense pair under monotonic aligned embeddability.*

Proof. Fix an enumeration $t_0, t_1, \ldots$ of $\bigcup_{n \in \mathbb{N}} 2^n \times 2^n$ and set $x_0 = x$. Given x_k, if t_k is avoidable in x_k, then Proposition 6.5 ensures that there is a dense sequence x_{k+1} such that $t_k \not\sqsubseteq x_{k+1}$ and there is an order-preserving aligned embedding of x_{k+1} into x_k. If t_k is unavoidable in x_k, then set $x_{k+1} = x_k$.

For each $n \in \mathbb{N}$, fix $k_n \in \mathbb{N}$ such that

$$\forall s \in 2^{n+1} \times 2^{n+1} \ (s \text{ occurs in } x_{k_n} \Leftrightarrow s \text{ is unavoidable in } x_{k_n}),$$

and fix $y(n) \in 2^{n+1} \times 2^{n+1}$ such that $y^0(\lfloor n/2 \rfloor) = 1$ and $y(n) \sqsubseteq x_{k_n}$. It is clear that y is dense.

Lemma 6.20. *There is an order-preserving aligned embedding of y into x_k.*

Proof. For each $n \in \mathbb{N}$, set $l_n = \max(k, k_n)$. As there is an order-preserving aligned embedding of x_{l_n} into x_{k_n}, Proposition 6.7 ensures that $y(n)$ is unavoidable in x_{l_n}. Proposition 6.6 then implies that $y(n)$ occurs recurrently in x_{l_n}. Appealing once more to the fact that there is an order-preserving aligned embedding of x_{l_n} into x_k, Proposition 6.7 implies that $y(n)$ occurs recurrently in x_k, and Proposition 6.4 implies that there is an order-preserving aligned embedding of y into x_k. □

In particular, Lemma 6.20 implies that there is an order-preserving aligned embedding of y into x. It remains to show that y is a minimal dense pair under monotonic aligned embeddability. By Proposition 6.9, it is enough to show that $y(n)$ is unavoidable in y, for each $n \in \mathbb{N}$. As $y(n) \sqsubseteq x_{k_n}$, it follows that $y(n)$ is unavoidable in x_{k_n}, thus $y(n)$ is unavoidable in y, by Proposition 6.7. □

As a corollary, we obtain the following:

Theorem 6.21. *Suppose that X and Y are Polish spaces, $R \subseteq X \times Y$, and there is a pair x such that $R \leq^r_{\mathrm{GB}} \mathcal{G}^x$. Then exactly one of the following holds:*

1. *R is potentially closed;*
2. *There is a dense pair y such that $\mathcal{G}^y \sqsubseteq^r_c R$ and $\mathcal{G}^y$ is minimal with respect to every quasi-order which lies between $\sqsubseteq^r_c$ and $\leq^r_{\mathrm{GB}}$.*

Proof. It is enough to show $\neg(1) \Rightarrow (2)$. By Theorem 5.13, there is a dense pair A which admits a monotonic aligned embedding into A^x and for which $\mathcal{G}^A \sqsubseteq^r_c R$. Proposition 6.1 implies that there exists z such that $A = A^z$, and Proposition 6.2 ensures that z is dense. By Proposition 6.19, there is a monotonic aligned embedding of y into z, for some y which is a minimal dense pair under monotonic aligned embeddability. Then $\mathcal{G}^y \sqsubseteq_c \mathcal{G}^z$, by Propositions 1.9, 5.2, and 6.3, thus $\mathcal{G}^y \sqsubseteq_c^r R$. Propositions 6.1, 6.2, and 6.3 imply that A^y is a minimal dense pair under monotonic aligned embeddability, and Theorem 5.13 and Proposition 5.14 imply that $\mathcal{G}^y$ is minimal with respect to every quasi-order which lies between $\sqsubseteq_c^r$ and $\leq_{\mathrm{GB}}^r$. □

Along similar lines, we have the following:

Theorem 6.22. *Suppose that $\mathcal{G}$ is an analytic directed graph on a Polish space and there is a pair x such that $\mathcal{G} \leq_{\mathrm{GB}} \mathcal{G}^x$. Then exactly one of the following holds:*

1. $\chi_B(\mathcal{G}) \leq \aleph_0$*;*
2. *There is a dense pair y such that $\mathcal{G}^y \sqsubseteq_c \mathcal{G}$ and $\mathcal{G}^y$ is minimal with respect to every quasi-order which lies between $\sqsubseteq_c$ and $\leq_{\mathrm{GB}}$.*

Proof. It is enough to show $\neg(1) \Rightarrow (2)$. By Theorem 5.5, there is a dense pair A which admits a monotonic aligned embedding into A^x and for which $\mathcal{G}^A \sqsubseteq_c \mathcal{G}$. Proposition 6.1 implies that there exists z such that $A = A^z$, and Proposition 6.2 ensures that z is dense. By Proposition 6.19, there is a monotonic aligned embedding of y into z, for some y which is a minimal dense pair under monotonic aligned embeddability. Then $\mathcal{G}^y \sqsubseteq_c \mathcal{G}^z$, by Propositions 1.9, 5.2, and 6.3, thus $\mathcal{G}^y \sqsubseteq_c \mathcal{G}$. Propositions 6.1, 6.2, and 6.3 imply that A^y is a minimal dense pair under monotonic aligned embeddability, and Theorem 5.5 and Proposition 5.14 imply that $\mathcal{G}^y$ is minimal with respect to every quasi-order which lies between $\sqsubseteq_c$ and $\leq_{\mathrm{GB}}$. □

7. The inexistence of antichain bases

Define $A_0 \in \mathcal{P}(2^{<\mathbb{N}}) \times \mathcal{P}(2^{<\mathbb{N}})$ by

$$(A_0)^0 = (A_0)^1 = \{s \in 2^{<\mathbb{N}} : \exists i \in \mathrm{supp}(s)\ (|\mathrm{supp}(s)| \equiv 2^{2i} \pmod{2^{2i+1}})\}.$$

Proposition 7.1. *Suppose that there is an aligned embedding of A into A_0. Then A is dense.*

Proof. Suppose that ψ is an aligned embedding of A into A_0. Given $r \in 2^n$, set $i = |\psi(r)| = n + k_0^\psi + \cdots + k_n^\psi$ and $k = 1 - \alpha^\psi(n)$, and fix $s \in 2^{<\mathbb{N}}$ of length 2^{2i+1} such that $|\mathrm{supp}(\psi(rks))| \equiv 2^{2i} \pmod{2^{2i+1}}$. As $[\psi(rks)](i) = 1$, it follows that $\psi(rks) \in (A_0)^0$, thus $rks \in A^0$. □

Proposition 7.2. *There is at most one monotonic aligned embedding of any pair A into A_0.*

Proof. Suppose that ψ is a monotonic aligned embedding of A into A_0, and let $\sigma = \alpha^\psi(0)$ denote the *orientation of* ψ. For each $j \in \{0,1\}$ and $m, n \in \mathbb{N}$, let S_{jmn} denote the finite set given by

$$S_{jmn} = \{l < m : \sigma^n j \bar{\jmath} 0^l 1^{m-l} \in A^0\}.$$

Through a series of lemmas, we will now show that these sets encode ψ:

Lemma 7.3. *The set $S_{\overline{\sigma}mn} \setminus S_{\sigma mn}$ consists of exactly those $l < m$ such that*

$$|\mathrm{supp}(\psi(\sigma^n\overline{\sigma}\sigma 0^l 1^{m-l}))| \equiv 2^{2(n+k_0^\psi+\cdots+k_n^\psi)} \ (mod\ 2^{2(n+k_0^\psi+\cdots+k_n^\psi)+1}),$$

and similarly, the set $S_{\sigma mn} \setminus S_{\overline{\sigma}mn}$ consists of exactly those $l < m$ such that

$$|\mathrm{supp}(\psi(0^n\sigma\overline{\sigma}0^l 1^{m-l}))| \equiv 2^{2(n+1+k_0^\psi+\cdots+k_{n+1}^\psi)} \ (mod\ 2^{2(n+1+k_0^\psi+\cdots+k_{n+1}^\psi)+1}).$$

Proof. If $l \in S_{\overline{\sigma}mn}$, then there exists $i \in \mathrm{supp}(\psi(\sigma^n\overline{\sigma}\sigma 0^l 1^{m-l}))$ such that

$$|\mathrm{supp}(\psi(\sigma^n\overline{\sigma}\sigma 0^l 1^{m-l}))| \equiv 2^{2i} \pmod{2^{2i+1}}.$$

As $\mathrm{supp}(\psi(\sigma^n\overline{\sigma}\sigma 0^l 1^{m-l})) \setminus \mathrm{supp}(\psi(\sigma^n\sigma\overline{\sigma}0^l 1^{m-l})) = \{n + k_0^\psi + \cdots + k_n^\psi\}$, it follows that $i = n + k_0^\psi + \cdots k_n^\psi$, thus

$$|\mathrm{supp}(\psi(\sigma^n\overline{\sigma}\sigma 0^l 1^{m-l}))| \equiv 2^{2(n+k_0^\psi+\cdots+k_n^\psi)} \pmod{2^{2(n+k_0^\psi+\cdots+k_n^\psi)+1}}.$$

Similarly, if $l \in S_{\sigma mn}$, then there exists $i \in \mathrm{supp}(\psi(\sigma^n\sigma\overline{\sigma}0^l 1^{m-l}))$ such that

$$|\mathrm{supp}(\psi(\sigma^n\sigma\overline{\sigma}0^l 1^{m-l}))| \equiv 2^{2i} \pmod{2^{2i+1}}.$$

As $\mathrm{supp}(\psi(\sigma^n\sigma\overline{\sigma}0^l 1^{m-l})) \setminus \mathrm{supp}(\psi(\sigma^n\overline{\sigma}\sigma 0^l 1^{m-l})) = \{n + 1 + k_0^\psi + \cdots + k_{n+1}^\psi\}$, it follows that $i = n + 1 + k_0^\psi + \cdots k_{n+1}^\psi$, thus

$$|\mathrm{supp}(\psi(\sigma^n\sigma\overline{\sigma}0^l 1^{m-l}))| \equiv 2^{2(n+1+k_0^\psi+\cdots+k_{n+1}^\psi)} \pmod{2^{2(n+1+k_0^\psi+\cdots+k_{n+1}^\psi)+1}},$$

and this completes the proof of the lemma. □

Lemma 7.4. *$\exists n \in \mathbb{N}\,\forall m \geq n\ (|S_{\sigma m0}| < |S_{\overline{\sigma}m0}|)$.*

Proof. Simply observe that if $m \geq 2^{2k_0^\psi+2}$, then the set of $l < m$ such that

$$|\mathrm{supp}(\psi(\overline{\sigma}\sigma 0^l 1^{m-l}))| \equiv 2^{2(1+k_0^\psi+k_1^\psi)} \pmod{2^{2(1+k_0^\psi+k_1^\psi)+1}}$$

is strictly smaller than the set of $l < m$ such that

$$|\mathrm{supp}(\psi(\overline{\sigma}\sigma 0^l 1^{m-l}))| \equiv 2^{2k_0^\psi} \pmod{2^{2k_0^\psi+1}}.$$

Then $|S_{\sigma m0} \setminus S_{\overline{\sigma}m0}| < |S_{\overline{\sigma}m0} \setminus S_{\sigma m0}|$ by Lemma 7.3, so $|S_{\sigma m0}| < |S_{\overline{\sigma}m0}|$. □

Lemma 7.5. *Suppose that $k, n \in \mathbb{N}$. Then the following are equivalent:*

1. *$k \leq n + k_0^\psi + \cdots + k_n^\psi$;*
2. *$\forall m \in \mathbb{N}\,\forall l_1, l_2 \in S_{\overline{\sigma}mn} \setminus S_{\sigma mn}\ (l_1 \equiv l_2\ (mod\ 2^{2k}))$.*

Proof. To see (1) ⇒ (2), suppose that $k \leq n + k_0^\psi + \cdots + k_n^\psi$, and observe that if $m \in \mathbb{N}$ and $l_1, l_2 \in S_{\overline{\sigma}mn} \setminus S_{\sigma mn}$, then Lemma 7.3 implies that

$$|\mathrm{supp}(\psi(\sigma^n\overline{\sigma}\sigma 0^{l_1} 1^{m-l_1})| \equiv |\mathrm{supp}(\psi(\sigma^n\overline{\sigma}\sigma 0^{l_2} 1^{m-l_2})| \pmod{2^{2(n+k_0^\psi+\cdots+k_n^\psi)+1}},$$

so $l_1 \equiv l_2 \pmod{2^{2(n+k_0^\psi+\cdots+k_n^\psi)+1}}$, thus $l_1 \equiv l_2 \pmod{2^{2k}}$.

To see $\neg(1) \Rightarrow \neg(2)$, suppose that $k > n + k_0^\psi + \cdots + k_n^\psi$, set $m = 2^{2(n+k_0^\psi+\cdots+k_n^\psi)+2}$, and fix $l_1, l_2 < m$ such that

$$|\mathrm{supp}(\psi(\sigma^n\overline{\sigma}\sigma 0^{l_1}1^{m-l_1}))| \equiv 2^{2(n+k_0^\psi+\cdots+k_n^\psi)} \pmod{2^{2(n+k_0^\psi+\cdots+k_n^\psi)+2}}$$

and

$$|\mathrm{supp}(\psi(\sigma^n\overline{\sigma}\sigma 0^{l_2}1^{m-l_2}))| \equiv 3 \cdot 2^{2(n+k_0^\psi+\cdots+k_n^\psi)} \pmod{2^{2(n+k_0^\psi+\cdots+k_n^\psi)+2}}.$$

Then Lemma 7.3 implies that $l_1, l_2 \in S_{\overline{\sigma}mn} \setminus S_{\sigma mn}$ and

$$l_1 \not\equiv l_2 \pmod{2^{2(n+k_0^\psi+\cdots+k_n^\psi)+2}},$$

thus $l_1 \not\equiv l_2 \pmod{2^{2k}}$. □

Lemma 7.6. *Suppose that $n \in \mathbb{N}$ and $m = 2^{2(n+k_0^\psi+\cdots+k_n^\psi)}$. Then*

$$0 \leq |S_{0mn}| - \sum_{i\in\mathrm{supp}(u_0^\psi 0\cdots 0u_n^\psi)} m/2^{2i+1} \leq 1.$$

Proof. Fix $l_0 < m$ such that $|\mathrm{supp}(\psi(\sigma^n 010^{l_0}1^{m-l_0}))| \equiv 0 \pmod{m}$. Fix $l < m$ such that $l \neq l_0$, set $t_l = \sigma^n 010^l 1^{m-l}$, and observe that

$$\begin{aligned} t_l \in A^0 &\Leftrightarrow \psi(t_l) \in (A_0)^0 \\ &\Leftrightarrow \exists i \in \mathrm{supp}(\psi(t_l))\ (|\mathrm{supp}(\psi(t_l))| \equiv 2^{2i} \pmod{2^{2i+1}}) \\ &\Leftrightarrow \exists i \in \mathrm{supp}(u_0^\psi 0 \ldots 0u_n^\psi)\ (|\mathrm{supp}(\psi(t_l))| \equiv 2^{2i} \pmod{2^{2i+1}}), \end{aligned}$$

thus $|S_{0mn} \setminus \{l_0\}| = \sum_{i\in\mathrm{supp}(u_0^\psi 0\ldots 0u_n^\psi)} m/2^{2i+1}$. □

Suppose now that ψ' is another monotonic aligned embedding of A into A_0. Lemma 7.4 ensures that ψ and ψ' have the same orientation, so Lemma 7.5 ensures that $n + k_0^\psi + \cdots + k_n^\psi = n + k_0^{\psi'} + \cdots + k_n^{\psi'}$, for all $n \in \mathbb{N}$, thus $k_n^\psi = k_n^{\psi'}$, for all $n \in \mathbb{N}$. It only remains to observe that if $u_n^\psi \neq u_n^{\psi'}$ and $m = 2^{2(n+k_0^\psi+\cdots+k_n^\psi)}$, then

$$\left| \sum_{i\in\mathrm{supp}(u_0^\psi 0\ldots 0u_n^\psi)} m/2^{2i+1} - \sum_{i\in\mathrm{supp}(u_0^{\psi'} 0\ldots 0u_n^{\psi'})} m/2^{2i+1} \right| \geq 2,$$

which contradicts Lemma 7.6. □

Let P denote the set of all triples (S, f, i), where $S \subseteq \mathbb{N}$ is co-infinite, $f : S \to \{0, 1\}$, and $i \in \{0, 1\}$. Define $\leq$ on P by

$$(S, f, i) \leq (T, g, j) \Leftrightarrow T \subseteq S \text{ and } g = f|T.$$

Proposition 7.7. *The restriction of monotonic aligned embeddability below A_0 is isomorphic to $(P, \leq)$.*

Proof. Given $(S,f,i) \in P$, let $k_0^S, k_1^S, \ldots$ denote the increasing enumeration of $\mathbb{N} \setminus S$, let $\psi_{(S,f,i)}$ denote the nicely aligned function given by $u_0^f = f|k_0^S$, $u_{n+1}^f = f|(k_n^S, k_{n+1}^S)$, and $\alpha^i = i^\infty$, and define $A_{(S,f,i)}$ by

$$(A_{(S,f,i)})^0 = (A_{(S,f,i)})^1 = \psi_{(S,f,i)}^{-1}((A_0)^0).$$

Suppose now that $(S,f,i) \leq (T,g,j)$, let $i_0, i_1, \ldots$ denote the increasing enumeration of the indices i such that $k_i^T \in \mathbb{N} \setminus S$, and let ψ denote the nicely aligned function given by $u_0^\psi = |f(k_0^T) - j| \ldots |f(k_{i_0-1}^T) - j|$, $u_{n+1}^\psi = |f(k_{i_n+1}^T) - j| \ldots |f(k_{i_{n+1}-1}^T) - j|$, and $\alpha^\psi = |i-j|^\infty$. If $s \in 2^n$, then

$$\begin{aligned}\psi(s) = {} & |f(k_0^T) - j| \ldots |f(k_{i_0-1}^T) - j||s(0) - |i-j|| \ldots \\ & |s(n-1) - |i-j|||f(k_{i_{n-1}+1}^T) - j| \ldots |f(k_{i_n-1}^T) - j|.\end{aligned}$$

As $|k - |i-j|| = ||k-i| - j|$, it follows that

$$\begin{aligned}\psi_{(T,g,j)} \circ \psi(s) &= (g|k_0^T) f(k_0^T) \ldots f(k_{i_0-1}^T)(g|(k_{i_0-1}^T, k_{i_0}^T))|s(0) - i| \ldots \\ &\quad |s(n-1) - i|(g|(k_{i_{n-1}}^T, k_{i_{n-1}+1}^T)) f(k_{i_{n-1}+1}^T) \ldots \\ &\quad f(k_{i_n-1}^T)(g|(k_{i_n-1}^T, k_{i_n}^T)) \\ &= (f|k_{i_0}^T)|s(0) - i| \ldots |s(n-1) - i|(f|(k_{i_{n-1}}^T, k_{i_n}^T)) \\ &= \psi_{(S,f,i)}(s),\end{aligned}$$

and this implies that

$$\begin{aligned}s \in (A_{(S,f,i)})^0 &\Leftrightarrow \psi_{(S,f,i)}(s) \in (A_0)^0 \\ &\Leftrightarrow \psi_{(T,g,j)} \circ \psi(s) \in (A_0)^0 \\ &\Leftrightarrow \psi(s) \in (A_{(T,g,j)})^0,\end{aligned}$$

thus ψ is a monotonic aligned embedding of $A_{(S,f,i)}$ into $A_{(T,g,j)}$.

Conversely, if ψ is a monotonic aligned embedding of $A_{(S,f,i)}$ into $A_{(T,g,j)}$, then Proposition 7.2 ensures that $\psi_{(S,f,i)} = \psi_{(T,g,j)} \circ \psi$. Suppose that $n \in T$. If $n \in T \setminus S$, then there exists $m \in \mathbb{N}$ such that $n = k_m^S$, in which case $\psi_{(S,f,i)}(0^m 0)$ and $\psi_{(S,f,i)}(0^m 1)$ differ on their n^{th} coordinate, but $\psi_{(T,g,j)} \circ \psi(0^m 0)$ and $\psi_{(T,g,j)} \circ \psi(0^m 1)$ agree on their n^{th} coordinate, a contradiction. Then $n \in S$, and since $\psi_{(S,f,i)}(0^n) = \psi_{(T,g,j)} \circ \psi(0^n)$, it follows that $f(n) = g(n)$, thus $(S,f,i) \leq (T,g,j)$. □

In particular, it follows that the restriction of monotonic aligned embeddability below A_0 has the same properties as $(P, \leq_P)$. For example, it is homogeneous, in the sense that it is isomorphic to its restriction to any initial segment. More central to our concerns here is the following:

Proposition 7.8. *Suppose that there is a monotonic aligned embedding of A into A_0. Then there is a strong antichain of continuum-many pairs which admit monotonic aligned embeddings into A.*

Proof. By Proposition 7.7, it is enough to prove the corresponding fact for $(P, \leq)$. Towards this end, suppose that $(S,f,i) \in P$, fix an infinite, co-infinite set $T \subseteq \mathbb{N} \setminus S$, and observe that the set of tuples of the form $(S \cup T, g, i)$, where $g : S \cup T \to \{0, 1\}$ and $g|S = f$, is as desired. $\square$

Remark 7.9. It is worth noting that there is a more direct proof of Proposition 7.8 which does not rely upon Proposition 7.7. We have included the above proof here in the hope that the graph $\mathcal{G}^{A_0}$ becomes useful in future work, in which case a knowledge of the restriction of monotonic aligned embeddability below A_0 could well become important.

At long last, we are now ready to prove our main results:

Theorem 7.10. *Suppose that $R \leq^r_{\mathrm{GB}} \mathcal{G}^{A_0}$ is Borel. Then exactly one of the following holds:*

1. *R is potentially closed;*
2. *There is a strong $\leq^r_{\mathrm{GB}}$-antichain of continuum-many $D_2(\mathbf{\Sigma}^0_1)$ subsets of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ which are $\sqsubseteq^r_c$-below R.*

In particular, if $\preceq$ is a quasi-order which lies between $\sqsubseteq^r_c$ and $\leq^r_{\mathrm{GB}}$, then no weak $\preceq$-antichain is a basis for the class of non-potentially closed Borel sets under $\preceq$, thus the class of $\preceq$-minimal sets is not a basis for the class of non-potentially closed Borel sets under $\preceq$.

Proof. It is enough to show $\neg(1) \Rightarrow (2)$. By Theorem 5.13, there is a dense pair A which admits a monotonic aligned embedding into A_0 and for which $\mathcal{G}^A \sqsubseteq^r_c R$. By Proposition 7.8, there is a strong antichain of continuum-many dense sets A_α which admit monotonic aligned embeddings into A. Set $R_\alpha = \mathcal{G}^{A_\alpha}$. Propositions 1.9 and 5.2 imply that $R_\alpha \sqsubseteq_c \mathcal{G}^A$, thus $R_\alpha \sqsubseteq^r_c R$. Propositions 1.1 and 5.6 imply that R_α is not potentially closed, and Theorem 5.13 and Proposition 5.14 imply that these sets form a strong $\leq^r_{\mathrm{GB}}$-antichain. $\square$

Along similar lines, we have the following:

Theorem 7.11. *Suppose that $\mathcal{G} \leq_{\mathrm{GB}} \mathcal{G}^{A_0}$ is analytic. Then exactly one of the following holds:*

1. *$\chi_B(\mathcal{G}) \leq \aleph_0$;*
2. *There is a strong $\leq_{\mathrm{GB}}$-antichain of continuum-many $D_2(\mathbf{\Sigma}^0_1)$ graphs on $2^{\mathbb{N}}$ which are $\sqsubseteq_c$-below $\mathcal{G}$.*

In particular, if $\preceq$ is a quasi-order which lies between $\sqsubseteq_c$ and $\leq_{\mathrm{GB}}$, then no weak $\preceq$-antichain is a basis for the class of analytic graphs of uncountable Borel chromatic number under $\preceq$, thus the class of $\preceq$-minimal graphs is not a basis for the class of analytic graphs of uncountable Borel chromatic number under $\preceq$.

Proof. It is enough to show $\neg(1) \Rightarrow (2)$. By Theorem 5.5, there is a dense pair A which admits a monotonic aligned embedding into A_0 and for which $\mathcal{G}^A \sqsubseteq_c \mathcal{G}$.

By Proposition 7.8, there is a strong antichain of continuum-many dense sets A_α which admit monotonic aligned embeddings into A. Set $\mathcal{G}_\alpha = \mathcal{G}^{A_\alpha}$. Propositions 1.9 and 5.2 imply that $\mathcal{G}_\alpha \sqsubseteq_c \mathcal{G}^A$, thus $\mathcal{G}_\alpha \sqsubseteq_c \mathcal{G}$. Proposition 1.1 implies that $\chi_B(\mathcal{G}_\alpha) > \aleph_0$, and Theorem 5.5 and Proposition 5.14 imply that these graphs form a strong $\leq_{\mathrm{GB}}$-antichain. □

Remark 7.12. Define $A^{\rightarrow}$ by $(A^{\rightarrow})^0 = A^0$ and $(A^{\rightarrow})^1 = \emptyset$. The analog of Theorem 7.11 for oriented graphs can be easily obtained by using $(A_0)^{\rightarrow}$ in place of A_0.

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