Complexity of Borel sets in product spaces

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These talks will be in the context of Descriptive Set Theory, where we study the topological complexity of definable subsets of Polish spaces, i.e., of separable and completely metrizable spaces, even if some people now consider the non separable case. Actual products of Polish spaces will be of particular interest for us, and we will denote them $\Pi_{i\in d}\ X_i$, where the number of factors d is any $dimension$ making sense in the context of Descriptive Set Theory. In particular, we will have $2\leq d\leq\omega$ since for example 2^{ω_1} is not metrizable. We are looking for dichotomy results of the following form, quite standard in Descriptive Set Theory: a set is either simple, or more complicated than a typical example. The theory in product spaces is strongly related to the theory in dimension one, even if it is much more complicated. We will come back to this, but we first recall some "dimension one" results.

1 Some results in dimension one

The most classical hierarchy of topological complexity is the $Borel\ hierarchy$:

$$\begin{array}{lllll} & \mathbf{\Sigma}^0_1=\text{open sets} \quad \mathbf{\Sigma}^0_2=F_\sigma \text{ sets} \quad \ldots \quad \mathbf{\Sigma}^0_\omega=(\bigcup_{n<\omega}\mathbf{\Pi}^0_n)_\sigma \quad \ldots & & & \\ \mathbf{\Delta}^0_1=\text{clopen sets} & \mathbf{\Delta}^0_2=\mathbf{\Sigma}^0_2\cap\mathbf{\Pi}^0_2 \quad \ldots & \mathbf{\Delta}^0_\omega=\mathbf{\Sigma}^0_\omega\cap\mathbf{\Pi}^0_\omega \quad \ldots & & \mathbf{\Delta}^1_1=\text{Borel sets} \\ & \mathbf{\Pi}^0_1=\text{closed sets} \quad \mathbf{\Pi}^0_2=G_\delta \text{ sets} \quad \ldots \quad \mathbf{\Pi}^0_\omega=\check{\mathbf{\Sigma}}^0_\omega \quad \ldots & & & \end{array}$$

A classical instance of the process we described is Hurewicz's Theorem, stating that a Borel subset B of a Polish space is not $\mathbf{\Pi}^0_2$ exactly when there is a copy of the Cantor space on which B is homeomorphic to the rationals. This result has been generalized to all Baire classes. We state this generalization in two parts:

Part 1. The reduction theorem (see [Lo-SR1]):

Theorem 1.1 *(Louveau-Saint Raymond) Let $1\leq\xi<\omega_1$, $C\in\mathbf{\Sigma}^0_\xi(2^\omega)$, X a Polish space, and A_0, A_1 disjoint analytic subsets of X. Then one of the following holds:*

(a) The set A_0 is separable from A_1 by a $\mathbf{\Pi}^0_\xi$ set.

(b) There is $u:2^\omega\to X$ continuous with $C\subseteq u^{-1}(A_0)$ and $2^\omega\setminus C\subseteq u^{-1}(A_1)$ (if $\xi\geq 3$, then we can have u one-to-one).

If moreover we assume that C is not separable from $2^\omega\setminus C$ by a $\mathbf{\Pi}^0_\xi$ set, then this is a dichotomy.

Part 2. There are some typical examples (see 22.4 in [K]):

Theorem 1.2 *There is a concrete example of a set $C_\xi\in\mathbf{\Sigma}^0_\xi(2^\omega)$ which is not separable from $2^\omega\setminus C_\xi$ by a $\mathbf{\Pi}^0_\xi$ set, for each $1\leq\xi<\omega_1$.*

A consequence of these two results is the following result:

Corollary 1.3 *Let $1\leq\xi<\omega_1$. Then there is a Borel subset C_ξ of 2^ω such that for any Polish space X, for any Borel subset A of X, exactly one of the following holds:*

(a) The set A is $\mathbf{\Pi}^0_\xi$-A is simple.

(b) There is $u:2^\omega\to X$ continuous with $C_\xi=u^{-1}(A)$-A is more complicated than the typical example C_ξ.

So the scheme of comparison is as follows:

$$\begin{array}{|r|ccc|r|} \hline C_\xi & & \dashrightarrow & & A \\ \hline 2^\omega\setminus C_\xi & & \dashrightarrow & & X\setminus A \\ \hline \end{array}$$

The notion of comparison is the $Wadge\ reduction$ (see 21.13 in [K]): we set, for $A\subseteq X$ and $B\subseteq Y$,

$$A\leq_W B\ \Leftrightarrow\ \exists u\!:\!X\to Y \text{ continuous with } A=u^{-1}(B).$$

This notion essentially makes sense in $zero\text{-}dimensional$ spaces, i.e., in spaces having a basis of the topology made of clopen sets, to ensure the existence of enough continuous functions. This leads to the following definition:

Definition 1.4 *Let $\mathbf{\Gamma}$ be a class of subsets of zero-dimensional Polish spaces. We say that $\mathbf{\Gamma}$ is a $Wadge$ $class$ if there is a zero-dimensional Polish space X_0, and a Borel subset A_0 of X_0 such that*

$$\mathbf{\Gamma}=\{u^{-1}(A_0)\mid u\ \textit{ continuous}\}.$$

One can prove that the classes $\mathbf{\Sigma}^0_\xi$, $\mathbf{\Pi}^0_\xi$, and also $\mathbf{\Delta}^0_1$, are Wadge classes. So the Wadge hierarchy, given by the inclusion of classes, essentially extends the Borel hierarchy. A. Louveau and J. Saint Raymond extended the previous results to Wadge classes (see [Lo-SR2]). Let us give some examples of Wadge classes that are not Borel classes.

Examples. The difference hierarchy (see 22.E in [K]).

Let $\eta<\omega_1$, and $(A_\theta)_{\theta<\eta}$ a non-decreasing sequence of subsets of some set. Then

$$D_\eta\big((A_\theta)_{\theta<\eta}\big):=\bigcup_{\theta<\eta,\ \text{parity}(\theta)\neq\text{parity}(\eta)} A_\theta\setminus(\bigcup_{\zeta<\theta}A_\zeta).$$

Then $D_\eta(\mathbf{\Sigma}^0_\xi):=\big\{D_\eta\big((A_\theta)_{\theta<\eta}\big)\mid(A_\theta)_{\theta<\eta}$ non decreasing, $A_\theta\in\mathbf{\Sigma}^0_\xi\big\}$ is a Wadge class. For example, $D_2(\mathbf{\Sigma}^0_\xi)$ is the class of the differences of two $\mathbf{\Sigma}^0_\xi$ sets, and the class of its complements is

$$\check D_2(\mathbf{\Sigma}^0_\xi):=\{A\cup B\mid A\in\mathbf{\Sigma}^0_\xi\ \text{ and }\ B\in\mathbf{\Pi}^0_\xi\}.$$

The difference hierarchy refines the Borel hierarchy as follows:

$$\begin{array}{cccccc} \mathbf{\Sigma}^0_\xi=D_1(\mathbf{\Sigma}^0_\xi) & D_2(\mathbf{\Sigma}^0_\xi) & \ldots & D_\omega(\mathbf{\Sigma}^0_\xi) & \ldots & \\ & & & & & \mathbf{\Delta}^0_{\xi+1} \\ \mathbf{\Pi}^0_\xi=\check D_1(\mathbf{\Sigma}^0_\xi) & \check D_2(\mathbf{\Sigma}^0_\xi) & \ldots & \check D_\omega(\mathbf{\Sigma}^0_\xi) & \ldots & \end{array}$$

The class $\oplus_{\xi<\lambda}\ \mathbf{\Pi}^0_\xi$ (for λ limit) is the class of sets of the form $\bigcup_n\ A_n$, where $A_n\in\bigcup_{\xi<\lambda}\ \mathbf{\Pi}^0_\xi$ and there is a partition (X_n) of the space into $\mathbf{\Delta}^0_1$ sets with $A_n\subseteq X_n$.

2 Complexity and comparison in products

Our goal is to extend 1.1-1.3 to higher dimensions. In the second talk, we will extend Theorem 1.2. In the third talk, we will extend Theorem 1.1 and Corollary 1.3. In this talk, we will give some basic definitions and results, and we will see that the situation in products is much more complicated than the one in dimension one. But we first have to give some motivation, and to define a notion of complexity and a notion of comparison in products. The last two things are actually very much related. The motivation comes from the theory of Borel (or analytic) equivalence relations, widely studied during the last decades.

The usual notion of comparison between two Borel equivalence relations $E \subseteq X^2$ and $F \subseteq Y^2$ on Polish spaces is the $Borel\ reducibility\ quasi\text{-}order$ (recall that a quasi-order is a reflexive and transitive relation):

$$E \leq_B F \Leftrightarrow \exists u : X \rightarrow Y \text{ Borel with } E = (u \times u)^{-1}(F)$$

This means that X/E embeds into Y/F in a "Borel" way. The scheme is the same as before. Note that this makes sense even if E, F are not equivalence relations. A. Louveau studied this quasi-order on other structures than equivalence relations (partial orders, quasi-orders). The notion of complexity that we will consider is a natural invariant for $\leq_B$. To introduce it, recall the following fact (see 13.5 in [K]):

Theorem 2.1 *(Kuratowski) Let X be a Polish space, and (B_n) a sequence of Borel subsets of X. Then there is a finer zero-dimensional Polish topology on X (and thus having the same Borel sets) making the B_n's clopen.*

Assume that u is a witness for $E \leq_B F$, and let σ be a finer zero-dimensional Polish topology on X making u continuous. If F is in some Wadge class $\mathbf{\Gamma}$, then $E \in \mathbf{\Gamma}\big((X, \sigma)^2\big)$. This motivates the following definition (see [Lo]):

Definition 2.2 *(Louveau) Let $(X_i)_{i \in d}$ be a sequence of Polish spaces, A a Borel subset of $\Pi_{i \in d}\ X_i$, and $\mathbf{\Gamma}$ a Borel class or a Wadge class. We say that A is $potentially\ in$ $\mathbf{\Gamma}$ $\big($denoted $A \in pot(\mathbf{\Gamma})\big)$ iff for each $i \in d$ there is a finer zero-dimensional Polish topology σ_i on X_i such that $A \in \mathbf{\Gamma}\big(\Pi_{i \in d}\ (X_i, \sigma_i)\big)$.*

This is an invariant for $\leq_B$: if F is pot($\mathbf{\Gamma}$) and $E \leq_B F$, then E is pot($\mathbf{\Gamma}$) too. This is the notion of complexity in products that we will study. Note that this notion depends only on the Borel structure of the X_i's, and not on their topology. Theorem 2.1 shows that any Borel subset of a Polish space can be made clopen by refining the Polish topology. This is not the case with potential complexity (see [L1]):

Theorem 2.3 *Let $\mathbf{\Gamma}$ be a Wadge class. Then there is $A \in \mathbf{\Gamma}\big((\omega^\omega)^2\big)$ such that $A \notin pot(\mathbf{\Gamma}')$ if $\mathbf{\Gamma}' \subset_{\neq} \mathbf{\Gamma}$ is a Wadge class.*

For example, if $\mathbf{\Gamma}$ is not self-dual, then there is $A \in \mathbf{\Gamma}\big((\omega^\omega)^2\big)$ which is not pot($\check{\mathbf{\Gamma}}$) (take a universal set). The simplest example is the diagonal $\Delta(2^\omega) := \{(\alpha, \alpha) \mid \alpha \in 2^\omega\}$, which is closed, but not potentially open since it is not a countable union of Borel rectangles. So this notion of complexity makes sense for product topologies. Using this notion, A. Louveau proved that the collection of $\mathbf{\Sigma}^0_\xi$ equivalence relations is not cofinal for $\leq_B$, and deduces from this the non existence of a maximum Borel equivalence relation for $\leq_B$. For equivalence relations again, we have the following result (see [H-K-Lo]):

Theorem 2.4 *(Harrington-Kechris-Louveau) Let X be a Polish space, E a Borel equivalence relation on X, and $E_0 := \{(\alpha, \beta) \in (2^\omega)^2 \mid \exists n \in \omega\ \ \forall m \geq n\ \ \alpha(m) = \beta(m)\}$. Then exactly one of the following holds:*

(a) The relation E is pot($\mathbf{\Pi}^0_1$).

(b) $E_0 \leq_B E$ (with u continuous and one-to-one).

The following result is proved in [Hj-K-Lo]:

Theorem 2.5 *(Hjorth-Kechris-Louveau) The potential Wadge classes of Borel equivalence relations induced by Borel actions of closed subgroups of the symmetric group are the following: $\mathbf{\Delta}^0_1$, $\mathbf{\Pi}^0_1$, $\mathbf{\Sigma}^0_2$, $\mathbf{\Pi}^0_n$, $D_2(\mathbf{\Pi}^0_n)$ ($n \geq 3$), $\oplus_{\xi < \lambda}\ \mathbf{\Pi}^0_\xi$, $\mathbf{\Pi}^0_\lambda$, $\mathbf{\Sigma}^0_{\lambda+1}$, $\mathbf{\Pi}^0_{\lambda+n}$, $D_2(\mathbf{\Pi}^0_{\lambda+n})$ (λ limit, $n \geq 2$).*

We want to extend Theorem 2.4 to arbitrary Borel sets in products. To do this, we introduce the rectangular version of $\leq_B$: if $A \subseteq X_0 \times X_1$ and $B \subseteq Y_0 \times Y_1$, then

$$A \leq^r_B B \Leftrightarrow \forall i \in 2\ \ \exists u_i : X_i \rightarrow Y_i \text{ Borel with } A = (u_0 \times u_1)^{-1}(B).$$

The following result is proved in [L1]:

Theorem 2.6 *Let $L_0 := \{(\alpha,\beta)\in(2^\omega)^2 \mid \alpha <_{lex} \beta\}$, Y_0, Y_1 be Polish spaces, and B a pot$\big(\check D_2(\mathbf{\Sigma}^0_1)\big)$ subset of $Y_0\times Y_1$. Then exactly one of the following holds:*

(a) The set A is pot$(\mathbf{\Pi}^0_1)$.

(b) $\neg\Delta(2^\omega)\leq^r_B B$ or $L_0\leq^r_B B$ (with u_i continuous and one-to-one).

The class $\check D_2(\mathbf{\Sigma}^0_1)$ is the limit level: things become much more complicated at the dual level $D_2(\mathbf{\Sigma}^0_1)$ (see [L5]):

Theorem 2.7 *(a) There is a perfect $\leq^r_B$-antichain $(A_\alpha)_{\alpha\in 2^\omega}\subseteq D_2(\mathbf{\Sigma}^0_1)\big((2^\omega)^2\big)$ such that for any $\alpha\in 2^\omega$, A_α is $\leq^r_B$-minimal among $\mathbf{\Delta}^1_1\backslash$pot$(\mathbf{\Pi}^0_1)$ sets.*

(b) There is a perfect $\leq_B$-antichain made of sets $\leq_B$-minimal among $\Gamma\cap\mathbf{\Delta}^1_1\setminus pot(\mathbf{\Pi}^0_1)$, where Γ is one of the following classes:

- Graphs (i.e., irreflexive and symmetric relations).

- Oriented graphs (i.e., irreflexive and antisymmetric relations).

- Quasi-orders (i.e., reflexive and transitive relations).

- Partial orders (i.e., reflexive, antisymmetric and transitive relations).

In other words, the case of equivalence relations, for which we have Theorem 2.4, is very specific. This theorem says, among other things, that the mixture between symmetry and transitivity is very strong. Theorem 2.7 shows that the classical notions of reduction (on the whole product) do not work, at least at the first level. So we must find another notion of comparison.

3 Some typical examples in products

We just saw that a reduction on the whole product is not possible. The idea to get the good notion of comparison is to keep product functions, but the reduction will hold only on a subset of the product. For the non self-dual Borel classes, the reduction will hold on a closed set independent of the classes, that can be seen as the set of branches of some tree. We now describe the properties of the tree ensuring the possibility of the reduction. This tree has to be small enough since we cannot have a reduction on the whole product. But as the same time it has to be big enough to ensure the existence of complicated sets in the set of its branches. We first describe the notions of smallness. The following definition is basic (see 2.1 in [K]):

Definition 3.1 *Let S be a set. A $tree$ on S is a set T of finite sequences of elements of S such that*

$$\forall s,t\in S^{<\omega}\ \ (s\subseteq t\ \text{ and }\ t\in T)\Rightarrow s\in T.$$

A $branch$ of T is an infinite sequence $\alpha\in S^\omega$ such that $\alpha|n\in T$ for each integer n. The set of branches of T is denoted $\lceil T\rceil$.

Notation. If $\mathcal{X}$ is a set, then $\vec{x}:=(x_i)_{i\in d}$ is an arbitrary element of $\mathcal{X}^d$. If $\mathcal{T}\subseteq\mathcal{X}^d$, then we denote by $G^{\mathcal{T}}$ the graph with set of vertices $\mathcal{T}$, and with set of edges

$$\Big\{\{\vec{x},\vec{y}\}\subseteq\mathcal{T}\mid\vec{x}\neq\vec{y}\ \text{ and }\ \exists i\in d\ \ x_i=y_i\Big\}.$$

Definition 3.2 *(a) We say that $\mathcal{T}$ is $one\text{-}sided$ if the following holds:*

$$\forall\vec{x},\vec{y}\in\mathcal{T}\ \ \forall i,j\in d\ \ \big(\ (\vec{x}\neq\vec{y}\ \text{ and }\ x_i=y_i\ \text{ and }\ x_j=y_j)\ \Rightarrow\ i=j\ \big).$$

(b) We say that $\mathcal{T}$ is $almost\ acyclic$ if for every $G^{\mathcal{T}}$-cycle $(\overrightarrow{x^n})_{n\leq L}$ there are $i\in d$ and $k<m<n<L$ such that $x^k_i=x^m_i=x^n_i$.

(c) We say that a tree T on d^d is a $tree\ with\ finite\ one\text{-}sided\ almost\ acyclic\ levels$ if, for each integer l, the set $\mathcal{T}^l:=T\cap(d^d)^l\subseteq(d^d)^l\equiv(d^l)^d$ is finite, one-sided and almost acyclic.

Now we can state the generalization of Theorem 1.2:

Theorem 3.3 *There are concrete examples of a tree T_d with finite one-sided almost acyclic levels, together with, for each non self-dual Wadge class $\mathbf{\Gamma}$ and for each $1\leq\xi<\omega_1$,*
(1) Some set $\mathbb{S}^d_{\mathbf{\Gamma}}\in\mathbf{\Gamma}(\lceil T_d\rceil)$ which is not separable from $\lceil T_d\rceil\setminus\mathbb{S}^d_{\mathbf{\Gamma}}$ by a pot($\check{\mathbf{\Gamma}}$) set.
(2) Some disjoint sets $\mathbb{S}^0_\xi,\mathbb{S}^1_\xi\in\mathbf{\Sigma}^0_\xi(\lceil T_d\rceil)$ such that $\mathbb{S}^0_\xi$ is not separable from $\mathbb{S}^1_\xi$ by a pot($\mathbf{\Delta}^0_\xi$) set.

Notation. Let $\varphi:\omega\rightarrow\omega^2$ be the natural bijection. More precisely, we set, for $q\in\omega$,

$$M(q):=\text{max}\{m\in\omega/\Sigma_{k\leq m}\ k\leq q\}.$$

Then we define $\varphi(q)=\big((q)_0,(q)_1\big):=\big(M(q)-q+(\Sigma_{k\leq M(q)}\ k),q-(\Sigma_{k\leq M(q)}\ k)\big)$. One can check that $<i,j>:=\varphi^{-1}(i,j)=(\Sigma_{k\leq i+j}\ k)+j$. More concretely, we get

$$\varphi[\omega]=\{(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\ldots\}.$$

Definition 3.4 *We say that $E\subseteq\bigcup_{l\in\omega}\ (d^l)^d$ is an $effective\ frame$ if*
(a) $\forall l\in\omega\ \exists!(s^i_l)_{i\in d}\in E\cap(d^l)^d$.
(b) $\forall p,q,r\in\omega\ \forall t\in d^{<\omega}\ \exists N\in\omega\ (s^i_qit0^N)_{i\in d}\in E$, $(|s^0_q0t0^N|-1)_0=p$ and $\big((|s^0_q0t0^N|-1)_1\big)_0=r$.
(c) $\forall l>0\ \exists q<l\ \exists t\in d^{<\omega}\ \forall i\in d\ \ s^i_l=s^i_qit$.
(d) The map $l\mapsto(s^i_l)_{i\in d}$ can be coded by a recursive map from ω into ω^d.

We will call T_d the $tree\ on\ d^d\ associated\ with\ an\ effective\ frame$ $E=\{(s^i_l)_{i\in d}\mid l\in\omega\}$:

$$T_d:=\Big\{\vec{s}\in(d^d)^{<\omega}\mid(\forall i\in d\ \ s_i=\emptyset)\ \textit{ or }\ \big(\exists l\in\omega\ \exists t\in d^{<\omega}\ \forall i\in d\ \ s_i=s^i_lit\textit{ and }\forall n<|s_0|\ s_0(n)\leq n\big)\Big\}.$$

The uniqueness condition in (a) and Condition (c) ensure that T_d is small enough, and also the almost acyclicity. The definition of T_d ensures that T_d has finite levels. The existence condition in (a) and Condition (b) ensure that T_d is big enough. More specifically, if (X,τ) is a Polish space and σ a finer Polish topology on X, then there is a dense G_δ subset of (X,τ) on which τ and σ coincide. The first part of Condition (b) ensures the possibility to get inside products of dense G_δ sets. The typical examples are build using the examples due to A. Louveau and J. Saint Raymond that we mentioned earlier. Conditions on verticals are involved, and the second part of Condition (b) gives a control on the choice of verticals. The very last part of Condition (b) is not necessary to get typical examples for the Borel classes, but is useful to get typical examples for the non self-dual Wadge classes of Borel sets.

Proposition 3.5 *The tree T_d associated with an effective frame is a tree with finite one-sided almost acyclic levels.*

Example. (a) Note first that there is a concrete example of a bijection $b_d:\omega\rightarrow d^{<\omega}$ with $|b_d(n)|\leq n$:

• If $d<\omega$, then $b_d(0):=\emptyset$ is the sequence of length 0, $b_d(1):=<0>$, ..., $b_d(d):=<d-1>$ are the sequences of length 1 in the lexicographical ordering, and so on.

• If $d=\omega$, then let $(p_n)_{n\in\omega}$ be the sequence of prime numbers, and $\mathcal{I}:\omega^{<\omega}\rightarrow\omega$ defined by $\mathcal{I}(\emptyset):=1$, and $\mathcal{I}(s):=p_0^{s(0)+1}...p_{|s|-1}^{s(|s|-1)+1}$ if $s\neq\emptyset$. Note that $\mathcal{I}$ is one-to-one, so that there is an increasing bijection $\imath:\text{Seq}:=\mathcal{I}[\omega^{<\omega}]\rightarrow\omega$. We set $b_\omega:=(\imath\circ\mathcal{I})^{-1}:\omega\rightarrow\omega^{<\omega}$.

(b) We now give a concrete example of an effective frame. Fix $i\in d$. We set $s^i_0=\emptyset$, and

$$s^i_{l+1}:=s^i_{(((l)_1)_1)_0}\ i\ b_d\Bigg(\Big(\big((l)_1\big)_1\Big)_1\Bigg)\ 0^{l-(((l)_1)_1)_0-|b_d((((l)_1)_1)_1)|}.$$

Now we want to get some typical examples inside the set of branches of the previous tree. The idea is to reduce the problem to a problem a dimension one. So let us describe the examples due to A. Louveau and J. Saint Raymond that we mentioned earlier.

Notation. A. Louveau and J. Saint Raymond introduced in [Lo-SR1] the following map $\rho_0:2^\omega\rightarrow 2^\omega$:

$$\rho_0(\varepsilon)(i):=\left\{\begin{array}{l} 1 \text{ if } \varepsilon(<i,j>)=0, \text{ for each } j\in\omega,\\ 0 \text{ otherwise.}\end{array}\right.$$

They also proved that $C_{n+1}:=\neg(\rho_0^n)^{-1}(\{0^\infty\})$ is $\mathbf{\Sigma}^0_{n+1}\setminus\mathbf{\Pi}^0_{n+1}$ for each integer n.

• The map $\mathcal{S}:2^\omega\rightarrow 2^\omega$ is the shift map: $\mathcal{S}(\alpha)(m):=\alpha(m+1)$.

• The map $\Delta:(d^\omega)^2\rightarrow 2^\omega$ is the symmetric difference. So, for $m\in\omega$,

$$\Delta(\alpha,\beta)(m):=(\alpha\Delta\beta)(m)=1\ \Leftrightarrow\ \alpha(m)\neq\beta(m).$$

• Let T_d be the tree associated with an effective frame, and $C\subseteq 2^\omega$. We put

$$S^d_C:=\Big\{\vec{\alpha}\in\lceil T_d\rceil\mid\mathcal{S}(\alpha_0\Delta\alpha_1)\in C\Big\}.$$

Fix $1\leq\xi<\omega$. Then $\mathbb{S}^d_{C_\xi}\in\mathbf{\Sigma}^0_\xi(\lceil T_d\rceil)$ and one can prove that $\mathbb{S}^d_{C_\xi}$ is not separable from $\lceil T_d\rceil\setminus\mathbb{S}^d_{C_\xi}$ by a pot($\mathbf{\Pi}^0_\xi$) set.

Let us go deeper into the proof of this.

Notation. We define $p:\omega^{<\omega}\setminus\{\emptyset\}\rightarrow\omega$. We define $p(s)$ by induction on $|s|=n+1$:

$$p(s):=\left\{\begin{array}{l} s(0) \text{ if } n=0,\\ \\ <p(s|n),s(n)> \text{ otherwise.}\end{array}\right.$$

Lemma 3.6 *Let T_d be the tree associated with an effective frame and, for each $i\in d$, G_i a dense G_δ subset of $\Pi''_i\lceil T_d\rceil$. Then there are $\alpha_0\in G_0$ and $F:2^\omega\rightarrow\Pi_{0<i<d}\ G_i$ continuous such that, for $\alpha\in 2^\omega$,*
(a) $\big(\alpha_0,F(\alpha)\big)\in\lceil T_d\rceil$.
(b) For each $s\in\omega^{<\omega}$, and each $m\in\omega$,
(i) $\alpha\big(p(sm)\big)=1\ \Rightarrow\ \exists m'\in\omega\ \ \mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\big(p(sm')\big)=1$.
(ii) $\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\big(p(sm)\big)=1\ \Rightarrow\ \exists m'\in\omega\ \ \alpha\big(p(sm')\big)=1$.
If moreover $s=\emptyset$ and $\alpha\in 2^\omega$, then there is an increasing bijection

$$B_\alpha:\{m\in\omega\mid\alpha(m)=1\}\rightarrow\{m'\in\omega\mid\mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)(m')=1\}$$

such that $(m)_0=\big(B_\alpha(m)\big)_0$ and $\big((m)_1\big)_0=\Big(\big(B_\alpha(m)\big)_1\Big)_0$ if $\alpha(m)=1$.

Now we come to the general condition to get some typical examples as in the statement of Theorem 3.3. Recall that $\omega-1:=\omega$.

Definition 3.7 *We say that $C\subseteq 2^\omega$ is $compatible\ with\ comeager\ sets$ if*

$$\alpha\in C\ \Leftrightarrow\ \mathcal{S}\big(\alpha_0\Delta F_0(\alpha)\big)\in C,$$

for each $\alpha_0\in d^\omega$ and $F:2^\omega\rightarrow(d^\omega)^{d-1}$ satisfying the conclusion of Lemma 3.6.(b).

Recall that $\Delta(\mathbf{\Gamma}):=\mathbf{\Gamma}\cap\check{\mathbf{\Gamma}}$.

Lemma 3.8 *Let T_d be the tree associated with an effective frame, and $\mathbf{\Gamma}$ a non self-dual Wadge class.*

(1) Assume that $C\in\mathbf{\Gamma}\setminus\check{\mathbf{\Gamma}}$ is compatible with comeager sets. Then $S^d_C\in\mathbf{\Gamma}(\lceil T_d\rceil)$ is a Borel subset of $(d^\omega)^d$, and is not separable from $\lceil T_d\rceil\setminus S^d_C$ by a pot$(\check{\mathbf{\Gamma}})$ set.

(2) Assume that C^0, $C^1\in\mathbf{\Gamma}$ are disjoint, compatible with comeager sets, and that C^0 is not separable from C^1 by a $\Delta(\mathbf{\Gamma})$ set. Then $S^d_{C^0},S^d_{C^1}\in\mathbf{\Gamma}(\lceil T_d\rceil)$ are disjoint Borel subsets of $(d^\omega)^d$, and $S^d_{C^0}$ is not separable from $S^d_{C^1}$ by a pot$\big(\Delta(\mathbf{\Gamma})\big)$ set.

4 The reduction theorem in products

Let us state the generalizations of 1.1 and 1.3:

Theorem 4.1 *Let T_d be a tree with finite one-sided almost acyclic levels, $1\leq\xi<\omega_1$, $(X_i)_{i\in d}$ a sequence of Polish spaces, and A_0, A_1 disjoint analytic subsets of $\Pi_{i\in d}\ X_i$.*

(1) (Debs-Lecomte when $d=2$) Let $S\in\mathbf{\Sigma}^0_\xi(\lceil T_d\rceil)$. Then one of the following holds:

(a) The set A_0 is separable from A_1 by a pot$(\mathbf{\Pi}^0_\xi)$ set.

(b) For each $i\in d$ there is $u_i:d^\omega\rightarrow X_i$ continuous such that the equalities $S\subseteq(\Pi_{i\in d}\ u_i)^{-1}(A_0)$ and $\lceil T_d\rceil\setminus S\subseteq(\Pi_{i\in d}\ u_i)^{-1}(A_1)$ hold.

If we moreover assume that S is not separable from $\lceil T_d\rceil\setminus S$ by a pot$(\mathbf{\Pi}^0_\xi)$ set, then this is a dichotomy.

(2) Let $S^0,S^1\in\mathbf{\Sigma}^0_\xi(\lceil T_d\rceil)$ disjoint. Then one of the following holds:

(a) The set A_0 is separable from A_1 by a pot$(\mathbf{\Delta}^0_\xi)$ set.

(b) For each $i\in d$ there is $u_i:d^\omega\rightarrow X_i$ continuous such that $S_\varepsilon\subseteq(\Pi_{i\in d}\ u_i)^{-1}(A_\varepsilon)$ for each $\varepsilon\in 2$.

If we moreover assume that S^0 is not separable from S^1 by a pot$(\mathbf{\Delta}^0_\xi)$ set, then this is a dichotomy.

Note that we can deduce Theorem 1.1 from the proof of Theorem 4.1, without game theory. Theorem 4.1 is the version of Theorem 1.1 for products.

Corollary 4.2 *Let $\mathbf{\Gamma}$ be a Borel class. Then there are Borel subsets $\mathbb{S}^0_{\mathbf{\Gamma}}$, $\mathbb{S}^1_{\mathbf{\Gamma}}$ of $(d^\omega)^d$ such that for any sequence of Polish spaces $(X_i)_{i\in d}$, and for any disjoint analytic subsets A_0, A_1 of $\Pi_{i\in d}\ X_i$, exactly one of the following holds:*

(a) The set A_0 is separable from A_1 by a pot$(\mathbf{\Gamma})$ set.

(b) For each $i\in d$ there is $u_i:d^\omega\rightarrow X_i$ continuous such that $\mathbb{S}^\varepsilon_{\mathbf{\Gamma}}\subseteq(\Pi_{i\in d}\ u_i)^{-1}(A_\varepsilon)$ for each $\varepsilon\in 2$.

Concerning the non self-dual Borel classes when $d=2$, this corollary has initially been shown by D. Lecomte when ξ is a successor ordinal. Then G. Debs proved it when ξ is a limit ordinal. Note that Theorem 4.1 and Corollary 4.2 can be extended to the difference hierarchy. One can prove that a reduction on the whole product is not possible, for acyclicity reasons:

Theorem 4.3 *In Corollary 4.2,*

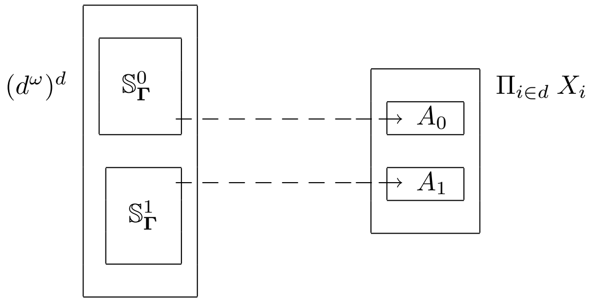
(a) (Debs, see [L7]) We cannot replace $\mathbb{S}^1_{\mathbf{\Gamma}}$ with $\neg\mathbb{S}^0_{\mathbf{\Gamma}}$.

(b) (Debs) We can ensure that the u_i's are one-to-one when $\xi\geq 3$ and $d=2$.

(c) We can ensure that $(\Pi_{i\in d}\ u_i)_{|\lceil T_d\rceil}$ is one-to-one when $\xi\geq 3$, for any d.

However, we can replace $\mathbb{S}^1_{\mathbf{\Gamma}}$ with $\overline{\mathbb{S}^0_{\mathbf{\Gamma}}}\setminus\mathbb{S}^0_{\mathbf{\Gamma}}$ if $\mathbf{\Gamma}$ is not self-dual.

So the good scheme of comparison is



One of the important tools to prove Theorem 4.1 is the representation theorem for Borel sets by G. Debs and J. Saint Raymond (see [D-SR]). It specifies the classical result by Lusin and Souslin asserting that any Borel set in a Polish space is the bijective continuous image of a closed subset of the Baire space (see 13.7 in [K]). We now give some material from [D-SR].

Definition 4.4 *(Debs-Saint Raymond) Let* c *be a countable set. A partial order relation* R *on* $c^{<\omega}$ *is a* $tree\ relation$ *if, for* $t\in c^{<\omega}$*,*

(a) $\emptyset\ R\ t$.

(b) The set $P_R(t):=\{s\in c^{<\omega}\mid s\ R\ t\}$ *is finite and linearly ordered by* R.

For instance, the non strict extension relation $\subseteq$ *is a tree relation.*

• *Let* R *be a tree relation. An* R-$branch$ *is an* $\subseteq$*-maximal subset of* $c^{<\omega}$ *linearly ordered by* R*. We denote by* $[R]$ *the set of all infinite* R*-branches.*

We equip $(c^{<\omega})^\omega$ *with the product of the discrete topology on* $c^{<\omega}$*. If* R *is a tree relation, then the space* $[R]\subseteq(c^{<\omega})^\omega$ *is equipped with the topology induced by that of* $(c^{<\omega})^\omega$*. The map* $h:c^\omega\rightarrow[\subseteq]$ *defined by* $h(\gamma):=(\gamma|j)_{j\in\omega}$ *is an homeomorphism.*

• *Let* R*,* S *be tree relations with* $R\subseteq S$*. The* $canonical\ map$ $\Pi:[R]\rightarrow[S]$ *is defined by*

$$\Pi(\mathcal{B}):=\text{the unique } S\text{-branch containing } \mathcal{B}.$$

• *Let* S *be a tree relation. We say that* $R\subseteq S$ *is* $distinguished$ *in* S *if*

$$\forall s,t,u\in c^{<\omega}\qquad \left.\begin{array}{c} s\ S\ t\ S\ u\\ s\ R\ u\end{array}\right\}\ \Rightarrow\ s\ R\ t.$$

For example, let C *be a closed subset of* c^ω*, and define*

$$s\ R\ t\ \Leftrightarrow\ s\subseteq t\ \text{ and }\ N_t\cap C\neq\emptyset.$$

Then R *is distinguished in* $\subseteq$*. In this case, the distinction expresses the fact that "when we leave the closed set, it is for ever".*

• *Let* $\eta<\omega_1$*. A family* $(R^{(\rho)})_{\rho\leq\eta}$ *of tree relations is a* $resolution\ family$ *if*

(a) $R^{(\rho+1)}$ *is a distinguished subtree of* $R^{(\rho)}$*, for all* $\rho<\eta$.

(b) $R^{(\lambda)}=\bigcap_{\rho<\lambda}\ R^{(\rho)}$*, for all limit* $\lambda\leq\eta$.

Theorem 4.5 *(Debs-Saint Raymond) Let* $\eta<\omega_1$*,* R *a tree relation,* $(I_n)_{n\in\omega}$ *a sequence of* $\mathbf{\Pi}^0_{\eta+1}$ *subsets of* $[R]$*. Then there is a resolution family* $(R^{(\rho)})_{\rho\leq\eta}$ *with*

(a) $R^{(0)}=R$.

(b) The canonical map $\Pi:[R^{(\eta)}]\rightarrow[R]$ *is a continuous bijection.*

(c) The set $\Pi^{-1}(I_n)$ *is a closed subset of* $[R^{(\eta)}]$ *for each integer* n.

Another important tool is Effective Descriptive Set Theory (see [M] for the basic notions, and also Section 5). Of particular importance are the following topologies:

Notation. Let X be a recursively presented Polish space X.

• The topology Δ_X on X is generated by $\Delta^1_1(X)$.

• The $Gandy$-$Harrington\ topology$ Σ_X on X is generated by $\Sigma^1_1(X)$.

Theorem 4.6 *Let X be a recursively presented Polish space X.*

(a) (Louveau, see [Lo]) The topology Δ_X is Polish.

(b) (Gandy, see [S]) The set $\Omega_X:=\{x\in X \mid \omega_1^x=\omega_1^{\text{CK}}\}$ is a dense basic open set for Σ_X.

(c) (see [L8]) The topology Σ_X is non metrizable in general. But (Ω_X,Σ_X) is a zero-dimensional Polish space. In fact, the intersection of Ω_X with any Σ^1_1 subset of X is a clopen subset of (Ω_X,Σ_X).

Notation. We set $\tau_1:=(\Delta_{\omega^\omega})^d$. If $2\leq\xi<\omega_1^{\text{CK}}$, then we denote by τ_ξ the topology generated by $\Sigma^1_1\big((\omega^\omega)^d\big)\cap\mathbf{\Pi}^0_{<\xi}(\tau_1)$. We have $\mathbf{\Sigma}^0_1(\tau_\xi)\subseteq\mathbf{\Sigma}^0_\xi(\tau_1)$, so that $\mathbf{\Pi}^0_1(\tau_\xi)\subseteq\mathbf{\Pi}^0_\xi(\tau_1)$.

Key idea of the proof of Theorem 4.1.(1) when $\xi=\eta+1<\omega_1^{\text{CK}}$, $X_i=\omega^\omega$ and $A_\varepsilon\in\Sigma^1_1$.

• We have $E:=\theta[\lceil T_d\rceil\setminus S]$ is $\mathbf{\Pi}^0_{\eta+1}([\subseteq])$. Theorem 4.5 provides a resolution family. We put

$$D:=\{\vec{s}\in T_d \mid \exists\mathcal{B}\in\Pi^{-1}(E)\ \ \vec{s}\in\mathcal{B}\}.$$

• We assume that (a) is not satisfied, so that $\overline{A_0}^{\tau_\xi}\cap A_1$ is not empty. We fix a complete metric d (resp., a metric δ) on $(\Omega_{(\omega^\omega)^d},\Sigma_{(\omega^\omega)^d})$ (resp., ω^ω equipped with its usual topology).

• We construct $(x^i_s)_{i\in d,s\in\Pi''_iT_d}\subseteq\omega^\omega$, $(U_{\vec{s}})_{\vec{s}\in T_d}\subseteq\Sigma^1_1\big((\omega^\omega)^d\big)$ with

$$\begin{array}{ll}
(1) & (x^i_{s_i})\in U_{\vec{s}}\subseteq\Omega_{(\omega^\omega)^d}.\\
(2) & \text{diam}_d(U_{\vec{s}})\leq 2^{-|\vec{s}|},\ \delta(x^i_s,x^i_{s\varepsilon})\leq 2^{-|s|}.\\
(3) & U_{\vec{s}}\subseteq\overline{A_0}^{\tau_\xi}\cap A_1 \text{ if } \vec{s}\in D.\\
(4) & U_{\vec{s}}\subseteq A_0 \text{ if } \vec{s}\notin D.\\
(5) & (1\leq\rho\leq\eta \text{ and } \vec{s}\ R^{(\rho)}\ \vec{t})\ \Rightarrow\ U_{\vec{t}}\subseteq\overline{U_{\vec{s}}}^{\tau_\rho}.\\
(6) & \big((\vec{s}\in D\ \Leftrightarrow\ \vec{t}\in D) \text{ and } \vec{s}\ R^{(\eta)}\ \vec{t}\big)\ \Rightarrow\ U_{\vec{t}}\subseteq U_{\vec{s}}.
\end{array}$$

All the conditions but (5) are quite natural to get (b), and they are sufficient. But to ensure that the construction is possible, we need an additional condition, namely (5). This is the key idea: there is a deep link between the smaller and smaller relations $R^{(\rho)}$ on one side, and the smaller and smaller closures $\overline{U_{\vec{s}}}^{\tau_\rho}$ on the other side.

Proof of $\neg$(a) $\Rightarrow$ (b) in Corollary 4.2 when $d=2$, $\mathbf{\Gamma}=\mathbf{\Pi}^0_1$, $X_i=\omega^\omega$ and $A_\varepsilon\in\Sigma^1_1$.

Let $\mathbb{S}:=\{\vec{\alpha}\in\lceil T_2\rceil \mid \mathcal{S}(\alpha_0)\neq\mathcal{S}(\alpha_1)\}$. We want to find u_0 and u_1 continuous with $\mathbb{S}\subseteq(u_0\times u_1)^{-1}(A_0)$ and $\overline{\mathbb{S}}\setminus\mathbb{S}\subseteq(u_0\times u_1)^{-1}(A_1)$.

One proves that $\overline{A_0}^{\tau_1}\in\Sigma^1_1\big((\omega^\omega)^2\big)$. We assume that (a) is not satisfied, so that $N:=\overline{A_0}^{\tau_1}\cap A_1$ is a nonempty Σ^1_1 subset of $(\omega^\omega)^2$. The key property is the following:

$$\forall U,V\in\Sigma^1_1\ \ \big(N\cap(U\times V)\neq\emptyset\ \Rightarrow\ A_0\cap(U\times V)\neq\emptyset\big).$$

Indeed, $A_0\cap(U\times V)=\emptyset$ implies the existence of U', $V'\in\Delta^1_1$ with $A_0\cap(U'\times V')=\emptyset$, $U\subseteq U'$ and $V\subseteq V'$ by the separation theorem. So we get $N\cap(U\times V)\subseteq N\cap(U'\times V')\subseteq\overline{A_0}^{\tau_1}\setminus\overline{A_0}^{\tau_1}$, which is absurd.

We define, for $\vec{s}=(s_0,s_1)\in T_2$,

$$(s_0,s_1)^*:=\begin{cases}(s_0,s_1) \ \text{ if } \ |\vec{s}|\leq 1 \ \text{ or} \\ \qquad \big(|\vec{s}|\geq 2 \ \text{ and } \ \mathcal{S}(s_0)(|\vec{s}|-2)\neq\mathcal{S}(s_1)(|\vec{s}|-2) \ \text{ and } \ \mathcal{S}(s_0)|(|\vec{s}|-2)=\mathcal{S}(s_1)|(|\vec{s}|-2)\big), \\ (s_0,s_1)|(|\vec{s}|-1) \text{ otherwise.}\end{cases}$$

We construct sequences $(x^i_s)_{i\in 2,s\in\Pi''_iT_2}\subseteq\omega^\omega$ and $(U_{\vec{s}})_{\vec{s}\in T_2}\subseteq\varSigma^1_1\big((\omega^\omega)^2\big)$ with

(1) $(x^0_{s_0},x^1_{s_1})\in U_{\vec{s}}\subseteq U_{\vec{s}^*}\subseteq\Omega_{(\omega^\omega)^2}$.
(2) $\text{diam}_d(U_{\vec{s}})\leq 2^{-|\vec{s}|}$.
(3) $U_{\vec{s}}\subseteq N$ if $|\vec{s}|=0$ or $\mathcal{S}(s_0)=\mathcal{S}(s_1)$.
(4) $U_{\vec{s}}\subseteq A_0$ if $\mathcal{S}(s_0)\neq\mathcal{S}(s_1)$.

• Assume that this construction is achieved. Then $\big((x^0_{\alpha_0|n},x^1_{\alpha_1|n})\big)_n$ is a Cauchy sequence in $(\Omega_{(\omega^\omega)^2},d)$ for each $\vec{\alpha}\in\lceil T_2\rceil$. So that we can define $F(\vec{\alpha})$ as the limit of it, and a function $F:\lceil T_2\rceil\rightarrow\Omega_{(\omega^\omega)^2}$. Notice that $d\big((x^0_{0\alpha|n},x^1_{1\alpha|n}),F(0\alpha,1\alpha)\big)\leq 2^{-n}$, so that $F_{|\lceil T_2\rceil\setminus\mathbb{S}}$ is continuous. We put $u_i(j\alpha):=\Pi_i\big(F(0\alpha,1\alpha)\big)$. This defines continuous maps. Notice that $u_i(j\alpha)$ is the limit of $(x^i_{i\alpha|n})_{n\in\omega}$, by continuity of the first projection.

If $\vec{\alpha}\in\mathbb{S}$, then $\mathcal{S}(\alpha_0|n)\neq\mathcal{S}(\alpha_1|n)$ if n is big enough. In this case, $(x^0_{\alpha_0|n},x^1_{\alpha_1|n})\in A_0\cap\Omega_{(\omega^\omega)^2}$ which is a closed subset of $(\Omega_{X\times Y},d)$, so that $F(\vec{\alpha})\in A_0$. But $\Pi_i\big(F(\vec{\alpha})\big)$ is the limit of $(x^i_{\alpha_i|n})_{n\in\omega}$, and is $u_i(\alpha_i)$. Thus $\big(u_0(\alpha_0),u_1(\alpha_1)\big)=F(\vec{\alpha})\in A_0$. Now if $\alpha\in 2^\omega$ then $\big(u_0(0\alpha),u_1(1\alpha)\big)=F(0\alpha,1\alpha)\in N\subseteq A_1$. This proves that $\overline{\mathbb{S}}\setminus\mathbb{S}\subseteq\lceil T_2\rceil\setminus\mathbb{S}\subseteq(u_0\times u_1)^{-1}(A_1)$.

• So let us show that the construction is possible. Let $(x^0_\emptyset,x^1_\emptyset)\in N\cap\Omega_{(\omega^\omega)^2}$. Then we choose $U_{\vec{\emptyset}}\in\varSigma^1_1$ with diameter at most 2^{-1} such that $(x^0_\emptyset,x^1_\emptyset)\in U_{\vec{\emptyset}}\subseteq N\cap\Omega_{(\omega^\omega)^2}$. Then we set $x^i_i:=x^i_\emptyset$, $U_{(0,1)}:=U_{\vec{\emptyset}}$. Assume that $(x^i_s)_{|s|\leq l}$ and $(U_{\vec{s}})_{|\vec{s}|\leq l}$ satisfying Conditions (1)-(4) have been constructed, which is the case for $l\leq 1$. If $\mathcal{S}(s^0_l)\neq\mathcal{S}(s^1_l)$, then we set $x^i_{s\varepsilon}:=x^i_s$, and we simply reduce the diameters of the $U_{\vec{s}}$'s. So assume that $\mathcal{S}(s^0_l)=\mathcal{S}(s^1_l)$. We put, for $i\in 2$,

$$U_i:=\{\overline{x}^i_{s^i_l}\in\omega^\omega\mid\exists(\overline{x}^i_s)_{s\in\Pi''_iT_2\cap 2^l\setminus\{s^i_l\}}\subseteq\omega^\omega \ \ \exists(\overline{x}^{1-i}_s)_{s\in\Pi''_{1-i}T_2\cap 2^l}\subseteq\omega^\omega \ \ \forall\vec{s}\in T_2\cap(2^l)^2 \ \ (\overline{x}^0_{s_0},\overline{x}^1_{s_1})\in U_{\vec{s}}\}.$$

The sets U_i are $\varSigma^1_1$, and $(x^0_{s^0_l},x^1_{s^1_l})\in U_0\times U_1$. In particular, $A_0\cap(U_0\times U_1)$ is not empty and $\varSigma^1_1$, so we can find $(x^0_{s^0_l0},x^1_{s^1_l1})\in A_0\cap(U_0\times U_1)\cap\Omega_{(\omega^\omega)^2}$. We choose witnesses $(x^i_{si})_{s\in\Pi''_iT_2\cap 2^l\setminus\{s^i_l\}}$, $(x^{1-i}_{si})_{s\in\Pi''_{1-i}T_2\cap 2^p}$ for the fact that $x^i_{s^i_li}\in U_i$. Then we choose $U_{(s_0\varepsilon_0,s_1\varepsilon_1)}\in\varSigma^1_1$ with diameter at most 2^{-l-1} such that $(x^0_{s_0\varepsilon_0},x^1_{s_1\varepsilon_1})$ is in $U_{(s_0\varepsilon_0,s_1\varepsilon_1)}\subseteq U_{\vec{s}}$ if $\varepsilon_0=\varepsilon_1$, and $(x^0_{s^0_l0},x^1_{s^1_l1})\in U_{(s^0_l0,s^1_l1)}\subseteq A_0\cap\Omega_{(\omega^\omega)^2}$. $\square$

5 Appendix: Effective Descriptive Set Theory

Effective Descriptive Set Theory is based on the notion of recursive function. This theory allows us to prove some results of classical type. For some of them, we do not know any proof of classical type. The material in this section can be found in [M], except where indicated.

Definition 5.1 *Let $\omega := \mathbb{N}$. The class of $recursive\ functions$ is the smallest class of functions from a space ω^k into ω ($k \geq 1$) such that*

- The constant functions are recursive: $\forall k \geq 1\ \forall n \in \omega\ C^k_n(\overline{x}) = n$ ($\overline{x} := (x_1, \ldots, x_k) \in \omega^k$).

- The projections are recursive: $\forall k \geq 1$, $1 \leq i \leq k$, $P^k_i(\overline{x}) = x_i$.

- The successor function is recursive: $S(n) := n+1$.

- The class is closed under composition: if $g : \omega^m \to \omega$, $h_i : \omega^k \to \omega$ ($1 \leq i \leq m$) are recursive, then f defined by $f(\overline{x}) := g\big(h_1(\overline{x}), \ldots, h_m(\overline{x})\big)$ is recursive.

- The class is closed under recursion: if $g : \omega^k \to \omega$, $h : \omega^{k+2} \to \omega$ are recursive, $f : \omega^{k+1} \to \omega$ defined by $f(0, \overline{x}) := g(\overline{x})$ and $f(n+1, \overline{x}) := h(f(n, \overline{x}), n, \overline{x})$ is recursive (k may be 0).

- The class is closed under minimalization: if $g : \omega^{k+1} \to \omega$ is recursive, and if $\forall \overline{x} \in \omega^k\ \exists n\ g(\overline{x}, n) = 0$, then f defined by $f(\overline{x}) := min\{n \in \omega \mid g(\overline{x}, n) = 0\}$ is recursive.

Note that the class of recursive functions is countable.

Examples. Let $(p_n)_{n \geq 1}$ be the sequence of prime numbers. The function i defined by

$$i(\overline{x}) := p_1^{x_1+1} \ldots p_k^{x_k+1}$$

is recursive. So is $c_j : \omega \to \omega$ defined by $c_j(m) := (m)_j := x_j$ if $m = i(\overline{x})$ and $1 \leq j \leq k$ (0 otherwise).

Definition 5.2 *(a) We say that $E \subseteq \omega^k$ is $recursive$ if its characteristic function is recursive.*

(b) We say that $\big(X, (x_n), d\big)$ (X for simple) is a $recursively\ presented$ Polish space if (x_n) is a dense subsequence of X, and d is a complete compatible distance on X such that the following sets are recursive: $\{(i, j, k, l) \in \omega^4 \mid d(x_i, x_j) \leq \frac{k}{l+1}\}$ and $\{(i, j, k, l) \in \omega^4 \mid d(x_i, x_j) < \frac{k}{l+1}\}$.

We enumerate a basis for the topology of X as follows: $B(X, n) := B(x_{(n)_0}, \frac{(n)_1}{(n)_2+1})$.

Definition 5.3 *Let X be a recursively presented Polish space. We say that $E \subseteq X$ is $semi\text{-}recursive$ if there is $\varepsilon : \omega \to \omega$ recursive such that $E = \bigcup_n\ B\big(X, \varepsilon(n)\big)$.*

One proves that $E \subseteq \omega^k$ is recursive if and only if E and $\neg E$ are semi-recursive.

Notation. If $E \subseteq X \times Y$, then $\exists^Y E := \{x \in X \mid \exists y \in Y\ (x, y) \in E\}$. If $\mathit{\Gamma}$ is a class of sets, then $\exists^Y \mathit{\Gamma} := \{\exists^Y E \mid E \in \mathit{\Gamma}\}$. The Kleene classes are defined as follows, for $n \geq 1$: $\Sigma^0_1 :=$ semi-recursive sets, $\Pi^0_n := \check{\Sigma}^0_n$, $\Sigma^0_{n+1} := \exists^\omega \Pi^0_n$, $\Delta^0_n := \Sigma^0_n \cap \Pi^0_n$, $\Sigma^1_1 := \exists^{\omega^\omega} \Pi^0_1$, $\Pi^1_1 := \check{\Sigma}^1_1$, $\Delta^1_1 := \Sigma^1_1 \cap \Pi^1_1$.

Theorem 5.4 *The class Σ^1_1 is closed under $\vee$, $\wedge$, $\forall^\omega$, and $\exists^X$ if X is recursively presented.*

The link with the Borel classes is made with the following notion of relativization. Note that the inclusions between classes remain true for their effective counterpart.

Definition 5.5 *Let $\mathit{\Gamma}$ be a class of sets, X and Y recursively presented Polish spaces, and $y \in Y$. We say that $E \subseteq X$ is in the $relativized$ at y class $\mathit{\Gamma}(y)$ of $\mathit{\Gamma}$ if there is $F \in \mathit{\Gamma}(X \times Y)$ with $E = \{x \in X \mid (x, y) \in F\}$.*

We will apply this definition to the Kleene classes of the form Π or Σ. For the classes Δ, we use the following definitions: $\Delta^0_n(y) := \Sigma^0_n(y) \cap \Pi^0_n(y)$ and $\Delta^1_1(y) := \Sigma^1_1(y) \cap \Pi^1_1(y)$.

The fundamental link with the usual classes is the following:

Theorem 5.6 *Let $\mathit{\Gamma}$ be a Kleene class. Then $\mathbf{\Gamma} = \bigcup_{y \in \omega^\omega}\ \mathit{\Gamma}(y)$.*

The Polish spaces are recursively in some y-presented.

There is an effective version of the separation theorem:

Theorem 5.7 *Let X be a recursively presented Polish space, and A, A' be two disjoint Σ^1_1 subsets of X. Then there is $B\in\Delta^1_1$ with $A\subseteq B\subseteq\neg A'$.*

One can code the Borel subsets of a recursively presented Polish space (see [H-K-Lo]):

Theorem 5.8 *Let X be a recursively presented Polish space. There are $W^X\in\Pi^1_1(\omega)$ and $C^X\in\Pi^1_1(\omega\times X)$ such that $\Delta^1_1(X)=\{C^X_n \mid n\in W^X\}$ and $\{(n,x)\in\omega\times X \mid n\in W^X$ and $(n,x)\notin C^X\}$ are $\Pi^1_1(\omega\times X)$.*

Notation. Let $\varepsilon\in\omega^\omega$. We define a binary relation $\leq_\varepsilon$ on ω as follows:

$$m\leq_\varepsilon n \ \Leftrightarrow\ \varepsilon\big(i(m,n)\big)=1$$

Let $WO:=\{\varepsilon\in\omega^\omega \mid\leq_\varepsilon$ is a well order$\}$. If $\varepsilon\in WO$, then we denote by $|\varepsilon|$ the order type of $\leq_\varepsilon$.

Definition 5.9 *We say that $\varepsilon\in\omega^\omega$ is $recursive$ if $B_\varepsilon:=\{n\in\omega \mid \varepsilon\in B(\omega^\omega,n)\}$ is Δ^0_1. This can be relativized: let Y be a recursively presented Polish space, and $y\in Y$. We say that ε is recursive in y if B_ε is $\Delta^0_1(y)$.*

Notation. Let $\omega_1^{\text{CK}}:=\sup\{|\varepsilon| \mid \varepsilon\in WO$ and ε is recursive$\}$. There is a relativized version:

$$\omega_1^y:=\sup\{|\varepsilon| \mid \varepsilon\in WO \text{ and } \varepsilon \text{ is recursive in } y\}.$$

6 References

[D-SR] G. Debs and J. Saint Raymond, Borel liftings of Borel sets: some decidable and undecidable statements, *Mem. Amer. Math. Soc.* 187, 876 (2007)

[H-K-Lo] L. A. Harrington, A. S. Kechris and A. Louveau, A Glimm-Effros dichotomy for Borel equivalence relations, *J. Amer. Math. Soc.* 3 (1990), 903-928

[Hj-K-Lo] G. Hjorth, A. S. Kechris and A. Louveau, Borel equivalence relations induced by actions of the symmetric group, *Ann. Pure Appl. Logic* 92 (1998), 63-112

[K] A. S. Kechris, *Classical Descriptive Set Theory,* Springer-Verlag, 1995

[L1] D. Lecomte, Classes de Wadge potentielles et théorèmes d'uniformisation partielle, *Fund. Math.* 143 (1993), 231-258

[L2] D. Lecomte, Uniformisations partielles et critères à la Hurewicz dans le plan, *Trans. Amer. Math. Soc.* 347, 11 (1995), 4433-4460

[L3] D. Lecomte, Tests à la Hurewicz dans le plan, *Fund. Math.* 156 (1998), 131-165

[L4] D. Lecomte, Complexité des boréliens à coupes dénombrables, *Fund. Math.* 165 (2000), 139-174

[L5] D. Lecomte, On minimal non potentially closed subsets of the plane, *Topology Appl.* 154, 1 (2007) 241-262

[L6] D. Lecomte, Hurewicz-like tests for Borel subsets of the plane, *Electron. Res. Announc. Amer. Math. Soc.* 11 (2005)

[L7] D. Lecomte, How can we recognize potentially $\mathbf{\Pi}^0_\xi$ subsets of the plane?, *submitted* (2006)

[L8] D. Lecomte, A dichotomy characterizing analytic digraphs of uncountable Borel chromatic number in any dimension, *to appear in Trans. Amer. Math. Soc. (see arXiv)*

[L9] D. Lecomte, Potential Wadge classes in any dimension, *preprint* (2009)

[Lo] A. Louveau, Ensembles analytiques et boréliens dans les espaces produit, *Astérisque (S. M. F.)* 78 (1980)

[Lo-SR1] A. Louveau and J. Saint Raymond, Borel classes and closed games: Wadge-type and Hurewicz-type results, *Trans. Amer. Math. Soc.* 304 (1987), 431-467

[Lo-SR2] A. Louveau and J. Saint Raymond, The strength of Borel Wadge determinacy, *Cabal Seminar 81-85, Lecture Notes in Math.* 1333 (1988), 1-30

[M] Y. N. Moschovakis, *Descriptive set theory,* North-Holland, 1980

[S] G. E. Sacks, *Higher Recursion Theory,* Springer-Verlag, 1990