How can we recognize potentially $\mathbf{\Pi}^0_\xi$ subsets of the plane?

Dominique LECOMTE

June 2006

• Université Paris 6, Equipe d'Analyse Fonctionnelle, tour 46-0, boîte 186,
4, place Jussieu, 75 252 Paris Cedex 05, France.
lecomte@moka.ccr.jussieu.fr

• Université de Picardie, I.U.T. de l'Oise, site de Creil,
13, allée de la faïencerie, 60 107 Creil, France.

Abstract. Let $\xi \geq 1$ be a countable ordinal. We study the Borel subsets of the plane that can be made $\mathbf{\Pi}^0_\xi$ by refining the Polish topology on the real line. These sets are called potentially $\mathbf{\Pi}^0_\xi$. We give a Hurewicz-like test to recognize potentially $\mathbf{\Pi}^0_\xi$ sets.

2000 Mathematics Subject Classification. Primary: 03E15, Secondary: 54H05

Keywords and phrases. Potentially, Baire classes, Reduction, Hurewicz's Theorem

Acknowledgements. I would like to thank G. Debs for his nice contributions to this paper. The main results proved in this paper are the solution to the main question (partially solved) in my thesis. This question was asked by A. Louveau fifteen years ago. I would like to thank him very much for that. I also thank them for their remarks during the lectures I gave in the descriptive set theory seminar of the University Paris 6, where I proved some of the results in this paper. These remarks improved the quality of this article.

1 Introduction.

The reader should see [K] for the descriptive set theoretic notation used in this paper. This work is the continuation of a study started in [L1]-[L5], and is announced in [L6]. The usual notion of comparison for Borel equivalence relations $E\subseteq X^2$ and $E'\subseteq {X'}^2$ on Polish spaces is the Borel reducibility quasi-order:

$$E\leq_B E'\ \Leftrightarrow\ \exists u\colon X\rightarrow X' \text{ Borel with } E=(u\times u)^{-1}(E')$$

(recall that a quasi-order is a reflexive and transitive relation). Note that this makes sense even if E, E' are not equivalence relations. It is known that if (B_n) is a sequence of Borel subsets of X, then there is a finer Polish topology on X making the B_n's clopen (see exercise 13.5 in [K]). So assume that $E\leq_B E'$, and let σ be a finer Polish topology on X making u continuous. If moreover E' is in some Baire class Γ, then $E\in\Gamma([X,\sigma]^2)$. This motivates the following (see [Lo2]):

Definition 1.1 *(Louveau) Let X, Y be Polish spaces, A a Borel subset of $X\times Y$, and Γ a Baire (or Wadge) class. We say that A is potentially in Γ (denoted $A\in pot(\Gamma)$) iff there is a finer Polish topology σ (resp., τ) on X (resp., Y) with $A\in\Gamma([X,\sigma]\times[Y,\tau])$.*

This notion is a natural invariant for $\leq_B$: if E' is pot(Γ) and $E\leq_B E'$, then E is pot(Γ). Using this notion, A. Louveau proved that the collection of $\mathbf{\Sigma}^0_\xi$ equivalence relations is not cofinal for $\leq_B$, and deduces from this the non existence of a maximum Borel equivalence relation for $\leq_B$ (this non existence result is due to H. Friedman and L. Stanley). More recently, G. Hjorth, A. Kechris and A. Louveau determined the potential classes of the Borel equivalence relations induced by Borel actions of closed subgroups of the symmetric group (see [Hj-K-Lo]).

A standard way to see that a set is complicated is to note that it is more complicated than a well-known example. For instance, we have the following (see [SR]):

Theorem 1.2 *(Hurewicz) Let $P_f:=\{\alpha\in 2^\omega\mid\exists n\in\omega\ \forall m\geq n\ \ \alpha(m)=0\}$, X be a Polish space, and A a Borel subset of X. Then exactly one of the following holds:*
(a) The set A is $\mathbf{\Pi}^0_2(X)$.
(b) There is $u\colon 2^\omega\rightarrow X$ continuous and one-to-one with $P_f=u^{-1}(A)$.

This result has been generalized to all Baire classes (see [Lo-SR]). We state this generalization in two parts:

Theorem 1.3 *(Louveau-Saint Raymond) Let $\xi<\omega_1$, $S\in\mathbf{\Sigma}^0_{1+\xi}(2^\omega)$, X be a Polish space, and A, B disjoint analytic subsets of X. Then one of the following holds:*
(a) The set A is separable from B by a $\mathbf{\Pi}^0_{1+\xi}(X)$ set.
(b) There is $u\colon 2^\omega\rightarrow X$ continuous with $S\subseteq u^{-1}(A)$ and $2^\omega\setminus S\subseteq u^{-1}(B)$.
If we moreover assume that $S\notin\mathbf{\Pi}^0_{1+\xi}$, then this is a dichotomy.

Note that in this dichotomy, we can have u one-to-one if $\xi\geq 2$. This is not possible if $\xi<2$.

Theorem 1.4 *There is a concrete example of a set $S_{1+\xi}\in\mathbf{\Sigma}^0_{1+\xi}(2^\omega)\setminus\mathbf{\Pi}^0_{1+\xi}(2^\omega)$, for each $\xi<\omega_1$.*

We try to adapt these results to the Borel subsets of the plane.

The following result is proved in [H-K-Lo]:

Theorem 1.5 *(Harrington-Kechris-Louveau) Let X be a Polish space, E a Borel equivalence relation on X, and $E_0:=\{(\alpha,\beta)\in 2^\omega\times 2^\omega \mid \exists n\in\omega\ \ \forall m\geq n\ \ \alpha(m)=\beta(m)\}$. Then exactly one of the following holds:*
(a) The relation E is pot($\mathbf{\Pi}^0_1$).
(b) $E_0\leq_B E$ (with u continuous and one-to-one).

For the Borel subsets of the plane, we need some other notions of comparison. Let X, Y, X', Y' be Polish spaces, and A (resp., A') a Borel subset of $X\times Y$ (resp., $X'\times Y'$). We set

$$A\leq^r_B A'\ \Leftrightarrow\ \exists u\!:\!X\rightarrow X'\ \ \exists v\!:\!Y\rightarrow Y'\ \text{ Borel with } A=(u\times v)^{-1}(A').$$

The following result is proved in [L1]:

Theorem 1.6 *Let $\Delta(2^\omega):=\{(\alpha,\beta)\in 2^\omega\times 2^\omega \mid \alpha=\beta\}$, $L_0:=\{(\alpha,\beta)\in 2^\omega\times 2^\omega \mid \alpha<_{lex}\beta\}$, X, Y be Polish spaces, and A a pot($\check{D}_2(\mathbf{\Sigma}^0_1)$) subset of $X\times Y$. Then exactly one of the following holds:*
(a) The set A is pot($\mathbf{\Pi}^0_1$).
(b) $\neg\Delta(2^\omega)\leq^r_B A$ or $L_0\leq^r_B A$ (with u, v continuous and one-to-one).

The class $\check{D}_2(\mathbf{\Sigma}^0_1)$ is the class of unions of a closed set and of an open set. Things become more complicated at the level $D_2(\mathbf{\Sigma}^0_1)$ of differences of two open sets (see [L5]):

Theorem 1.7 *(a) There is a perfect $\leq^r_B$-antichain $(A_\alpha)_{\alpha\in 2^\omega}\subseteq D_2(\mathbf{\Sigma}^0_1)(2^\omega\times 2^\omega)$ such that A_α is $\leq^r_B$-minimal among $\mathbf{\Delta}^1_1\backslash$pot($\mathbf{\Pi}^0_1$) sets, for any $\alpha\in 2^\omega$.*
(b) There is a perfect $\leq_B$-antichain $(R_\alpha)_{\alpha\in 2^\omega}$ such that R_α is $\leq_B$-minimal among $\mathbf{\Delta}^1_1\backslash$pot($\mathbf{\Pi}^0_1$) sets, for any $\alpha\in 2^\omega$. Moreover, $(R_\alpha)_{\alpha\in 2^\omega}$ can be taken to be a subclass of any of the following classes:
- Graphs (i.e., irreflexive and symmetric relations).
- Oriented graphs (i.e., irreflexive and antisymmetric relations).
- Quasi-orders.
- Partial orders (i.e., reflexive, antisymmetric and transitive relations).

In other words, the case of equivalence relations, for which we have a unique (up to bi-reducibility) minimal non potentially closed element with Theorem 1.5, is very specific. Theorem 1.7.(b) says, among other things, that the mixture between symmetry and transitivity is very strong. Theorem 1.7.(a) shows that the classical notions of reduction (on the whole product) don't work, at least at the first level. So we must find another notion of comparison. The following result is proved in [L5]:

Theorem 1.8 *There is $S_1\in\mathbf{\Delta}^1_1(2^\omega\times 2^\omega)$ such that for any Polish spaces X, Y, and for any Borel subset A of $X\times Y$, exactly one of the following holds:*
(a) The set A is pot($\mathbf{\Pi}^0_1$).
(b) There are $u\!:\!2^\omega\rightarrow X$ and $v\!:\!2^\omega\rightarrow Y$ continuous satisfying the inclusions $S_1\subseteq(u\times v)^{-1}(A)$ and $\overline{S_1}\backslash S_1\subseteq(u\times v)^{-1}(\neg A)$.
Moreover, we can neither replace $\overline{S_1}\backslash S_1$ with $\neg S_1$, nor ensure that u and v are one-to-one.

So we get a minimum non-potentially closed set if we do not ask for a reduction on the whole product. We will show that this dichotomy is true for each countable ordinal $\xi \geq 1$. The result is actually stronger than that. First the A_ξ's are concrete examples. Secondly it is better to state that the reduction in condition (b) holds in the set $\lceil T \rceil$ of the branches of some tree T that does not depend on ξ, rather than $\overline{A_\xi}$. Finally, to get the full strength of the result, it is better to split it in two parts. We need some notation and a definition:

Notation. If $\mathcal{F}_0$, $\mathcal{F}_1$ are finite sets and $\mathcal{T} \subseteq \mathcal{F}_0 \times \mathcal{F}_1$, we denote by $G_\mathcal{T}$ the bipartite graph with set of vertices the sum $\mathcal{F}_0 \oplus \mathcal{F}_1$, and with set of edges

$$\Big\{\{(f_0, 0), (f_1, 1)\} \subseteq \mathcal{F}_0 \oplus \mathcal{F}_1 \mid (f_0, f_1) \in \mathcal{T}\Big\}.$$

(see [B] for basic notions about graphs). In the sequel, we will denote $\overline{f_\varepsilon} := (f_\varepsilon, \varepsilon)$.

Definition 1.9 *We say that a tree T on 2×2 is a $\underline{\textit{tree with acyclic levels}}$ if, for each integer p, the graph $G_{\mathcal{T}_p}$, associated with $\mathcal{T}_p := T \cap (2^p \times 2^p) \subseteq 2^p \times 2^p$, is acyclic.*

Now we can state the main results proved in this paper:

Theorem 1.10 *(Debs-Lecomte) Let T be a tree with acyclic levels, $\xi < \omega_1$, $S \in \mathbf{\Sigma}^0_{1+\xi}(\lceil T \rceil)$, X, Y Polish spaces, and A, B disjoint analytic subsets of $X \times Y$. Then one of the following holds:*
(a) The set A is separable from B by a pot($\mathbf{\Pi}^0_{1+\xi}$) set.
(b) There are $u : 2^\omega \rightarrow X$ and $v : 2^\omega \rightarrow Y$ continuous with $S \subseteq (u \times v)^{-1}(A)$ and $\lceil T \rceil \setminus S \subseteq (u \times v)^{-1}(B)$.
If we moreover assume that $S \notin$ pot($\mathbf{\Pi}^0_{1+\xi}$), then this is a dichotomy.

Note that we can deduce Theorem 1.3 from the proof of Theorem 1.10. Theorem 1.10 is the analogous of Theorem 1.3 in dimension two. The proofs of Theorem 1.3 in [Lo-SR], and also Theorem III-2.1 in [D-SR], use games. This is not the case here, so that we get a new proof of Theorem 1.3.

Theorem 1.11 *We can find concrete examples of a tree T with acyclic levels, together with sets $S_{1+\xi} \in \mathbf{\Sigma}^0_{1+\xi}(\lceil T \rceil) \setminus$pot($\mathbf{\Pi}^0_{1+\xi}$), for each $\xi < \omega_1$.*

The following corollary has initially been shown by D. Lecomte when $1+\xi$ is a successor ordinal. Then G. Debs showed it when $1+\xi$ is a limit ordinal.

Corollary 1.12 *(Debs-Lecomte) Let $\xi < \omega_1$. There is $S \in \mathbf{\Delta}^1_1(2^\omega \times 2^\omega)$ such that for any Polish spaces X, Y, and for any disjoint analytic subsets A, B of $X \times Y$, exactly one of the following holds:*
(a) The set A is separable from B by a pot($\mathbf{\Pi}^0_{1+\xi}$) set.
(b) There are $u : 2^\omega \rightarrow X$ and $v : 2^\omega \rightarrow Y$ continuous with $S \subseteq (u \times v)^{-1}(A)$ and $\overline{S} \setminus S \subseteq (u \times v)^{-1}(B)$.

Theorem 1.8 shows that we cannot replace $\overline{S} \setminus S$ with $\neg S$ in Corollary 1.12 when $\xi = 0$. G. Debs found a simpler proof, which moreover works in the general case:

Theorem 1.13 *(Debs) We cannot replace $\overline{S} \setminus S$ with $\neg S$ in Corollary 1.12.*

Once again, some cycles are involved, so that the acyclicity is essentially necessary and sufficient in Corollary 1.12 (even if we have two different notions of acyclicity). G. Debs proved very recently that we can have u and v one-to-one in Corollary 1.12 if $\xi \geq 2$. This is not possible if $\xi < 2$ (see Theorem 1.8 when $\xi = 0$, and Theorem 15 in [L4] when $\xi = 1$).

This paper is organized as follows:

- In Section 2 we recall the material used to state the representation theorem of Borel sets proved in [D-SR]. We use it to prove Theorem 1.10, also in this section. To do this we assume some results proved in [Lo2]. We also prove Theorem 1.13.

- In Section 3 we prove Theorem 1.11.

- We use some tools of effective descriptive set theory (the reader should see [M] for the basic notions about it). In Section 4 we give an alternative proof of the results in [Lo2] that we assumed in Section 2. This leads to the following:

Theorem 1.14 *(Debs-Lecomte-Louveau) Let T given by Theorem 1.11, $\xi<\omega_1^{CK}$, S given by Theorem 1.11, X, Y be recursively presented Polish spaces, and A, B disjoint Σ^1_1 subsets of $X\times Y$. Then the following are equivalent:*
(a) The set A cannot be separated from B by a pot$(\mathbf{\Pi}^0_{1+\xi})$ set.
(b) The set A cannot be separated from B by a $\Delta^1_1\cap$ pot$(\mathbf{\Pi}^0_{1+\xi})$ set.
(c) There are $u:2^\omega\to X$ and $v:2^\omega\to Y$ continuous with $S\subseteq(u\times v)^{-1}(A)$ and $\lceil T\rceil\setminus S\subseteq(u\times v)^{-1}(B)$.

The equivalence between (a) and (b) is proved in [Lo2]. We will actually prove more than Theorem 1.14, with some additional notation that will be introduced later. Among other things, we will use the fact that the set of codes for Δ^1_1 and pot$(\mathbf{\Pi}^0_{1+\xi})$ sets is Π^1_1.

2 Proof of Theorem 1.10.

2.1 Acyclicity.

In this subsection we prove a result that will be used later to show Theorem 1.10. This is the place where the essence of the notion of a tree with acyclic levels is really used. We will also prove that we cannot have a reduction on the whole product, using some cycles. Some of the arguments used in the initial proof of Corollary 1.12 by D. Lecomte (when $1+\xi$ is a successor ordinal) are replaced here by Lemma 2.1.2 below.

Definition 2.1.1 *(Debs) Let $\mathcal{F}_0$, $\mathcal{F}_1$, X_0, X_1 be sets, $\mathcal{T}\subseteq\mathcal{F}_0\times\mathcal{F}_1$ and $\Psi:\mathcal{F}_0\times\mathcal{F}_1\to 2^{X_0\times X_1}$. We say that $\psi=\psi_0\times\psi_1:\mathcal{F}_0\times\mathcal{F}_1\to X_0\times X_1$ is a $\underline{\pi-selector\ on\ \mathcal{T}\ for\ \Psi}$ if:*
(a) $\psi(f_0,f_1)=[\psi_0(f_0),\psi_1(f_1)]$, for each $(f_0,f_1)\in\mathcal{F}_0\times\mathcal{F}_1$.
(b) $\psi(t)\in\Psi(t)$, for each $t\in\mathcal{T}$.

Notation. Let X be a recursively presented Polish space. We denote by Δ_X the topology on X generated by $\Delta^1_1(X)$. This topology is Polish (see (iii)⇒(i) in the proof of Theorem 3.4 in [Lo2]). We set $\tau_1:=\Delta_X\times\Delta_Y$ if Y is also a recursively presented Polish space.

Lemma 2.1.2 *(Debs) Let $\mathcal{F}_0$, $\mathcal{F}_1$ be finite sets, $\mathcal{T}\subseteq\mathcal{F}_0\times\mathcal{F}_1$ such that the graph $G_{\mathcal{T}}$ associated with $\mathcal{T}$ is acyclic, X_0, X_1 recursively presented Polish spaces, $\Psi:\mathcal{F}_0\times\mathcal{F}_1\to\Sigma^1_1(X_0\times X_1)$, and $\overline{\Psi}:\mathcal{F}_0\times\mathcal{F}_1\to\Sigma^1_1(X_0\times X_1)$ defined by $\overline{\Psi}(t):=\overline{\Psi(t)}^{\tau_1}$. Then Ψ admits a π-selector on $\mathcal{T}$ if $\overline{\Psi}$ does.*

Proof. (a) Let $t_0:=(f_0,f_1)\in\mathcal{T}$, and $\Phi:\mathcal{F}_0\times\mathcal{F}_1\to\Sigma^1_1(X_0\times X_1)$. We assume that $\Phi(t)=\Psi(t)$ if $t\neq t_0$, and that $\Phi(t_0)\subseteq\overline{\Psi(t_0)}^{\tau_1}$. We first prove that Ψ admits a π-selector on $\mathcal{T}$ if Φ does.

• Fix a π-selector $\tilde\varphi$ on $\mathcal{T}$ for Φ. We define Σ^1_1 sets U_ε, for $\varepsilon\in 2$, by

$$U_\varepsilon:=\{\ x\in X_\varepsilon\mid\exists\varphi:\mathcal{F}_0\times\mathcal{F}_1\to X_0\times X_1\quad x=\varphi_\varepsilon(f_\varepsilon)\ \text{ and }\ \forall t\in\mathcal{T}\ \varphi(t)\in\Phi(t)\ \}.$$

As $\tilde\varphi(t_0)=[\tilde\varphi_0(f_0),\tilde\varphi_1(f_1)]\in\Phi(t_0)\cap(U_0\times U_1)$ we get $\emptyset\neq\Phi(t_0)\cap(U_0\times U_1)\subseteq\overline{\Psi(t_0)}^{\tau_1}\cap(U_0\times U_1)$. By the separation theorem this implies that $\Psi(t_0)\cap(U_0\times U_1)$ is not empty and contains some point (x_0,x_1). Fix $\varepsilon\in 2$. As $x_\varepsilon\in U_\varepsilon$ there is $\psi^\varepsilon:\mathcal{F}_0\times\mathcal{F}_1\to X_0\times X_1$ such that $x_\varepsilon=\psi^\varepsilon_\varepsilon(f_\varepsilon)$ and $\psi^\varepsilon(t)\in\Phi(t)$, for each $t\in\mathcal{T}$.

• If $e_0\neq e_0'\in\mathcal{F}_0$ and $[(\tilde e_i,j_i)]_{i\leq l}$ is a path in $G_\mathcal{T}$ with $(\tilde e_0,j_0)=\overline{e_0}$ and $(\tilde e_l,j_l)=\overline{e_0'}$, then it is unique by Theorem I.2.5 in [B]. We call it $p_{e_0,e_0'}$. We will define a partition of $\mathcal{F}_0\times\mathcal{F}_1$. We put

$$N:=\{\ (e_0,e_1)\in\mathcal{F}_0\times\mathcal{F}_1\setminus\{t_0\}\mid(e_0,e_1)\notin\mathcal{T}\ \text{ or }\ [e_0\neq f_0\text{ and }p_{e_0,f_0}\text{ does not exist}]\ \},$$

$$H:=\{\ (e_0,e_1)\in\mathcal{T}\setminus\{t_0\}\mid e_0\neq f_0\text{ and }p_{e_0,f_0}(|p_{e_0,f_0}|-2)=\overline{f_1}\ \},$$

$$V:=\{\ (e_0,e_1)\in\mathcal{T}\setminus\{t_0\}\mid e_0=f_0\ \text{ or }\ [e_0\neq f_0\text{ and }p_{e_0,f_0}(|p_{e_0,f_0}|-2)\neq\overline{f_1}]\ \}.$$

The definition of H means that if we view the graph $G_\mathcal{T}$ as $\mathcal{T}$ itself in the product $\mathcal{F}_0\times\mathcal{F}_1$ instead of seeing it in the sum $\mathcal{F}_0\oplus\mathcal{F}_1$, then the last edge in the path from (e_0,e_1) to t_0 is horizontal (and vertical in V). So we defined a partition $(\{t_0\},N,H,V)$ of $\mathcal{F}_0\times\mathcal{F}_1$.

• Let us show that $\Pi_{\mathcal{F}_\varepsilon}[H]\cap\Pi_{\mathcal{F}_\varepsilon}[V]=\emptyset$, for each $\varepsilon\in 2$.

We may assume that $\varepsilon=1$. We argue by contradiction. This gives $e_1\in\Pi_{\mathcal{F}_1}[H]\cap\Pi_{\mathcal{F}_1}[V]$, and also e_0 (resp., e_0') such that $(e_0,e_1)\in H$ (resp., $(e_0',e_1)\in V$). Note that $e_0\neq f_0$, and also that $e_1\neq f_1$ (by contradiction, we get $e_0'\neq f_0$ since $(e_0',e_1)\neq t_0$, and $p_{e_0',f_0}=(\overline{e_0'},\overline{f_1},\overline{f_0})$, which is absurd). If $e_0'=f_0$, then $\overline{e_1}^\frown p_{e_0,f_0}{}^\frown\overline{e_1}$ gives a cycle, which is absurd. If $e_0'\neq f_0$, then $\overline{e_1}^\frown p_{e_0,f_0}$ and $\overline{e_1}^\frown p_{e_0',f_0}$ give two different pathes from $\overline{e_1}$ to $\overline{f_0}$, which is also absurd.

• Now we can define $\psi_\varepsilon:\mathcal{F}_\varepsilon\to X_\varepsilon$. We put

$$\psi_0(e_0):=\begin{cases}x_0\ \text{ if }\ e_0=f_0,\\[2ex]\psi^1_0(e_0)\ \text{ if }\ e_0\in\Pi_{\mathcal{F}_0}[H],\\[2ex]\psi^0_0(e_0)\ \text{ otherwise,}\end{cases}$$

$$\psi_1(e_1):=\begin{cases}x_1\ \text{ if }\ e_1=f_1,\\[2ex]\psi^1_1(e_1)\ \text{ if }\ e_1\in\Pi_{\mathcal{F}_1}[H]\setminus\{f_1\},\\[2ex]\psi^0_1(e_1)\ \text{ otherwise.}\end{cases}$$

Then we set $\psi(e_0,e_1):=[\psi_0(e_0),\psi_1(e_1)]$.

• It remains to see that $\psi(t)\in\Psi(t)$, for each $t\in\mathcal{T}$. Notice first that $\psi(t_0)=(x_0,x_1)\in\Psi(t_0)$. If $t:=(e_0,e_1)\in V$ and $e_0\neq f_0$, then we get

$$\psi(t)=[\psi_0(e_0),\psi_1(e_1)]=[\psi_0^0(e_0),\psi_1^0(e_1)]=\psi^0(t)\in\Phi(t)=\Psi(t).$$

Now if $t\in V$ and $e_0=f_0$, then we get

$$\psi(t)=[x_0,\psi_1^0(e_1)]=[\psi_0^0(f_0),\psi_1^0(e_1)]=[\psi_0^0(e_0),\psi_1^0(e_1)]=\psi^0(t)\in\Phi(t)=\Psi(t).$$

We argue similarly if $t\in H$.

If $t\in N\cap\mathcal{T}$, then $e_0\neq f_0$. If moreover $e_1\notin(\{f_1\}\cup\Pi_{\mathcal{F}_1}[H])$, then we get

$$\psi(t)=[\psi_0(e_0),\psi_1(e_1)]=[\psi_0^0(e_0),\psi_1^0(e_1)]=\psi^0(t)\in\Phi(t)=\Psi(t).$$

If $e_1=f_1$, then $p_{e_0,f_0}=(\overline{e_0},\overline{e_1},\overline{f_0})$ exists, which is absurd. If $e_1\in\Pi_{\mathcal{F}_1}[H]\setminus\{f_1\}$, let $e_0'\in\mathcal{F}_0$ with $(e_0',e_1)\in H$. The sequence $(\overline{e_0},\overline{e_1},\overline{e_0'},\ldots,\overline{f_1},\overline{f_0})$ shows that p_{e_0,f_0} exists, which is absurd again.

(b) Write $\mathcal{T}:=\{t_1,\ldots,t_n\}$, and set $\Phi_0:=\overline{\Psi}$. We define $\Phi_{j+1}:\mathcal{F}_0\times\mathcal{F}_1\rightarrow\Sigma_1^1(X_0\times X_1)$ as follows. We put $\Phi_{j+1}(t):=\Phi_j(t)$ if $t\neq t_{j+1}$, and $\Phi_{j+1}(t_{j+1}):=\Psi(t_{j+1})$, for $j<n$. The result now follows from an iterative application of (a). □

Proof of Theorem 1.13. We argue by contradiction. This gives a Borel set S'. Consider first that $A:=S'$ and $B:=\neg S'$. Then (b) holds with $u=v=\mathrm{Id}_{2^\omega}$. So (a) does not hold and S' is not pot($\mathbf{\Pi}^0_{1+\xi}$).

Consider now that $A:=S$ and $B:=\lceil T\rceil\setminus S$, where T and S are given by Theorem 1.11. As (a) does not hold, (b) holds. This gives continuous maps u, v with

$$S'\subseteq(u\times v)^{-1}(S)\subseteq(u\times v)^{-1}(\lceil T\rceil),$$

$$\neg S'\subseteq(u\times v)^{-1}(\lceil T\rceil\setminus S)\subseteq(u\times v)^{-1}(\lceil T\rceil).$$

Claim. There is a Borel subset A of 2^ω with $S'=A\times 2^\omega$ or $S'=2^\omega\times A$.

• We argue by contradiction to prove the claim. There are $\alpha\in 2^\omega$, and $\beta\neq\beta'\in 2^\omega$ such that $(\alpha,\beta)\in S'$ and $(\alpha,\beta')\notin S'$ (otherwise $A:=(S')^{0^\infty}\in\mathbf{\Delta}^1_1(2^\omega)$ and satisfies $S'=A\times 2^\omega$). Then $(u(\alpha),v(\beta))\in S$ and $(u(\alpha),v(\beta'))\notin S$, thus $v(\beta)\neq v(\beta')$.

• Note that $(\alpha',\beta)\in S'$, for each $\alpha'\in 2^\omega$. Indeed, we argue by contradiction. This gives α' with $(u(\alpha'),v(\beta))\notin S$. Thus $u(\alpha)\neq u(\alpha')$, and $(u(\alpha),v(\beta))$, $(u(\alpha'),v(\beta))$, $(u(\alpha),v(\beta'))$, $(u(\alpha'),v(\beta'))$ are in $\lceil T\rceil$. Let $p\in\omega$ with $e_0:=u(\alpha)\lceil p\neq e_0':=u(\alpha')\lceil p$ and $e_1:=v(\beta)\lceil p\neq e_1':=v(\beta')\lceil p$. Then (e_0,e_1), (e_0',e_1), (e_0,e_1'), $(e_0',e_1')\in\mathcal{T}_p$, and the sequence $(\overline{e_0},\overline{e_1},\overline{e_0'},\overline{e_1'},\overline{e_0})$ is a cycle, which is absurd.

• Let $\gamma\in S'_\alpha$. We have $(\alpha',\gamma)\in S'$, for each $\alpha'\in 2^\omega$, as before. Conversely, assume that $(\alpha',\gamma)\in S'$. Then $\gamma\in S'_\alpha$, as before. Thus $S'=2^\omega\times S'_\alpha$, which is absurd. This proves the claim. ◇

Now the claim contradicts the fact that S' is not pot($\mathbf{\Pi}^0_{1+\xi}$). □

2.2 The topologies.

In this subsection we prove another result that will be used to show Theorem 1.10. Some topologies are involved, and this is the place where we use some results in [Lo2].

Notation. Let X, Y be recursively presented Polish spaces.

• Recall the existence of Π^1_1 sets $W^X\subseteq\omega$, $C^X\subseteq\omega\times X$ with $\Delta^1_1(X)=\{C^X_n\mid n\in W^X\}$ and $\{(n,x)\in\omega\times X\mid n\in W^X \text{and } x\notin C^X_n\}\in\Pi^1_1(\omega\times X)$ (see Theorem 3.3.1 in [H-K-Lo]).

• Set pot$(\mathbf{\Pi}^0_0):=\mathbf{\Delta}^1_1(X)\times\mathbf{\Delta}^1_1(Y)$ and, for $\xi<\omega_1^{\text{CK}}$,

$$W^{X\times Y}_\xi:=\{p\in W^{X\times Y}\mid C^{X\times Y}_p\in\text{pot}(\mathbf{\Pi}^0_\xi)\}.$$

We also set $W^{X\times Y}_{<\xi}:=\bigcup_{\eta<\xi}\ W^{X\times Y}_\eta$.

The following result is essentially proved in [Lo2]. However, the statement is not in it, so we give a proof, which uses several statements in [Lo2]. Recall that τ_1 is defined before Lemma 2.1.2.

Theorem 2.2.1 *(Louveau) Let $\xi<\omega_1^{CK}$, X, Y be recursively presented Polish spaces. Then $W^{X\times Y}_\xi$ and $W^{X\times Y}_{<\xi}$ are Π^1_1. If moreover A, B are disjoint Σ^1_1 subsets of $X\times Y$, then the following are equivalent:*
(a) The set A is separable from B by a pot$(\mathbf{\Pi}^0_{1+\xi})$ set.
(b) The set A is separable from B by a $\Delta^1_1\cap$ pot$(\mathbf{\Pi}^0_{1+\xi})$ set.
(c) The set A is separable from B by a $\mathbf{\Pi}^0_{1+\xi}(\tau_1)$ set.

Proof. By the second paragraph page 44 in [Lo2], $\mathbf{\Delta}^1_1(X)$ and $\mathbf{\Delta}^1_1(Y)$ are regular families (see Definition 2.7 in [Lo2] for the definition of a regular family). By Theorem 2.12 in [Lo2], the family $\Phi:=$ pot$(\mathbf{\Pi}^0_0)$ is regular too. We define a sequence $(\Phi_\xi)_{\xi<\omega_1^{\text{CK}}}$ of families as follows (see Corollary 2.10.(v) in [Lo2]):

$$\begin{aligned}\Phi_0 &:=\Phi,\\ \Phi_{\xi+1}&:=(\Phi_\xi)_{\sigma\text{c}},\\ \Phi_\lambda &:=\textstyle\bigcup_{\xi<\lambda}\ \Phi_\xi\ \text{ if }\ 0<\lambda<\omega_1^{\text{CK}}\text{ is a limit ordinal.}\end{aligned}$$

By Corollary 2.10.(v) in [Lo2], Φ_ξ is a regular family for each $\xi<\omega_1^{\text{CK}}$. In particular, the set $W_{\Phi_\xi}:=\{p\in W^{X\times Y}\mid C^{X\times Y}_p\in\Phi_\xi\}$ is $\Pi^1_1(\omega)$. By Theorem 2.8 in [Lo2], the family $\Phi_{\xi+1}$ is a separating family (see Definition 2.1 in [Lo2] for the definition of a separating family), for each $\xi<\omega_1^{\text{CK}}$. An easy induction on ξ shows the following facts:

$$\begin{aligned}\Phi_\xi &=\text{pot}(\mathbf{\Pi}^0_\xi)\ \text{ if }\ \xi<\omega,\\ \Phi_\xi &=\textstyle\bigcup_{\eta<\xi}\ \text{pot}(\mathbf{\Pi}^0_\eta)\ \text{ if }\ 0<\xi<\omega_1^{\text{CK}}\text{ is a limit ordinal,}\\ \Phi_{\xi+1}&=\text{pot}(\mathbf{\Pi}^0_\xi)\ \text{ if }\ \omega\leq\xi<\omega_1^{\text{CK}}.\end{aligned}$$

This shows that $W^{X\times Y}_\xi=W_{\Phi_\xi}$ is Π^1_1 if $\xi<\omega$, $W^{X\times Y}_\xi=W_{\Phi_{\xi+1}}$ is Π^1_1 if $\omega\leq\xi<\omega_1^{\text{CK}}$. If $0<\xi<\omega_1^{\text{CK}}$ is a limit ordinal, then $W^{X\times Y}_{<\xi}=W_{\Phi_\xi}$ is Π^1_1.

(b) $\Rightarrow$ (c) follows from Theorem 3.4 in [Lo2].

(c) $\Rightarrow$ (a) follows from the fact that $\varDelta_X$ and $\varDelta_Y$ are Polish.

(a) $\Rightarrow$ (b) Assume first that $\xi < \omega$. Then $\text{pot}(\mathbf{\Pi}^0_{1+\xi}) = \Phi_{1+\xi} = \Phi_{\xi+1}$ is a separating family. So A and B are separable by a $\varDelta^1_1 \cap \Phi_{\xi+1} = \varDelta^1_1 \cap \text{pot}(\mathbf{\Pi}^0_{1+\xi})$ set. If $\omega \leq \xi < \omega_1^{\text{CK}}$, then we use the fact that $\text{pot}(\mathbf{\Pi}^0_{1+\xi}) = \text{pot}(\mathbf{\Pi}^0_{\xi}) = \Phi_{\xi+1}$. $\square$

Notation. Let X, Y be recursively presented Polish spaces.

• We will use the Gandy-Harrington topology $\varSigma_X$ on X generated by $\varSigma^1_1(X)$. Recall that the set $\Omega_X := \{x \in X \mid \omega_1^x = \omega_1^{\text{CK}}\}$ is Borel and $\varSigma^1_1$, that $[\Omega_X, \varSigma_X]$ is a 0-dimensional Polish space (the intersection of Ω_X with any nonempty $\varSigma^1_1$ set is a nonempty clopen subset of $[\Omega_X, \varSigma_X]$) (see [L1]).

• Let $2 \leq \xi < \omega_1^{\text{CK}}$. The topology τ_ξ is generated by $\varSigma^1_1(X \times Y) \cap \mathbf{\Pi}^0_{<\xi}(\tau_1)$. We have the inclusion $\mathbf{\Sigma}^0_1(\tau_\xi) \subseteq \mathbf{\Sigma}^0_\xi(\tau_1)$, so that $\mathbf{\Pi}^0_1(\tau_\xi) \subseteq \mathbf{\Pi}^0_\xi(\tau_1)$. These topologies are similar to the ones considered in [Lo1] (see Definition 1.5).

Lemma 2.2.2 *Let X, Y be recursively presented Polish spaces, and $\xi < \omega_1^{CK}$.*
(a) Fix $S \in \varSigma^1_1(X \times Y)$. Then $\overline{S}^{\tau_{1+\xi}} \in \varSigma^1_1(X \times Y)$.
(b) Let $n \geq 1$, $1 \leq \xi_1 < \xi_2 < \ldots < \xi_n \leq 1+\xi$, and $S_1, \ldots, S_n$ be $\varSigma^1_1$ sets. Assume that $S_{n'} \subseteq \overline{S_{n'+1}}^{\tau_{\xi_{n'+1}}}$ for $1 \leq n' < n$. Then $S_n \cap \bigcap_{1 \leq i < n} \overline{S_i}^{\tau_{\xi_i}}$ is τ_1-dense in $\overline{S_1}^{\tau_1}$.

Proof. (a) This is essentially proved in [Lo2] (see the proof of Theorem 2.8 in [Lo2]). We emphasize the fact that the analogous version of (a) in [Lo2] and the assertions of Theorem 2.2.1 are proved simultaneously by induction on ξ, and interact. Assume first that $\xi = 0$. Then

$$\begin{array}{rl} (x,y) \notin \overline{S}^{\tau_1} \Leftrightarrow & \exists U \in \varDelta^1_1(X)\ \exists V \in \varDelta^1_1(Y)\ \ (x,y) \in U \times V \text{ and } (U \times V) \cap S = \emptyset \\ \Leftrightarrow & \exists m \in W^X\ \exists n \in W^Y\ (\ C^X_m(x) \text{ and } C^Y_n(y) \text{ and } \forall (x',y') \in X \times Y \\ & [(m \in W^X \text{ and } x' \notin C^X_m) \text{ or } (n \in W^Y \text{ and } y' \notin C^Y_n) \text{ or } (x',y') \notin S]\ ). \end{array}$$

So $\overline{S}^{\tau_1} \in \varSigma^1_1(X \times Y)$. Now assume that $\xi \geq 1$. We have, by Theorem 2.2.1:

$$\begin{array}{rl} (x,y) \notin \overline{S}^{\tau_{1+\xi}} \Leftrightarrow & \exists T \in \varSigma^1_1(X \times Y) \cap \mathbf{\Pi}^0_{<1+\xi}(\tau_1)\ \ (x,y) \in T \text{ and } T \cap S = \emptyset \\ \Leftrightarrow & \exists E \in \varDelta^1_1(X \times Y) \cap \text{pot}(\mathbf{\Pi}^0_{<1+\xi})\ \ (x,y) \in E \text{ and } E \cap S = \emptyset \\ \Leftrightarrow & \exists m \in W^{X \times Y}_{<1+\xi}\ \ (\ C^{X \times Y}_m(x,y) \text{ and } \forall (x',y') \in X \times Y \\ & [(m \in W^{X \times Y} \text{ and } (x',y') \notin C^{X \times Y}_m) \text{ or } (x',y') \notin S]\ ). \end{array}$$

By Theorem 2.2.1, $W^{X \times Y}_{<1+\xi} \in \varPi^1_1$ and we are done.

(b) Let U (resp., V) a $\varDelta^1_1(X)$ (resp., $\varDelta^1_1(Y)$) set with $\overline{S_1}^{\tau_1} \cap (U \times V) \neq \emptyset$. Then $S_1 \cap (U \times V) \neq \emptyset$, which proves the desired property for $n = 1$. Then we argue inductively on n. So assume that the property is proved for n. We have $S_n \subseteq \overline{S_{n+1}}^{\tau_{\xi_{n+1}}}$, and $S_n \cap \bigcap_{1 \leq i < n} \overline{S_i}^{\tau_{\xi_i}} \cap (U \times V) \neq \emptyset$, by induction assumption. Thus $\overline{S_{n+1}}^{\tau_{\xi_{n+1}}} \cap \bigcap_{1 \leq i \leq n} \overline{S_i}^{\tau_{\xi_i}} \cap (U \times V) \neq \emptyset$. As $\bigcap_{1 \leq i \leq n} \overline{S_i}^{\tau_{\xi_i}} \cap (U \times V)$ is $\mathbf{\Sigma}^0_1(\tau_{\xi_{n+1}})$, we get $S_{n+1} \cap \bigcap_{1 \leq i \leq n} \overline{S_i}^{\tau_{\xi_i}} \cap (U \times V) \neq \emptyset$. $\square$

2.3 Representation of Borel sets.

Now we come to the representation theorem of Borel sets by G. Debs and J. Saint Raymond (see [D-SR]). It specifies the classical result of Lusin asserting that any Borel set in a Polish space is the bijective continuous image of a closed subset of the Baire space. The following definitions can be found in [D-SR]:

Definition 2.3.1 *(Debs-Saint Raymond) Let a be a finite set. A partial order relation R on $a^{<\omega}$ is a* $\underline{tree\ relation}$ *if, for $t\in a^{<\omega}$,*

(a) $\emptyset\ R\ t$.

(b) The set $P_R(t):=\{s\in a^{<\omega}\mid s\ R\ t\}$ is finite and linearly ordered by R.

For instance, the non strict extension relation $\prec$ is a tree relation.

• *Let R be a tree relation. An* $\underline{R-branch}$ *is an $\subseteq$-maximal subset of $a^{<\omega}$ linearly ordered by R. We denote by $[R]$ the set of all infinite R-branches.*

We equip $(a^{<\omega})^\omega$ with the product of the discrete topology on $a^{<\omega}$. If R is a tree relation, the space $[R]\subseteq(a^{<\omega})^\omega$ is equipped with the topology induced by that of $(a^{<\omega})^\omega$. The map $\theta : a^\omega\rightarrow[\prec]$ defined by $\theta(\gamma):=[\gamma\lceil j]_{j\in\omega}$ is an homeomorphism.

• *Let R, S be tree relations with $R\subseteq S$. The* $\underline{canonical\ map}$ *$\Pi : [R]\rightarrow[S]$ is defined by*

$$\Pi(A):=\text{the unique } S\text{-branch containing } A.$$

• *Let S be a tree relation. We say that $R\subseteq S$ is* $\underline{distinguished}$ *in S if*

$$\forall s,t,u\in a^{<\omega}\quad \left.\begin{array}{c} s\ S\ t\ S\ u\\ \\ s\ R\ u\end{array}\right\}\ \Rightarrow\ s\ R\ t.$$

For example, let C be a closed subset of a^ω, and define:

$$s\ R\ t\ \Leftrightarrow\ s\prec t \text{ and } N_t\cap C\neq\emptyset.$$

Then R is distinguished in $\prec$. In this case, the distinction expresses the fact that “when we leave the closed set, it is for ever”.

• *Let $\eta<\omega_1$. A family $(R^{(\rho)})_{\rho\leq\eta}$ of tree relations is a* $\underline{resolution\ family}$ *if:*

(a) $R^{(\rho+1)}$ is a distinguished subtree of $R^{(\rho)}$, for all $\rho<\eta$.

(b) $R^{(\lambda)}=\bigcap_{\rho<\lambda}\ R^{(\rho)}$, for all limit $\lambda\leq\eta$.

We will use the following extension of the property of distinction:

Lemma 2.3.2 *Let $\eta<\omega_1$, $(R^{(\rho)})_{\rho\leq\eta}$ a resolution family with $R^{(0)}=\prec$, and $\rho<\eta$. Assume that $s\prec s'\ R^{(\rho)}\ s''$ and $s\ R^{(\rho+1)}\ s''$. Then $s\ R^{(\rho+1)}\ s'$.*

Proof. We argue by induction on ρ. Assume that the property is proved for $\mu<\rho$. As $s'\ R^{(\rho)}\ s''$ and $R^{(\rho+1)}$ is distinguished in $R^{(\rho)}$ we have $s\ R^{(\rho+1)}\ s'$. $\square$

Notation. Let $\eta<\omega_1$, $(R^{(\rho)})_{\rho\leq\eta}$ a resolution family with $R^{(0)}=\prec$, $\rho\leq\eta$ and $z\in a^{<\omega}\setminus\{\emptyset\}$. We set

$$z^\rho := z\lceil \max\{r<|z| \mid z\lceil r\ R^{(\rho)}\ z\}.$$

We enumerate $\{z^\rho \mid \rho\leq\eta\}$ by $\{z^{\xi_i} \mid 1\leq i\leq n\}$, where $1\leq n\in\omega$ and $\xi_1<\ldots<\xi_n=\eta$. We can write $z^{\xi_n}\prec_{\neq} z^{\xi_{n-1}}\prec_{\neq}\ldots\prec_{\neq} z^{\xi_2}\prec_{\neq} z^{\xi_1}\prec_{\neq} z$. By Lemma 2.3.2 we have $z^{\xi_{i+1}}\ R^{(\xi_i+1)}\ z^{\xi_i}$ for each $1\leq i<n$.

Lemma 2.3.3 *Let $\eta<\omega_1$, $(R^{(\rho)})_{\rho\leq\eta}$ a resolution family with $R^{(0)}=\prec$, $z\in a^{<\omega}\setminus\{\emptyset\}$ and $1\leq i<n$.*
(a) Set $\eta_i:=\{\rho\leq\eta \mid z^{\xi_i}\prec z^\rho\}$. Then η_i is a successor ordinal.
(b) We may assume that $z^{\xi_i+1}\prec_{\neq} z^{\xi_i}$.

Proof. (a) First notice that η_i is an ordinal. Note that $\xi_i+1\leq\eta_i\leq\eta+1$. We argue by contradiction, so that $\eta_i\leq\eta$. Let $\xi_i\leq\rho<\eta_i$. Then we have $z^{\xi_i}=z^\rho$, $z^{\xi_i}\ R^{(\rho)}\ z$, $z^{\xi_i}\ R^{(\eta_i)}\ z$, and $z^{\xi_i}\prec z^{\eta_i}$. As $\eta_i\leq\eta$, we get $\eta_i\in\eta_i$, which is absurd.

(b) So we can write $\eta_i=\nu_i+1$. Note that $z^{\nu_i}=z^{\xi_i}$ since $\xi_i\leq\nu_i$. If $\nu_i+1\leq\eta$ we get $z^{\nu_i+1}\prec_{\neq} z^{\nu_i}$, so we may assume that $\xi_i=\nu_i$. If $\nu_i+1=\eta+1$ we get $\nu_i=\eta$ and $z^{\xi_i}=z^{\nu_i}=z^\eta=z^{\xi_n}$, which is absurd.□

The following is part of Theorem I-6.6 in [D-SR].

Theorem 2.3.4 *(Debs-Saint Raymond) Let $\eta<\omega_1$, E be a $\mathbf{\Pi}^0_{\eta+1}$ subset of $[\prec]$. Then there is a resolution family $(R^{(\rho)})_{\rho\leq\eta}$ with:*
(a) $R^{(0)}=\prec$.
(b) The canonical map $\Pi:[R^{(\eta)}]\to[\prec]$ is a bijection.
(c) The set $\Pi^{-1}(E)$ is a closed subset of $[R^{(\eta)}]$.

Now we come to the actual proof of Theorem 1.10.

2.4 Proof of Theorem 1.10.

Theorem 2.4.1 *Let T be a tree with acyclic levels, $\xi<\omega_1^{CK}$ such that $1+\xi$ is a successor ordinal, $S\in\mathbf{\Sigma}^0_{1+\xi}(\lceil T\rceil)$, X, Y recursively presented Polish spaces, and A, B disjoint Σ^1_1 subsets of $X\times Y$. Then one of the following holds:*
(a) $\overline{A}^{\tau_{1+\xi}}\cap B=\emptyset$.
(b) There are $u:2^\omega\to X$ and $v:2^\omega\to Y$ continuous with $S\subseteq(u\times v)^{-1}(A)$ and $\lceil T\rceil\setminus S\subseteq(u\times v)^{-1}(B)$.

Proof. Fix $\eta<\omega_1^{\rm CK}$ with $1+\xi=\eta+1$.

• We identify $(2\times 2)^Q$ with $2^Q\times 2^Q$, for $Q\leq\omega$. With the notation of Definition 2.3.1 and $a:=2\times 2$, we get $E:=\theta[\lceil T\rceil\setminus S]\in\mathbf{\Pi}^0_{\eta+1}([\prec])$. Theorem 2.3.4 provides a resolution family. We put

$$D:=\{(s,t)\in T \mid \exists\gamma\in\Pi^{-1}(E)\ \ (s,t)\in\gamma\}.$$

For example, we may assume that $(\emptyset,\emptyset)\in D$.

• We set $N:=\overline{A}^{\tau_{1+\xi}}\cap B$. Applying Lemma 2.2.2.(a), we see that N is Σ^1_1. We assume that N is not empty. Recall that $[\Omega_{X\times Y},\Sigma_{X\times Y}]$ is a Polish space (see the notation before Lemma 2.2.2). We fix a complete metric d (resp., metrics δ_X, δ_Y) on $[\Omega_{X\times Y},\Sigma_{X\times Y}]$ (resp., X, Y equipped with the initial topologies).

• We construct $(x_s)_{s\in\Pi_0[T]}\subseteq X$, $(y_t)_{t\in\Pi_1[T]}\subseteq Y$, $(U_{(s,t)})_{(s,t)\in T}\subseteq\Sigma^1_1(X\times Y)$ with:

(i) $(x_s,y_t)\in U_{(s,t)}\subseteq\Omega_{X\times Y}$.

(ii) $\text{diam}_d(U_{(s,t)})\leq 2^{-|s|}$, $\delta_X(x_s,x_{s\varepsilon})\leq 2^{-|s|}$, $\delta_Y(y_t,y_{t\varepsilon})\leq 2^{-|t|}$.

(iii) $U_{(s,t)}\subseteq N$ if $(s,t)\in D$.

(iv) $U_{(s,t)}\subseteq A$ if $(s,t)\notin D$.

(v) $[1\leq\rho\leq\eta \text{ and } (s,t)\ R^{(\rho)}\ (s',t')]\ \Rightarrow\ U_{(s',t')}\subseteq\overline{U_{(s,t)}}^{\tau_\rho}$.

(vi) $[((s,t)\in D\ \Leftrightarrow\ (s',t')\in D) \text{ and } (s,t)\ R^{(\eta)}\ (s',t')]\ \Rightarrow\ U_{(s',t')}\subseteq U_{(s,t)}$.

• Let us show that this construction is sufficient to get the theorem. If $(\alpha,\beta)\in\lceil T\rceil$, then we can define $(j_i)_{i\in\omega}:=(j_i^{\alpha,\beta})_{i\in\omega}$ by $\Pi^{-1}([(\alpha,\beta)\lceil j]_{j\in\omega})=[(\alpha,\beta)\lceil j_i]_{i\in\omega}$, where $j_i<j_{i+1}$. In particular, we have $(\alpha,\beta)\lceil j_i\ R^{(\eta)}\ (\alpha,\beta)\lceil j_{i+1}$. We have the following:

$$(\alpha,\beta)\in S\ \Leftrightarrow\ \theta(\alpha,\beta)=[(\alpha,\beta)\lceil j]_{j\in\omega}\notin E\ \Leftrightarrow\ [(\alpha,\beta)\lceil j_i]_{i\in\omega}\notin\Pi^{-1}(E)$$

$$\Leftrightarrow\ \exists i_0\in\omega\ \forall i\geq i_0\ \ (\alpha,\beta)\lceil j_i\notin D$$

since $\Pi^{-1}(E)$ is a closed subset of $[R^{(\eta)}]$. Similarly, $(\alpha,\beta)\in\lceil T\rceil\setminus S$ is equivalent to the existence of $i_0\in\omega$ such that $(\alpha,\beta)\lceil j_i\in D$ for each $i\geq i_0$ (with $i_0=0$).

Therefore $U_{(\alpha,\beta)\lceil j_{i+1}}\subseteq U_{(\alpha,\beta)\lceil j_i}\subseteq\Omega_{X\times Y}$ if $i\geq i_0$ and $(\alpha,\beta)\in\lceil T\rceil$. Thus $(U_{(\alpha,\beta)\lceil j_i})_{i\geq i_0}$ is a decreasing sequence of nonempty clopen subsets of $[\Omega_{X\times Y},d]$ whose diameters tend to 0. Therefore $\{F(\alpha,\beta)\}=\bigcap_{i\geq i_0}\ U_{(\alpha,\beta)\lceil j_i}$ defines $F(\alpha,\beta)$ in $\Omega_{X\times Y}$. Note that $F(\alpha,\beta)$ is the limit of the sequence $((x_{\alpha\lceil j_i},y_{\beta\lceil j_i}))_{i\in\omega}$.

Let $\alpha\in\Pi_0(\lceil T\rceil)$, and β_α such that $(\alpha,\beta_\alpha)\in\lceil T\rceil$. We set $u(\alpha):=\Pi_X(F(\alpha,\beta_\alpha))$. Note that $u(\alpha)$ is the limit of some subsequence of $(x_{\alpha\lceil i})_{i\in\omega}$, by continuity of the projection. As $\delta_X(x_s,x_{s\varepsilon})\leq 2^{-|s|}$, $u(\alpha)$ is also the limit of $(x_{\alpha\lceil i})_{i\in\omega}$. Thus $u(\alpha)$ does not depend on the choice of β_α. This also shows that u is continuous on $\Pi_0(\lceil T\rceil)$. As $\Pi_0(\lceil T\rceil)$ is a closed subset of 2^ω, we can find a continuous retraction r_0 from 2^ω onto $\Pi_0(\lceil T\rceil)$ (see Proposition 2.8 in [K]). We set $u(\alpha):=u(r_0(\alpha))$, so that u is continuous on 2^ω.

Similarly, we define a continuous map $v:2^\omega\rightarrow Y$ such that $v(\beta)$ is the limit of $(y_{\beta\lceil i})_{i\in\omega}$ if β is in $\Pi_1(\lceil T\rceil)$. This implies that $F(\alpha,\beta)=(u(\alpha),v(\beta))$ if $(\alpha,\beta)\in\lceil T\rceil$.

If $(\alpha,\beta)\in S$ (resp., $\lceil T\rceil\setminus S$), then $F(\alpha,\beta)\in A$ (resp., N). This shows that $S\subseteq(u\times v)^{-1}(A)$ and $\lceil T\rceil\setminus S\subseteq(u\times v)^{-1}(B)$.

• So let us show that the construction is possible. Fix $(x_\emptyset,y_\emptyset)\in N\cap\Omega_{X\times Y}$, which is not empty since $N\neq\emptyset$ is Σ^1_1. Then we choose $U_{(\emptyset,\emptyset)}\in\Sigma^1_1$ with diameter at most 1 with $(x_\emptyset,y_\emptyset)\in U_{(\emptyset,\emptyset)}\subseteq N\cap\Omega_{X\times Y}$. Assume that $(x_s)_{|s|\leq p}$, $(y_t)_{|t|\leq p}$, $(U_{(s,t)})_{|s|\leq p}$ satisfying conditions (i)-(vi) have been constructed, which is the case for $p=0$.

- Let $s\in\Pi_0[T]\cap 2^p$ (resp., $t\in\Pi_1[T]\cap 2^p$), and X_s (resp., Y_t) be a Δ^1_1 neighborhood of x_s (resp., y_t) with δ_X-diameter (resp., δ_Y-diameter) at most 2^{-p}.

- If $we:=(s\varepsilon,t\varepsilon')\in T\cap(2\times 2)^{p+1}$ ($w:=(s,t)\in(2\times 2)^p$ and $e:=(\varepsilon,\varepsilon')\in 2\times 2$), then we set

$$(we)^{\eta+1}:=\begin{cases}(we)^\eta \text{ if there is } r\leq p \text{ with } [\ w\lceil r\in D\ \Leftrightarrow\ we\in D\ ] \text{ and } w\lceil r\ R^{(\eta)}\ we,\\ we \text{ otherwise.}\end{cases}$$

Note that $(we)^\eta\in D$ if $we\in D$, so that $we\notin D$ if $(we)^{\eta+1}=we$. Note also the equivalence between the fact that $we\in D$, and the fact that $(we)^{\eta+1}\in D$. Indeed, we may assume that $we\notin D$ and $(we)^{\eta+1}=(we)^\eta$. So that there is $r\leq p$ with $w\lceil r\notin D$ and $w\lceil r\ R^{(\eta)}\ we$. By Lemma 2.3.2 we have $w\lceil r\ R^{(\eta)}\ (we)^\eta$, so that $(we)^{\eta+1}=(we)^\eta\notin D$. The conclusions in the assertions (a) and (b) in the following claim do not really depend on their respective assumptions, but we will use these assertions later in this form.

Claim. Assume that $\eta>0$.
(a) $A\cap\bigcap_{1\leq\rho\leq\eta}\overline{U_{(we)^\rho}}^{\tau_\rho}\cap(X_s\times Y_t)$ is τ_1-dense in $\overline{U_{(we)^1}}^{\tau_1}\cap(X_s\times Y_t)$ if $(we)^{\eta+1}=we$.
(b) $U_{(we)^\eta}\cap\bigcap_{1\leq\rho<\eta}\overline{U_{(we)^\rho}}^{\tau_\rho}\cap(X_s\times Y_t)$ is τ_1-dense in $\overline{U_{(we)^1}}^{\tau_1}\cap(X_s\times Y_t)$ if $(we)^{\eta+1}\neq we$.

Indeed, we use the notation before Lemma 2.3.3 with $z:=we$. By Lemma 2.3.3 we may assume that $z^{\xi_i+1}\prec_{\neq}z^{\xi_i}$ if $1\leq i<n$. We set $S_i:=U_{z^{\xi_i}}$, for $1\leq\xi_i\leq\eta$. We have $S_i\subseteq\overline{S_{i+1}}^{\tau_{\xi_{i+1}}}$, for $1\leq\xi_i<\eta$, by induction assumption, since $z^{\xi_{i+1}}\ R^{(\xi_i+1)}\ z^{\xi_i}$. Moreover, the inclusion $S_n\subseteq\overline{A}^{\tau_{\eta+1}}$ holds. Thus $A\cap\bigcap_{1\leq\xi_i\leq\eta}\overline{U_{(we)^{\xi_i}}}^{\tau_{\xi_i}}\cap(X_s\times Y_t)$ (respectively, $U_{(we)^\eta}\cap\bigcap_{1\leq\xi_i<\eta}\overline{U_{(we)^{\xi_i}}}^{\tau_{\xi_i}}\cap(X_s\times Y_t)$) is τ_1-dense in the set $\overline{U_{(we)^1}}^{\tau_1}\cap(X_s\times Y_t)$ if $(we)^{\eta+1}=we$ (respectively, $(we)^{\eta+1}\neq we$), by Lemma 2.2.2.(b). But if $1\leq\rho\leq\eta$, then there is $1\leq i\leq n$ with $(we)^\rho=(we)^{\xi_i}$. And $\rho\leq\xi_i$ since we have $(we)^{\xi_i+1}\prec_{\neq}(we)^{\xi_i}$ if $1\leq i<n$. Thus we are done since $\bigcap_{1\leq\rho\leq\eta}\overline{U_{(we)^\rho}}^{\tau_\rho}=\bigcap_{1\leq\xi_i\leq\eta}\overline{U_{(we)^{\xi_i}}}^{\tau_{\xi_i}}$ and $U_{(we)^\eta}\cap\bigcap_{1\leq\rho<\eta}\overline{U_{(we)^\rho}}^{\tau_\rho}=U_{(we)^\eta}\cap\bigcap_{1\leq\xi_i<\eta}\overline{U_{(we)^{\xi_i}}}^{\tau_{\xi_i}}$. ◇

- Let $\mathcal{F}_0:=\mathcal{F}_1:=2^{p+1}$, $\mathcal{T}:=T\cap(\mathcal{F}_0\times\mathcal{F}_1)$, $\Psi:\mathcal{F}_0\times\mathcal{F}_1\rightarrow\Sigma^1_1(X\times Y)$ defined on $\mathcal{T}$ by

$$\Psi(we):=\begin{cases}A\cap\bigcap_{1\leq\rho\leq\eta}\overline{U_{(we)^\rho}}^{\tau_\rho}\cap(X_s\times Y_t)\cap\Omega_{X\times Y} \text{ if } (we)^{\eta+1}=we,\\ U_{(we)^\eta}\cap\bigcap_{1\leq\rho<\eta}\overline{U_{(we)^\rho}}^{\tau_\rho}\cap(X_s\times Y_t) \text{ if } (we)^{\eta+1}\neq we.\end{cases}$$

By the claim, $\Psi(we)$ is τ_1-dense in $\overline{U_{(we)^1}}^{\tau_1}\cap(X_s\times Y_t)$ if $\eta>0$. As $(we)^1\prec w\prec we$ and $R^{(1)}$ is distinguished in $\prec$ we get $(we)^1\ R^{(1)}\ w$ and $U_w\subseteq\overline{U_{(we)^1}}^{\tau_1}$, by induction assumption. Thus $\overline{U_w}^{\tau_1}\cap(X_s\times Y_t)\subseteq\overline{U_{(we)^1}}^{\tau_1}\cap(X_s\times Y_t)\subseteq\overline{\Psi}(we)$. Thus (x_s,y_t) is in $U_w\cap(X_s\times Y_t)\subseteq\overline{\Psi}(we)$ (even if $\eta=0$). Therefore $\overline{\Psi}$ admits a π-selector on $\mathcal{T}$. By Lemma 2.1.2, Ψ admits a π-selector ψ on $\mathcal{T}$. We set $x_{s\varepsilon}:=\psi_0(s\varepsilon)$, $y_{t\varepsilon'}:=\psi_1(t\varepsilon')$, and choose Σ^1_1 sets U_{we} with d-diameter at most 2^{-p-1} such that $\psi(we)\in U_{we}\subseteq\Psi(we)$. This finishes the proof since $(s,t)\ R^{(\rho)}\ we$ and $(s,t)\neq we$ imply that $(s,t)\ R^{(\rho)}\ (we)^\rho\ R^{(\rho)}\ we$, by Lemma 2.3.2. □

Now we come to the limit case. We need some more definitions that can be found in [D-SR].

Definition 2.4.2 *(Debs-Saint Raymond) Let a be a finite set.*

• *Let R be a tree relation on $a^{<\omega}$. If $t\in a^{<\omega}$, then $h_R(t)$ is the number of strict R-predecessors of t. So we have $h_R(t)=Card(P_R(t))-1$.*

• *Let $\xi<\omega_1$ be an infinite limit ordinal. We say that a resolution family $(R^{(\rho)})_{\rho\leq\xi}$ is $\underline{uniform}$ if*

$$\forall k\in\omega\ \exists\eta_k<\xi\ \forall s,t\in a^{<\omega}\ \ [min(h_{R^{(\xi)}}(s),h_{R^{(\xi)}}(t))\leq k \text{ and } s\ R^{(\eta_k)}\ t]\ \Rightarrow\ s\ R^{(\xi)}\ t.$$

We may (and will) assume that $\eta_k\geq 1$.

The following is part of Theorem I-6.6 in [D-SR].

Theorem 2.4.3 *(Debs-Saint Raymond) Let $\xi<\omega_1$ be an infinite limit ordinal, E a $\mathbf{\Pi}^0_\xi$ subset of $[\prec]$. Then there is a uniform resolution family $(R^{(\rho)})_{\rho\leq\xi}$ with:*
(a) $R^{(0)}=\prec$.
(b) The canonical map $\Pi:[R^{(\xi)}]\rightarrow[\prec]$ is a bijection.
(c) The set $\Pi^{-1}(E)$ is a closed subset of $[R^{(\xi)}]$.

Theorem 2.4.4 *(Debs-Lecomte) Let T be a tree with acyclic levels, $\xi<\omega_1^{CK}$ an infinite limit ordinal, $S\in\mathbf{\Sigma}^0_\xi(\lceil T\rceil)$, X, Y recursively presented Polish spaces, and A, B disjoint Σ^1_1 subsets of $X\times Y$. Then one of the following holds:*
(a) $\overline{A}^{\tau_\xi}\cap B=\emptyset$.
(b) There are $u:2^\omega\rightarrow X$ and $v:2^\omega\rightarrow Y$ continuous with $S\subseteq(u\times v)^{-1}(A)$ and $\lceil T\rceil\setminus S\subseteq(u\times v)^{-1}(B)$.

Proof. Let us indicate the differences with the proof of Theorem 2.4.1.

• The set $E:=\theta[\lceil T\rceil\setminus S]$ is $\mathbf{\Pi}^0_\xi([\prec])$. Theorem 2.4.3 provides a uniform resolution family.

• If $w\in(2\times 2)^{<\omega}$ then we set

$$\eta(w):=\max\{\eta_{h_{R^{(\xi)}}(w')+1}\mid w'\prec w\}.$$

Note that $\eta(w')\leq\eta(w)$ if $w'\prec w$.

• Conditions (v) and (vi) become

(v′) $\quad[1\leq\rho\leq\eta(s,t)$ and $(s,t)\ R^{(\rho)}\ (s',t')]\ \Rightarrow\ U_{(s',t')}\subseteq\overline{U_{(s,t)}}^{\tau_\rho}$.

(vi′) $\quad[((s,t)\in D\ \Leftrightarrow\ (s',t')\in D)$ and $(s,t)\ R^{(\xi)}\ (s',t')]\ \Rightarrow\ U_{(s',t')}\subseteq U_{(s,t)}$.

• If $we:=(s\varepsilon,t\varepsilon')\in T\cap(2\times 2)^{p+1}$, then we set

$$(we)^{\xi+1}:=\begin{cases}(we)^\xi & \text{if there is } r\leq p \text{ with } [\ w\lceil r\in D\ \Leftrightarrow\ we\in D\ ] \text{ and } w\lceil r\ R^{(\xi)}\ we,\\ we & \text{otherwise.}\end{cases}$$

Note that $we\notin D$ if $(we)^{\xi+1}=we$. Note also the equivalence between the fact that $we\in D$ and the fact that $(we)^{\xi+1}\in D$.

Claim 1. Assume that $(we)^\rho \neq (we)^\xi$. Then $\rho+1 \leq \eta((we)^{\rho+1})$.

We argue by contradiction. We get

$$\rho+1 > \rho \geq \eta((we)^{\rho+1}) \geq \eta_{h_{R(\xi)}((we)^\xi)+1} = \eta_{h_{R(\xi)}(we)}.$$

As $(we)^\rho\ R^{(\rho)}\ we$ we get $(we)^\rho\ R^{(\xi)}\ we$ and $(we)^\rho = (we)^\xi$, which is absurd. $\diamond$

Note that $\xi_{n-1} < \xi_{n-1}+1 \leq \eta((we)^{\xi_{n-1}+1}) \leq \eta(we)$. Thus $(we)^{\eta(we)} = (we)^\xi$.

Claim 2. (a) $A \cap \bigcap_{1\leq\rho\leq\eta(we)} \overline{U_{(we)^\rho}}^{\tau_\rho} \cap (X_s \times Y_t)$ is τ_1-dense in $\overline{U_{(we)^1}}^{\tau_1} \cap (X_s \times Y_t)$ if $(we)^{\xi+1} = we$.
(b) $U_{(we)^\xi} \cap \bigcap_{1\leq\rho<\eta(we)} \overline{U_{(we)^\rho}}^{\tau_\rho} \cap (X_s \times Y_t)$ is τ_1-dense in $\overline{U_{(we)^1}}^{\tau_1} \cap (X_s \times Y_t)$ if $(we)^{\xi+1} \neq we$.

Indeed, we set $S_i := U_{z^{\xi_i}}$, for $1 \leq \xi_i \leq \xi$. By Claim 1 we can apply Lemma 2.2.2.(b) and we are done. $\diamond$

• Let $\mathcal{F}_0 := \mathcal{F}_1 := 2^{p+1}$, $\mathcal{T} := T \cap (\mathcal{F}_0 \times \mathcal{F}_1)$, $\Psi : \mathcal{F}_0 \times \mathcal{F}_1 \to \Sigma^1_1(X \times Y)$ defined on $\mathcal{T}$ by

$$\Psi(we) := \begin{cases} A \cap \bigcap_{1\leq\rho\leq\eta(we)} \overline{U_{(we)^\rho}}^{\tau_\rho} \cap (X_s \times Y_t) \cap \Omega_{X\times Y} \text{ if } (we)^{\xi+1} = we, \\ \\ U_{(we)^\xi} \cap \bigcap_{1\leq\rho<\eta(we)} \overline{U_{(we)^\rho}}^{\tau_\rho} \cap (X_s \times Y_t) \ \text{ if } (we)^{\xi+1} \neq we. \end{cases}$$

We conclude as in the proof of Theorem 2.4.1, using the facts that $\eta_k \geq 1$ and $\eta(.)$ is increasing. $\square$

Proof of Theorem 1.10. We may assume that $\xi < \omega_1^{\text{CK}}$, X, Y are recursively presented, and A, B are Σ^1_1. We assume that A is not separable from B by a pot($\mathbf{\Pi}^0_{1+\xi}$) set, and set $N := \overline{A}^{\tau_{1+\xi}} \cap B$. Then N is not empty since $\mathbf{\Pi}^0_1(\tau_{1+\xi}) \subseteq \mathbf{\Pi}^0_{1+\xi}(\tau_1) \subseteq$ pot($\mathbf{\Pi}^0_{1+\xi}$). So (b) holds, by Theorems 2.4.1 and 2.4.3.

So (a) or (b) holds. If $D \in$ pot($\mathbf{\Pi}^0_{1+\xi}$) separates A from B and (b) holds, then $S \in$ pot($\mathbf{\Pi}^0_{1+\xi}$), since $S = (u \times v)^{-1}(D) \cap \lceil T \rceil$, which is absurd. $\square$

3 Proof of Theorem 1.11.

We have seen that we cannot have a reduction on the whole product in Theorem 1.13. We have seen that it is possible to have it on the set of branches of some tree with acyclic levels. We now build an example of such a tree. This tree has to be small enough since we cannot have a reduction on the whole product. But as the same time it has to be big enough to ensure the existence of complicated sets, as in the statement of Theorem 1.11.

Notation. Let $\varphi : \omega \to \omega^2$ be the natural bijection. More precisely, we set, for $q \in \omega$,

$$M(q) := \max\{m \in \omega \mid \Sigma_{k\leq m}\ k \leq q\}.$$

Then we define $\varphi(q) = ((q)_0, (q)_1) := (M(q) - q + (\Sigma_{k\leq M(q)}\ k), q - (\Sigma_{k\leq M(q)}\ k))$. One can check that $<n, p> := \varphi^{-1}(n, p) = (\Sigma_{k\leq n+p}\ k) + p$. More concretely, we get

$$\varphi[\omega] = \{(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), \ldots\}.$$

Definition 3.1 *We say that* $E \subseteq \bigcup_{q\in\omega} 2^q \times 2^q$ *is a* <u>*test*</u> *if:*

(a) $\forall q\in\omega\ \exists!(s_q, t_q)\in E\cap(2^q\times 2^q)$.

(b) $\forall m, p\in\omega\ \forall u\in 2^{<\omega}\ \exists v\in 2^{<\omega}\ (s_p0uv, t_p1uv)\in E$ *and* $(|t_p1uv|-1)_0 = m$.

(c) $\forall n>0\ \exists q<n\ \exists w\in 2^{<\omega}\ \ s_n = s_q0w$ and $t_n = t_q1w$.

We will call T *the tree generated by a test* $E = \{(s_q, t_q) \mid q\in\omega\}$*:*

$$T := \{(s,t)\in 2^{<\omega}\times 2^{<\omega} \mid s=t=\emptyset \text{ or } \exists q\in\omega\ \exists w\in 2^{<\omega}\ \ (s,t) = (s_q0w, t_q1w)\}.$$

The uniqueness condition in (a) and condition (c) ensure that T is small enough, and also the acyclicity. The existence condition in (a) and condition (b) ensure that T is big enough. More specifically, if X is a Polish space and σ a finer Polish topology on X, then there is a dense G_δ subset of X on which the two topologies coincide. The first part of condition (b) ensures the possibility to get inside the square of a dense G_δ subset of 2^ω. The examples in Theorem 1.11 are build using the examples in [Lo-SR]. Conditions on verticals are involved, and the second part of condition (b) gives a control on the choice of verticals.

Proposition 3.2 *The tree* T *associated with a test is a tree with acyclic levels.*

Proof. Fix $p\in\omega$. Let us show that $G_{\mathcal{T}_p}$ is acyclic. We argue by contradiction. Let $(\tilde{e}_i, j_i)_{i\leq l}$ be a cycle in $G_{\mathcal{T}_p}$, and $n<p$ maximal such that the sequence $(\tilde{e}_i(n))_{i\leq l}$ is not constant. There is i_1 minimal with $\tilde{e}_{i_1}(n)\neq\tilde{e}_{i_1+1}(n)$. We have $\tilde{e}_{i_1}(n) = \tilde{e}_0(n) = \tilde{e}_l(n)$. There is $i_2 > i_1+1$ minimal with $\tilde{e}_{i_1+1}(n)\neq\tilde{e}_{i_2}(n)$. Then $\tilde{e}_{i_1}(n) = \tilde{e}_{i_2}(n)$, and in fact $\tilde{e}_{i_1} = \tilde{e}_{i_2}$ because of the uniqueness condition in (a), and $\tilde{e}_{i_1+1} = \tilde{e}_{i_2-1}$. If $j_{i_1} = j_{i_2}$, then $i_1 = 0$ and $i_2 = l$. But $j_{i_1+1} = 1-j_{i_1} = 1-j_{i_2} = j_{i_2-1}$, which is absurd. If $j_{i_1}\neq j_{i_2}$, then for example $j_{i_1} = 0 = 1-j_{i_2}$. If $p>0$, then $\tilde{e}_{i_1}(0) = 0 = 1-\tilde{e}_{i_2}(0)$, which contradicts $\tilde{e}_{i_1} = \tilde{e}_{i_2}$. If $p=0$, then $\tilde{e}_0 = \emptyset = \tilde{e}_2$, which is absurd. $\square$

Notation. Let $\psi : \omega\rightarrow 2^{<\omega}$ be the natural bijection ($\psi(0)=\emptyset$, $\psi(1)=0$, $\psi(2)=1$, $\psi(3)=0^2$, $\psi(4)=01$, $\psi(5)=10$, $\psi(6)=1^2$, ...). Note that $|\psi(q)|\leq q$.

Lemma 3.3 *There exists a test.*

Proof. We set $s_0 = t_0 := \emptyset$, and

$$s_{q+1} := s_{[(q)_1]_0}\ 0\ \psi([(q)_1]_1)\ 0^{q-[(q)_1]_0-|\psi([(q)_1]_1)|},$$

$$t_{q+1} := t_{[(q)_1]_0}\ 1\ \psi([(q)_1]_1)\ 0^{q-[(q)_1]_0-|\psi([(q)_1]_1)|}.$$

Note that $(q)_0+(q)_1 = M(q)\leq\Sigma_{k\leq M(q)}\ k\leq q$, so that s_q, t_q are well defined and we have the equality $|s_q| = |t_q| = q$, by induction on q. It remains to check that condition (b) in the definition of a test is fullfilled. Set $n := \psi^{-1}(u)$, $r := <p,n>$ and $q := <m,r>$. It remains to put $v := 0^{q-p-|u|}$: $(s_p0uv, t_p1uv) = (s_{q+1}, t_{q+1})$. $\square$

Now we come to the lemma crucial for the proof of Theorem 1.11.

Notation. (a) We define $p\!:\!\omega^{<\omega}\!\setminus\!\{\emptyset\}\!\rightarrow\!\omega$. We define $p(s)$ by induction on $|s|$:

$$p(s)\!:=\!\begin{cases} s(0) \text{ if } |s|\!=\!1,\\ \\ <\!p(s\lceil(|s|\!-\!1)), s(|s|\!-\!1)\!> \text{ otherwise.}\end{cases}$$

Note that $p_{|\omega^n}\!:\!\omega^n\!\rightarrow\!\omega$ is a bijection, for each $n\!\geq\!1$.

(b) The map $\Delta\!:\!2^\omega\!\times\!2^\omega\!\rightarrow\!2^\omega$ is the symmetric difference. So, for $m\!\in\!\omega$,

$$\Delta(\alpha,\beta)(m)\!=\!(\alpha\Delta\beta)(m)\!=\!1\;\Leftrightarrow\;\alpha(m)\!\neq\!\beta(m).$$

Lemma 3.4 *Let G be a dense G_δ subset of 2^ω, and T the tree associated with a test. Then there are $\alpha_0\!\in\!G$ and $f\!:\!2^\omega\!\rightarrow\!G$ continuous such that, for each $\alpha\!\in\!2^\omega$,*
(a) $(\alpha_0,f(\alpha))\!\in\![T]$.
(b) For each $t\!\in\!\omega^{<\omega}$, and each $m\!\in\!\omega$,
(i) $\alpha(p(tm))\!=\!1\;\Rightarrow\;\exists m'\!\in\!\omega\;\;(\alpha_0\Delta f(\alpha))(p(tm')\!+\!1)\!=\!1$.
(ii) $(\alpha_0\Delta f(\alpha))(p(tm)\!+\!1)\!=\!1\;\Rightarrow\;\exists m'\!\in\!\omega\;\;\alpha(p(tm'))\!=\!1$.

Proof. Let (O_q) be a sequence of dense open subsets of 2^ω with $G=\bigcap_q\;O_q$. By density we get: $\forall q,l\!\in\!\omega\;\exists u_{q,l}\!\in\!2^{<\omega}\;\forall s\!\in\!2^l\;\;N_{su_{q,l}}\!\subseteq\!O_q$.

• We construct finite approximations of α_0 and f. The idea is to linearize the binary tree $2^{<\omega}$. So we will use the bijection ψ defined before Lemma 3.3. To construct $f(\alpha)$ we have to imagine, for each length l, the different possibilities for $\alpha\lceil l$. More precisely, we construct subsequences of $2^{<\omega}$, namely $(v_w)_{w\in 2^{<\omega}}$, $(s_w)_{w\in 2^{<\omega}}$ and $(t_w)_{w\in 2^{<\omega}}$, satisfying the following conditions:

(1) $(s_w,t_w)\!\in\!E\!\setminus\!\{(\emptyset,\emptyset)\}$, and $(|t_w|\!-\!1)_0\!=\!(|w|)_0$, for each $w\!\in\!2^{<\omega}$.

$$(2)\quad\begin{cases} s_\emptyset\;\;=\!0\;u_{0,1}\;v_\emptyset,\\ \\ s_{w\varepsilon}\!=\!s_{\psi(\psi^{-1}(w\varepsilon)-1)}\;0\;u_{\psi^{-1}(w\varepsilon),|s_{\psi(\psi^{-1}(w\varepsilon)-1)}|+1}\;v_{w\varepsilon}.\end{cases}$$

$$(3)\quad\begin{cases} t_\emptyset\;\;=\!1\;u_{0,1}\;v_\emptyset,\\ \\ t_{w\varepsilon}\!=\!t_w\;\varepsilon\;[\frown_{\;\psi^{-1}(w)<i<\psi^{-1}(w\varepsilon)}\;u_{i,|s_{\psi(i-1)}|+1}\;v_{\psi(i)}\;0]\;u_{\psi^{-1}(w\varepsilon),|s_{\psi(\psi^{-1}(w\varepsilon)-1)}|+1}\;v_{w\varepsilon}.\end{cases}$$

We show the existence of the three subsequences inductively on $\psi^{-1}(w)$. We choose $v_\emptyset\in 2^{<\omega}$ with $(0\;u_{0,1}\;v_\emptyset,1\;u_{0,1}\;v_\emptyset)\in E$ and $(|1\;u_{0,1}\;v_\emptyset|-1)_0=0$. Assume that $(v_w)_{\psi^{-1}(w)\leq r}$, $(s_w)_{\psi^{-1}(w)\leq r}$, $(t_w)_{\psi^{-1}(w)\leq r}$ satisfying properties (1)-(3) have been constructed, which is the case for $r\!=\!0$.

Fix $w\in 2^{<\omega}$ and $\varepsilon\in 2$ with $\psi(r+1)=w\varepsilon$. We choose $v_{w\varepsilon}\in 2^{<\omega}$ such that $(s_{w\varepsilon},t_{w\varepsilon})\in E$ and $(|t_{w\varepsilon}|\!-\!1)_0\!=\!(|w|\!+\!1)_0$. Let us show that this is possible. We want that

$$(s_{\psi(\psi^{-1}(w\varepsilon)-1)}\;0\;u_{\psi^{-1}(w\varepsilon),|s_{\psi(\psi^{-1}(w\varepsilon)-1)}|+1}\;v_{w\varepsilon}\;,\;t_w\;\varepsilon\;u_{\psi^{-1}(w)+1,|s_w|+1}\;v_{\psi(\psi^{-1}(w)+1)}\;0\ldots$$
$$u_{\psi^{-1}(w\varepsilon)-1,|s_{\psi(\psi^{-1}(w\varepsilon)-2)}|+1}\;v_{\psi(\psi^{-1}(w\varepsilon)-1)}\;0\;u_{\psi^{-1}(w\varepsilon),|s_{\psi(\psi^{-1}(w\varepsilon)-1)}|+1}\;v_{w\varepsilon})\!\in\!E.$$

It is enough to see that $(s_{\psi(\psi^{-1}(w\varepsilon)-1)}\;0,t_w\;\varepsilon\ldots v_{\psi(\psi^{-1}(w\varepsilon)-1)}\;0)\!\in\!T$.

But

$$\begin{array}{rl} & s_{\psi(\psi^{-1}(w\varepsilon)-1)}\ 0 \\ = & s_{\psi(\psi^{-1}(w\varepsilon)-2)}\ 0\ u_{\psi^{-1}(w\varepsilon)-1,|s_{\psi(\psi^{-1}(w\varepsilon)-2)}|+1}\ v_{\psi(\psi^{-1}(w\varepsilon)-1)}\ 0 \\ = & \ldots \\ = & s_w\ 0\ u_{\psi^{-1}(w)+1,|s_w|+1}\ v_{\psi(\psi^{-1}(w)+1)}\ 0 \ldots\ u_{\psi^{-1}(w\varepsilon)-1,|s_{\psi(\psi^{-1}(w\varepsilon)-2)}|+1}\ v_{\psi(\psi^{-1}(w\varepsilon)-1)}\ 0. \end{array}$$

We are done since $(s_w, t_w)\in E$.

• So this defines sequences $(v_w)_{w\in 2^{<\omega}}$, $(s_w)_{w\in 2^{<\omega}}$ and $(t_w)_{w\in 2^{<\omega}}$. As $s_{\psi(q)} \prec_{\neq} s_{\psi(q+1)}$ we can define $\alpha_0 := \sup_q s_{\psi(q)}$. Similarly, we set $f(\alpha) := \sup_m t_{\alpha\lceil m}$, and f is continuous.

• Let us show that $\alpha_0 \in G$. By definition of $s_{w\varepsilon}$ we get $s_{\psi(q)} 0 u_{q+1,|s_{\psi(q)}|+1} \prec s_{\psi(q+1)}$, for each q. This implies that $\alpha_0 \in \bigcap_q O_q = G$ since $0u_{0,1} \prec \alpha_0$.

• Now fix $\alpha\in 2^\omega$. Let us show that $f(\alpha)\in G$. Fix $q\in\omega$, and $m\in\omega$ such that

$$\psi^{-1}(\alpha\lceil m) < q+1 \leq \psi^{-1}(\alpha\lceil(m+1)).$$

Again it is enough to show the existence of $s\in 2^{<\omega}$ with $su_{q+1,|s|} \prec t_{\alpha\lceil(m+1)}$. Set

$$s := t_{\alpha\lceil m}\ \alpha(m)\ u_{\psi^{-1}(\alpha\lceil m)+1,|s_{\alpha\lceil m}|+1}\ v_{\psi(\psi^{-1}(\alpha\lceil m)+1)}\ 0\ \ldots\ u_{q,|s_{\psi(q-1)}|+1}\ v_{\psi(q)}\ 0.$$

By definition of $t_{\alpha\lceil(m+1)}$ we have $su_{q+1,|s_{\psi(q)}|+1} \prec t_{\alpha\lceil(m+1)}$. But the construction of $t_{w\varepsilon}$ shows that $|s_{\psi(q)}|+1 = |s|$. So s is suitable.

(a) Moreover, $(\alpha_0, f(\alpha)) \in \lceil T\rceil$. Indeed, fix $r\in\omega$. There is $m\in\omega$ with $l := |t_{\alpha\lceil m}| \geq r$. We get $(\alpha_0, f(\alpha))\lceil l = (s_{\alpha\lceil m}, t_{\alpha\lceil m}) \in E \subseteq T$. Thus $(\alpha_0, f(\alpha))\lceil r\in T$, and $(\alpha_0, f(\alpha))$ is in $\lceil T\rceil$.

(b).(i) We set $w := \alpha\lceil p(tm)$, so that $t_w 1 \prec t_{w1} = t_{\alpha\lceil[p(tm)+1]} \prec f(\alpha)$. As $(|t_w|-1)_0 = p(t)$, there is m' with $|t_w| = p(tm')+1$. But $s_w 0 \prec s_{\psi(\psi^{-1}(w)+1)}$, so that $\alpha_0(|t_w|) \neq f(\alpha)(|t_w|)$.

(b).(ii) First notice that the only coordinates where α_0 and $f(\alpha)$ can differ are 0 and the $|t_{\alpha\lceil q}|$'s. Therefore there is an integer q with $p(tm)+1 = |t_{\alpha\lceil q}|$. In particular $(|t_{\alpha\lceil q}|-1)_0 = p(t)$ and $(q)_0 = p(t)$. Thus there is m' with $q = p(tm')$. We have $\alpha_0(|t_{\alpha\lceil q}|) = 0 \neq f(\alpha)(|t_{\alpha\lceil q}|) = \alpha(q)$. So $\alpha(p(tm')) = 1$.□

Now we come to the existence of complicated sets, as in the statement of Theorem 1.11.

Notation. In [Lo-SR], Lemma 3.3, the map $\rho_0 : 2^\omega \rightarrow 2^\omega$ defined as follows is introduced:

$$\rho_0(\varepsilon)(i) := \begin{cases} 1 \text{ if } \varepsilon(<i,j>) = 0, \text{ for each } j\in\omega, \\ 0 \text{ otherwise.} \end{cases}$$

In this paper, $\rho_0^\xi:2^\omega\rightarrow 2^\omega$ is also defined for $\xi<\omega_1$ as follows, by induction on ξ (see the proof of Theorem 3.2). We put:
- $\rho_0^0:=\mathrm{Id}_{2^\omega}$.
- $\rho_0^{\eta+1}:=\rho_0\circ\rho_0^\eta$.
- If $\lambda>0$ is limit, fix $(\xi_k^\lambda)\subseteq\lambda\setminus\{0\}$ with $\Sigma_k\ \xi_k^\lambda=\lambda$. For $\varepsilon\in 2^\omega$ and $k\in\omega$ we define $(\varepsilon)^k\in 2^\omega$ by $(\varepsilon)^k(i):=\varepsilon(i+k)$. We also define $\rho_0^{(k,k+1)}:2^\omega\rightarrow 2^\omega$ by

$$\rho_0^{(k,k+1)}(\varepsilon)(i):=\begin{cases}\varepsilon(i)\text{ if }i<k,\\ \\ \rho_0^{\xi_k^\lambda}((\varepsilon)^k)(i-k)\text{ if }i\geq k.\end{cases}$$

We set $\rho_0^{(0,k+1)}:=\rho_0^{(k,k+1)}\circ\rho_0^{(k-1,k)}\circ\ldots\circ\rho_0^{(0,1)}$ and $\rho_0^\lambda(\varepsilon)(k):=\rho_0^{(0,k+1)}(\varepsilon)(k)$.

The set $H_{1+\xi}:=(\rho_0^\xi)^{-1}(\{0^\infty\})$ is also introduced, and the authors show that $H_{1+\xi}$ is $\mathbf{\Pi}^0_{1+\xi}\setminus\mathbf{\Sigma}^0_{1+\xi}$ (see Theorem 3.2).

• The map $\mathcal{S}:2^\omega\rightarrow 2^\omega$ is the shift map: $\mathcal{S}(\alpha)(m):=\alpha(m+1)$.

• Let T be the tree generated by a test. We put, for $\xi<\omega_1$,

$$S_{1+\xi}:=\{(\alpha,\beta)\in 2^\omega\times 2^\omega\mid(\alpha,\beta)\in\lceil T\rceil\text{ and }\mathcal{S}(\alpha\Delta\beta)\notin H_{1+\xi}\}.$$

Theorem 3.5 *Let $\xi<\omega_1$. The set $\lceil T\rceil\setminus S_{1+\xi}$ is $\mathbf{\Pi}^0_{1+\xi}(2^\omega\times 2^\omega)\setminus$pot$(\mathbf{\Sigma}^0_{1+\xi})$, and $S_{1+\xi}$ is not pot$(\mathbf{\Pi}^0_{1+\xi})$.*

Proof. As $H_{1+\xi}$ is $\mathbf{\Pi}^0_{1+\xi}$ and Δ, $\mathcal{S}$ are continuous, $\lceil T\rceil\setminus S_{1+\xi}$ is $\mathbf{\Pi}^0_{1+\xi}(2^\omega\times 2^\omega)$.

• Let G be a dense G_δ subset of 2^ω. Lemma 3.4 provides $\alpha_0\in G$ and $f:2^\omega\rightarrow G$ continuous.

• Let us show that $\rho_0^\xi(\alpha)=\rho_0^\xi(\mathcal{S}[\alpha_0\Delta f(\alpha)])$, for each $1\leq\xi<\omega_1$ and $\alpha\in 2^\omega$. For $\xi=1$ we apply Lemma 3.4.(b) to $t\in\omega$. Then we have, by induction:

$$\rho_0^{\eta+1}(\alpha)=\rho_0(\rho_0^\eta(\alpha))=\rho_0\Big(\rho_0^\eta(\mathcal{S}[\alpha_0\Delta f(\alpha)])\Big)=\rho_0^{\eta+1}(\mathcal{S}[\alpha_0\Delta f(\alpha)]).$$

From this we deduce, by induction again, that

$$\rho_0^{(0,1)}(\alpha)=\rho_0^{\xi_0^\lambda}(\alpha)=\rho_0^{\xi_0^\lambda}(\mathcal{S}[\alpha_0\Delta f(\alpha)])=\rho_0^{(0,1)}(\mathcal{S}[\alpha_0\Delta f(\alpha)]).$$

Thus $\rho_0^{(0,k+1)}(\alpha)=\rho_0^{(0,k+1)}(\mathcal{S}[\alpha_0\Delta f(\alpha)])$, and

$$\rho_0^\lambda(\alpha)(k)=\rho_0^{(0,k+1)}(\alpha)(k)=\rho_0^{(0,k+1)}(\mathcal{S}[\alpha_0\Delta f(\alpha)])(k)=\rho_0^\lambda(\mathcal{S}[\alpha_0\Delta f(\alpha)])(k).$$

• This implies that $\alpha\in H_{1+\xi}$ is equivalent to $\mathcal{S}[\alpha_0\Delta f(\alpha)]\in H_{1+\xi}$ (for $\xi=0$ we apply Lemma 3.4.(b) to $t:=\emptyset$).

• We argue by contradiction to show that $\lceil T\rceil\setminus S_{1+\xi}$ (resp., $S_{1+\xi}$) is not pot$(\mathbf{\Sigma}^0_{1+\xi})$ (resp., pot$(\mathbf{\Pi}^0_{1+\xi})$): there is a dense G_δ subset G of 2^ω such that $(\lceil T\rceil\setminus S_{1+\xi})\cap G^2$ (resp., $S_{1+\xi}\cap G^2$) is a $\mathbf{\Sigma}^0_{1+\xi}$ (resp., $\mathbf{\Pi}^0_{1+\xi}$) subset of G^2. But by the previous point we get $H_{1+\xi}=f^{-1}([(\lceil T\rceil\setminus S_{1+\xi})\cap G^2]_{\alpha_0})$ (resp., $\neg H_{1+\xi}=f^{-1}([S_{1+\xi}\cap G^2]_{\alpha_0})$), which is absurd. □

4 Proof of Theorem 1.14.

As announced in the introduction, we show more than Theorem 1.14.

Notation. Let X, Y be recursively presented Polish spaces. We set

$$B_0^{X\times Y} := \{p\in W^{X\times Y} \mid \exists (m,n)\in W^X\times W^Y \ \ C_p^{X\times Y} = C_m^X\times C_n^Y\}.$$

Then we define an inductive operator Φ over ω (see [C]) as follows:

$$\Phi(A) := B_0^{X\times Y}\cup A\cup$$

$$\{p\in W^{X\times Y} \mid \exists\alpha\in\Delta^1_1\ \forall n\in\omega\ \ \alpha(n)\in W^{X\times Y}\cap A \text{ and } \neg C_p^{X\times Y} = \bigcup_n\ C_{\alpha(n)}^{X\times Y}\}.$$

Then Φ is clearly a Π^1_1 monotone inductive operator. We let, for any ordinal ξ,

$$B_\xi^{X\times Y} = \Phi^\xi := \Phi(\bigcup_{\eta<\xi}\ \Phi^\eta)$$

(which is coherent with the definition of $B_0^{X\times Y}$).

Theorem 4.1 *(Debs-Lecomte-Louveau) Let T given by Theorem 1.11, $\xi<\omega_1^{CK}$, S given by Theorem 1.11, and X, Y be recursively presented Polish spaces.*

• *Let A, B be disjoint Σ^1_1 subsets of $X\times Y$. The following are equivalent:*
(a) The set A cannot be separated from B by a pot($\mathbf{\Pi}^0_{1+\xi}$) set.
(b) The set A cannot be separated from B by a $\Delta^1_1\cap$ pot($\mathbf{\Pi}^0_{1+\xi}$) set.
(c) The set A cannot be separated from B by a $\mathbf{\Pi}^0_{1+\xi}(\tau_1)$ set.
(d) $\overline{A}^{\tau_{1+\xi}}\cap B\neq\emptyset$.
(e) There are $u:2^\omega\to X$ and $v:2^\omega\to Y$ continuous with $S\subseteq (u\times v)^{-1}(A)$ and $\lceil T\rceil\setminus S\subseteq (u\times v)^{-1}(B)$.

• *The sets $W_0^{X\times Y} = B_0^{X\times Y}$, $W_{1+\xi}^{X\times Y} = B_{1+\xi}^{X\times Y}$ and $W_{<1+\xi}^{X\times Y}$ are Π^1_1.*

Proof. The set $B_0^{X\times Y}$ is clearly Π^1_1 and a subset of $W_0^{X\times Y}$. Conversely, if p is in $W_0^{X\times Y}$, then $C_p^{X\times Y}$ is a Σ^1_1 rectangle, and a Δ^1_1 rectangle by reflection. So $p\in B_0^{X\times Y} = W_0^{X\times Y}$.

• We argue by induction on ξ. So assume that the result has been shown for $\eta<\xi$.

• Let us show that $W_{<1+\xi}^{X\times Y}$ is Π^1_1. We may assume that $\xi = 1+\xi$ is an infinite limit ordinal since $W_{<\eta+1}^{X\times Y} = W_\eta^{X\times Y}$. By Lemma 4.8 in [C] the following relation is Π^1_1:

$$R(p,\delta)\ \Leftrightarrow\ \delta\in\text{WO and } p\in\Phi^{|\delta|}.$$

The following argument can be found in [Lo1], Proposition 1.4. Let $\delta_\xi\in\text{WO}\cap\Delta^1_1$ with $|\delta_\xi| = \xi$, and δ_ξ^m be the restriction of the ordering δ_ξ to the δ_ξ-predecessors of m. We get, by induction assumption,

$$p\in W_{<1+\xi}^{X\times Y}\ \Leftrightarrow\ \exists\eta<\xi\ \ p\in W_\eta^{X\times Y}\ \Leftrightarrow\ \exists\eta<\xi\ \ p\in B_\eta^{X\times Y}$$

$$\Leftrightarrow\ \exists\eta<\xi\ \ p\in\Phi^\eta\ \Leftrightarrow\ \exists m\in\omega\ \ R(p,\delta_\xi^m).$$

This shows that $W_{<1+\xi}^{X\times Y}$ is Π^1_1.

(a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) are clear since $\mathit{\Delta}_X$ and $\mathit{\Delta}_Y$ are Polish.

(c) $\Rightarrow$ (d) This comes from the fact that $\mathbf{\Pi}^0_1(\tau_{1+\xi})\subseteq\mathbf{\Pi}^0_{1+\xi}(\tau_1)$.

(d) $\Rightarrow$ (e) This comes from Theorems 2.4.1 and 2.4.4 (Lemma 2.2.2 is at this moment true until the level $1+\xi$).

(e) $\Rightarrow$ (a) If $D\in$ pot($\mathbf{\Pi}^0_{1+\xi}$) separates A from B, then $S=(u\times v)^{-1}(D)\cap\lceil T\rceil$ is pot($\mathbf{\Pi}^0_{1+\xi}$), which contradicts Theorem 1.11.

(b) $\Rightarrow$ (d) We argue by contradiction, so that $\overline{A}^{\tau_{1+\xi}}$ separates A from B. By induction assumption and the first reflection theorem there is $\alpha\in\mathit{\Delta}^1_1$ with $\alpha(n)\in W^{X\times Y}_{<1+\xi}$ and $C^{X\times Y}_{\alpha(n)}\subseteq\neg A$, for each integer n, and $B\subseteq E:=\bigcup_n\ C^{X\times Y}_{\alpha(n)}$. But E is $\mathit{\Delta}^1_1\cap$ pot($\mathbf{\Sigma}^0_{1+\xi}$) and separates B from A, which is absurd.

• The proof of the implication (b) $\Rightarrow$ (d) imply that $W^{X\times Y}_{1+\xi}$ is $\mathit{\Pi}^1_1$ since $W^{X\times Y}_{<1+\xi}$ is $\mathit{\Pi}^1_1$ and

$$W^{X\times Y}_{1+\xi}=\{p\in W^{X\times Y}\mid\exists\alpha\in\mathit{\Delta}^1_1\ \forall n\in\omega\ \ \alpha(n)\in W^{X\times Y}_{<1+\xi}\ \text{ and }\ \neg C^{X\times Y}_p=\bigcup_n\ C^{X\times Y}_{\alpha(n)}\}.$$

• It remains to see that $W^{X\times Y}_{1+\xi}=B^{X\times Y}_{1+\xi}$. But by induction assumption we get

$$\begin{aligned}
&B^{X\times Y}_{1+\xi}\\
&=\Phi(\textstyle\bigcup_{\eta<1+\xi}\ \Phi^\eta)=\Phi(\bigcup_{\eta<1+\xi}\ B^{X\times Y}_\eta)\\
&=\textstyle\bigcup_{\eta<1+\xi}\ B^{X\times Y}_\eta\ \ \cup\{p\in W^{X\times Y}\mid\exists\alpha\in\mathit{\Delta}^1_1\ \forall n\in\omega\ \ \alpha(n)\in\bigcup_{\eta<1+\xi}\ B^{X\times Y}_\eta\ \text{ and }\\
&\hspace{20em}\neg C^{X\times Y}_p=\textstyle\bigcup_n\ C^{X\times Y}_{\alpha(n)}\}\\
&=W^{X\times Y}_{<1+\xi}\ \ \cup\{p\in W^{X\times Y}\mid\exists\alpha\in\mathit{\Delta}^1_1\ \forall n\in\omega\ \ \alpha(n)\in W^{X\times Y}_{<1+\xi}\ \text{ and }\ \neg C^{X\times Y}_p=\textstyle\bigcup_n\ C^{X\times Y}_{\alpha(n)}\}\\
&=W^{X\times Y}_{1+\xi}.
\end{aligned}$$

This finishes the proof. □

Remark. As we saw with Theorem 2.2.1, the equivalence between (a), (b) and (c) is essentially shown in [Lo2]. It is also essentially shown in [Lo2] that (a), (b) and (c) are equivalent to (d) (see the proof of Theorem 2.8, (a) page 25, in [Lo2]). An immediate consequence of Theorem 4.1 is the following, shown in [Lo2]:

Corollary 4.2 *(Louveau) Let $\xi<\omega_1^{CK}$, X, Y be recursively presented Polish spaces, and A a $\mathit{\Delta}^1_1$ subset of $X\times Y$. The following are equivalent:*
(a) The set A is pot($\mathbf{\Pi}^0_{1+\xi}$).
(b) The set A is $\mathbf{\Pi}^0_{1+\xi}(\tau_1)$.

5 References.

[B] B. Bollobás, *Modern graph theory,* Springer-Verlag, New York, 1998

[C] D. Cenzer, Monotone inductive definitions over the continuum, *J. Symbolic Logic* 41 (1976), 188-198

[D-SR] G. Debs and J. Saint Raymond, Borel liftings of Borel sets: some decidable and undecidable statements, *to appear in Mem. Amer. Math. Soc.*

[H-K-Lo] L. A. Harrington, A. S. Kechris and A. Louveau, A Glimm-Effros dichotomy for Borel equivalence relations, *J. Amer. Math. Soc.* 3 (1990), 903-928

[Hj-K-Lo] G. Hjorth, A. S. Kechris and A. Louveau, Borel equivalence relations induced by actions of the symmetric group, *Ann. Pure Appl. Logic* 92 (1998), 63-112

[K] A. S. Kechris, *Classical Descriptive Set Theory,* Springer-Verlag, 1995

[L1] D. Lecomte, Classes de Wadge potentielles et théorèmes d'uniformisation partielle, *Fund. Math.* 143 (1993), 231-258

[L2] D. Lecomte, Uniformisations partielles et critères à la Hurewicz dans le plan, *Trans. Amer. Math. Soc.* 347, 11 (1995), 4433-4460

[L3] D. Lecomte, Tests à la Hurewicz dans le plan, *Fund. Math.* 156 (1998), 131-165

[L4] D. Lecomte, Complexité des boréliens à coupes dénombrables, *Fund. Math.* 165 (2000), 139-174

[L5] D. Lecomte, On minimal non potentially closed subsets of the plane, *to appear in Topology Appl.*

[L6] D. Lecomte, Hurewicz-like tests for Borel subsets of the plane, *Electron. Res. Announc. Amer. Math. Soc.* 11 (2005)

[Lo1] A. Louveau, A separation theorem for Σ^1_1 sets, *Trans. Amer. Math. Soc.* 260 (1980), 363-378

[Lo2] A. Louveau, Ensembles analytiques et boréliens dans les espaces produit, *Astérisque (S. M. F.)* 78 (1980)

[Lo-SR] A. Louveau and J. Saint Raymond, Borel classes and closed games: Wadge-type and Hurewicz-type results, *Trans. Amer. Math. Soc.* 304 (1987), 431-467

[M] Y. N. Moschovakis, *Descriptive set theory,* North-Holland, 1980

[SR] J. Saint Raymond, La structure borélienne d'Effros est-elle standard ?, *Fund. Math.* 100 (1978), 201-210