International Journal of Number Theory

Highest perfect power of a product of integers less than x

Élie Goudout

Institut de Mathématiques de Jussieu-PRG,
Université Paris Diderot, Sorbonne Paris Cité,
75013 Paris, France
eliegoudout@hotmail.com

Received 13 December 2017
Accepted 16 January 2018

For $x \geq 3$, we define $w(x)$ as the highest integer w for which there exist integers $m, y \geq 1$ and $1 \leq n_1 < \cdots < n_m \leq x$ such that $n_1 \cdots n_m = y^w$. We show that

$$w(x) = x \exp\big(-(\sqrt{2}+o(1))\sqrt{\log x \log\log x}\big).$$

Keywords: Perfect powers; friable numbers.

Mathematics Subject Classification 2010: 11A05, 11N56

In [1], Skałba defines, for $x \geq 3$, $w(x)$ as the highest integer w for which there exist integers $m, y \geq 1$ and $1 \leq n_1 < \cdots < n_m \leq x$ such that $n_1 \cdots n_m = y^w$, and shows

$$x \exp\big(-(\sqrt{2}+o(1))L(x)\big) \leq w(x) \leq x \exp\big(-(\tfrac{1}{\sqrt{2}}+o(1))L(x)\big),$$

where $L(x) := \sqrt{\log x \log\log x}$, as $x \to +\infty$. In this paper, we improve the upper bound, making the result optimal up to the $\exp\big(o(L(x))\big)$.

Theorem 0.1. *When $x \to +\infty$,*

$$w(x) = x \exp\big(-(\sqrt{2}+o(1))L(x)\big).$$

Proof. For $3 \leq z \leq x$, let $\Psi(x,z) := \#\{n \leq x : \quad P^+(n) \leq z\}$ be the number of z-friable integers less than x. From [2], we have the classical estimate

$$\Psi(x,z) \ll x u^{-u} + \sqrt{x}. \qquad (u = \log x/\log z) \tag{0.1}$$

From now on, let $x \geq 3$, m and $n_1, \ldots, n_m \leq x$ denote integers such that $n_1 \cdots n_m$ is of the form $y^{w(x)}$. Let $q := P^+(y^{w(x)})$ denote the highest prime factor of $y^{w(x)}$ and $v_q(y^{w(x)})$ denote its q-valuation. Since the number of q-friable integers $n \leq x$ with $q|n$ is less than $\Psi(x/q,q)$, we have

$$w(x) \leq v_q(y^{w(w)}) \leq \Psi(x/q,q)\frac{\log x}{\log q}.$$

The estimate (0.1) yields that the maximum of the right member is attained for

$$q = \exp\big((\tfrac{1}{\sqrt{2}} + o(1))L(x)\big).$$

In this case, we get the desired upper bound, which, together with Skałba's lower bound, gives the result. □

Bibliography

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[2] G. Tenenbaum, Introduction à la théorie analytique et probabiliste des nombres, *Belin, quatrième édition* (2015).