The moduli space of the modular group in complex hyperbolic geometry *

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Abstract

We construct the space of discrete, faithful, type-preserving representations of the modular group into the isometry group of complex hyperbolic 2-space up to conjugacy. This is the first Fuchsian group for which the entire complex hyperbolic deformation space has been constructed. We also show how the $\mathbb{C}$-spheres of Falbel-Zocca are related to the $\mathbb{R}$-spheres (hybrid spheres) of Schwartz.

1 Introduction

Let $\Gamma = \mathbf{PSL}(2,\mathbb{Z})$ be the classical modular group. It is well known that Γ is generated by an element of order two and an element of order three whose product is parabolic. This is the point of view we take here. Our main goal is to completely describe, up to conjugacy, the space of discrete, faithful, type-preserving representations of Γ into $\mathbf{PU}(2,1)$, the holomorphic isometry group of complex hyperbolic 2-space. (Here type-preserving means that parabolic elements of $\mathbf{PSL}(2,\mathbb{Z})$ are represented by parabolic elements of $\mathbf{PU}(2,1)$.) Also, Γ is an index 2 subgroup of a group $\widehat{\Gamma}$ generated by three (antiholomorphic) involutions. These involutions correspond to reflections in the sides of a hyperbolic triangle with internal angles $\pi/2$, $\pi/3$ and 0, that is $\widehat{\Gamma}$ is a $(2,3,\infty)$-triangle group. We show that all the representations of Γ which we construct are contained as an index two subgroup in a representation, with anti-holomorphic generators, of $\widehat{\Gamma}$ into $\widehat{\mathbf{PU}}(2,1)$, the full isometry group of complex hyperbolic 2-space. This is the first Fuchsian group whose entire complex hyperbolic deformation space has been constructed. (For ideal triangle groups the papers [6] and [11] only construct the deformation space where the reflection generators are holomorphic.) For more general triangle groups there will be two dimensional components of the deformation space, see [2], and three dimensional components of the deformation space of their index 2 Fuchsian subgroup. In particular, there will be representations to $\mathbf{PU}(2,1)$ of the Fuchsian subgroup that are not contained in representations to $\widehat{\mathbf{PU}}(2,1)$ of the original triangle group.

Another theme that recurs throughout the paper is a duality between the complex and symplectic structures inherent in complex hyperbolic space. This theme could be thought of as a counterpart

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to ideas in mirror symmetry, where a similar duality is present in Calabi-Yau manifolds (see the introduction to [10]). In our paper, the first instance of this duality is the striking similarity between our Theorem 1.2 and Theorem 1.3 (the latter due to Goldman-Parker and Schwartz). This pair of results essentially have the same statement but for groups generated by real reflections and complex reflections. Furthermore, they describe a family of deformations beginning with groups preserving a complex line and a totally real subspace, respectively, and ending with groups that are commensurable, Theorem 1.4. This duality between the complex and symplectic structures appears in a second way. Namely, we will construct two different fundamental domains for the action of our groups, one using real objects and the other using complex objects. Complex hyperbolic isometries act as CR automorphisms of S^3, the boundary of complex hyperbolic 2-space, see page 42 of [9]. A common way to show that a group of complex hyperbolic isometries is discrete is to construct a fundamental domain for its action on S^3, see [6], [12] and [1]. This is the case for both the fundamental domains we construct. The first of these fundamental domains has faces that are $\mathbb{R}$-spheres as defined by Schwartz [12], that is they are spheres foliated by arcs of $\mathbb{R}$-circles around a $\mathbb{C}$-circle. The second fundamental domain has faces that are $\mathbb{C}$-spheres as defined by Falbel-Zocca [4], that is spheres foliated by $\mathbb{C}$-circles along an $\mathbb{R}$-circle. Both $\mathbb{R}$-spheres and $\mathbb{C}$-spheres are generalisations of bisectors and we explore how they are related.

Our main theorem is

Theorem 1.1 *Let* $\Gamma = \mathbf{PSL}(2,\mathbb{Z})$ *and let* $\rho : \Gamma \longrightarrow \mathbf{PU}(2,1)$ *be a discrete, faithful, type-preserving representation of* Γ *to* $\mathbf{PU}(2,1)$. *Then, up to conjugacy in* $\widehat{\mathbf{PU}}(2,1)$, *the group* $\rho(\Gamma)$ *lies in one of six components.*

- *Four of these components are points, each of which corresponds to a rigid* $\mathbb{C}$*-Fuchsian representation for which the elliptic elements of order 2 and 3 are complex reflections.*
- *One component is a one (real) dimensional closed analytic arc with one endpoint corresponding to a* $\mathbb{C}$*-Fuchsian representation and the other endpoint corresponding to an* $\mathbb{R}$*-Fuchsian representation.*
- *The remaining component is a one (real) dimensional analytic arc, with one endpoint corresponding to a* $\mathbb{C}$*-Fuchsian representation. This arc may be closed by the addition of an endpoint corresponding to a discrete, faithful representation with an additional class of parabolic elements.*

Moreover, each of these representations extends to a discrete, faithful, type-preserving representation $\hat{\rho}$ *of* $\widehat{\Gamma}$ *to* $\widehat{\mathbf{PU}}(2,1)$ *where the generators of* $\hat{\rho}(\widehat{\Gamma})$ *are (antiholomorphic) reflections in totally real planes.*

Let $\widehat{\Gamma}$ be the triangle group of type $(2,3,\infty)$, that is, the abstract group presented by

$$\langle \iota_0,\, \iota_1,\, \iota_2 : {\iota_0}^2 = 1,\, \iota_1^2 = 1,\, \iota_2^2 = 1,\, (\iota_0 \circ \iota_1)^2 = 1,\, (\iota_0 \circ \iota_2)^3 = 1 \rangle.$$

We will describe the Teichmüller space of faithful, discrete embeddings of $\widehat{\Gamma}$ modulo conjugation in $\widehat{\mathbf{PU}}(2,1)$, with anti-holomorphic generators and such that $\iota_1 \circ \iota_2$ is parabolic.

We will consider the following groups:

- $\widehat{\Gamma} = \langle \iota_0, \iota_1, \iota_2 \rangle$, the triangle modular group.
- $\Gamma = \langle \iota_0\iota_1, \iota_1\iota_2 \rangle \subset \widehat{\Gamma}$ of index 2 is the modular group $\mathbf{PSL}(2,\mathbb{Z})$.

- $G = \langle c_1 = \iota_0\iota_1, c_0 = \iota_0\iota_2\iota_0\iota_1\iota_2\iota_0, c_2 = \iota_2\iota_0\iota_0\iota_1\iota_0\iota_2 = \iota_2\iota_1\iota_0\iota_2\rangle \subset \Gamma$ of index 3.
- $\widehat{G} = \langle r_0 = \iota_1, r_1 = \iota_0\iota_2\iota_1\iota_2\iota_0, r_2 = \iota_2\iota_0\iota_1\iota_0\iota_2 = \iota_2\iota_1\iota_2\rangle \subset \widehat{\Gamma}$ of index 6.

Observe that the group $\langle c_1c_2, c_1c_0\rangle \subset G$ of index 2 is torsion free. The quotient of the discontinuity set in the boundary of $\mathbf{H}^2_{\mathbb{C}}$ by the group is a disc bundle over the torus with one point deleted. Also $\widehat{G}$ is an ideal triangle group, that is a group generated by reflections in the sides of a hyperbolic triangle all of whose vertices lie on the boundary of the hyperbolic plane.

Of particular interest will be the component in Theorem 1.1 which is not a closed arc. By passing to the index six subgroup $\hat{\rho}(\widehat{G})$ we have the following theorem:

Theorem 1.2 *Let R_0 be a totally real plane in $\mathbf{H}^2_{\mathbb{C}}$ and $g \in \mathbf{PU}(2,1)$ have order 3. Suppose that R_0, $R_1 = g(R_0)$ and $R_2 = g^2(R_0)$ are pairwise asymptotic. Let r_0, $r_1 = gr_0g^2$ and $r_2 = g^2r_0g$ be (antiholomorphic) involutions fixing R_0, R_1 and R_2. Also, suppose that there is a totally real plane R so that r, reflection in R, maps R_0 to itself and interchanges R_1 and R_2. Then the group generated by r_0, r_1 and r_2 is a discrete, faithful representation of $\widehat{G}$ if and only if $(r_0r_1r_2)^2$ is not elliptic. In addition, it is type-preserving provided $(r_0r_1r_2)^2$ is loxodromic.*

The additional condition about the existence of R in Theorem 1.2 ensures that $\langle r_0, r_1, r_2\rangle$ is a copy of $\widehat{G}$ (here $r = \iota_0$). In other words, this condition implies that the configuration of totally real planes R_0, R_1 and R_2 has an antiholomorphic symmetry, r, as well as a rotational symmetry, g. We conjecture that a similar result should be true when the configuration R_0, R_1 and R_2 only has rotational symmetry by g. Theorem 1.2 should be contrasted with the situation of representations of $\widehat{G}$ with holomorphic generators:

Theorem 1.3 (see [6, 11]) *Let C_0 be a complex line in $\mathbf{H}^2_{\mathbb{C}}$ and $g \in \mathbf{PU}(2,1)$ have order 3. Suppose that C_0, $C_1 = g(C_0)$ and $C_2 = g^2(C_0)$ are pairwise asymptotic. Let c_0, $c_1 = gc_0g^2$ and $c_2 = g^2c_0g$ be (holomorphic) involutions fixing C_0, C_1 and C_2. Then the group generated by c_0, c_1 and c_2 is a discrete, faithful representation of $\widehat{G}$ if and only if $c_0c_1c_2$ is not elliptic. In addition, it is type-preserving provided $c_0c_1c_2$ is loxodromic.*

We remark that the c_0, c_1 and c_2 in Theorem 1.3 are *not* the same as the generators of G defined above (since in G we do not have c_jc_k parabolic). This theorem is only included to provide contrast with Theorem 1.2 and, except for Theorem 1.4, will not be used later. Our final main result says that, after a suitable conjugation, the groups of Theorems 1.2 and 1.3 for which $(r_0r_1r_2)^2$ and $c_0c_1c_2$ are parabolic are commensurable.

Theorem 1.4 *Let $\tilde{\rho}$ be a representation of $\widehat{\Gamma}$ so that both $\iota_1\iota_2$ and $(\iota_0\iota_1\iota_2)^2$ are parabolic. Then $\tilde{\rho}(\widehat{\Gamma})$ contains as subgroups the group $\langle r_0, r_1, r_2\rangle$ from Theorem 1.2 where $(r_0r_1r_2)^2$ is parabolic and the group $\langle c_0, c_1, c_2\rangle$ from Theorem 1.3 where $c_0c_1c_2$ is parabolic.*

Proof: Writing r_0, r_1 and r_2 in terms of ι_0, ι_1 and ι_2 as above, we see that

$$\begin{aligned} r_0r_2 &= \iota_1\iota_2\iota_0\iota_1\iota_0\iota_2 = (\iota_1\iota_2)^2 \\ r_0r_1r_2 &= \iota_1\iota_0\iota_2\iota_1\iota_2\iota_0\iota_2\iota_1\iota_2 = (\iota_0\iota_1\iota_2)^3 \end{aligned}$$

Since we have c_jc_k and $c_0c_1c_2$ parabolic, the group from Theorem 1.3 becomes a special case of G, defined above. Therefore, we may write c_0, c_1 and c_2 in terms of ι_0, ι_1 and ι_2. Doing this, we see that

$$\begin{aligned} c_1c_2 &= \iota_0\iota_1\iota_2\iota_1\iota_0\iota_2 = (\iota_0\iota_1\iota_2)^2 \\ c_0c_1c_2 &= \iota_1\iota_0\iota_0\iota_2\iota_1\iota_0\iota_2\iota_0\iota_2\iota_0\iota_1\iota_0\iota_0\iota_2 = (\iota_1\iota_2)^3 \end{aligned}$$

The result follows easily. $\square$

The representations constructed in Theorems 1.2 and 1.3 are parametrised by the Cartan angular invariant $\mathbb{A}$ of the three classes of parabolic fixed point. The representations constructed in [6] and [11] (Theorem 1.3) are discrete and faithful whenever $0 \leq \tan(\mathbb{A}) \leq \sqrt{125/3}$. On the other hand, we will show that the representations generated by real reflections from Theorem 1.2 are discrete and faithful whenever $\sqrt{15} \leq \tan(\mathbb{A}) \leq \infty$ (that is $0 \leq \cos(\mathbb{A}) \leq 1/4$). Theorem 1.4 shows that the groups from these two families where $\tan(\mathbb{A}) = \sqrt{125/3}$ and $\tan(\mathbb{A}) = \sqrt{15}$, respectively, are commensurable. At first sight this may seem to be a contradiction. The explanation is that this group (as considered in [12]) has two triples of parabolic fixed points and the angular invariant of these triples is different.

We will prove Theorem 1.1 by constructing fundamental domains for the action of G on $S^3 = \partial \mathbf{H}^2_{\mathbb{C}}$. The construction of fundamental domains for $\widehat{\Gamma}$ as in [2] breaks down when the invariant $\mathbb{C}$-circles are linked. In a certain way, the subgroup gives the possibility to consider further "invariant" $\mathbb{C}$-circles which are not linked. It is possible that the fundamental domains for $\widehat{\Gamma}$ might use a combination of $\mathbb{C}$-spheres and $\mathbb{R}$-spheres.

2 Complex hyperbolic space

2.1 $\mathbf{PU}(2,1)$, $\widehat{\mathbf{PU}}(2,1)$ and the Heisenberg group

Let $\mathbb{C}^{2,1}$ denote the complex vector space equipped with the Hermitian form

$$\langle z, w \rangle = z_1\overline{w}_3 + z_2\overline{w}_2 + z_3\overline{w}_1.$$

Consider the following subspaces in $\mathbb{C}^{2,1}$:

$$\begin{aligned} V_+ &= \{ z \in \mathbb{C}^{2,1} \ : \ \langle z, z \rangle > 0 \ \}, \\ V_0 &= \{ z \in \mathbb{C}^{2,1} \setminus \{0\} \ : \ \langle z, z \rangle = 0 \ \}, \\ V_- &= \{ z \in \mathbb{C}^{2,1} \ : \ \langle z, z \rangle < 0 \ \}. \end{aligned}$$

Let $P : \mathbb{C}^{2,1} \setminus \{0\} \to \mathbb{C}P^2$ be the canonical projection onto complex projective space. Then $\mathbf{H}^2_{\mathbb{C}} = P(V_-)$ equipped with the Bergman metric is complex hyperbolic space. The boundary of complex hyperbolic space is $P(V_0) = \partial \mathbf{H}^2_{\mathbb{C}}$. The isometry group $\widehat{\mathbf{PU}}(2,1)$ of $\mathbf{H}^2_{\mathbb{C}}$ comprises holomorphic transformations in $\mathbf{PU}(2,1)$, the unitary group of $\langle \cdot, \cdot \rangle$, and anti-holomorphic transformations arising elements of $\mathbf{PU}(2,1)$ followed by complex conjugation.

The *Heisenberg group* $\mathfrak{N}$ is the set of pairs $(z,t) \in \mathbb{C} \times \mathbb{R}$ with the product

$$(z,t) \cdot (z',t') = (z + z', t + t' + 2\mathrm{Im}\, z\overline{z}').$$

Using stereographic projection, we can identify $\partial \mathbf{H}^2_{\mathbb{C}}$ with the one-point compactification $\overline{\mathfrak{N}}$ of $\mathfrak{N}$. The Heisenberg group acts on itself by left translations. Heisenberg translations by $(0,t)$ for $t \in \mathbb{R}$ are called *vertical translations*.

Define the inversion in the x-axis in $\mathbb{C} \subset \mathfrak{N}$ by

$$\iota_x : (z,t) \mapsto (\overline{z}, -t).$$

All these actions extend trivially to the compactification $\overline{\mathfrak{N}}$ of $\mathfrak{N}$ and represent transformations in $\widehat{\mathbf{PU}}(2,1)$ acting on the boundary of complex hyperbolic space (see [5]).

A point $p = (z, t)$ in the Heisenberg group and the point ∞ are lifted to the following points in $\mathbb{C}^{2,1}$:

$$\hat{p} = \begin{bmatrix} -|z|^2 + it \\ \sqrt{2}z \\ 1 \end{bmatrix} \quad \text{and} \quad \hat{\infty} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Given any three points p_1, p_2, p_3 in $\partial \mathbf{H}^2_{\mathbb{C}}$ we define *Cartan's angular invariant* $\mathbb{A}$ as

$$\mathbb{A}(p_1, p_2, p_3) = \arg(-\langle \hat{p}_1, \hat{p}_2 \rangle \langle \hat{p}_2, \hat{p}_3 \rangle \langle \hat{p}_3, \hat{p}_1 \rangle).$$

In the special case where $p_1 = \infty$, $p_2 = (0, 0)$ and $p_3 = (z, t)$ we simply get $\tan(\mathbb{A}) = t/|z|^2$.

2.2 $\mathbb{R}$-circles, $\mathbb{C}$-circles, $\mathbb{R}$-spheres and $\mathbb{C}$-spheres

There are two kinds of totally geodesic submanifolds of real dimension 2 in $\mathbf{H}^2_{\mathbb{C}}$: complex lines in $\mathbf{H}^2_{\mathbb{C}}$ are *complex geodesics* (represented by $\mathbf{H}^1_{\mathbb{C}} \subset \mathbf{H}^2_{\mathbb{C}}$) and Lagrangian planes in $\mathbf{H}^2_{\mathbb{C}}$ are *totally real geodesic 2-planes* (represented by $\mathbf{H}^2_{\mathbb{R}} \subset \mathbf{H}^2_{\mathbb{C}}$). Each of these totally geodesic submanifolds is a model of the hyperbolic plane.

A discrete subgroup of $\mathbf{PU}(2,1)$ preserving a complex line is called $\mathbb{C}$-*Fuchsian* and is isomorphic to a subgroup of $\mathbf{P}\big(\mathbf{U}(1) \times \mathbf{U}(1,1)\big) \subset \mathbf{PU}(2,1)$. A discrete subgroup of $\mathbf{PU}(2,1)$ preserving a Lagrangian plane is called $\mathbb{R}$-*Fuchsian* and is isomorphic to a subgroup of $\mathbf{SO}(2,1)$ included in $\mathbf{PU}(2,1)$ by the projectivisation of the obvious inclusion $\mathbf{SO}(2,1) \subset \mathbf{SU}(2,1)$.

Consider complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$ and its boundary $\partial \mathbf{H}^2_{\mathbb{C}}$. We define $\mathbb{C}$-*circles* in $\partial \mathbf{H}^2_{\mathbb{C}}$ to be the boundaries of complex geodesics in $\mathbf{H}^2_{\mathbb{C}}$. Analogously, we define $\mathbb{R}$-*circles* in $\partial \mathbf{H}^2_{\mathbb{C}}$ to be the boundaries of Lagrangian planes in $H^2_{\mathbb{C}}$.

Proposition 2.1 (see [5]) *In the Heisenberg model, $\mathbb{C}$-circles are either vertical lines or ellipses, whose projection on the z-plane are circles.*

Finite $\mathbb{C}$-circles are determined by a *centre* $M = (z = a + ib, c)$ and a *radius* R. They may also be described using polar vectors in $P(V_+)$ (see Goldman [5] page 129). If we use the Hermitian form $\langle \cdot, \cdot \rangle$, a finite chain with centre $(a + ib, c)$ and radius R has polar vector

$$\begin{bmatrix} R^2 - a^2 - b^2 + ic \\ \sqrt{2}(a + ib) \\ 1 \end{bmatrix}$$

The condition for self-intersection between the complex lines defined by polar vectors v_1 and v_2 is

$$L(v_1, v_2) = |\langle v_1, v_2 \rangle|^2 - \langle v_1, v_1 \rangle \langle v_2, v_2 \rangle < 0.$$

(Compare Goldman [5] page 100, but note we are using a different Hermitian form.) This condition was obtained independently in [1] as a non-linking condition for $\mathbb{C}$-circles. We now show these two are equivalent. Let C_1 and C_2 be two circles of centres $(a_1 + ib_1, c_1)$, $(a_2 + ib_2, c_2)$ and radii R_1 and R_2. Let d and h be the horizontal and vertical distances between centres

$$d = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}, \quad h = c_2 - c_1, \quad S = \frac{1}{2}(a_1 b_2 - a_2 b_1).$$

Proposition 2.2 (see [1]) *The $\mathbb{C}$-circles $\mathbb{C}_1$ and $\mathbb{C}_2$ (with polar vectors v_1 and v_2) are linked if and only if*

$$\begin{aligned} L(v_1, v_2) &= \left(d^2 - (R_1 + R_2)^2\right)\left(d^2 - (R_1 - R_2)^2\right) + (h + 4S)^2 \\ &= \left(d^2 - {R_2}^2 - {R_1}^2\right)^2 + (h + 4S)^2 - 4{R_1}^2{R_2}^2 < 0. \end{aligned}$$

Proof: This can be seen by putting

$$v_1 = \begin{bmatrix} {R_1}^2 - {a_1}^2 - {b_1}^2 + ic_1 \\ \sqrt{2}(a_1 + ib_1) \\ 1 \end{bmatrix} \quad \text{and } v_2 = \begin{bmatrix} {R_2}^2 - {a_2}^2 - {b_2}^2 + ic_2 \\ \sqrt{2}(a_2 + ib_2) \\ 1 \end{bmatrix}$$

in Goldman's formula given above. □

We observe that there is a similar linking function for $\mathbb{R}$-circles but it would be too complicated to use in practice. Namely, consider any two $\mathbb{R}$-circles. The product of inversion in them is an element of $\mathbf{PU}(2,1)$. Choosing a representative g of this map in $\mathbf{SU}(2,1)$. Then there is a function $f\big(\mathrm{tr}(g)\big)$ (Theorem 6.2.4 of [5]) so that g is loxodromic when $f\big(\mathrm{tr}(g)\big) > 0$, g is parabolic or boundary elliptic when $f\big(\mathrm{tr}(g)\big) = 0$, and g is regular elliptic when $f\big(\mathrm{tr}(g)\big) < 0$. This corresponds to the original $\mathbb{R}$-circles being unlinked, touching or linked respectively (see [4]).

Definition 2.3 *Allowing a point to be a (degenerate) $\mathbb{C}$-circle, we define*

- *A $\mathbb{C}$-sphere around an $\mathbb{R}$-circle ($\mathbb{C}$-circle) is a union of $\mathbb{C}$-circles invariant under inversion in the $\mathbb{R}$-circle ($\mathbb{C}$-circle), and which is homeomorphic to a sphere.*
- *An $\mathbb{R}$-sphere around an $\mathbb{C}$-circle ($\mathbb{R}$-circle) is a union of arcs of $\mathbb{R}$-circles invariant under inversion in the $\mathbb{C}$-circle ($\mathbb{R}$-circle), and which is homeomorphic to a sphere.*

The definition of a $\mathbb{C}$-sphere first appeared in [4]. Special cases of $\mathbb{R}$-spheres were considered by Richard Schwartz [11, 12] where they are called *hybrid spheres*. Moreover, spinal spheres see [5] are special cases of both $\mathbb{C}$-spheres around an $\mathbb{R}$-circle and $\mathbb{R}$-spheres around a $\mathbb{C}$-circle.

3 Embeddings of the $(2,3,\infty)$-triangle group

3.1 The different components

Consider the $(2,3,\infty)$-triangle group $\widehat{\Gamma}$. In this section we consider representations of this group into $\widehat{\mathbf{PU}}(2,1)$ with antiholomorphic generators (so that the index two subgroup of holomorphic motions is isomorphic to the modular group Γ). Our representations will be determined by three $\mathbb{R}$-circles. The configurations of these $\mathbb{R}$-circles are closely related to the different ways of representing the elliptic elements of order 2 and 3 in Γ. Specifically, we show that there are six components of the representation space.

In order to motivate our discussion of elliptic elements we first consider the more familiar case of an elliptic element of $\mathbf{PSL}(2,\mathbb{R})$. Such a map fixes a point p of $\mathbf{H}^1_{\mathbb{C}}$ and rotates the tangent space at this point through an angle 2θ. The elliptic map may be written as the product of reflections in any two geodesics through p which meet at angle θ. Likewise, an elliptic element g of $\mathbf{PU}(2,1)$ acts on the tangent space at a fixed point in $\mathbf{H}^2_{\mathbb{C}}$ as an element of $\mathbf{U}(2)$ with eigenvalues $e^{2i\theta_1}$ and $e^{2i\theta_2}$. If we can write this elliptic map g as the product of inversions in a pair of $\mathbb{R}$-circles R_1 and R_2, then R_1 and R_2 are said to have *configuration* (θ_1,θ_2). (This notation was introduced in [2] and the definition we give here may be deduced from Definition 3.3 and Proposition 4.2 of that paper.) If θ_1 and θ_2 are both non-zero (mod π) then R_1 and R_2 are linked. If either $\theta_1 = 0$ or $\theta_2 = 0$ then R_1 and R_2 intersect twice. In this case g fixes the complex line through these two points and g is complex reflection in this line. If $\theta_1 = \theta_2$ then g is complex reflection in a point. Our space of embeddings of the $(2,3,\infty)$-triangle group $\widehat{\Gamma}$ will have different components depending on the different configurations of the pairs of $\mathbb{R}$-circles associated to the elliptic elements of orders 2 and 3.

First, consider an element of order 2 in $\mathbf{U}(2,1)$. It must have eigenvalues $+1$ and -1, one of which is repeated. There are two possibilities (other than $\pm I$): either the two dimensional eigenspace does not intersect V_-, in which case the map is a complex reflection in a point of $\mathbf{H}^2_{\mathbb{C}}$; or else the two dimensional eigenspace intersects V_-, in which case the map is a complex reflection in a complex geodesic in $\mathbf{H}^2_{\mathbb{C}}$. The corresponding $\mathbb{R}$-circles have configurations $(\pi/2, \pi/2)$ and $(\pi/2, 0)$ respectively.

Secondly, consider an element of order 3 in $\mathbf{U}(2,1)$. It must have eigenvalues 1, $\omega = e^{2\pi i/3}$ and $\omega^2 = \omega^{-1}$. Other than scalar multiples of the identity there are three possibilities (and their inverses). If there is a repeated eigenvalue then we obtain a complex reflection of order 3 in a point of $\mathbf{H}^2_{\mathbb{C}}$ or a complex line in $\mathbf{H}^2_{\mathbb{C}}$ as before. In this case $\mathbb{R}$-circles have configuration $(\pi/3, \pi/3)$ and $(\pi/3, 0)$ respectively. The third possibility is that the eigenvalues should be distinct. In this case the map is not a complex reflection and corresponds to $\mathbb{R}$-circles with configuration $(\pi/3, 2\pi/3)$. This is the most interesting case.

We obtain the following classes

1. $(\pi/2,\ 0;\ \pi/3,\ 2\pi/3)$ order 2: complex reflection in a line; order 3: not complex reflection;
2. $(\pi/2,\ 0;\ \pi/3,\ \pi/3)$ order 2: complex reflection in a line; order 3: complex reflection in a point;
3. $(\pi/2,\ 0;\ \pi/3,\ 0)$ order 2: complex reflection in a line; order 3: complex reflection in a line;
4. $(\pi/2,\ \pi/2;\ \pi/3,\ 2\pi/3)$ order 2: complex reflection in a point; order 3: not complex reflection;
5. $(\pi/2,\ \pi/2;\ \pi/3,\ \pi/3)$ order 2: complex reflection in a point; order 3: complex reflection in a point;
6. $(\pi/2,\ \pi/2;\ \pi/3,\ 0)$ order 2: complex reflection in a point; order 3: complex reflection in a line.

In this paper we will determine the component corresponding to class 1. The classes 2-6 were analysed in [2, 3, 8]. We now summarise the results about these classes.

Proposition 3.1 *[2] Suppose that $\alpha : \Gamma \longrightarrow \mathbf{PU}(2,1)$ is a representation of the modular group $\Gamma = \mathbf{PSL}(2,\mathbb{Z})$ for which all elliptic elements are complex reflections (either in points or lines). Then $\alpha(\Gamma)$ is a rigid $\mathbb{C}$-Fuchsian representation, that is it stabilises a complex line.*

PROOF: A complex reflection in a point p preserves every complex line passing through p. A complex reflection in a complex line l preserves every complex line orthogonal to l. Suppose that $\rho(\Gamma)$ is generated by two complex reflections g_2 and g_3 of orders 2 and 3 respectively. Suppose they fix a point p_2 (or line l_2) and a point p_3 (or line l_3). Then there is a unique complex line C through p_2 (orthogonal to l_2) and through p_3 (orthogonal to l_3 respectively). This line C is preserved by both generators and so is preserved by the whole group. Finally, C is a copy of $\mathbf{H}^1_{\mathbb{C}}$ and $\rho(\Gamma)$ acts there as the standard action of $\mathbf{PSL}(2,\mathbb{Z})$ on $\mathbf{H}^1_{\mathbb{C}}$. It is well known that this action is rigid. $\square$

This proposition deals with cases 2, 3, 5 and 6. We remark that that these four cases are not isomorphic. In order to see this, observe that in each case there is a complex line C preserved by the group and in each case the action there is the same. The elliptic maps g_2 and g_3, of orders 2 and 3 respectively, generating the modular group each have a unique fixed point in C, called p_2 and p_3 respectively. Consider the action of g_2 and g_3 on the complex lines C_2 and C_3 passing through these points and orthogonal to C. In each of the four cases g_j preserves C_j. When g_j is a complex reflection in a point then g_j acts on C_j as an elliptic map of order j. On the other hand, when g_j is a complex reflection in a line then g_j fixes C_j pointwise.

Proposition 3.2 *[3, 8] There is a one real dimensional space of discrete, faithful type preserving representations (up to conjugacy in $\widehat{\mathbf{PU}}(2,1)$) $\alpha_t : \Gamma \longrightarrow \mathbf{PU}(2,1)$ with the following properties. Each group $\alpha_t(\Gamma)$ is generated by a complex reflection of order 2 fixing a point and a map of order 3 that is not a complex reflection. The product of these two generators is parabolic. No two of these representations are conjugate in $\widehat{\mathbf{PU}}(2,1)$. These representations interpolate between a $\mathbb{C}$-Fuchsian representation (preserving a complex geodesic) and an $\mathbb{R}$-Fuchsian representation (preserving a Lagrangian plane). Furthermore, each representation α_t extends to a representation $\hat{\alpha}_t : \widehat{\Gamma} \longrightarrow \widehat{\mathbf{PU}}(2,1)$ with antiholomorphic generators.*

Throughout the rest of the paper we will consider the case where the generator $\iota_1\iota_0$ of order 2 is a complex reflection in a complex line and where the generator $\iota_0\iota_2$ of order 3 is not a complex reflection. Using the notation described above, this is the class $(\pi/2, 0; \pi/3, 2\pi/3)$.

3.2 Normalisation and notation

Conjugating if necessary, we now give the particular normalisation of the group which we choose to work with. Following the usual description of the modular group acting on the upper half plane, we normalise so that the fixed point of the parabolic map $\iota_1\iota_2$ is ∞ and the reflection ι_0 fixes the standard imaginary $\mathbb{R}$-circle R_0. This $\mathbb{R}$-circle is defined by the following equations, see equation (4.17) on page 141 of Goldman [5]:

$$R_0 = \left\{(x+iy,t) \in \mathfrak{N} : (x^2+y^2)^2 + x^2 - y^2 = yt - x(1+x^2+y^2) = 0\right\}.$$

This normalisation automatically means that the $\mathbb{R}$-circle R_1 is the y-axis

$$R_1 = \{(x+iy,t) \in \mathfrak{N} : x = t = 0\}.$$

(Compare this with [8] where R_0 is as above but R_1 is the x-axis.) The inversions ι_0 and ι_1 are then given by

$$\begin{aligned} \iota_0(x+iy,t) &= \left(\frac{-x+iy}{x^2+y^2+it}, \frac{t}{(x^2+y^2)^2+t^2}\right), \\ \iota_1(x+iy,t) &= (-x+iy,-t). \end{aligned}$$

As maps on $\mathbb{C}^{2,1}$ these inversions are given by

$$\iota_0 : \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \longrightarrow \begin{bmatrix} \overline{Z} \\ \overline{Y} \\ \overline{X} \end{bmatrix}, \qquad \iota_1 : \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \longrightarrow \begin{bmatrix} \overline{X} \\ -\overline{Y} \\ \overline{Z} \end{bmatrix}. \tag{3.1}$$

Thus, the order 2 generator $\iota_1\iota_0 = c_1$ is given by the following element of $\mathbf{PU}(2,1)$

$$\iota_1\iota_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Observe that $\iota_1\iota_0 = c_1$ fixes each point of the *equator* C_1, that is the $\mathbb{C}$-circle with radius 1 and centred at $(0,0)$. The polar vector of this $\mathbb{C}$-circle is

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

There is a one parameter family of $\mathbb{R}$-circles R_2 so that $\iota_0\iota_2$ has order three, where ι_2 is inversion in R_2. Before writing down expressions for these $\mathbb{R}$-circles, we explain our parametrisation.

Consider the point $p_1 = \infty$ fixed by the parabolic map $\iota_1\iota_2$ and its images p_2 and p_3 under $\iota_0\iota_2$ and $(\iota_0\iota_2)^2 = \iota_2\iota_0$. These are three points of $\partial\mathbf{H}^2_{\mathbb{C}}$ and have an angular invariant $\mathbb{A}$. This angular invariant will be our parameter. First, observe that

$$(\iota_0\iota_2)\iota_1\iota_2(\iota_2\iota_0) = \iota_0(\iota_2\iota_1)\iota_0$$

and its fixed point is $p_2 = \iota_0(\infty) = (0,0)$. Now consider $(\iota_2\iota_0)\iota_1\iota_2(\iota_0\iota_2)$. Its fixed point is $p_3 = \iota_0\iota_2(0,0) = \iota_2\iota_0(\infty) = \iota_2(0,0)$. Thus p_3 is fixed by ι_0 and so lies on R_0. This determines p_3 in terms of $\mathbb{A} = \mathbb{A}(p_1,p_2,p_3)$ as

$$p_3 = \Big(ie^{-i\mathbb{A}/2}\sqrt{\cos(\mathbb{A})}, \sin(\mathbb{A})\Big).$$

(There is also an ambiguity in the sign of the z coordinate of p_3, which corresponds to a further conjugation in $\mathbf{PU}(2,1)$.) We also have

$$\hat{p}_3 = \begin{bmatrix} -e^{-i\mathbb{A}} \\ ie^{-i\mathbb{A}/2}\sqrt{2\cos(\mathbb{A})} \\ 1 \end{bmatrix}.$$

From this we see that the third $\mathbb{R}$-circle R_2 must be given by

$$R_2 = \left\{ (x+iy,y) \in \mathfrak{N} : \begin{matrix} 0 = 2x\sin(\mathbb{A}/2) + 2y\cos(\mathbb{A}/2) - \sqrt{\cos(\mathbb{A})} \\ 0 = t - x\cos(\mathbb{A}/2)\sqrt{\cos(\mathbb{A})} + y\sin(\mathbb{A}/2)\sqrt{\cos(\mathbb{A})} - \sin(\mathbb{A})/2 \end{matrix} \right\}.$$

Inversion in R_2 is given by

$$\iota_2(x+iy,t) = \Big((x-iy)e^{-i\mathbb{A}} + ie^{i\mathbb{A}/2}\sqrt{\cos(\mathbb{A})}, -t + 2\big(x\cos(\mathbb{A}/2) - y\sin(\mathbb{A}/2)\big)\sqrt{\cos(\mathbb{A})} + \sin(\mathbb{A})\Big),$$

or

$$\iota_2 : \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \longmapsto \begin{bmatrix} \overline{X} + ie^{-i\mathbb{A}/2}\sqrt{2\cos(\mathbb{A})}\,\overline{Y} - e^{-i\mathbb{A}}\overline{Z} \\ e^{-i\mathbb{A}}\overline{Y} + ie^{-i\mathbb{A}/2}\sqrt{2\cos(\mathbb{A})}\,\overline{Z} \\ \overline{Z} \end{bmatrix}. \tag{3.2}$$

Using this, we see that

$$\begin{aligned} \iota_0\iota_2 &= \begin{bmatrix} 0 & 0 & -e^{-i\mathbb{A}/3} \\ 0 & -e^{2i\mathbb{A}/3} & ie^{i\mathbb{A}/6}\sqrt{2\cos(\mathbb{A})} \\ -e^{-i\mathbb{A}/3} & ie^{i\mathbb{A}/6}\sqrt{2\cos(\mathbb{A})} & e^{2i\mathbb{A}/3} \end{bmatrix}, \\ \iota_1\iota_2 &= \begin{bmatrix} -e^{-i\mathbb{A}/3} & ie^{i\mathbb{A}/6}\sqrt{2\cos(\mathbb{A})} & e^{2i\mathbb{A}/3} \\ 0 & e^{2i\mathbb{A}/3} & -ie^{i\mathbb{A}/6}\sqrt{2\cos(\mathbb{A})} \\ 0 & 0 & -e^{-i\mathbb{A}/3} \end{bmatrix}. \end{aligned}$$

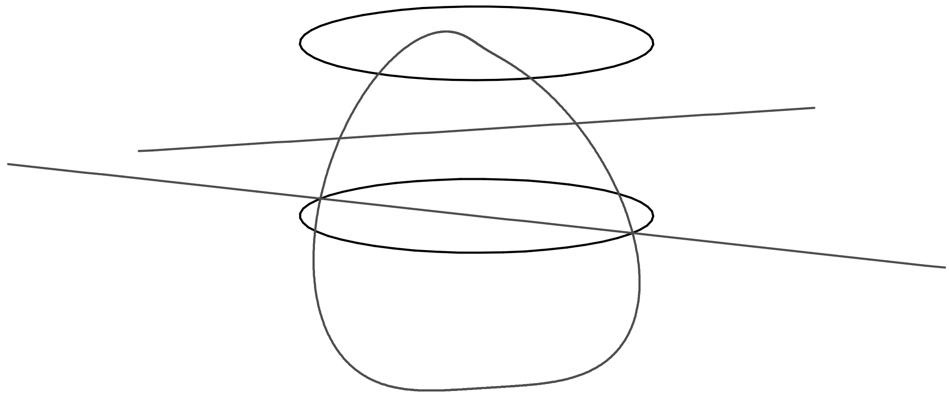

Figure 1: *Configuration of the three* $\mathbb{R}$*-circles* R_0, R_1 *and* R_2 *when* $\mathbb{A} = \pi/2$.

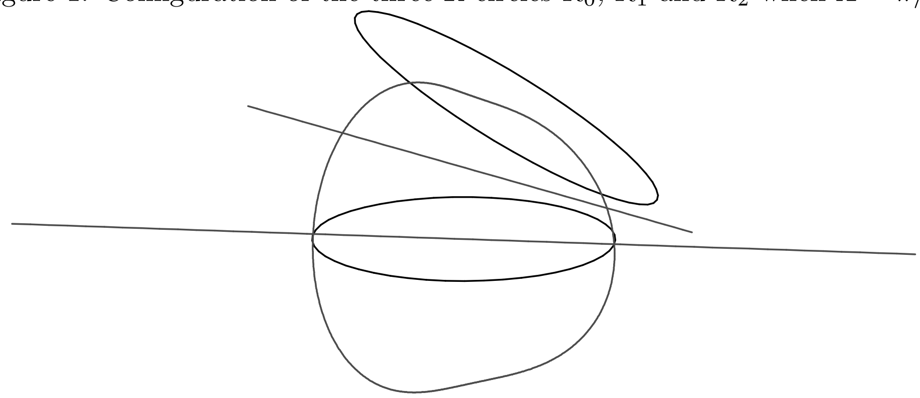

Figure 2: *Configuration of the three* $\mathbb{R}$*-circles* R_0, R_1 *and* R_2 *when* $\mathbb{A} = 1.4$.

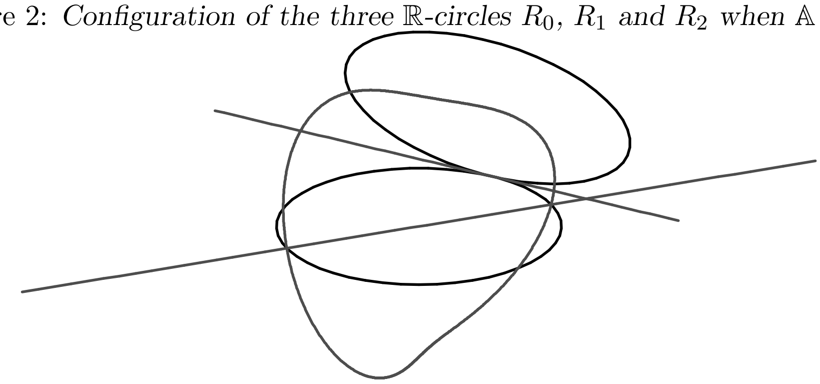

Figure 3: *Configuration of the three* $\mathbb{R}$*-circles* R_0, R_1 *and* R_2 *when* $\cos \mathbb{A} = 1/4$.

Remark 3.3 *Another description of the deformation is obtained by parametrising the point of R_2 that is a centre of an invariant $\mathbb{C}$-circle under both inversions ι_0 and ι_2 (see [2] for details). Using polar coordinates we obtain the following expressions for the projection of the centre, $r(\theta)$, and the angle α between the x-axis and the projection of $\mathbb{R}_2$:*

$$\alpha = 3\theta - \pi/2, \qquad r(\theta)^2 = \frac{4\sin^2 2\theta - 1}{4\cos 2\theta} = -\frac{\cos 6\theta}{4\cos^2 2\theta}$$

for $\pi/12 \le \theta \le \pi/6$.

A brief calculation using Cartan's invariant shows that these two parametrisations are related by $\mathbb{A} = -2\alpha = \pi - 6\theta$.

By construction we know that $\iota_0\iota_1$ has order 2, $\iota_0\iota_2$ has order 3 and $\iota_1\iota_2$ is parabolic. Therefore we have the following result:

Proposition 3.4 *For each $\mathbb{A} \in [-\pi/2, \pi/2]$ the map $\hat{\rho} : \widehat{\Gamma} \longrightarrow \widehat{\mathbf{PU}}(2,1)$ given by (3.1) and (3.2) is a representation of the $(2,3,\infty)$-triangle group.*

3.3 Further properties

We have already found the fixed $\mathbb{C}$-circle of $\iota_1\iota_0$. We now find the invariant $\mathbb{C}$-circles of $\iota_0\iota_2$.

Lemma 3.5 *The three eigenspaces of $\iota_2\iota_0$ are spanned by*

$$q_j = \begin{bmatrix} -e^{-i(\mathbb{A}+2j\pi)/3} \\ -ie^{-i(\mathbb{A}+2j\pi)/6}\sqrt{2\cos(\mathbb{A})}/2\cos\big((\mathbb{A}+2j\pi)/3\big) \\ 1 \end{bmatrix}$$

for $j = 0, \pm 1$. Moreover, q_0 lies in V_- and is the fixed point of $\iota_2\iota_0$ in $\mathbf{H}^2_{\mathbb{C}}$ and $q_{\pm 1}$ lie in V_+ and are polar to the two invariant $\mathbb{C}$-circles of $\iota_0\iota_2$.

PROOF: We know that $\iota_2\iota_0$ has eigenvalues $e^{2j\pi i/3}$ and so the expressions for q_j follow easily. We now calculate $\langle q_j, q_j\rangle$:

$$\begin{aligned} \langle q_j, q_j\rangle &= -2\cos\big((\mathbb{A}+2j\pi)/3\big) + \frac{\cos(\mathbb{A})}{2\cos^2\big((\mathbb{A}+2j\pi)/3\big)} \\ &= \frac{-3}{2\cos\big((\mathbb{A}+2j\pi)/3\big)} \end{aligned}$$

where we have used $\cos(\mathbb{A}) = 4\cos^3\big((\mathbb{A}+2j\pi)/3\big) - 3\cos\big((\mathbb{A}+2j\pi)/3\big)$. Since $\mathbb{A} \in [-\pi/2, \pi/2]$ we see that

$$\mathbb{A}/3 \in [-\pi/6, \pi/6], \qquad (\mathbb{A}+2\pi)/3 \in [\pi/2, 5\pi/6], \qquad (\mathbb{A}-2\pi)/3 \in [-5\pi/6, -\pi/2].$$

Therefore $q_0 \in V_-$ and $q_{\pm 1} \in V_+$ as required. □

We now show why the methods used in [2] to construct fundamental domains do not work in our case. There it was crucial that the fixed $\mathbb{C}$-circles of the generators were not linked.

Proposition 3.6 *For all $\mathbb{A} \in [0, \pi/2]$ the $\mathbb{C}$-circle polar to q_{-1} is linked with C_1, the fixed $\mathbb{C}$-circle of $\iota_1\iota_0$. On the other hand, the $\mathbb{C}$-circle polar to q_{+1} is not linked with C_1 if and only if $0 \le \cos(\mathbb{A}) < (19 - 7\sqrt{7})/2 \sim 0.2399\ldots$.*

PROOF: For $j = \pm 1$ we calculate

$$\begin{aligned}
& L\left(\begin{bmatrix} -e^{-i(\mathbb{A}+2j\pi)/3} \\ -ie^{-i(\mathbb{A}+2j\pi)/6}\sqrt{2\cos(\mathbb{A})}/2\cos\big((\mathbb{A}+2j\pi)/3\big) \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) \\
= & \left|1 - e^{-i(\mathbb{A}+2j\pi)/3}\right|^2 - \frac{-3}{2\cos\big((\mathbb{A}+2j\pi)/3\big)} \cdot 2 \\
= & \frac{2\cos^2\big((\mathbb{A}+2j\pi)/3\big) - 2\cos\big((\mathbb{A}+2j\pi)/3\big) - 3}{-\cos\big((\mathbb{A}+2j\pi)/3\big)}.
\end{aligned}$$

The denominator is always positive. The numerator is positive when

$$\cos\big((\mathbb{A}+2j\pi)/3\big) < (1-\sqrt{7})/2.$$

This never happens for $j = -1$ and $\mathbb{A} \in [0, \pi/2]$. For $j = +1$ this condition is equivalent to

$$\begin{aligned}
\cos(\mathbb{A}) &= 4\cos^3\big((\mathbb{A}+2j\pi)/3\big) - 3\cos\big((\mathbb{A}+2j\pi)/3\big) \\
&< \frac{(1-\sqrt{7})^3}{2} - \frac{3(1-\sqrt{7})}{2} \\
&= \frac{19 - 7\sqrt{7}}{2}
\end{aligned}$$

The result follows. □

We remark that for $\mathbb{A} \in [-\pi/2, 0]$ the result is the same except that q_{+1} and q_{-1} are the other way around. The importance of this proposition is that, in order to follow the construction of a fundamental domain for $\langle \iota_0, \iota_1, \iota_2 \rangle$ bounded by $\mathbb{C}$-spheres in [2], it is necessary that C_1 should not be linked with (at least) one of the two invariant $\mathbb{C}$-circles of $\iota_2\iota_0$. Therefore the construction of [2] could only be followed for groups with

$$0 \leq \cos(\mathbb{A}) \leq (19 - 7\sqrt{7})/2.$$

As we shall see in the next sections, the group remains discrete for for $\mathbb{A}$ beyond the end of this interval. In fact, we shall see that the group $\langle \iota_0,\ \iota_1,\ \iota_2 \rangle$ is discrete provided the $\mathbb{R}$-circle R_2 and the equator C_1 are not linked. We now show that this condition holds whenever $0 \leq \cos(\mathbb{A}) \leq 1/4$ and we observe that $(19 - 7\sqrt{7})/2 < 1/4$.

Lemma 3.7 *The $\mathbb{R}$-circle R_2 and the equator C_1 are not linked whenever* $0 \leq \cos(\mathbb{A}) \leq 1/4$.

PROOF: We know that the $\mathbb{R}$-circle R_2 is given by the equations

$$\begin{aligned}
x\sin(\mathbb{A}/2)\sqrt{\cos(\mathbb{A})} + y\cos(\mathbb{A}/2)\sqrt{\cos(\mathbb{A})} &= \frac{\cos(\mathbb{A})}{2} \\
t - x\cos(\mathbb{A}/2)\sqrt{\cos(\mathbb{A})} + y\sin(\mathbb{A}/2)\sqrt{\cos(\mathbb{A})} &= \frac{\sin(\mathbb{A})}{2}
\end{aligned}$$

Setting $t = 0$, squaring both equations and adding we obtain

$$(x^2 + y^2)\cos(\mathbb{A}) = \frac{\cos^2(\mathbb{A}) + \sin^2(\mathbb{A})}{4} = \frac{1}{4}.$$

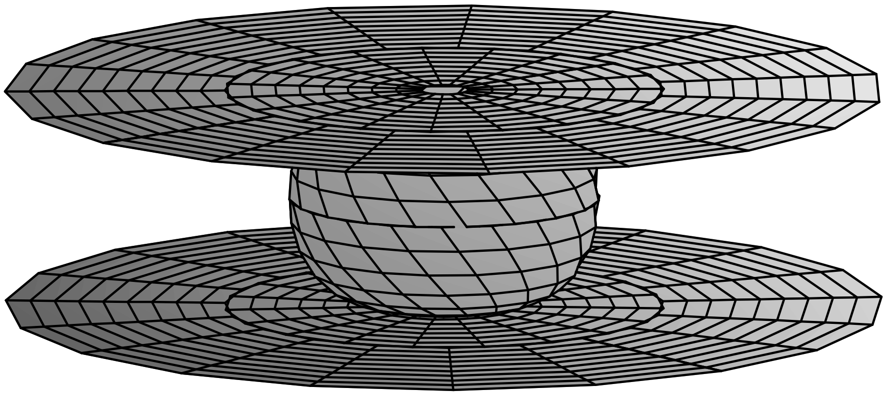

Figure 4: *Surfaces bounding a fundamental domain when* $\mathbb{A} = \pi/2$.

In other words, R_2 intersects the $\mathbb{C}$-circle centred at $(0,0)$ with radius $1/\big(2\sqrt{\cos(\mathbb{A})}\big)$. It is easy to see that R_2 and C_1 are not linked whenever this radius is greater than 1. This gives the result. □

The element $(\iota_0\iota_1\iota_2)^2$ is reflection in R_2 followed by reflection in $c_1(R_2)$. This lemma says that when $\cos(\mathbb{A}) = 1/4$ these two $\mathbb{R}$-circles touch and so $(\iota_0\iota_1\iota_2)^2$ has become parabolic (see [4]). Another way to see this is to observe that a brief calculation gives

$$\mathrm{tr}\big((\iota_0\iota_1\iota_2)^2\big) = 4 - 4\cos(\mathbb{A}).$$

Therefore, $(\iota_0\iota_1\iota_2)^2$ is loxodromic for $\cos(\mathbb{A}) < 1/4$, parabolic for $\cos(\mathbb{A}) = 1/4$ and elliptic for $\cos(\mathbb{A}) > 1/4$. If $(\iota_0\iota_1\iota_2)^2$ is elliptic then either it has infinite order, which means that $\hat{\rho}$ is not discrete, or else it has finite order, in which case $\hat{\rho}$ is not faithful (compare [6]). Thus we have:

Proposition 3.8 *For each* $\mathbb{A}$ *satisfying* $1/4 < \cos(\mathbb{A}) \leq 1$ *the representation* $\hat{\rho} : \widehat{\Gamma} \longrightarrow \widehat{\mathbf{PU}}(2,1)$ *given by (3.1) and (3.2) is either not discrete or not faithful.*

In the next section we show that for $0 \leq \cos(\mathbb{A}) \leq 1/4$ the representation $\hat{\rho}$ is both discrete and faithful. This will complete the proof of our main theorems.

4 Fundamental domains

4.1 The general setup

The strategy we adopt is to construct a fundamental domain for the subgroup G generated by $c_1 = \iota_1\iota_0$, $c_0 = (\iota_0\iota_2)\iota_1\iota_0(\iota_2\iota_0)$ and $c_2 = (\iota_2\iota_0)\iota_1\iota_0(\iota_0\iota_2) = \iota_2\iota_1\iota_0\iota_2$. This fundamental domain will consist of three topological spheres S_j each of which is preserved by one of the involutions c_j. We give two different constructions. The first construction follows Schwartz [12] and uses $\mathbb{R}$-spheres. Indeed, for the limiting case when $\cos(\mathbb{A}) = 1/4$ our fundamental domain will coincide with his. Our second construction uses $\mathbb{C}$-spheres and may be compared to [1, 8].

The basic idea is, for each $\mathbb{A}$ with $0 \leq \cos(\mathbb{A}) \leq 1/4$, to construct an $\mathbb{R}$-sphere (respectively a $\mathbb{C}$-sphere) S_1 with the following properties. First, S_1 is mapped to itself by $c_1 = \iota_1\iota_0$ and secondly, for $\cos(\mathbb{A}) < 1/4$, only intersects S_2, its image under $\iota_2\iota_1$, at the point at infinity. For $\mathbb{A} = \arccos(1/4)$ there will be another intersection point which is $C_1 \cap R_2$. This point is clearly invariant under $\iota_1\iota_0$ and ι_2. (For the $\mathbb{R}$-sphere fundamental domain there will be additional points of tangency along an arc of R_2.)

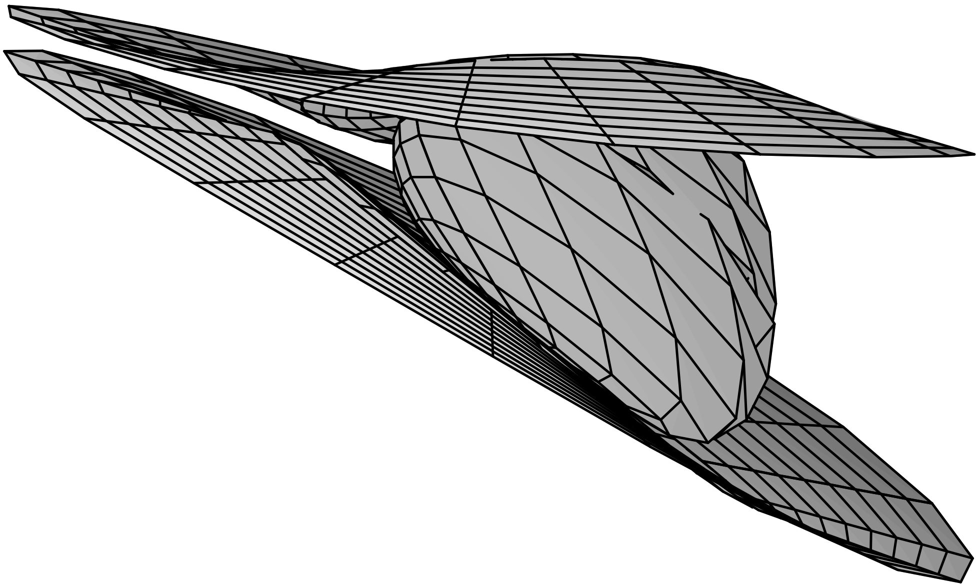

Figure 5: $\mathbb{R}$-*spheres bounding a fundamental domain when* $\mathbb{A} = 1.4$.

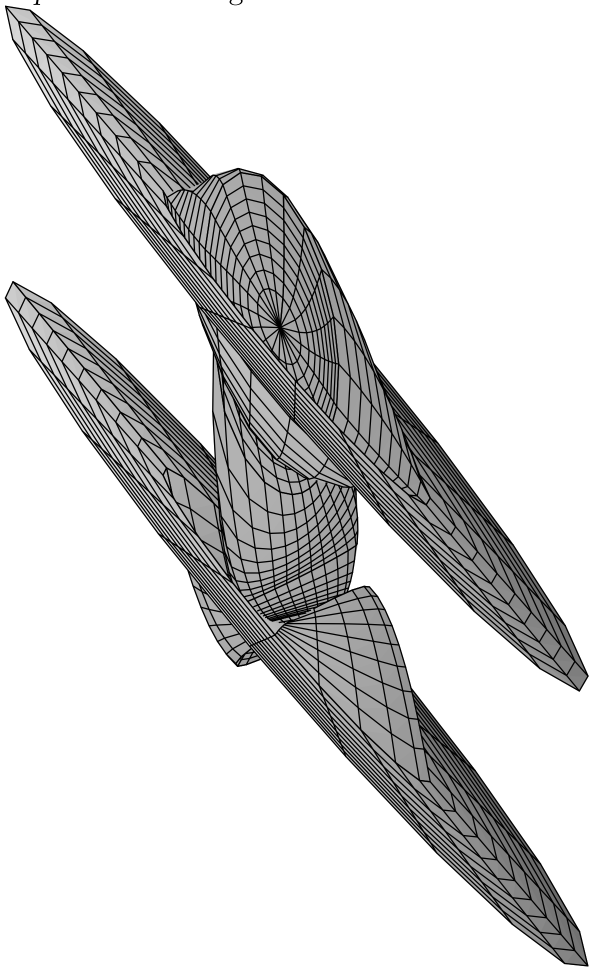

Figure 6: $\mathbb{C}$-*spheres bounding a fundamental domain when* $\mathbb{A} = 1.5$.

Using the symmetry of the construction under $\iota_2\iota_0$ we now have $\mathbb{R}$-spheres (respectively $\mathbb{C}$-spheres) S_0, S_1 and S_2. The sphere S_j consists of two hemispheres which are interchanged by the complex reflection c_j fixing the $\mathbb{C}$-circle $C_j \subset S_j$. The spheres S_j are disjoint except for points of tangency and have a common exterior D. In order to show that $\langle c_0, c_1, c_2\rangle$ is discrete we show that D is a fundamental polyhedron with side pairing maps c_0, c_1 and c_2.

This construction uses Poincaré's polyhedron theorem. For the case of polyhedra constructed from $\mathbb{R}$-spheres this has not been written down. For the polyhedra we consider this would consist of writing out Theorem 6.2 of [7] word for word except that bisector should be replaced with $\mathbb{R}$-sphere. This is straightforward provided one is careful about points of tangency. A version of Poincaré's polyhedron theorem for polyhedra bounded by $\mathbb{C}$-spheres is given in [4].

Alternatively, we could follow Schwartz and use a the ping-pong lemma, that is a topological version of Klein's combination theorem in its simplest form (Corollary 4.2 of [12]). This shows that the representation is discrete, faithful and type preserving.

4.2 A fundamental domain bounded by $\mathbb{R}$-spheres

The main result of this section will be

Theorem 4.1 *For each $\mathbb{A}$ with $0 \leq \cos(\mathbb{A}) \leq 1/4$ there exist $\mathbb{R}$-spheres S_0, S_1, S_2 so that*

- *S_j contains C_j the fixed $\mathbb{C}$-circle of c_j;*
- *$c_j(S_j) = S_j$;*
- *S_0, S_1 and S_2 are the faces of a fundamental domain for the group G generated by the side pairing maps c_0, c_1 and c_2;*
- *for $0 \leq \cos(\mathbb{A}) < 1/4$ the $\mathbb{R}$-spheres S_j and S_k are disjoint except for points of tangency at parabolic fixed points,*
- *for $\cos(\mathbb{A}) = 1/4$ the $\mathbb{R}$-spheres S_j and S_k are disjoint except for an arc of tangency contained in an $\mathbb{R}$-circle.*

We remark that when $\cos(\mathbb{A}) = 1/4$ the group $\langle c_0, c_1, c_2\rangle$ is the "last ideal triangle group" considered by Schwartz in [12] and the fundamental domain we construct in this section is the same as the one constructed by Schwartz. Notice that the deformation of this group considered in this paper is different from the deformations considered in [12] (see also [6, 11]). In our groups $c_1c_0c_2$ remains parabolic throughout whereas c_1c_2 is loxodromic for all groups except the last one.

The theorem will follow using the threefold symmetry given by the map $\iota_0\iota_2$ and the following results.

Proposition 4.2 *The $\mathbb{R}$-sphere S_1 is a topological sphere which may be decomposed into two hemispheres S_1^+ and S_1^- with common boundary C_1. Each of these hemispheres is mapped to itself under ι_1 and the two hemispheres are interchanged by ι_0.*

Proposition 4.3 *Let $S_2 = \iota_2\iota_0(S_1)$. Then, for $\cos(\mathbb{A}) < 1/4$, the $\mathbb{R}$-spheres S_1 and S_2 are disjoint except for a single point of tangency at ∞, the fixed point of the parabolic map $c_1c_0c_2 = (\iota_1\iota_2)^3$. For $\cos(\mathbb{A}) = 1/4$, the $\mathbb{R}$-spheres have an additional line of tangency along an arc of the $\mathbb{R}$-circle R_2 joining ∞ to the fixed point of the parabolic map $c_1c_2 = (\iota_0\iota_1\iota_2)^2$.*

We remark that this construction generalises the natural fundamental domain for the $\mathbb{C}$-Fuchsian representation bounded by three bisectors. Specifically, we shall see that when $\mathbb{A} = \pi/2$ (that is $\cos(\mathbb{A}) = 0$) the $\mathbb{R}$-sphere S_1 is the bisector with vertices $(0,0)$ and ∞ (that is the plane $t = 0$) and the bisector S_2 is the bisector with vertices $(0,1)$ and ∞ (that is the plane $t = 1$). Both Proposition 4.2 and Proposition 4.3 are clear for this case.

We now begin our construction. The axis A_0 of the parabolic map $\iota_1\iota_2$ is the vertical $\mathbb{C}$-circle passing through finite points of both R_1 and R_2. It is given by

$$A_0 = \left\{ (x+iy,t) \in \mathfrak{N} : x = 0, y = y_0 = \frac{\sqrt{\cos(\mathbb{A})}}{2\cos(\mathbb{A}/2)} \right\}.$$

Observe that $y_0 \leq 1/\sqrt{2}$ and so the $\mathbb{C}$-circles A_0 and C_1 are obviously linked. (In fact, for $0 \leq \cos(\mathbb{A}) \leq 1/4$ we have $0 \leq y_0 \leq 1/\sqrt{10}$.)

Following Section 3.2 of Schwartz [12], we define the $\mathbb{R}$-sphere S_1 as follows. Consider the fixed $\mathbb{C}$-circle of $\iota_1\iota_0$. This is the equator C_1. For any point $p_\alpha = (e^{i\alpha}, 0)$ of C_1 we define an $\mathbb{R}$-arc I_α (that is an arc of an $\mathbb{R}$-circle) as follows. There is a unique $\mathbb{R}$-circle passing through ∞, p_α and a finite point of A_0. The points ∞ and p_α divide this $\mathbb{R}$-circle into two $\mathbb{R}$-arcs. One of these $\mathbb{R}$-arcs does not meet the finite part of A_0. We define this $\mathbb{R}$-arc to be I_α. The hemisphere S_1^+ is defined to be the union over all points $p_\alpha \in C_1$ of the $\mathbb{R}$-arcs I_α. The hemisphere S_1^- is defined to be the image of S_1^+ under inversion in C_1, that is the map $c_1 = \iota_1\iota_0$. The $\mathbb{R}$-sphere S_1 is defined to be the union of S_1^+ and S_1^-. This construction is a little clearer if we consider the image of S_1 under vertical projection $\Pi(x+iy,t) = (x+iy)$. The vertical projection of C_1 is the unit circle in $\mathbb{C}$ and the vertical projection of A_0 is the point iy_0 in its interior. Consider the family of lines in $\mathbb{C}$ passing through iy_0. Each of these lines intersects the unit circle in two points $e^{i\alpha}$ and $e^{i\alpha'}$. (These angles are related by $\sin(\alpha - \alpha') = y_0(\cos\alpha' - \cos\alpha)$.) The vertical projection of I_α (or $I_{\alpha'}$) is the infinite segment of this line outside the unit disc with endpoint $e^{i\alpha}$ (or $e^{i\alpha'}$ respectively). The union of these infinite arcs gives a foliation of the exterior of the unit disc. As vertical projection is injective on a single infinite $\mathbb{R}$-circle it is clear that S_1^+ is a homeomorphic to a disc. As this disc lies outside the unit Heisenberg sphere (that is the points $(x+iy,t) \in \mathfrak{N}$ with $(x^2+y^2)^2 + t^2 = 1$) and this sphere is preserved by ι_0 we see that S_1^- is a topological disc disjoint from S_1^+. Thus we have proved the following lemma (see Lemma 3.2 of [12]):

Lemma 4.4 *The $\mathbb{R}$-sphere $S_1 = S_1^+ \cup S_1^-$ is a topological sphere.*

It is clear from the construction that S_1 is invariant under $c_1 = \iota_1\iota_0$. We claim, moreover, that S_1 is invariant under ι_1. We know that the $\mathbb{R}$-circle R_1 intersects C_1 at the points $(\pm i, 0)$ and intersects A_0 at $(iy_0, 0)$. Therefore R_1 contains the $\mathbb{R}$-arcs $I_{\pm\pi/2}$. Since R_1 is invariant under c_1, it is clear that the whole of R_1 is contained in S_1. Moreover, the $\mathbb{R}$-arc I_α is just the image of $I_{\pi-\alpha}$ under ι_1. Hence S_1 is invariant under ι_1 which, moreover, preserves the two hemispheres. Because S_1 is preserved by $c_1 = \iota_1\iota_0$ we see that it is also invariant under ι_0 (and this map interchanges the two hemispheres). This has proved Proposition 4.2.

We now define $S_2 = \iota_2\iota_0(S_1) = \iota_2(S_1)$. This consists of a hemisphere S_2^+ foliated by $\mathbb{R}$-arcs joining ∞ to C_2, the fixed $\mathbb{C}$-circle of $c_2 = (\iota_2\iota_0)\iota_1\iota_0(\iota_0\iota_2)$. These arcs are contained in $\mathbb{R}$-circles passing through ∞, C_2 and A_0. The other hemisphere S_2^- of S_2 is the image of S_2^+ under c_2.

We want to prove Proposition 4.3. Our proof will follow the proof of Lemma 4.1 of [12] and will rely on one of Schwartz's results about $\mathbb{R}$-spheres (Proposition 4.6).

Following Schwartz, we define a projection along the $\mathbb{R}$-arcs I_α defined above. The simplest way to define this projection is to apply the Heisenberg translation $T_{-iy_0,0}$ sending A_0 to the t-axis

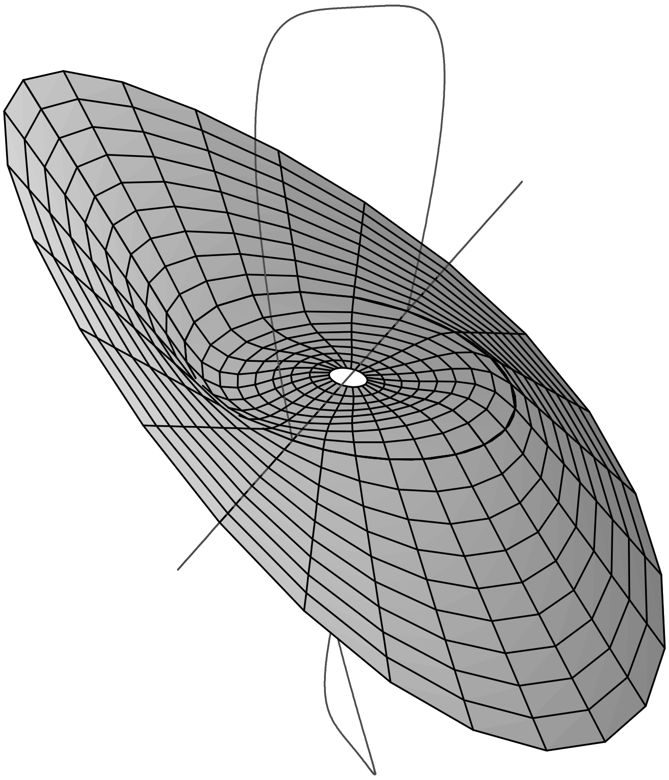

Figure 7: S_1 *when* $\mathbb{A} = 1.4$.

in the Heisenberg group. Then putting cylindrical polar coordinates $(R, \varphi, t) \in \mathbb{R}_+ \times S^1 \times \mathbb{R}$ on $\mathfrak{N}$ (here $\varphi \in [0, 2\pi]$ is an angle parameter for S^1) we define a projection $\xi : \mathfrak{N} \longrightarrow S^1 \times \mathbb{R}$ by $\xi(R, \varphi, t) = (\varphi, t)$ (see Section 2.4 of [12]). We now make this construction explicit. The projection we use is

$$\xi_0 = \xi \circ T_{-iy,0} : \mathfrak{N} \longrightarrow S^1 \times \mathbb{R}.$$

The fibres of this projection are arcs of $\mathbb{R}$-circles joining points of A_0 to ∞. Specifically, each infinite $\mathbb{R}$-circle that intersects A_0 corresponds to a pair of points in $S^1 \times \mathbb{R}$ which differ by addition of π to the S^1 coordinate. In particular, it is not hard to show that $\xi_0(R_1)$ comprises the two points $(\pi/2, 0)$ and $(3\pi/2, 0)$. Thus $\xi_0 \iota_1 \xi_0^{-1}$ is the rotation of the cylinder $S^1 \times \mathbb{R}$ fixing these points given by $(\varphi, t) \longmapsto (\pi - \varphi, -t)$. Similarly, $\xi_0(R_2)$ comprises the two points $\bigl(-\mathbb{A}/2, \tan(\mathbb{A}/2)/2\bigr)$ and $\bigl(\pi - \mathbb{A}/2, \tan(\mathbb{A}/2)/2\bigr)$ in $S^1 \times \mathbb{R}$. Thus $\xi_0 \iota_2 \xi_0^{-1}$ is a rotation of the cylinder $S^1 \times \mathbb{R}$ fixing these two points given by $(\varphi, t) \longmapsto \bigl(-\varphi - \mathbb{A}, \tan(\mathbb{A}/2) - t\bigr)$.

It is also easy to see that the arcs I_α (and $\iota_2(I_\alpha)$) each project under ξ_0 to the point $\xi_0(p_\alpha)$ (and $\xi_0(\iota_2 p_\alpha)$ respectively) in $S^1 \times \mathbb{R}$. We now find the images of C_1 and C_2 under ξ_0. This is essentially an application of Lemma 2.1 of [12]. We include all the details for clarity. The Heisenberg translation $T_{-iy_0,0}$ sending A_0 to the t-axis is given in Heisenberg coordinates by

$$T_{-iy_0,0}(x + iy, t) = (x + iy - iy_0, t - 2y_0 x).$$

Let $p_\alpha = (e^{i\alpha}, 0)$ be a point on C_1. Then

$$T_{-iy_0,0}\bigl(e^{i\alpha}, 0\bigr) = \bigl(e^{i\alpha} - iy_0, -2y_0 \cos(\alpha)\bigr).$$

In cylindrical polar coordinates, this point is given by

$$\begin{aligned} R\cos(\varphi) &= \cos(\alpha), \\ R\sin(\varphi) &= \sin(\alpha) - y_0, \\ t &= -2y_0\cos(\alpha). \end{aligned}$$

In order to find $\xi_0(C_1)$ we need to eliminate R and α from the above equations. This is done as follows

$$1 = \cos^2(\alpha) + \sin^2(\alpha) = R^2 + 2Ry_0\sin(\varphi) + {y_0}^2.$$

Taking the positive root of this equation gives

$$R = -y_0\sin(\varphi) + \sqrt{1 - {y_0}^2\cos^2(\varphi)}.$$

We may now express t as a function $t = t_1(\varphi)$ of φ depending only on φ:

$$t_1(\varphi) = -2y_0R\cos(\varphi) = 2{y_0}^2\cos(\varphi)\sin(\varphi) - 2y_0\cos(\varphi)\sqrt{1 - {y_0}^2\cos^2(\varphi)}.$$

Then $\xi_0(C_1)$ is the curve in $S^1 \times \mathbb{R}$ defined by $t = t_1(\varphi)$.

Similarly, let $\big(e^{i\beta} + iy_0(1+e^{-i\mathbb{A}}), 2y_0\cos(\beta+\mathbb{A}) + 2y_0\cos(\beta) + \sin(\mathbb{A})\big)$ be any point on C_2. Then the image of this point under $T_{-iy_0,0}$ is

$$\big(e^{i\beta} + iy_0e^{-i\mathbb{A}}, 2y_0\cos(\beta+\mathbb{A}) + \tan(\mathbb{A}/2)\big).$$

In cylindrical polar coordinates this point is given by

$$\begin{aligned} R\cos(\varphi) &= \cos(\beta) + y_0\sin(\mathbb{A}) \\ R\sin(\varphi) &= \sin(\beta) + y_0\cos(\mathbb{A}) \\ t &= 2y_0\cos(\mathbb{A})\cos(\beta) - 2y_0\sin(\mathbb{A})\sin(\beta) + \tan(\mathbb{A}/2). \end{aligned}$$

Eliminating β as before we obtain

$$1 = R^2 - 2Ry_0\sin(\varphi + \mathbb{A}) + {y_0}^2.$$

Taking the positive root we find

$$R = y_0\sin(\varphi + \mathbb{A}) + \sqrt{1 - {y_0}^2\cos^2(\varphi + \mathbb{A})}.$$

Finally, this enables us to substitute for β and R to obtain t as a function $t_2(\varphi)$ of φ. This is

$$t_2(\varphi) = 2{y_0}^2\cos(\varphi + \mathbb{A})\sin(\varphi + \mathbb{A}) + 2y_0\cos(\varphi + \mathbb{A})\sqrt{1 - {y_0}^2\cos^2(\varphi + \mathbb{A})} + \tan(\mathbb{A}/2).$$

Then $\xi_0(C_2)$ is the curve in $S^1 \times \mathbb{R}$ given by $t = t_2(\varphi)$.

The following lemma gives some important properties of the functions $t_1(\varphi)$ and $t_2(\varphi)$ which are immediately clear from the definitions. (These essentially say that $t_1(\varphi)$ should be invariant under $\xi_0\iota_1{\xi_0}^{-1}$ and that $\xi_0\iota_2{\xi_0}^{-1}$ should map $t_1(\varphi)$ to $t_2(\varphi)$.)

Lemma 4.5 *The functions $t_1(\varphi)$ and $t_2(\varphi)$ are periodic with period 2π. They also satisfy*

$$t_1(\pi - \varphi) = -t_1(\varphi), \qquad t_2(\varphi) = t_1(\varphi + \mathbb{A} - \pi) + \tan(\mathbb{A}/2).$$

The proof of Proposition 4.3 will depend crucially on the following lemma of Schwartz. One of the hypotheses in [12] is that the $\mathbb{R}$-sphere should be *tame*. In our notation this is equivalent to $y_0 = \sqrt{\cos(\mathbb{A})}/2\cos(\mathbb{A}/2) < 1/2$. This is satisfied when $\mathbb{A} \neq 0$ and so for all the values of $\mathbb{A}$ we will be interested in.

Proposition 4.6 *(Lemma 4.5 of [12]) The image of S_1^- under ξ_0 is contained in the region between the curve $t = t_1(\varphi)$ and the line $t = 0$. The image of S_2^- under ξ_0 is contained in the region between the curve $t = t_2(\varphi)$ and the line $t = \tan(\mathbb{A}/2)$.*

We claim that the intersection result will follow from this proposition and the following result

Proposition 4.7 *For all $0 \leq \cos(\mathbb{A}) \leq 1/4$ and $0 \leq \varphi \leq 2\pi$ we have*

$$\max\Big\{t_1(\varphi), 0\Big\} \leq \min\Big\{t_2(\varphi), \tan(\mathbb{A}/2)\Big\}$$

with equality if and only if both $\cos(\mathbb{A}) = 1/4$ and $\varphi = \pi - \mathbb{A}/2$.

The proof of this result is a straightforward, but lengthy, exercise in calculus. We break it down into a series of simple lemmas

Lemma 4.8

- *$t_1(\varphi) \leq 0$ for $-\pi/2 \leq \varphi \leq \pi/2$ and $t_1(\varphi) \geq 0$ for $\pi/2 \leq \varphi \leq 3\pi/2$,*
- *$t_1(\varphi)$ attains its maximum of $2y_0$ when $\varphi = \arctan(y_0) \in [\pi, 3\pi/2]$,*
- *$t_1(\varphi)$ attains its minimum of $-2y_0$ when $\varphi = \arctan(-y_0) \in [\pi/2, \pi]$,*
- *$t_2(\varphi) \leq \tan(\mathbb{A}/2)$ for $\varphi \in [\pi/2-\mathbb{A}, 3\pi/2-\mathbb{A}]$ and $t_2(\varphi) \geq \tan(\mathbb{A}/2)$ for $\varphi \in [-\pi/2-\mathbb{A}, \pi/2-\mathbb{A}]$,*
- *$t_2(\varphi)$ attains its maximum of $2y_0 + \tan(\mathbb{A}/2)$ when $\varphi = -\mathbb{A} + \arctan(y_0) \in [-\mathbb{A}, \pi/2 - \mathbb{A}]$,*
- *$t_1(\varphi)$ attains its minimum of $-2y_0 + \tan(\mathbb{A}/2)$ when $\varphi = -\mathbb{A} + \arctan(-y_0) \in [-\pi/2 - \mathbb{A}, -\mathbb{A}]$.*

PROOF: For the first assertion, using $y_0 \leq 1/\sqrt{2}$ we have $y_0 \sin(\varphi) < \sqrt{1 - {y_0}^2\cos^2(\varphi)}$ and so $t_1(\varphi)$ has the same sign as $-\cos(\varphi)$.

A simple calculation shows that $t_1'(\varphi) = 0$ if and only if $\tan^2(\varphi) = {y_0}^2$. The second and third assertions follow by examining $t_1(\varphi)$ at these values and being careful about the sign of $\sqrt{1 - {y_0}^2\cos^2(\varphi)} = \pm\cos(\varphi) > 0$.

The final three assertions follow using $t_2(\varphi) = t_1(\varphi + \mathbb{A} - \pi) + \tan(\mathbb{A}/2)$. □

This lemma means that we can prove Proposition 4.7 for certain values of φ. We formulate this as a separate lemma.

Lemma 4.9 *For $0 \leq \cos(\mathbb{A}) < 1/3$ and on the interval $-\pi/2 - \mathbb{A} \leq \varphi \leq \pi/2$ we have*

$$\max\Big\{t_1(\varphi), 0\Big\} \leq \min\Big\{t_2(\varphi), \tan(\mathbb{A}/2)\Big\}.$$

PROOF: First observe that for $0 \leq \cos(\mathbb{A}) < 1/3$ we have

$$2y_0 = \frac{\sqrt{\cos(\mathbb{A})}}{\cos(\mathbb{A}/2)} < \frac{\sqrt{(1 - \cos(\mathbb{A}))/2}}{\cos(\mathbb{A}/2)} = \tan(\mathbb{A}/2).$$

Figure 8: *The functions* $t_1(\varphi)$ *and* $t_2(\varphi)$ *when* $\mathbb{A} = 1.55$.

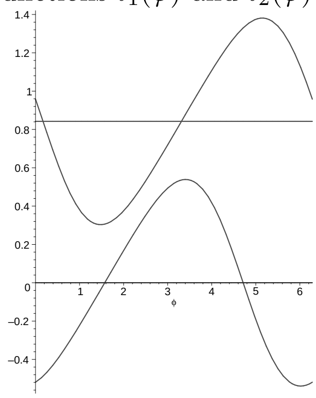

Figure 9: *The functions* $t_1(\varphi)$ *and* $t_2(\varphi)$ *when* $\mathbb{A} = 1.4$.

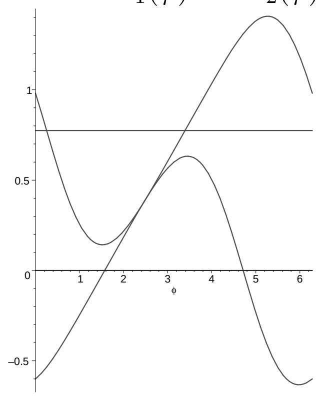

Figure 10: *The functions* $t_1(\varphi)$ *and* $t_2(\varphi)$ *when* $\cos \mathbb{A} = 1/4$.

Thus, for $-\pi/2 \le \varphi \le \pi/2$ we have

$$t_1(\varphi) \le 0 < \tan(\mathbb{A}/2) - 2y_0 \le t_2(\varphi).$$

Similarly, for $-\pi/2 - \mathbb{A} \le \varphi \le \pi/2 - \mathbb{A}$ we have

$$t_1(\varphi) \le 2y_0 < \tan(\mathbb{A}/2) \le t_2(\varphi).$$

$\square$

Therefore we need to only consider the interval $\pi/2 < \varphi < 3\pi/2 - \mathbb{A}$. For this interval we use the mean value theorem. The proof requires the following two lemmas.

Lemma 4.10 *For $0 \le \cos(\mathbb{A}) \le 1/4$ we have $t_1(\pi - \mathbb{A}/2) \le t_2(\pi - \mathbb{A}/2)$ with equality if and only $\cos(\mathbb{A}) = 1/4$. Also, $t_2'(\pi - \mathbb{A}/2) = t_1'(\pi - \mathbb{A}/2)$.*

Proof: From the two expressions in Lemma 4.5 we see that

$$t_2(\varphi) = t_1(\varphi + \mathbb{A} - \pi) + \tan(\mathbb{A}/2) = -t_1(2\pi - \varphi - \mathbb{A}) + \tan(\mathbb{A}/2).$$

The second assertion follows by differentiating this with respect to φ and then setting $\varphi = \pi - \mathbb{A}/2$.

Setting $\varphi = \pi - \mathbb{A}/2$ in the above expression we obtain

$$t_1(\pi - \mathbb{A}/2) + t_2(\pi - \mathbb{A}/2) = \tan(\mathbb{A}/2).$$

Thus the first assertion is equivalent to showing $2t_1(\pi - \mathbb{A}/2) \le \tan(\mathbb{A}/2)$ with equality only when $\cos(\mathbb{A}) = 1/4$. Using $y_0 = \sqrt{\cos(\mathbb{A})}/(2\cos(\mathbb{A}/2))$, we see that

$$2t_1(\pi - \mathbb{A}/2) = \tan(\mathbb{A}/2) - \sin(\mathbb{A}) + \sqrt{4\cos(\mathbb{A}) - \cos^2(\mathbb{A})} \le \tan(\mathbb{A}/2).$$

The result follows. $\square$

Lemma 4.11 *For $0 \le \cos(\mathbb{A}) \le 1/4$ we have $t_1''(\varphi) < 0$ for $\varphi \in (\pi/2, 3\pi/2)$, and $t_2''(\varphi) > 0$ for $\varphi \in (\pi/2 - \mathbb{A}, 3\pi/2 - \mathbb{A})$.*

Proof: The second assertion follows from the first and Lemma 4.5. Thus it suffices to prove $t_1''(\varphi) < 0$ for $\varphi \in (\pi/2, 3\pi/2)$.

Throughout this proof we have $\varphi \in (\pi/2, 3\pi/2)$ and so $\cos(\varphi) < 0$. Let $\epsilon = +1$ for $\varphi \in (\pi/2, \pi)$ and $\epsilon = -1$ for $\varphi \in (\pi, 3\pi/2)$ (thus ϵ is the sign of $\sin(\varphi)$). Then

$$t_1(\varphi) = 2{y_0}^2\cos(\varphi)\Big(\epsilon\sqrt{1-\cos^2(\varphi)} - \sqrt{{y_0}^{-2} - \cos^2(\varphi)}\Big) = 2{y_0}^2\Big(F({y_0}^{-2}, \varphi) - \epsilon F(1, \varphi)\Big)$$

where $F(a, \varphi) = -\cos(\varphi)\sqrt{a - \cos^2(\varphi)}$. When $0 \le \cos(\mathbb{A}) \le 1/4$ we have ${y_0}^{-2} \ge 10$. Thus we are only interested in the range $a \ge 10$. A brief calculation shows that

$$\frac{\partial^2 F}{\partial \varphi^2}(a, \varphi) = \frac{\cos(\varphi)\big(a^2 + 3a - (6a+2)\cos^2\varphi + 4\cos^4(\varphi)\big)}{\big(a - \cos^2(\varphi)\big)^{3/2}}.$$

We claim that this is a decreasing function of a provided $a > 3$. In order to see this we calculate

$$\frac{\partial^3 F}{\partial a \varphi^2}(a, \varphi) = \frac{\cos(\varphi)\big(a^2 - 3a + 2a\cos^2(\varphi)\big)}{2\big(a - \cos^2(\varphi)\big)^{5/2}}$$

which is negative for $a > 3$. Thus for $a \geq 10$

$$\frac{\partial^2 F}{\partial \varphi^2}(a,\varphi) \leq \frac{\partial^2 F}{\partial \varphi^2}(10,\varphi).$$

It is easy to check that

$$\frac{\partial^2 F}{\partial \varphi^2}(10,\varphi) < \frac{\partial^2 F}{\partial \varphi^2}(1,\varphi) < 0.$$

Thus

$$\begin{aligned} t_1''(\varphi) &= 2{y_0}^2\left(\frac{\partial^2 F}{\partial \varphi^2}({y_0}^{-2},\varphi) - \epsilon\frac{\partial^2 F}{\partial \varphi^2}(1,\varphi)\right) \\ &\leq 2{y_0}^2\left(\frac{\partial^2 F}{\partial \varphi^2}(10,\varphi) - \frac{\partial^2 F}{\partial \varphi^2}(1,\varphi)\right) \\ &< 0. \end{aligned}$$

$\square$

Therefore as φ varies in the interval $(\pi/2, 3\pi/2 - \mathbb{A})$ the function $t_2'(\varphi) - t_1'(\varphi)$ has positive derivative, and equals zero when $\varphi = \pi - \mathbb{A}/2$. By the mean value theorem, on this interval we have

$$t_2(\varphi) - t_1(\varphi) \geq t_2(\pi - \mathbb{A}/2) - t_1(\pi - \mathbb{A}/2)$$

with equality only when $\varphi = \pi - \mathbb{A}/2$. Combining this with Lemma 4.10 we have completed the proof of Proposition 4.7.

Proposition 4.3 now follows from Proposition 4.7 in the same way that Lemma 4.1 of [12] follows from Lemmas 4.3 of [12] using the Schwartz's projection lemma, Proposition 4.6. For completeness we now give this argument. Suppose that S_1 and S_2 intersect in a point p other than ∞. This means that $\xi_0(p) \in \xi_0(S_1) \cap \xi_0(S_2)$. Now $\xi_0(S_1)$ is contained in the finite region bounded by the curves $t = t_1(\varphi)$ and $t = 0$. Similarly $\xi_0(S_2)$ is contained in the region bounded by the curves $t = t_2(\varphi)$ and $t = \tan(\mathbb{A}/2)$ (Proposition 4.6). The finite regions bounded by these curves are disjoint when $0 \leq \cos(\mathbb{A}) < 1/4$. intersect when $\cos(\mathbb{A}) = 1/4$. When $\cos(\mathbb{A}) = 1/4$ the only point of intersection between the two regions is $\varphi = \pi - \mathbb{A}/2$ and $t = \tan(\mathbb{A}/2)/2$. The inverse image of this point under ξ_0 is the $\mathbb{R}$-circle R_2. By construction the arc of R_2 between ∞ and C_1 that does not meet A_0 is contained in both S_1^+ and S_1^-.

Theorem 4.1 now follows from Propositions 4.2 and 4.3.

4.3 A fundamental domain bounded by $\mathbb{C}$-spheres.

The main result of this section will be

Theorem 4.12 *For each $\mathbb{A}$ with $0 \leq \cos(\mathbb{A}) \leq 1/4$ there exist $\mathbb{C}$-spheres S_0, S_1, S_2 so that*

- *S_j contains C_j the fixed $\mathbb{C}$-circle of c_j;*
- *$c_j(S_j) = S_j$;*
- *S_0, S_1 and S_2 are the faces of a fundamental domain for the group G generated by the side pairing maps c_0, c_1 and c_2;*

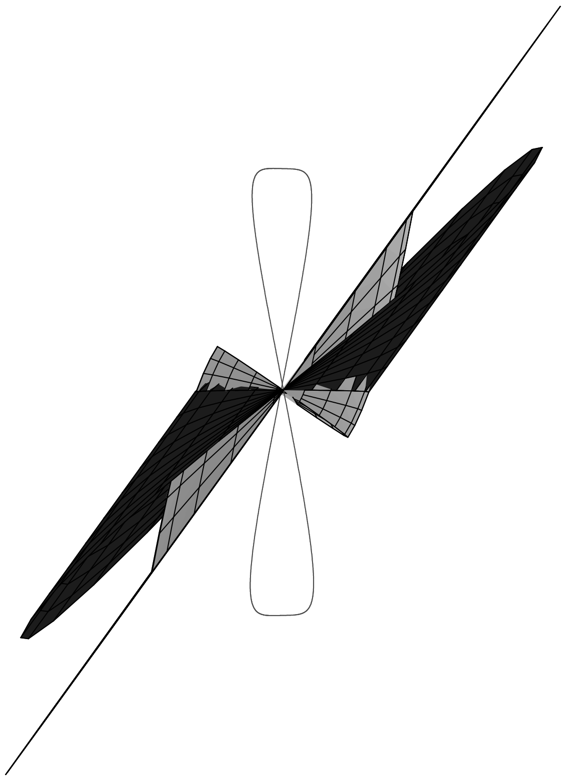

Figure 11: *A comparison between S_1 constructed using $\mathbb{C}$-spheres and $\mathbb{R}$-spheres*

- *S_j and S_k are disjoint except for points of tangency at parabolic fixed points.*

This result will follow using the threefold symmetry given by the map $\iota_0\iota_2$ and the following results.

Proposition 4.13 *The $\mathbb{C}$-sphere S_1 may be decomposed into two hemispheres S_1^+ and S_1^- with common boundary C_1. Each of these hemispheres is mapped to itself under ι_1 and the two hemispheres are interchanged by ι_0.*

Proposition 4.14 *Let $S_2 = \iota_2\iota_0(S_1)$. Then, for $\cos(\mathbb{A}) < 1/4$, the $\mathbb{C}$-spheres S_1 and S_2 are disjoint except for a single point of tangency at the fixed point of the parabolic map $c_1c_0c_2 = (\iota_1\iota_2)^3$. For $\cos(\mathbb{A}) = 1/4$, the $\mathbb{C}$-spheres have an additional point of tangency at the fixed point of the parabolic map $c_1c_2 = (\iota_0\iota_1\iota_2)^2$.*

In the previous section we constructed an $\mathbb{R}$-sphere around the $\mathbb{C}$-circle C_1. In this section we construct a $\mathbb{C}$-sphere around the $\mathbb{R}$-circle R_1. A finite $\mathbb{C}$-circle is invariant under ι_1 if and only if its centre lies on R_1, or equivalently, if its polar vector is invariant under ι_1.

A $\mathbb{C}$-circle with centre $(ib, 0)$ on R_1 and radius R has polar vector

$$\begin{bmatrix} R^2 - b^2 \\ \sqrt{2}ib \\ 1 \end{bmatrix}.$$

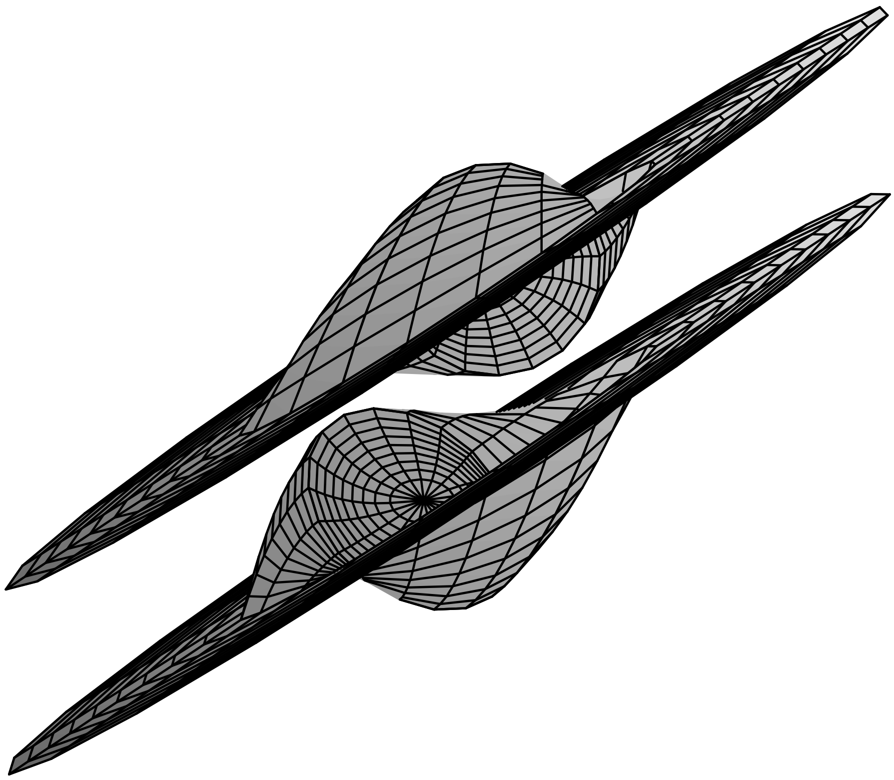

Figure 12: $\mathbb{C}$-*spheres* S_1 *and* S_2 *when* $\mathbb{A} = 1.4$.

We may specify our $\mathbb{C}$-sphere by giving a one parameter family of such polar vectors. The image of this polar vector under ι_0 is

$$\begin{bmatrix} 1 \\ -\sqrt{2}ib \\ R^2 - b^2 \end{bmatrix} \approx \begin{bmatrix} \bigl(R/(R^2-b^2)\bigr)^2 - \bigl(b/(R^2-b^2)\bigr)^2 \\ \sqrt{2}i\bigl(-b/(R^2-b^2)\bigr) \\ 1 \end{bmatrix}.$$

Thus ι_0 sends a $\mathbb{C}$-circle with centre $(ib, 0)$ and radius R to a $\mathbb{C}$-circle with centre $\bigl(-ib/(R^2-b^2), 0\bigr)$ and radius $R/|R^2-b^2|$. In order for our $\mathbb{C}$-sphere S_1 to be invariant under $c_1 = \iota_1\iota_0$ then this map should permute the slices of S_1.

First, consider the structure of S_1 near ∞. We want S_1 to be disjoint from its image under ι_2. One way to ensure that this is the case is to require that, on some neighbourhood of infinity, S_1 is a piece of a bisector with one vertex at ∞ and the other on A_0, the axis of $\iota_1\iota_2$. This means that, close to infinity, S_1 and $\iota_2(S_1) = S_2$ are contained in parallel planes (see Figure 12). This condition is the same as requiring that all slices of S_1 with sufficiently large radius should have the same centre, $(iy_0, 0)$.

For each $\mathbb{C}$-circle C centred at $(iy_0, 0)$ so that C and C_1 are disjoint and unlinked there exists a unique bisector having C and C_1 as slices (see Theorem 9.1.4 of [5]). The $\mathbb{C}$-sphere S_1 will consist of three pieces of bisector joined along common slices. We begin with $\mathbb{C}$-circles all centred at $(iy_0, 0)$ with radii larger than a particular $\tilde{R}$. Let $\tilde{C}$ be the bisector centred at $(iy_0, 0)$ with radius $\tilde{R}$. The next piece of the $\mathbb{C}$-sphere will consist of that part of the bisector that has slices $\tilde{C}$ and C_1 lying between $\tilde{C}$ and $c_1(\tilde{C})$. The remaining piece consists of the image of the first piece under c_1.

We choose the radius $\tilde{R}$ so that when $\cos(\mathbb{A}) = 1/4$ this bisector is tangent to R_2. For $0 \le \cos(\mathbb{A}) < 1/4$ we have a certain amount of flexibility. We will make a choice that simplifies the calculations.

Specifically, we let the slices have the following polar vectors

$$p_\lambda = \begin{bmatrix} \lambda \\ \sqrt{2}iy_0 \\ 1 \end{bmatrix} \text{ for } \lambda \geq 2;$$

$$p_\lambda = \begin{bmatrix} \lambda \\ \sqrt{2}iy_0(\lambda-1) \\ 1 \end{bmatrix} \text{ for } 1/2 \leq \lambda \leq 2;$$

$$p_\lambda = \begin{bmatrix} \lambda \\ -\sqrt{2}iy_0\lambda \\ 1 \end{bmatrix} \text{ for } 0 \leq \lambda \leq 1/2.$$

It is clear that $\iota_0(p_\lambda) = p_{1/\lambda}$. We need to show that $L(\iota_2(p_\lambda), p_\mu) \geq 0$ for all $\lambda, \mu \in \mathbb{R}_+$ and all $0 \leq \cos(\mathbb{A}) \leq 1/4$. Once we have shown this it will be clear that S_1 and $\iota_2(S_1) = S_2$ are disjoint.

As in the previous section, let A_0 be the axis of $\iota_1\iota_2$. That is

$$A_0 = \left\{ (x+iy,t) \in \mathfrak{N} : x = 0, y = y_0 = \frac{\sqrt{\cos(\mathbb{A})}}{2\cos(\mathbb{A}/2)} \right\}.$$

The polar vector to the $\mathbb{C}$-circle centred at $(iby_0, 0)$ and radius R is

$$\begin{bmatrix} R^2 - b^2{y_0}^2 \\ \sqrt{2}iby_0 \\ 1 \end{bmatrix}.$$

For convenience we write $x = R^2 - b^2{y_0}^2$ define $P(b,x)$ by

$$P(b,x) = \begin{bmatrix} x \\ \sqrt{2}iby_0 \\ 1 \end{bmatrix}.$$

The image of $P(b,x)$ under ι_0 is

$$\iota_0(P(b,x)) = \begin{bmatrix} 1 \\ -\sqrt{2}iby_0 \\ x \end{bmatrix} \approx \begin{bmatrix} 1/x \\ -\sqrt{2}i(b/x)y_0 \\ 1 \end{bmatrix} = P(-b/x, 1/x).$$

The image of $P(b,x)$ under ι_2 is

$$\iota_2(P(b,x)) = \begin{bmatrix} x + 2by_0{}^2(1+e^{-i\mathbb{A}}) - e^{-i\mathbb{A}} \\ -\sqrt{2}iby_0e^{-i\mathbb{A}} + \sqrt{2}iy_0(1+e^{-i\mathbb{A}}) \\ 1 \end{bmatrix}.$$

Given real numbers b, x, b' and x' and $\mathbb{A} \in [0, \pi/2]$ we define

$$f_{\mathbb{A}}(b,x,b',x') = \big(1+\cos(\mathbb{A})\big)L\big(\iota_2P(b,x), P(b',x')\big).$$

Using the expressions above we find that

$$
\begin{aligned}
f_{\mathbb{A}}(b,x,b',x') &= \big(1+\cos(\mathbb{A})\big)L\big(\iota_2 P(b,x),P(b',x')\big) \\
&= \big(1+\cos(\mathbb{A})\big)\Big|x+x'+2(b+b'){y_0}^2(1+e^{-i\mathbb{A}})-(1+2bb'{y_0}^2)\Big|^2 \\
&\qquad -4\big(1+\cos(\mathbb{A})\big)(x+b^2{y_0}^2)(x'+{b'}^2{y_0}^2) \\
&= \big(1+\cos(\mathbb{A})\big)\Big(x+x'+2(b+b'){y_0}^2\Big)^2 \\
&\qquad +2\big(1+\cos(\mathbb{A})\big)\cos(\mathbb{A})\Big(x+x'+2(b+b'){y_0}^2\Big)\Big(2(b+b'){y_0}^2-2bb'{y_0}^2-1\Big) \\
&\qquad +\big(1+\cos(\mathbb{A})\big)\Big(2(b+b'){y_0}^2-2bb'{y_0}^2-1\Big)^2 \\
&\qquad -4\big(1+\cos(\mathbb{A})\big)(x+b^2{y_0}^2)(x'+{b'}^2{y_0}^2) \\
&= \big(1+\cos(\mathbb{A})\big)\big((x-x')^2+1\big) \\
&\qquad +2\big(1+\cos(\mathbb{A})\big)\cos(\mathbb{A})\Big((x+x')(b+b')-(x+x')-(b+b')\Big) \\
&\qquad +2\cos(\mathbb{A})\Big(-x{b'}^2-x'b^2+bb'\Big) \\
&\qquad +2\cos^2(\mathbb{A})\Big((b+b')^2-(b+b')bb'-(x+x')bb'\Big).
\end{aligned}
$$

We have used $2{y_0}^2=\cos(\mathbb{A})/\big(1+\cos(\mathbb{A})\big)$. Observe that $f_{\mathbb{A}}(b,x,b',x')=f_{\mathbb{A}}(b',x',b,x)$.

We now use this expression to show that, for $0\le\cos(\mathbb{A})<1/4$, the $\mathbb{C}$-spheres S_1 and S_2 are disjoint except for an isolated point of tangency at ∞, and also for $\cos(\mathbb{A})=1/4$ the $\mathbb{C}$-spheres are disjoint except for isolated points of tangency at ∞ and at the fixed point of c_1c_2.

We constructed the $\mathbb{C}$-sphere S_1 so that its slices have polar vectors p_λ for $\lambda\in\mathbb{R}_+$ where

$$
p_\lambda=\begin{cases} P(-\lambda,\lambda) & \text{if } 0<\lambda\le 1/2,\\ P(\lambda-1,\lambda) & \text{if } 1/2\le\lambda\le 2,\\ P(1,\lambda) & \text{if } 2\le\lambda. \end{cases}
$$

Therefore the result will follow from the following proposition.

Proposition 4.15 *For $0\le\mathbb{A}\le 1/4$ the following functions are non-negative*

- $f_{\mathbb{A}}(-\lambda,\lambda,-\mu,\mu)$ *for* $0<\lambda,\mu\le 1/2$,
- $f_{\mathbb{A}}(-\lambda,\lambda,\mu-1,\mu)$ *for* $0<\lambda\le 1/2\le\mu\le 2$,
- $f_{\mathbb{A}}(-\lambda,\lambda,1,\mu)$ *for* $0<\lambda\le 1/2$, $2\le\mu$,
- $f_{\mathbb{A}}(\lambda-1,\lambda,\mu-1,\mu)$ *for* $1/2\le\lambda,\mu\le 2$,
- $f_{\mathbb{A}}(\lambda-1,\lambda,1,\mu)$ *for* $1/2\le\lambda\le 2\le\mu$,
- $f_{\mathbb{A}}(1,\lambda,1,\mu)$ *for* $2\le\lambda,\mu$.

Moreover, the functions are strictly positive unless $\cos(\mathbb{A})=1/4$ *and* $\lambda=\mu=1$.

PROOF: In each case we express $f_{\mathbb{A}}(b,x,b',x')$ as a sum of squares. In each case we write c for $\cos(\mathbb{A})$. We treat these six cases separately.

(i)

$$\begin{aligned}
f_{\mathbb{A}}(-\lambda,\lambda,-\mu,\mu) = \;& (c+1)(\lambda-\mu)^2 + 2c(\lambda+\mu)(-\lambda-\mu)) \\
& -2c(\lambda+\mu-\lambda-\mu))(1+c(1+\lambda)(1+\mu)) \\
& +(c+1) + 2c\lambda\mu - 2c\lambda\mu^2 - 2c\mu\lambda^2 \\
= \;& (1-c)(\lambda-\mu)^2 + (1-c) \\
& +2c(1-4\lambda\mu) + 2c\lambda\mu(1-\lambda-\mu) > 0
\end{aligned}$$

for $0 \le \lambda, \mu \le 1/2$.

(ii)

$$\begin{aligned}
f_{\mathbb{A}}(-\lambda,\lambda,\mu-1,\mu) = \;& (c+1)(\lambda-\mu)^2 + 2c(\lambda+\mu)(-\lambda+\mu-1) \\
& -2c(\lambda+\mu-\lambda+\mu-1)(1+c(1+\lambda))(2-\mu)) \\
& +(c+1) - 2c\lambda(\mu-1) - 2c\lambda(\mu-1)^2 - 2c\mu\lambda^2 \\
= \;& (1-4c)(\lambda-\mu)^2 + (1-4c) \\
& +c(1-2c)(\lambda+1)(2-\mu)(2\mu-1) \\
& +c\mu(\lambda+7)(1-2\lambda) + 3c\lambda^2 + 9c(\mu-1)^2 > 0
\end{aligned}$$

for $0 \le \lambda \le 1/2 \le \mu \le 2$.

(iii)

$$\begin{aligned}
f_{\mathbb{A}}(-\lambda,\lambda,1,\mu) = \;& (c+1)(\lambda-\mu)^2 + 2c(\lambda+\mu)(-\lambda+1) \\
& -2c(\lambda+\mu-\lambda+1)(1+c(1+\lambda)(1-1)) \\
& +(c+1) - 2c\lambda - 2c\lambda - 2c\mu\lambda^2 \\
= \;& (1-c)(\lambda-\mu)^2 + (1-c) \\
& +c\mu(\lambda+4)(1-2\lambda) + c(2\mu+\lambda)(\mu-2) > 0
\end{aligned}$$

for $\mu \ge 2$ and $\lambda \le 1/2$.

(iv)

$$\begin{aligned}
f_{\mathbb{A}}(\lambda-1,\lambda,\mu-1,\mu) = \;& (c+1)(\lambda-\mu)^2 + 2c(\lambda+\mu)(\lambda-1+\mu-1) \\
& -2c(\lambda+\mu+\lambda-1+\mu-1)(1+c(2-\lambda)(2-\mu)) \\
& +(c+1) + 2c(\lambda-1)(\mu-1) - 2c\lambda(\mu-1)^2 - 2c\mu(\lambda-1)^2 \\
= \;& (1-4c)(\lambda-\mu)^2 + (1-4c) + c(1-4c)(\lambda+\mu-1)(2-\lambda)(2-\mu) \\
& +3c(\lambda-1)^2(2-\mu) + 3c(\mu-1)^2(2-\lambda) \\
& +3c(\lambda-1)^2 - 3c(\lambda-1)(\mu-1) + 3c(\mu-1)^2 \ge 0
\end{aligned}$$

for $1/2 \le \lambda, \mu \le 2$ with equality only when $c = 1/4$ and $\lambda = \mu = 1$.

(v)

$$\begin{aligned}
f_{\mathbb{A}}(\lambda-1,\lambda,1,\mu) = \;& (c+1)(\lambda-\mu)^2 + 2c(\lambda+\mu)(\lambda-1+1) \\
& -2c(\lambda+\mu+\lambda-1+1)(1+c(2-\lambda)(1-1)) \\
& +(c+1) + 2c(\lambda-1) - 2c\lambda - 2c\mu(\lambda-1)^2 \\
= \;& (1-c)(\lambda-\mu)^2 + (1-c) \\
& +2c(\mu-2)^2 + 2c(\mu-2)(\lambda+1)(2-\lambda) > 0
\end{aligned}$$

for $1/2 \le \lambda \le 2 \le \mu$.

(vi)

$$\begin{aligned}
f_{\mathbb{A}}(1,\lambda,1,\mu) = \;& (c+1)(\lambda-\mu)^2 + 2c(\lambda+\mu)(1+1) \\
& -2c(\lambda+\mu+1+1)(1+c(1-1)(1-1)) \\
& +(c+1) + 2c - 2c\lambda - 2c\mu \\
= \;& (c+1)(\lambda-\mu)^2 + (1-c) > 0
\end{aligned}$$

for $\lambda, \mu \ge 2$.

This completes the proof. □

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