Exam : Introduction to Riemann Surfaces

Exercise 1. (Separating points on a Riemann surface) Le X be a compact Riemann surface.

1. Let $f : X \to \mathbb{C}P^1$ be a non-constant holomorphic function and consider a point $z \in \mathbb{C}P^1$ which is not a ramification point. Show that, if $g : X \to \mathbb{C}P^1$ is a holomorphic function that separates the points of $f^{-1}(z)$, then $\mathcal{M}(X) = \mathbb{C}(f, g)$.
2. Show that for any collection $\{z_1, \cdots, z_n\}$ of distinct points in X and $\{w_1, \cdots, w_n\}$ distinct points of $\mathbb{C}$, there exists a meromorphic function $g \in \mathcal{M}(X)$ such that $g(z_i) = w_i$, for all $1 \leq i \leq n$.

Exercise 2. (Sums of divisors and Clifford theorem on special divisors) Let X be a connected compact Riemann surface.

1. Show that if D_1 and D_2 are effective divisors on X with *disjoint supports* then the inequality $\ell(D_1) + \ell(D_2) - 1 \leq \ell(D_1 + D_2)$ holds.
2. Let D be a divisor on X that satisfies $\ell(D) > 0$. Show that there exists a finite set $F \subset X$ such that $L(D - [p]) \subsetneq L(D)$ for every $p \in X \setminus F$.
3. Let D be a divisor on X satisfying $n := \ell(D) > 0$ and let $U \subset X$ be any non-empty open set. Show that there exists an effective divisor D' on X and $n-1$ points $p_1, \ldots, p_{n-1} \in U$ such that D is linearly equivalent to $D' + [p_1] + \cdots + [p_{n-1}]$ and
$$\mathbb{C} = L(D') \subsetneq L(D' + [p_1]) \subsetneq L(D' + [p_1] + [p_2]) \subsetneq \cdots \subsetneq L(D' + [p_1] + \cdots + [p_{n-1}]).$$
4. Let D be a divisor on X that satisfies $\ell(D) > 0$. Show that, for every non-empty open set $U \subset X$, there exists an effective divisor $\widetilde{D}$ on X linearly equivalent to D such that every non-constant $f \in L(\widetilde{D})$ has a pole in U.
5. Let D_1 and D_2 be two divisors on X that satisfy $\ell(D_1) > 0$ and $\ell(D_2) > 0$. Show that $\ell(D_1) + \ell(D_2) - 1 \leq \ell(D_1 + D_2)$.
6. (Clifford theorem on special divisors) Let D be a divisor on X satisfying $\ell(D) > 0$ and $\ell(K - D) > 0$ (called a special divisor), where K is the divisor of a meromorphic one-form on X (i.e., a canonical divisor). Show that
$$\ell(D) \leq 1 + \frac{\deg(D)}{2}.$$

Exercise 3. (Fermat curves) Let

$$V_n = \{\ [x,y,z] \in \mathbb{C}P^2 \mid F(x,y,z) = 0\ \}$$

be the projective plane curve defined by $F(x,y,z) = x^n + y^n + z^n$, $n \geq 3$. The goal of the exercise is to obtain an explicit basis of holomorphic forms on V_n.

1. Show that V_n is smooth.
2. Let L be the projective line defined by $z = 0$ and

$$\pi : \mathbb{C}P^2 - \{[0,0,1]\} \to L,$$

the projection defined by $\pi([x,y,z]) = [x,y,0]$. Show that $\pi_{|V_n}$ is a ramified cover of L. What is its degree ?

3. Determine the intersection $L \cap V_n$.
4. Show that the branching points of π is the set $\{\ [1,\zeta,0] \mid \zeta^n = -1\ \}$. Compute the ramification points of π and their multiplicities.
5. Determine the genus of V_n for all $n > 0$.
6. Let $\Omega^1(V_n)$ be the vector space of holomorphic forms on V_n. What is its dimension ?
7. Compute the intersection divisors of the coordinate functions $(x,y,z) \to x$, $(x,y,z) \to y$ and $(x,y,z) \to z$. Show that the functions defined by restricting to V_n the quotients of the coordinates, $X = x/z$ and $Y = y/z$, over $\mathbb{C}P^2$ are meromorphic functions with simple poles. Compute $div(X)$ and $div(Y)$.
8. Show that the meromorphic forms on V_n, dX et dY, satisfy

$$\frac{dX}{Y^{n-1}} = -\frac{dY}{X^{n-1}}.$$

9. Show that the forms

$$\omega_{i,j} = X^{i-1}Y^{j-1}\frac{dX}{Y^{n-1}}$$

are holomorphic at all points in V_n such that X et Y are holomorphic.

10. If $p \in V_n$ is a pole of X, show that $ord_p(dX) = -2$, and then that

$$ord_p(\omega_{i,j}) = n - 1 - i - j.$$

11. Determine a basis of holomorphic forms among the forms defined above.

Exercise 4. (A Fermat curve which is not hyperelliptique) Recall that a Riemann surface is called *hyperelliptic* if there exists a a holomorphic map of degree two into $\mathbb{C}P^1$. Let

$$V_4 = \{\ [x,y,z] \in \mathbb{C}P^2 \mid x^4 + y^4 + z^4 = 0\ \}.$$

In this exercise you will show that V_4 is not hyperelliptic.

From the previous exercise,assume that V_4 is a genus 3 Riemann surface, that $X = x/z$ and $Y = y/z$ define meromorphic functions on V_4, and consider the following basis of holomorphic forms :

$$\omega_1 = \frac{dX}{Y^3},\quad \omega_2 = X\frac{dX}{Y^3},\quad \omega_3 = Y\frac{dX}{Y^3}.$$

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1. A point $p \in \Sigma$ (Σ a compact Riemann surface) is a Weierstrass point if there exists a meromorphic function with a unique pole at p of order at most g. Show that p is a Weierstrass point if and only if there exists a holomorphic form of order at least g at p.
2. Show that if $p \in V_4$ is such that $Y(p) \neq 0, \infty$ then X is a local coordinate. We write any holomorphic form as
$$\omega = (af_1(X) + bf_2(X) + cf_3(X))dX,$$
where $f_1(X) = \frac{1}{Y^3(X)}$,$f_2(X) = \frac{X}{Y^3(X)}$ et $f_3(X) = \frac{1}{Y^2(X)}$ and $Y(X)$ is defined implicitly by the equation $X^4 + Y^4 + 1 = 0$.
3. Show that $p \in V_4$ (avec $Y(p) \neq 0, \infty$) is a Weierstrass point if and only if $\det M = 0$ where
$$M = \begin{pmatrix} f_1(X(p)) & f_2(X(p)) & f_3(X(p)) \\ f_1'(X(p)) & f_2'(X(p)) & f_3'(X(p)) \\ f_1''(X(p)) & f_2''(X(p)) & f_3''(X(p)) \end{pmatrix}$$
4. Show that
$$\det M = \frac{3X^2}{Y^{16}}.$$
5. Show that the Weierstrass points of V_4 with $y, z \neq 0$ are of the form $[0, 1, \zeta]$ where $\zeta^4 = -1$. Compute all Weierstrass points.
6. Show that there exists a meromorphic function with a unique pole of order two at $p \in V_4$ (p as before with $Y(p) \neq 0, \infty$) if and only if the rank of the matrix
$$\begin{pmatrix} f_1(X(p)) & f_2(X(p)) & f_3(X(p)) \\ f_1'(X(p)) & f_2'(X(p)) & f_3'(X(p)) \end{pmatrix}$$
is equal to one.
7. Show that V_4 is not hyper-elliptic.