Geometric structures associated to triangulations as fixed point sets of involutions

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Abstract

We establish that hyperbolic structures and spherical CR structures on a three dimensional manifold are contained in fixed point sets of a larger class of structures associated to a triangulation of the manifold. We generalize the 5 term relation to this setting.

1 Introduction

Thurston's first examples of hyperbolic structures on the complement of links were obtained by using topological ideal triangulations, that is, triangulations with removed vertices [T]. That idea was the source of a huge number of examples and the proof of the hyperbolic Dehn surgery theorem. Recently, examples of spherical CR structures (locally modelled on the Heisenberg group, see section 5) obtained by triangulation were constructed in [F] namely for the figure eight knot complement and the Whitehead link (see also [S] for different constructions and for a Dehn surgery construction in spherical CR geometry). Both geometries are deeply connected to 3-manifolds although their relation remains elusive.

The main goal of this paper is to give a common framework to both geometries in order to make explicit a relation between these geometries. We will start with a triangulation of a 3-manifold and associate complex invariants to each simplex in the triangulation. Imposing certain algebraic equations on those invariants we obtain for each triangulation a set of solutions which are referred to as **T-structures**. It turns out that (ideal) hyperbolic structures and spherical CR structures are contained in fixed point sets of two different involutions in the space of **T**-structures (Propositions 4.1 and 4.2, Theorem 4.3). This can be seen even for a single simplex and we shall describe the hyperbolic and CR simplices (hyperbolic tetrahedra are parametrized by the cross-ratio and CR-tetrahedra are parametrized by a generalized cross-ratio defined in [KR]).

Although **T**-structures are more general than geometric structures, the most natural instances arise from hyperbolic or CR structures on manifolds and, more generally, from representations of the fundamental group of 3-manifolds to $PSL(2,\mathbb{C})$ or $PU(2,1)$. In fact, constructing appropriate triangulations of those manifolds and associating to each vertex a point in S^2 (in the case of hyperbolic structures) or S^3 (in the CR case) defines **T**-structures. The problem of deciding if a particular representation of the fundamental group in $PU(2,1)$ is the holonomy representation of a spherical CR-structure on a manifold is difficult, as opposed to the hyperbolic case. In fact, once a representation is given, one has to construct 3-dimensional spherical CR tetrahedra which are glued to form a manifold. This involves defining edges and faces which are carried by the set of vertices which can be associated to a representation into $PU(2,1)$. This problem was studied for the figure eight knot in [F] but is not adressed in the paper.

The special examples arising from real hyperbolic structures (see [T]) and spherical CR structures obtained by triangulation in [F] for the figure eight knot complement and the Whitehead link seem to have an intimate relation as their holonomies are defined over the same ring of integers. This paper offers a first explanation of this fact in the sense that each of the two structures are points in a complex subspace (corresponding to real hyperbolic structures) and a Lagrangian subspace (corresponding to spherical CR strucutres) of the moduli of $\mathbf{T}$-structures on a particular triangulation of the manifold.

Another motivation for this paper, as suggested by R. Benedetti, is the recent work on simplicial formulae for the Cheeger-Chern-Simons class and its generalized quantum invariants of complements of links which uses triangulations by hyperbolic tetrahedra (see [N, BB]). I thank R. Benedetti and Julien Marché for inumerable discussions and comments on earlier versions of the work. I also thank Pierre-Vincent Koseleff and Pierre Will for discussions on earlier drafts.

2 Triangulations and Simplicial Cross Ratios

Condider an ideal triangulation of a three manifold with cusps. By this we mean a simplicial complex where the underlying topological space is a manifold if the vertices is deleted.

Definition 2.1 *A simplicial cross ratio structure associated to a triangulation of a 3-manifold is a function $\mathbf{X}$ which, to each four ordered vertices in a simplex, associates a value in $\mathbb{C}$ satisfying the following axioms*

1. *(basic symmetry) If $[u_0, u_1, u_2, u_3]$ is a simplex then*

$$\mathbf{X}[u_0, u_1, u_2, u_3] = \frac{1}{\mathbf{X}[u_1, u_0, u_2, u_3]} = \frac{1}{\mathbf{X}[u_1, u_0, u_2, u_3]}$$

2. *(edge compatibility) If $[u_0, u_1, u_3, u_2], [u_0, u_1, u_4, u_3], \cdots, [u_0, u_1, u_{n+1}, u_2]$ are n simplices which have a common edge $[u_0, u_1]$ then*

$$\mathbf{X}[u_0, u_1, u_2, u_3]\mathbf{X}[u_0, u_1, u_3, u_4] \cdots \mathbf{X}[u_0, u_1, u_{n+1}, u_2] = 1,$$

$$\mathbf{X}[u_2, u_3, u_0, u_1]\mathbf{X}[u_3, u_4, u_0, u_1] \cdots \mathbf{X}[u_{n+1}, u_2, u_0, u_1] = 1$$

and

$$(1+\mathbf{X}[u_0, u_3, u_1, u_2](\mathbf{X}[u_3, u_1, u_0, u_2]-1)) \cdots (1+\mathbf{X}[u_0, u_2, u_1, u_{n+1}](\mathbf{X}[u_2, u_1, u_0, u_{n+1}]-1)) =$$

$$(\mathbf{X}[u_0, u_1, u_2, u_3]\mathbf{X}[u_0, u_3, u_1, u_2]\mathbf{X}[u_3, u_1, u_0, u_2] + 1) \cdots$$

$$(\mathbf{X}[u_0, u_1, u_{n+1}, u_2]\mathbf{X}[u_0, u_2, u_1, u_{n+1}]\mathbf{X}[u_2, u_1, u_0, u_{n+1}] + 1)$$

3. *(face compatibility) If $[u_0, u_1, u_2, u_3]$ and $[u_1, u_2, u_3, u_4]$ are simplices with a common face $[u_1, u_2, u_3]$ then*

$$\mathbf{X}[u_0, u_1, u_2, u_3]\mathbf{X}[u_0, u_3, u_1, u_2]\mathbf{X}[u_0, u_2, u_3, u_1]$$

$$= \mathbf{X}[u_4,u_1,u_2,u_3]\mathbf{X}[u_4,u_3,u_1,u_2]\mathbf{X}[u_4,u_2,u_3,u_1]$$

and

$$\mathbf{X}[u_1,u_2,u_3,u_0]\mathbf{X}[u_3,u_1,u_2,u_0]\mathbf{X}[u_2,u_3,u_1,u_0]$$
$$= \mathbf{X}[u_1,u_2,u_3,u_4]\mathbf{X}[u_3,u_1,u_2,u_4]\mathbf{X}[u_2,u_3,u_1,u_4]$$

In view of the symmetries of the cross ratio, for each simplex, we organize the relevant 6 cross ratios out of the 24 permutations and we write

$$[[u_1,u_2,u_3,u_4]] = \begin{pmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \omega_0' \\ \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} \mathbf{X}[u_1,u_2,u_3,u_4] \\ \mathbf{X}[u_1,u_4,u_2,u_3] \\ \mathbf{X}[u_1,u_3,u_4,u_2] \\ \mathbf{X}[u_3,u_4,u_1,u_2] \\ \mathbf{X}[u_2,u_3,u_1,u_4] \\ \mathbf{X}[u_4,u_2,u_1,u_3] \end{pmatrix} \tag{1}$$

Remark 1: In the case of points in $P^1\mathbb{C} = \partial H^3_{\mathbb{R}}$ the definition of cross ratio as

$$\mathbf{X}[u_0,u_1,u_2,u_3] = \frac{(u_2-u_1)(u_3-u_0)}{(u_3-u_1)(u_2-u_0)}$$

yields a cross ratio defined on ideal triangulations of real hyperbolic manifolds. In particular, the last condition is trivialy satified as $\mathbf{X}[u_0,u_1,u_2,u_3]\mathbf{X}[u_0,u_3,u_1,u_2]\mathbf{X}[u_0,u_2,u_3,u_3] = -1$. Moreover, we have an aditional symmetry as $\mathbf{X}[u_0,u_1,u_2,u_3] = \mathbf{X}[u_2,u_3,u_0,u_1]$.

Remark 2: In the case of points in $S^3 = \partial H^4_{\mathbb{R}}$, the conformal sphere, one can consider for each quadruple of points, the 2-sphere which contains them and define then the cross ratio as in the $P^1\mathbb{C}$ case. The ideal triangulations of hyperbolic geometry are a special case of a conformal triangulation. Note however that one cannot reconstruct the conformal structure from the cross ratios. The reason is that they do not detect the position between two 2-spheres where two adjacent simplices live. The edge compatibility relation does not need to be satisfied in the conformal case.

Remark 3: (see section 5) In the case of points in $S^3 = \partial H^2_{\mathbb{C}}$ the definition of cross ratio as ([KR])

$$\mathbf{X}[u_0,u_1,u_2,u_3] = \frac{\langle u_3,u_1\rangle\langle u_2,u_0\rangle}{\langle u_2,u_1\rangle\langle u_3,u_0\rangle}.$$

yields a cross ratio defined on triangulations of spherical CR manifolds. In that case we have the extra symmetry $\mathbf{X}[u_0,u_1,u_2,u_3] = \overline{\mathbf{X}[u_2,u_3,u_0,u_1]}$.

Remark 4: If a triangulation has oriented edges one can associate complex numbers to the edges using a simple convention (cf. [BB]). For instance, in the simplex $[0,1,2,3]$, the edge $[0,1]$ (if it is oriented in that order will have the complex number $\mathbf{X}[0,1,2,3]$ if the the edges in the face $[1,2,3]$ are in majority induced by the orientation of the order $[1,2,3]$. In the case of branched triangulations (cf. [BB]) it is easy to keep track of the orientations of the edges. We obtain for each simplex (with oriented edges) 6 complex numbers and one can consider a cross ratio as a function associating to each edge of a triangulation (with oriented edges) a complex number. In our paper we prefer to work directly with the 0-skeleton of the simplices which seems more appropriate in the context of cross-ratios. For difficulties related to the 1-skeleton and the 2-skeleton see [F].

Remark 5: A large class of cross-ratio simplicial structures are constructed associating to a $PSL(2,\mathbb{C})$-valued (or $PU(2,1)$-valued as in the last section) representation of the fundamental group of the (cusped) 3-manifold a symplicial 1-cocycle. In fact, a 1-cocycle defined on a symplex defines a configuration of 4 points in $\mathbb{C}P^1$ (or S^3) by identifying one of the vertices to a point and then using the 1-cocycle to obtain the other three (see [BB] and [N]) . A generic choice of the first point (and a generic choice of the 1-cocycle up to a 1-coboundary) will give rise to a configuration of 4 distinct points. By the same procedure, following all the edges of the simplicial structure we obtain, generically, a configuration of points in $\mathbb{C}P^1$ (or S^3) which is in correspondence to the vertices of the simplicial structure (this process is called idealization in [BB]). Using that correspondence, the cross-ratios defined in remarks 1 and 3 for the configuration of points in S^2 or S^3 define a cross ratio structure on the original simplicial space equipped with a generic 1-cocycle with values in $PSL(2,\mathbb{C})$ or $PU(2,1)$ respectively.

In order to deal with $2 \longleftrightarrow 3$ moves we will impose moreover the following conditions

Definition 2.2 *Let $[u_1,u_2,u_3,u_4]$ and $[u_0,u_2,u_3,u_4]$ be simplices with the common face $[u_2,u_3,u_4]$, one considers the $2 \to 3$ move obtained by decomposition of the union of those simplices in three simplices $[u_0,u_1,u_3,u_4]$, $[u_0,u_1,u_2,u_4]$ and $[u_0,u_1,u_2,u_3]$. A mobile simplicial cross ratio structure along this move is a simplicial cross ratio structure satifying*

- *One can define a new simplicial cross ratio structure which has the same value in all simplices of the original triangulation and in the three new ones it has values satisfying the following relations*

$$\mathbf{X}[u_0,u_1,u_2,u_3]\mathbf{X}[u_0,u_1,u_3,u_4]\mathbf{X}[u_0,u_1,u_4,u_2] = 1,$$

$$\mathbf{X}[u_2,u_3,u_0,u_1]\mathbf{X}[u_3,u_4,u_0,u_1]\mathbf{X}[u_4,u_2,u_0,u_1] = 1,$$

$$\mathbf{X}[u_0,u_1,u_3,u_4]\mathbf{X}[u_1,u_2,u_3,u_4]\mathbf{X}[u_2,u_0,u_3,u_4] = 1,$$

$$\mathbf{X}[u_3,u_4,u_0,u_1]\mathbf{X}[u_3,u_4,u_1,u_2]\mathbf{X}[u_3,u_4,u_2,u_0] = 1.$$

Remark: The conditions above for a mobile cross ratio structure along the move $2 \to 3$ can be interpreted as compatibility conditions for a mobile structure along each bistellar move and we will not repeat the definition for each of them. In fact, the bistellar move along a face common to two simplices was described above whereas the bistellar move along a simplex corresponds to a creation of a new vertex and substitution of this simplex by four others. It involves precisely the 5 abstract simplices above with the interpretation of u_0 as the created vertex. Of course, the inverse moves impose the same relations.

Remark: If one starts with a cross ratio defined over a closed triangulation (each face is contained in two simplices) satifying only the first axiom of a simplicial cross ratio structure, the conditions for a mobile cross ratio structure along all $2 \to 3$ moves imply the other two conditions for a simplicial structure as it is shown in the next section.

Definition 2.3 *Referring to notation as in equation 1, a simplicial cross ratio structure is said to be a* $\mathbf{T}-$ *structure if, restricted to each simplex the following relations hold*

$$\omega_0\omega_1\omega_2\omega_0'\omega_1'\omega_2' = 1 \quad (2)$$

$$\big((\omega_0-1)(\omega_0'-1)-1\big)+\omega_0\omega_0'\big((\omega_1-1)(\omega_1'-1)-1\big)+\omega_0\omega_0'\omega_1\omega_1'\big((\omega_2-1)(\omega_2'-1)-1\big)=0 \quad (3)$$

The definition might seem awkward at first sight but it is justified by the existence of two involutions and the analysis of their fixed point sets.

Remark: For a mobile simplicial cross ratio structure the first equation is automaticaly verified as long as each simplex is not isolated as shown in the next section.

Remark: Special cases include real hyperbolic structures defined by a triangulation by ideal tetrahedra and spherical CR triangulations. In particular, for each 1-cocycle with values in $PSL(2,\mathbb{C})$ (or $PU(2,1)$) one associates an idealization as in remark **5** and that idealization defines a **T**-structure. If the cocycle has values in the parabolic subgrup $\mathbb{R} \subset PSL(2,\mathbb{C})$ it can be interpreted as having values in the center of the parabolic subgroup of $PU(2,1)$ and in this case the **T**-structure can be interpreted as being carried by both hyperbolic and spherical CR simplices. I thank R. Benedetti for discussions concerning that remark. In the case of a topological ideal triangulation with torus boundary components with n tetrahedra, there are n edge equations and $2n$ face equations, that is $3n$ equations. In principle for each simplex there is a 4 complex dimensional admissible subvariety of $(\mathbb{C}\setminus\{0\})^6$ making a total of $4n$ variables. We have enough room for many solutions. In the hyperbolic case, the face equations are trivial ($x_0x_1x_2 = -1$) and there is only one variable for each simplex. In the CR case there are essentially 2 complex variables for each simplex, the face equations correspond to equality of Cartan's invariant for each pair of identified faces.

3 The general 5 term relation

Let $[u_1,u_2,u_3,u_4]$ and $[u_0,u_2,u_3,u_4]$ be two simplices with a common face $[u_2,u_3,u_4]$ in a mobile simplicial cross-ratio structure. One considers then the decomposition of the union of those simplices in three simplices $[u_0,u_1,u_3,u_4]$, $[u_0,u_1,u_2,u_4]$ and $[u_0,u_1,u_2,u_3]$. A straightforward computation using the symmetries of a mobile cross ratio structure above gives the following proposition.

Proposition 3.1 *If*

$$[[u_1,u_2,u_3,u_4]] = \begin{pmatrix} x_0\\ x_1\\ x_2\\ x_0'\\ x_1'\\ x_2'\end{pmatrix} \quad \textit{and} \quad [[u_0,u_2,u_3,u_4]] = \begin{pmatrix} y_0\\ y_1\\ y_2\\ y_0'\\ y_1'\\ y_2'\end{pmatrix}$$

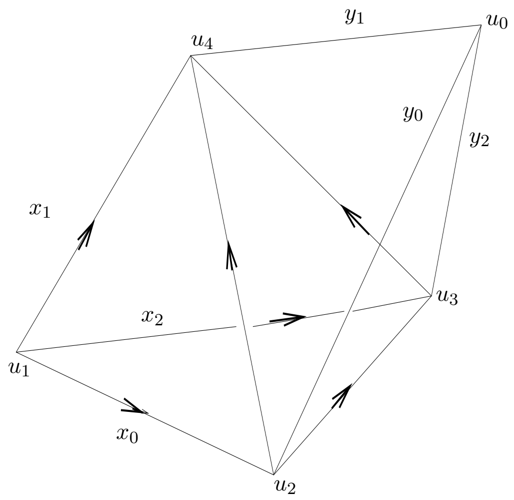

Figure 1: *The 2 tetrahedra of the 5 term relation. The relation $x_0x_1x_2 = y_0y_1y_2$ should hold.*

then

$$[[u_0, u_1, u_3, u_4]] = \begin{pmatrix} \frac{y_0}{x_0} \\ \alpha_1 \\ \frac{1}{y_0 x_1' x_2' \alpha_1} \\ \frac{y_0'}{x_0'} \\ \alpha_1' \\ \frac{x_0 y_1' y_2'}{\alpha_1'} = \frac{1}{y_0' x_1 x_2 \alpha_1'} \end{pmatrix},$$

$$[[u_0, u_1, u_2, u_4]] = \begin{pmatrix} \frac{x_2}{y_2} \\ \frac{\alpha_1}{y_1} \\ \frac{1}{y_0 x_2' \alpha_1} \\ \frac{x_2'}{y_2'} \\ \frac{\alpha_1'}{y_1'} \\ \frac{1}{y_0' x_2 \alpha_1'} \end{pmatrix} \quad [[u_0, u_1, u_2, u_3]] = \begin{pmatrix} \frac{y_1}{x_1} \\ \frac{\alpha_1}{x_0' y_1} = y_2 y_0 x_1' x_2' \alpha_1 \\ \frac{1}{x_2' \alpha_1} \\ \frac{y_1'}{x_1'} \\ \frac{\alpha_1'}{x_0 y_1'} \\ \frac{1}{x_2 \alpha_1'} \end{pmatrix}$$

Proof. A convenient way to organize the computations is to start with an unknown element, say $\mathbf{X}(u_0, u_1, u_3, u_4)$. Using the two relations obtained by fixing the edges (u_0, u_1) and (u_3, u_4) respectively we obtain:

$$\mathbf{X}[u_0, u_1, u_3, u_4]\mathbf{X}[u_0, u_1, u_4, u_2]\mathbf{X}[u_0, u_1, u_2, u_3] = 1,$$

and

$$\mathbf{X}[u_0, u_1, u_3, u_4]\mathbf{X}[u_1, u_2, u_3, u_4]\mathbf{X}[u_2, u_0, u_3, u_4] = 1.$$

The first equation gives imediately that

$$\mathbf{X}(u_0, u_1, u_3, u_4) = \frac{y_0}{x_0}.$$

The proof follows by writing all relations in this manner.

□

We also obtain the following relations

$$\frac{y_0}{x_0}\frac{y_1}{x_1}\frac{y_2}{x_2} = 1 \tag{4}$$

That is $x_0x_1x_2 = y_0y_1y_2$, and

$$x_0'x_1'x_2'y_0y_1y_2 = 1 \qquad x_0x_1x_2y_0'y_1'y_2' = 1. \tag{5}$$

That implies the following

Proposition 3.2 *The cross ratios of the simplices $[u_1, u_2, u_3, u_4]$ and $[u_0, u_2, u_3, u_4]$ satify $x_0x_1x_2x_0'x_1'x_2' = 1$ and $y_0y_1y_2y_0'y_1'y_2' = 1$. The same property is valid for the simplices $[u_0, u_1, u_3, u_4]$, $[u_0, u_1, u_2, u_4]$ and $[u_0, u_1, u_2, u_3]$.*

Observe that the variables α_1 and α_1' are not determined but they could be fixed by imposing a further condition on the simplices, for instance that they be geometric as shown in the next section in the CR case. It is interesting to note that a positive move in the triangulation, that is a move that augments the number of simplices, introduces a two complex parameter family of indetermination in the simplicial cross ratio.

4 Geometric structures

We consider the two involutions of $\mathbb{C}^6$:

$$H : \begin{pmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \omega_0' \\ \omega_1' \\ \omega_2' \end{pmatrix} \longrightarrow \begin{pmatrix} \omega_0' \\ \omega_1' \\ \omega_2' \\ \omega_0 \\ \omega_1 \\ \omega_2 \end{pmatrix}$$

and

$$A : \begin{pmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \omega_0' \\ \omega_1' \\ \omega_2' \end{pmatrix} \longrightarrow \begin{pmatrix} \bar{\omega}_0' \\ \bar{\omega}_1' \\ \bar{\omega}_2' \\ \bar{\omega}_0 \\ \bar{\omega}_1 \\ \bar{\omega}_2 \end{pmatrix}$$

Observe that both involutions are defined on Ω. We explicit the fixed set for each of the involutions in the following

Proposition 4.1 *One component of the fixed set of H corresponds to triples $(\omega_0, \omega_1, \omega_2)$ satisfying*

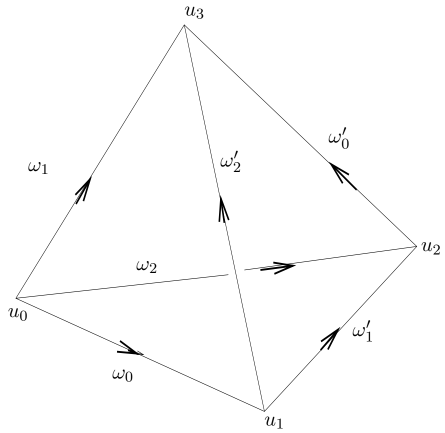

Figure 2: *Parameters for a simplex*

- $\omega_0\omega_1\omega_2 = -1$
-
$$\omega_{i+1} = 1 - \frac{1}{\omega_i}$$

We call a simplex with moduli in the above component *ideal hyperbolic simplex* or *conformal simplex*.

PROOF. The fixed points of H satisfy $\omega_i = \omega'_i$. Substituting in the second formula 2.3 we obtain

$$2\omega_0 = \frac{1}{\omega_2^2}\left((1-\frac{1}{\omega_1})^2 - 1\right) + (1-\frac{1}{\omega_2})^2.$$

The component we are interested is the one where $\omega_0\omega_1\omega_2 = -1$. Substituting $\omega_1 = -1/\omega_0\omega_2$ we get

$$2\omega_0 = \frac{1}{\omega_2^2}\left((1+\omega_0\omega_2)^2 - 1\right) + (1-\frac{1}{\omega_2})^2.$$

and then

$$2\omega_0 = \frac{2\omega_0}{\omega_2} + \omega_0^2 + (1-\frac{1}{\omega_2})^2$$

and simplifying

$$(\omega_0 - (1-\frac{1}{\omega_2}))^2 = 0$$

which gives the result. □

Remark that the conventions we used make the moduli ω_i correspond to the inverse of the moduli of a hyperbolic tetrahedron with positive orientation as in Thurston's conventions.

Proposition 4.2 *The fixed set of A corresponds to triples $(\omega_0, \omega_1, \omega_2)$ satisfying*

- $|\omega_0\omega_1\omega_2| = 1$
- $$((\omega_0 - 1)(\bar{\omega}_0 - 1) - 1) + \omega_0\bar{\omega}_0 \left((\omega_1 - 1)(\bar{\omega}_1 - 1) - 1\right) + \omega_0\bar{\omega}_0\omega_1\bar{\omega}_1 \left((\omega_2 - 1)(\bar{\omega}_2 - 1) - 1\right) = 0$$

We call a simplex with moduli in that fixed set a *CR simplex*. The justification of that definition will follow from Proposition 5.4.

More generally we say a $\mathbf{T}-$*structure* is hyperbolic or CR if each simplex is hyperbolic or CR.

Theorem 4.3 *Consider the space of $\mathbf{T}-$structures associated to a triangulation. Then hyperbolic triangulations are fixed by a holomorphic involution and spherical CR triangulations are fixed by an anti-holomorphic involution.*

Proof. The proof follows by combining the involutions defined for each simplex in the triangulation. □

4.1 Hyperbolic five term relations

Imposing the extra symmetry on the cross ratio given by $\mathbf{X}[u_0, u_1, u_2, u_3] = \mathbf{X}[u_2, u_3, u_0, u_1]$, implies that $(x_0x_1x_2)^2 = 1$. One of the connected components of solutions to this relation is compatible with the hyperbolic cross ratio, that is, $x_0x_1x_2 = -1$. But apparently no further relations are obtained from the compatibility equations. In order to obtain the other relation ($x_1 = 1 - 1/x_0$), one can impose that the simplices be geometric or consider the 5-term relation (taking into accounts the symmetries, we only need the first three componets of the cross ratio, moreover the third component is determined by the relation $x_0x_1x_2 = -1$):

$$[[u_1, u_2, u_3, u_4]] - [[u_0, u_2, u_3, u_4]] + [[u_0, u_1, u_3, u_4]] - [[u_0, u_1, u_2, u_4]] + [[u_0, u_1, u_2, u_3]]$$

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \frac{y_0}{x_0} \\ \alpha \\ \frac{1}{x_1x_2y_0\alpha} \end{pmatrix} - \begin{pmatrix} \frac{x_2}{y_2} \\ \frac{\alpha}{y_1} \\ \frac{1}{x_2y_0\alpha} \end{pmatrix} + \begin{pmatrix} \frac{y_1}{x_1} \\ \frac{\alpha}{x_0y_1} \\ \frac{1}{x_2\alpha} \end{pmatrix}$$

where α is an arbitrary complex number. We write the 5 term as an alternate sum for convenience but in this paper we do not deal with Bloch groups and its generalizations.

Proposition 4.4 *If we impose that the second component of each cross ratio is given by a universal Möbius transformation M applied to the first term , that is $z_1 = M(z_0)$ where z_0 and z_1 are any of the the first and second component of each of the 5 terms, then $M(z) = 1 - 1/z$.*

Proof. The proof follows by eliminating α from the 5 equations obtained imposing the general form $M(z) = \frac{az+b}{cz+d}$.

□

In general, a way to obtain a restricted number of possibilities for α would be to impose that a polynomial in two variables $p(z_1, z_2)$ verifies simultaneously the following relations:

$$p(x_0, x_1) = p(y_0, y_1) = p(\frac{y_0}{x_0}, \alpha) = p(\frac{y_0 y_1}{x_0 x_1}, \frac{\alpha}{y_1}) = p(\frac{y_1}{x_1}, \frac{\alpha}{x_0 y_1}) = 0.$$

I do not know what are the possible polynomials.

5 CR geometry (see [G] or [J] for details)

CR geometry is modeled on the *Heisenberg group* $\mathfrak{N}$, the set of pairs $(z, t) \in \mathbb{C} \times \mathbb{R}$ with the product

$$(z, t) \cdot (z', t') = (z + z', t + t' + 2\mathrm{Im}\, z\overline{z}').$$

The one point compactification of the Heisenberg group, $\overline{\mathfrak{N}}$, of $\mathfrak{N}$ can be interpreted as S^3 which , in turn, can be identified to the boundary of Complex Hyperbolic space.

We consider the group $U(2, 1)$ preserving the Hermitian form $\langle z, w\rangle = w^* J z$ defined by the matrix

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and the following subspaces in $\mathbb{C}^3$:

$$V_0 = \left\{ z \in \mathbb{C}^3 - \{0\} \quad : \quad \langle z, z\rangle = 0 \right\},$$

$$V_- = \left\{ z \in \mathbb{C}^3 \quad : \quad \langle z, z\rangle < 0 \right\}.$$

Let $\mathbb{P} : \mathbb{C}^3 \setminus \{0\} \to \mathbb{C}P^2$ be the canonical projection. Then $\mathbf{H}^2_{\mathbb{C}} = \mathbb{P}(V_-)$ is the complex hyperbolic space and $S^3 = \mathbf{H}^2_{\mathbb{C}} = \mathbb{P}(V_0)$ can be identified to $\overline{\mathfrak{N}}$.

The group of biholomorphic transformations of $\mathbf{H}^2_{\mathbb{C}}$ is then $PU(2, 1)$, the projectivization of $U(2, 1)$. It acts on S^3 by CR transformations. An involution in $PU(2, 1)$ has a fixed point in the interior of complex hyperbolic space. If it has fixed points in the boundary of complex hyperbolic space, one shows that the set of fixed points is a topological circle, called $\mathbb{C}$-*circle*. We can also define $\mathbb{C}$-circles as boundaries of complex lines in $\mathbf{H}^2_{\mathbb{C}}$. Using the identification $S^3 = \mathfrak{N} \cup \{\infty\}$ one can define alternatively a $\mathbb{C}$-circle as any circle in S^3 which is obtained from the vertical line $\{(0, t)\} \cup \{\infty\}$ in the compactified Heisenberg space by translation by an element of $PU(2, 1)$.

A point $p = (z, t)$ in the Heisenberg group and the point ∞ are lifted to the following points in $\mathbb{C}^{2,1}$:

$$\hat{p} = \begin{bmatrix} \frac{-|z|^2 + it}{2} \\ z \\ 1 \end{bmatrix} \quad \text{and} \quad \hat{\infty} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Definition 5.1 *Given any three ordered points p_1, p_2, p_3 in $\partial\mathbf{H}^2_{\mathbb{C}}$ we define Cartan's angular invariant $\mathbb{A}$ as*

$$\mathbb{A}(p_1, p_2, p_3) = \arg(-\langle\hat{p}_1, \hat{p}_2\rangle\langle\hat{p}_2, \hat{p}_3\rangle\langle\hat{p}_3, \hat{p}_1\rangle).$$

The Cartan's angular invariants classifies ordered triples of points in S^3:

Proposition 5.2 ([C], see also[G]) *There exists an element of $PU(2, 1)$ which translates an ordered triple of points in S^3 to another if and only if their corresponding Cartan's invariants are equal.*

The CR cross ratio is given by the Koranyi-Reimann invariant introduced in [KR] (see [KR] and [G] for its properties):

Definition 5.3 *The CR cross-ratio associated to four distinct points in S^3 is*

$$X[u_0,u_1,u_2,u_3]=\frac{\langle u_3,u_1\rangle\langle u_2,u_0\rangle}{\langle u_2,u_1\rangle\langle u_3,u_0\rangle}.$$

Here, we choose lifts for the points u_i which we denote by the same letter. The invariant does not depend on the choice of lifts.

Consider a generic configuration of four points in S^3 (any three of them not contained in a $\mathbb{C}$-circle) up to overall translation by an element of the automorphism group. One can always arrange them as the following configuration of distinct points in the Heisenberg group where $s,t\in\mathbb{R}$ and $z\in\mathbb{C}$.

$$u_0=\infty \quad u_1=0 \quad u_3=(1,t) \quad u_2=(z,s|z|^2).$$

(cf. [F, Wi] for different normalizations). Lifting those elements to $\mathbb{C}^{2,1}$ we may compute

$$\omega_0=X(u_0,u_1,u_2,u_3)=\frac{ti-1}{|z|^2(si-1)}$$

$$\omega_1=X(u_0,u_3,u_1,u_2)=\frac{1+ti-2z+|z|^2(1-si)}{1+ti}$$

$$\omega_2=X(u_0,u_2,u_3,u_1)=\frac{|z|^2(1+si)}{1-ti-2\bar{z}+|z|^2(1+si)}$$

The product of the three cross ratios gives the Cartan invariant (see [G])

$$X(u_0,u_1,u_2,u_3)X(u_0,u_3,u_1,u_2)X(u_0,u_2,u_3,u_1)=e^{2i\mathbb{A}(u_1,u_2,u_3)}$$

We prove the following

Proposition 5.4 *There exists a bijection between the set of distinct ordered four points (up to translation by elements of $PU(2,1)$) in S^3 not all contained in the same $\mathbb{C}$-circle and the set of solutions of the following equations (cf. 4.2) with ω_i not all real with $\omega_0\omega_1\omega_2=-1$.*

- $|\omega_0\omega_1\omega_2|=1$
- $$((\omega_0-1)(\bar{\omega}_0-1)-1)+\omega_0\bar{\omega}_0\left((\omega_1-1)(\bar{\omega}_1-1)-1\right)+\omega_0\bar{\omega}_0\omega_1\bar{\omega}_1\left((\omega_2-1)(\bar{\omega}_2-1)-1\right)=0$$

PROOF. 1) The generic case (no three points are contained in a $\mathbb{C}$-circle). One can use the expressions of $\omega_0,\omega_1,\omega_2$ in terms of z,t,s obtained above in the generic case to verify equations 4.2. Conversely, suppose that $\bar{\omega}_0\bar{\omega}_1\omega_2\neq -1$ and $\bar{\omega}_0\omega_1\bar{\omega}_2\neq -1$. The configuration will be generic in that case. In particular any three points are not contained on a common $\mathbb{C}-circle$. We have to solve for z,t,s in terms of $\omega_0,\omega_1,\omega_2$. We compute $\bar{\omega}_0\bar{\omega}_1\omega_2=\frac{i-t}{i+t}$. Therefore

$$t=-i\frac{\bar{\omega}_0\bar{\omega}_1\omega_2-1}{\bar{\omega}_0\bar{\omega}_1\omega_2+1}.$$

Also $\bar{\omega}_0\omega_1\bar{\omega}_2=\frac{1-is}{1+is}$, therefore

$$s=i\frac{\bar{\omega}_0\omega_1\bar{\omega}_2-1}{\bar{\omega}_0\omega_1\bar{\omega}_2+1}.$$

Substituting the values of t and s in the expression for ω_0 we obtain $|z|^2 = \frac{\bar{\omega}_0\bar{\omega}_2\omega_1+1}{\bar{\omega}_0(1+\omega_0\omega_1\bar{\omega}_2)}$. and substituting this value in the expression for ω_1 we obtain

$$z = \frac{1+\omega_1(\bar{\omega}_2-1)}{\omega_0\omega_1\bar{\omega}_2+1}.$$

Observe that the solution above holds if the following relations between the invariants are verified (there seems to have a slight mistake in the analogous formula 7.12 for the second equation in [G]) :

$$|\omega_0\omega_1\omega_2| = 1$$

$$\frac{1+\omega_1(\bar{\omega}_2-1)}{\omega_0\omega_1\bar{\omega}_2+1}.\overline{\left(\frac{1+\omega_1(\bar{\omega}_2-1)}{\omega_0\omega_1\bar{\omega}_2+1}\right)} = \frac{\bar{\omega}_0\bar{\omega}_2\omega_1+1}{\bar{\omega}_0(1+\omega_0\omega_1\bar{\omega}_2)}$$

The first equation is a necessary and sufficient condition for solving for t and s. The second equation follows from the compatibility condition comparing the expression for $|z|^2$ and the one for $z\bar{z}$. It can be simplified

$$|\omega_0|^2|1-\omega_1|^2 + \left(1 - 2\mathrm{Re}\,\frac{1}{\omega_2}\right) = \omega_0 + \bar{\omega}_0$$

or

$$|\omega_0|^2|1-\omega_1|^2 + \frac{|1-\omega_2|^2-1}{|\omega_2|^2} = \omega_0 + \bar{\omega}_0$$

By using $|\omega_0|^2 = |\omega_1\omega_2|^{-2}$ we obtain the following

$$2\mathrm{Re}\,\omega_0 = \frac{1}{|\omega_2|^2}\left(|1-\frac{1}{\omega_1}|^2 - 1\right) + |1-\frac{1}{\omega_2}|^2$$

which, in turn, is equivalent combined with $|\omega_0\omega_1\omega_2| = 1$ to

$$((\omega_0-1)(\bar{\omega}_0-1)-1) + \omega_0\bar{\omega}_0\left((\omega_1-1)(\bar{\omega}_1-1)-1\right) + \omega_0\bar{\omega}_0\omega_1\bar{\omega}_1\left((\omega_2-1)(\bar{\omega}_2-1)-1\right) = 0.$$

2) In order to treat the non-generic case we use more general coordinates

$$u_0 = \infty \quad u_1 = 0 \quad u_3 = (z,t) \quad u_2 = (w,s),$$

to obtain

$$\bar{\omega}_0\bar{\omega}_1\omega_2 = \frac{|z|^2+it}{|z|^2-it}$$

and

$$\omega_0\bar{\omega}_1\omega_2 = \frac{|w|^2+is}{|w|^2-is}.$$

This shows that $\bar{\omega}_0\bar{\omega}_1\omega_2 = -1$ if and only if $z = 0$ (in this case $t \neq 0$ in order that the points be distinct), that is, u_0, u_1, u_3 are in the same $\mathbb{C}$-circle and $\omega_0\bar{\omega}_1\omega_2 = -1$ if and only if $w = 0$ (in this case $s \neq 0$), that is, u_0, u_1, u_2 are in the same $\mathbb{C}$-circle.

We have

$$\omega_0 = X(u_0,u_1,u_2,u_3) = \frac{|z|^2-it}{|w|^2-is}$$

$$\omega_1 = X(u_0,u_3,u_1,u_2) = \frac{|w|^2+|z|^2+(t-s)i-2\bar{z}w}{|z|^2+ti}$$

$$\omega_2 = X(u_0,u_2,u_3,u_1) = \frac{|w|^2+si}{|w|^2+|z|^2-(t-s)i-2z\bar{w}}$$

By hypothesis the four points are not on the same $\mathbb{C}$-circle. Suppose that $w=0$ (the other case is similar). In that case we have $z\neq 0$ and we can normalize so that $z=1$ and $\omega_0\bar{\omega}_1\omega_2=-1$.

We show that t and s are uniquely determined: The cross ratios are in this case:

$$\omega_0 = X(u_0,u_1,u_2,u_3) = \frac{1-it}{-is}$$

$$\omega_1 = X(u_0,u_3,u_1,u_2) = \frac{1+(t-s)i}{1+it}$$

$$\omega_2 = X(u_0,u_2,u_3,u_1) = \frac{si}{1-(t-s)i}$$

Now, we obtain from

$$\bar{\omega}_0\bar{\omega}_1\omega_2 = \frac{1+it}{1-it}$$

as before

$$t = -i\frac{\bar{\omega}_0\bar{\omega}_1\omega_2-1}{\bar{\omega}_0\bar{\omega}_1\omega_2+1}.$$

and from the expression of ω_0 we deduce s. □

Remark: In the previous version there was a mistake concerning degenerate configurations. The degenerate configurations of points contained in the same $\mathbb{C}$-circle up to $PU(2,1)$ have a $\mathbb{Z}_2$ action whose quotient is identified by the cross-ratios. I thank H. Cunha and N. Gusevskii (see arXiv:0812.2159) for pointing that out and to J. Genzmer, J. Parker, I. Platis and P. Will for several discussions about this variety. It seems important to note that the papers using this cross-ratio variety or other similar constructions as in [PP, Wi1] are not greatly affected by this mistake and a reader can easily handle the degenerate cases. Notably, in [F, FP], degenerate configurations are excluded from the beginning.

The case excluded in the above proposition can be easily handled and it is best stated in the following lemma.

Lemma 5.5 (Degenerate configurations) *There exists a 2 to 1 map from the set of distinct ordered four points (up to translation by elements of $PU(2,1)$) in S^3 contained in the same $\mathbb{C}$-circle and the set of solutions of the following equations (cf. 4.2) with all ω_i real.*

- $\omega_0\omega_1\omega_2 = -1$
- $$((\omega_0-1)(\bar{\omega}_0-1)-1)+\omega_0\bar{\omega}_0\left((\omega_1-1)(\bar{\omega}_1-1)-1\right)+\omega_0\bar{\omega}_0\omega_1\bar{\omega}_1\left((\omega_2-1)(\bar{\omega}_2-1)-1\right)=0$$

Proof. In fact the last equation is equivalent in that case to $\omega_2 = 1-1/\omega_1$. Using the coordinates as in the proof of the proposition (cf. also Cunha and Gusevskii) we obtain for $z=w=0$ that t can be normalized by a dilation either to 1 or to -1. We compute then, for $t=\pm 1$:

$$\omega_0 = X(u_0,u_1,u_2,u_3) = \pm\frac{1}{s},$$

$$\omega_1 = X(u_0, u_3, u_1, u_2) = (1 - (\pm s)),$$
$$\omega_2 = X(u_0, u_2, u_3, u_1) = \frac{\pm s}{\pm s - 1}.$$

So that each solution of the equation correspond to two configurations, one with $t = 1$ and the other with $t = -1$.

□

Observe that the Z_2 action on the space of degenerate configurations up to $PU(2,1)$ (which is a one dimensional space) is simply the diagonal action of an antiholomorphic involution in (the boundary of) complex hyperbolic space fixing the first two points.

Example: A special case of tetrahedra consists of those having a $\mathbb{Z}_2$ anti-symplectic symmetry (see [F, Wi]). Without loss of generality one can assume that the symmetry is $(0,1,2,3) \to (1,0,3,2)$ and a simple calculation shows that this is the case if and only if $\omega_0 \in \mathbb{R}^+$. In fact, symmetric tetrahedra can be characterized in the coordinates above as those with $t = s$ and therefore $\omega_0 = \frac{1}{|z|^2}$, $\omega_1 = \frac{1+ti-2z+|z|^2(1-ti)}{1+ti}$ and $\omega_2 = \frac{|z|^2(1+ti)}{1-ti-2\bar{z}+|z|^2(1+ti)}$. In particular for $z = \omega = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $t = \sqrt{3}$ we obtain $\omega_0 = 1$, $\omega_1 = -\omega$, $\omega_2 = 1$.

5.1 The CR five term relation

In this section we explicit a 5-term relation in the CR case. The five term relation corresponding to general tetrahedra is given by constructing the vectors in $\mathbb{C}^6$ corresponding to

$$[[u_1, u_2, u_3, u_4]] - [[u_0, u_2, u_3, u_4]] + [[u_0, u_1, u_3, u_4]] - [[u_0, u_1, u_2, u_4]] + [[u_0, u_1, u_2, u_3]]$$

where

$$[[u_1, u_2, u_3, u_4]] = \begin{pmatrix} \mathbf{X}[u_1, u_2, u_3, u_4] \\ \mathbf{X}[u_1, u_4, u_2, u_3] \\ \mathbf{X}[u_1, u_3, u_4, u_2] \\ \mathbf{X}[u_3, u_4, u_1, u_2] \\ \mathbf{X}[u_2, u_3, u_1, u_4] \\ \mathbf{X}[u_4, u_2, u_1, u_3] \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_0' \\ x_1' \\ x_2' \end{pmatrix}$$

In the hyperbolic case the first component of the vector in $\mathbb{C}^6$ determines the other 5 component. That is, we might write

$$[[u_1, u_2, u_3, u_4]]_H = \mathbf{X}[u_1, u_2, u_3, u_4]$$

In the CR case we might use only the first 3 components of the vector in $\mathbb{C}^6$, that is, we write

$$[[u_1, u_2, u_3, u_4]]_{CR} = \begin{pmatrix} \mathbf{X}[u_1, u_2, u_3, u_4] \\ \mathbf{X}[u_1, u_4, u_2, u_3] \\ \mathbf{X}[u_1, u_3, u_4, u_2] \end{pmatrix}$$

where each coefficient in the column vector is a KR invariant as defined before. Bellow we will write $[[u_1, u_2, u_3, u_4]]_{CR}$ without the subindex CR.

Theorem 5.6 *For a spherical CR structure the five term relation is*

$$[[u_1, u_2, u_3, u_4]] - [[u_0, u_2, u_3, u_4]] + [[u_0, u_1, u_3, u_4]] - [[u_0, u_1, u_2, u_4]] + [[u_0, u_1, u_2, u_3]]$$

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \frac{y_0}{x_0} \\ z_1 \\ \frac{1}{\bar{x}_1\bar{x}_2 y_0 z_1} \end{pmatrix} - \begin{pmatrix} \frac{x_2}{y_2} \\ \frac{z_1}{y_1} \\ \frac{1}{\bar{x}_2 y_0 z_1} \end{pmatrix} + \begin{pmatrix} \frac{y_1}{x_1} \\ \frac{z_1}{\bar{x}_0 y_1} \\ \frac{1}{\bar{x}_2 z_1} \end{pmatrix}$$

where

$$z_1 = 1 + \bar{x}_0 y_1 y_2 - \left(\overline{\frac{x_0x_1x_2 - x_0x_1 + x_0}{y_0y_1y_2 - y_0y_1 + y_0}} \right) (1 + \bar{y}_0 y_1 y_2)$$

Proof. The formula follows from the general 5 term relation by making $x_i' = \bar{x}_i$ and $y_i' = \bar{y}_i$.

It remains to determine $\alpha = z_1$. In order to do so we determine the tetrahedra $u_1 = (z, s|z|^2)$, $u_2 = (1, t)$, $u_3 = 0$, $u_4 = \infty$ and $u_0 = (z', s'|z'|^2)$. Observe that in that case $x_0x_1x_2 = y_0y_1y_2$ and solving for the variables t, s, z, s', z' we obtain

$$t = -i\frac{1 - x_0x_1x_2}{1 + x_0x_1x_2} \quad s = -i\frac{1 - \bar{x}_0x_1x_2}{1 + \bar{x}_0x_1x_2} \quad s' = -i\frac{1 - \bar{y}_0y_1y_2}{1 + \bar{y}_0y_1y_2}$$

$$\bar{z} = \frac{x_0x_1x_2 - x_0x_1 + x_0}{1 + x_0x_1x_2} \quad \bar{z}' = \frac{y_0y_1y_2 - y_0y_1 + y_0}{y_0y_1y_2 + 1}$$

On the other hand we compute

$$z_1 = \mathbf{X}[u_0, u_4, u_1, u_3] = \frac{z\bar{z}(i+s) - 2iz\bar{z}' + z'\bar{z}'(i - s')}{z'\bar{z}'(i - s')}$$

the expression for z_1 is obtained substituting in the last formula the equations above. $\square$

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