TETRAHEDRA OF FLAGS, VOLUME AND HOMOLOGY OF SL(3)

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Abstract. In the paper we define a "volume" for simplicial complexes of flag tetrahedra. This generalizes and unifies the classical volume of hyperbolic manifolds and the volume of CR tetrahedra complexes considered in [4, 6]. We describe when this volume belongs to the Bloch group and more generally describe a variation formula in terms of boundary data. In doing so, we recover and generalize results of Neumann-Zagier [13], Neumann [11], and Kabaya [10]. Our approach is very related to the work of Fock and Goncharov [7, 8].

Contents

1. Introduction

It follows from Mostow's rigidity theorem that the volume of a complete hyperbolic manifold is a topological invariant. In fact, it coincides with Gromov's purely topological definition of simplicial volume. If the complete hyperbolic manifold M has cusps, Thurston showed that one could obtain complete hyperbolic structures on manifolds obtained from M by Dehn surgery by gluing a solid torus with a sufficiently long geodesic. Thurston's framed his results for more general deformations

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which are not complete hyperbolic manifolds, the volume of the deformation being the volume of its metric completion. Neumann and Zagier [13] and afterwards Neumann [11] provided a deeper analysis of these deformations and their volume. In particular, they showed that the variation of the volume depends only on the geometry of the boundary and they gave a precise formula for that variation in terms of the boundary holonomy.

It is natural to consider an invariant associated to a hyperbolic structure defined on the pre-Bloch group $\mathcal{P}(k)$ which is defined as the abelian group generated by all the points in $k \setminus \{0,1\}$ where k is a field quotiented by the 5-term relations (see section 3 for definitions and references). The volume function is well defined as a map $\mathrm{Vol} : \mathcal{P}(k) \to \mathbb{R}$ using the dilogarithm. The Bloch group $\mathcal{B}(k)$ is a subgroup of the pre-Bloch group $\mathcal{P}(k)$. It is defined as the kernel of the map

$$\delta : \mathcal{P}(k) \to k^\times \wedge_{\mathbb{Z}} k^\times$$

given by $\delta([z]) = z \wedge (1-z)$. The volume and the Chern-Simons invariant can then be seen through a function (the Bloch regulator)

$$\mathcal{B}(k) \to \mathbb{C}/\mathbb{Q}.$$

The imaginary part being related to the volume and the real part related to Chern-Simons CS mod $\mathbb{Q}$ invariant.

Several extensions of Neumann-Zagier results were obtained. Kabaya [10] defined an invariant in $\mathcal{P}(\mathbb{C})$ associated to a hyperbolic 3-manifold with boundary and obtained a description of the variation of the volume function which depends only on the boundary data. Using different coordinates and methods Bonahon [3] showed a similar formula.

The volume function was extended in [4, 6] in order to deal with Cauchy-Riemann (CR) structures. More precisely, consider $\mathbb{S}^3 \subset \mathbb{C}^2$ with the contact structure obtained as the intersection $D = TS^3 \cap JTS^3$ where J is the multiplication by i in $\mathbb{C}^2$. The operator J restricted to D defines the standard CR structure on $\mathbb{S}^3$. The group of CR-automorphisms of $\mathbb{S}^3$ is $\mathrm{PU}(2,1)$ and we say that a manifold M has a spherical CR structure if it has a $(\mathbb{S}^3, \mathrm{PU}(2,1))$- geometric structure. Associated to a CR triangulation it was defined in [6] an invariant in $\mathcal{P}(\mathbb{C})$ which is in the Bloch group in case the structure has unipotent boundary holonomy. The definition of that invariant is valid for "cross-ratio structures" (which includes hyperbolic and CR structures) as defined in [4]. It turns out to be a coordinate description of the decorated triangulations described bellow and the invariant in $\mathcal{P}(k)$ coincides with the one defined before up to a multiple of four.

We consider in this paper a geometric framework which includes both hyperbolic structures and CR structures on manifolds. We are in fact dealing with representations of the fundamental group of the manifold in $\mathrm{PGL}(3,\mathbb{C})$ that are parabolic : the peripheral holonomy should preserve a flag in $\mathbb{C}^3$. Recall that a flag in $\mathbb{C}^3$ is a line in a plane of $\mathbb{C}^3$. The consideration of these representations links us to the work of Fock and Goncharov [7, 8]. Indeed we make an intensive use of their combinatorics on the space of representations of surface groups in $\mathrm{SL}(3,\mathbb{R})$.

As in the original work of Thurston and Neumann-Zagier, we work with decorated triangulations. Namely, let T be a triangulation of a 3-manifold M. To each tetrahedron we associate a quadruple of flags (corresponding to the four vertices) in $\mathbb{C}^3$. In the case of ideal triangulations, where the manifold M is obtained from the triangulation by deleting the vertices, we impose that the holonomy around each vertex preserves the flag decorating this vertex. Such a decorated triangulation gives a set of flag coordinates, and more precisely two sets: affine flag coordinates a and projective flag coordinates z. Those are, in the Fock and Goncharov setting, the a- and z-coordinates on the boundary of each tetrahedron, namely a four-holed sphere.

The main result in this paper is the construction of an element $\beta \in \mathcal{P}(\mathbb{C})$ associated to a decorated triangulation and a description of a precise formula for $\delta(\beta)$ in terms of boundary data. This formula is given in theorem 5.14.

The core of the proof of theorem 5.14 goes along the same lines of the homological proof of Neumann [11]. We nevertheless believe that the use of the combinatorics of Fock and Goncharov sheds some light on Neumann's work. The two theories fit well together, allowing a new understanding, in particular, of the "Neumann-Zagier" symplectic form.

The organisation of the paper is as follows. In section 2 we describe flags and configurations of flags. Following [7], we define a- and z-coordinates for configurations of flags. These data define a decorated tetrahedron. In section 3 we define an element in the pre-Bloch group associated to a decorated tetrahedron (cf. also [6]). We then define the volume of a decorated tetrahedron and show how previous definitions in hyperbolic and CR geometry are included in this context. We moreover relate our work to Suslin's work on K_3, showing that our volume map is essentially Suslin map from $H_3(\mathrm{SL}(3))$ to the Bloch group. This gives a geometric and intuitive construction of the latter. Here we are very close to the work of Zickert on the extended Bloch group [17]. In the next section 4 we associate to a decorated tetrahedron T the element $\delta(\beta(T))$ and compute it using both a-coodinates and z-coordinates.

This local work being done, we move on in section 5 to the framework of decorated simplicial complexes. The decoration consists of a-coordinates or z-coordinates associated to each tetrahedron and satisfying appropriate compatibility conditions along edges and faces. The main result is the computation of $W = \delta(\beta(M))$ which turns out to depend only on boundary data (Theorem 5.14).

We first give a proof of Theorem 5.14 when the decoration is unipotent. We then deal with the proof of the general case. In doing so we have to develop a generalization of the Neumann-Zagier bilinear relations to the $\mathrm{PGL}(3,\mathbb{C})$ case. In doing so the Goldman-Weil-Petersson form for tori naturally arises. We hope that our proof sheds some light on the classical $\mathrm{PGL}(2,\mathbb{C})$ case.

In section 10 we describe all unipotent decorations on the complement of the figure eight knot. It was proven by P.-V. Koseleff that there are a finite number of unipotent structures and all of them are either hyperbolic or CR. The natural question of the rigidity of unitotent representation will be investigated in a forthcoming paper [1] (see also [9]).

Finally in section 11, we describe applications of theorem 5.14. First, we follow again Neumann-Zagier and obtain an explicit formula for the variation of the volume function which depends on boundary data. Then, relying on remarks of Fock and Goncharov, we describe a 2-form on the space of representations of the boundary of our variety which coincides with Weil-Petersson form in some cases (namely for hyperbolic structures and unipotent decorations).

We thank J. Genzmer, P.-V. Koseleff and Q. Wang for fruitful discussions.

2. Configurations of flags and cross-ratios

We consider in this section the flag variety $\mathcal{F}l$ and the affine flag variety $\mathcal{AF}l$ of SL(3) over a field k. We define coordinates on the configurations of 4 flags (or affine flags), very similar to the coordinates used by Fock and Goncharov [7].

2.1. Flags, affine flags and their spaces of configuration.

We set up here notations for our objects of interest. Let k be a field and $V = k^3$. A flag in V is usually seen as a line and a plane, the line belonging to the plane. We give, for commodity reasons, the following alternative description using the dual vector space V^* and the projective spaces $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$:

We define the spaces of *affine flags* $\mathcal{AF}l(k)$ and *flags* $\mathcal{F}l(k)$ by the following:

$$\begin{aligned} \mathcal{AF}l(k) &= \{(x,f) \in (V \setminus \{0\}) \times (V^* \setminus \{0\}) \text{ such that } f(x)=0\} \\ \mathcal{F}l(k) &= \{([x],[f]) \in \mathbb{P}(V) \times \mathbb{P}(V^*) \text{ such that } f(x)=0\}. \end{aligned}$$

The space of flags $\mathcal{F}l(k)$ is identified with the homogeneous space $\mathrm{PGL}(3,k)/B$, where B is the Borel subgroup of upper-triangular matrices in $\mathrm{PGL}(3,k)$. Similarly, the space of affine flags $\mathcal{AF}l(k)$ is identified with the homogeneous space $\mathrm{SL}(3,k)/U$, where U is the subgroup of unipotent upper-triangular matrices in $\mathrm{SL}(3,k)$.

2.2. Given a G-space X, we classicaly define the configuration module of ordered points in X as follows. For $n \geq 0$, let $C_n(X)$ be the free abelian group generated by the set

$$(p_0, \cdots, p_n) \in X^{n+1}$$

of all ordered $(n+1)$ set of points in X. The group G acts on X and therefore also acts diagonally on $C_n(X)$ giving it a left G-module structure.

We define the differential $d_n : C_n(X) \to C_{n-1}(X)$ by

$$d_n(p_0, \ldots, p_n) = \sum_{i=0}^{n} (-1)^i (p_0, \ldots, \hat{p}_i, \ldots, p_n),$$

then we can check that every d_n is a G-module homomorphism and $d_n \circ d_{n+1} = 0$. Hence we have the G-complex

$$C_\bullet(X) : \cdots \to C_n(X) \to C_{n-1}(X) \to \cdots \to C_0(X).$$

The augmentation map $\epsilon : C_0(X) \to \mathbb{Z}$ is defined on generators by $\epsilon(p) = 1$ for each $p \in X$. If X is infinite, the augmentation complex is exact.

For a left G-module M, we denote M_G its group of co-invariants, that is,

$$M_G = M / \langle gm - m, \forall g \in G, m \in M \rangle.$$

Taking the co-invariants of the complex $C_\bullet(X)$, we get the induced complex:

$$C_\bullet(X)_G : \cdots \to C_n(X)_G \to C_{n-1}(X)_G \to \cdots \to C_0(X)_G,$$

with differential $\bar{d}_n : C_n(X)_G \to C_{n-1}(X)_G$ induced by d_n. We call $H_\bullet(X)$ the homology of this complex.

2.3. We let now $G = \mathrm{PGL}(3, k)$ and $X = \mathcal{F}l$. For every integer $n \geq 0$, the $\mathbb{Z}$-module of *coinvariant configurations of* $n+1$ *ordered flags* is defined by:

$$\mathcal{C}_\bullet(\mathcal{F}l) = C_\bullet(\mathcal{F}l)_G.$$

The natural projection $\pi : \mathrm{SL}(3) \to \mathrm{PGL}(3) \to \mathrm{PGL}(3)/B = \mathcal{F}l$ gives a map

$$\pi_* : H_3(\mathrm{SL}(3)) \to H_3(\mathcal{F}l).$$

We will study in this paper the homology groups $H_3(\mathrm{SL}(3, k))$ (which is the third group of discrete homology of $\mathrm{SL}(3, k)$), $H_3(\mathcal{AF}l)$ and $H_3(\mathcal{F}l)$.

It is usefull to consider a subcomplex of $\mathcal{C}_\bullet(\mathcal{F}l)$ of generic configurations which contains all the information about its homology. We leave to the reader the verification that indeed the definition below gives rise to subcomplexes of $C_3(\mathcal{F}l)$ and $\mathcal{C}_3(\mathcal{F}l)$

2.4. A *generic configuration of flags* $([x_i],[f_i])$, $1 \leq i \leq n+1$ is given by $n+1$ points $[x_i]$ in general position and $n+1$ lines Ker f_i in $\mathbb{P}(V)$ such that $f_j(x_i) \neq 0$ if $i \neq j$. We will denote $C_n^r(\mathcal{F}l) \subset C_n(\mathcal{F}l)$ and $\mathcal{C}_n^r(\mathcal{F}l) \subset \mathcal{C}_n(\mathcal{F}l)$ the corresponding module of configurations and its coinvariant module by the diagonal action by $\mathrm{SL}(3)$.

A configuration of ordered points in $\mathbb{P}(V)$ is said to be in *general position* when they are all distinct and no three points are contained in the same line. Observe that the genericity condition of flags does not imply that the lines are in a general position.

2.5. Since G acts transitively on $C_1^r\mathcal{F}l$, we see that $C_n^r(\mathcal{F}l)_G = \mathbb{Z}$ if $n \leq 1$, and the differential $\bar{d}_1 : C_1^r(\mathcal{F}l)_G \to C_0^r(\mathcal{F}l)_G$ is zero.

In order to describe $\mathcal{C}_2^r(\mathcal{F}l)$ consider a configuration of 3 generic flags $([x_i],[f_i])_{1\leq i\leq 3} \in \mathcal{C}_2^r(\mathcal{F}l)$. One can then define a projective coordinate system of $\mathbb{P}(\mathbb{C}^3)$: take the one where the point x_1 has coordinates $[1:0:0]^t$, the point x_2 has coordinates $[0:0:1]^t$, the point x_3 has coordinates $[1:-1:1]^t$ and the intersection of Kerf_1 and Kerf_2 has coordinates $[0:1:0]^t$. The line Kerf_3 then has coordinates $[z:z{+}1:1]$ where

$$z = \frac{f_1(x_2)f_2(x_3)f_3(x_1)}{f_1(x_3)f_2(x_1)f_3(x_2)} \in k^\times$$

is the *triple ratio*. We have $\mathcal{C}_2^r(\mathcal{F}l) = \mathbb{Z}[k^\times]$. Moreover the differential $\bar{d}_2 : C_2^r(\mathcal{F}l)_G \to C_1^r(\mathcal{F}l)_G$ is given on generators $z \in k^\times$ by $\bar{d}_2(z) = 1$ and therefore $H_1(\mathcal{F}l) = 0$.

We denote by z_{123} the triple ratio of a cyclically oriented triple of flags $([x_i],[f_i])_{i=1,2,3}$. Note that $z_{213} = 1/z_{123}$. Observe that when $z_{123} = -1$ the three lines are not in general position.

2.6. Coordinates for a tetrahedron of flags. We call a generic configuration of 4 flags a *tetrahedron of flags*. The coordinates we use for a tetrahedron of flags are the same as those used by Fock and Goncharov [7] to describe a flip in a triangulation. We may see it as a blow-up of the flip into a tetrahedron. They also coincide with coordinates used in [4] to describe a *cross-ratio structure* on a tetrahedron (see also section 3.8).

Let $([x_i],[f_i])_{1\leq i\leq 4}$ be an element of $\mathcal{C}_3(\mathcal{F}l)$. Let us dispose symbolically these flags on a tetrahedron 1234 (see figure 1). We define a set of 12 coordinates on the edges of the thetrahedron (1 for each oriented edge) and four coordinates associated to the faces.

To define the coordinate z_{ij} associated to the edge ij, we first define k and l such that the permutation $(1,2,3,4) \mapsto (i,j,k,l)$ is even. The pencil of (projective) lines through the point x_i is a projective line $\mathbb{P}_1(k)$. We naturally have four points in this projective line: the line $\ker(f_i)$ and the three lines through x_i and one of the x_l for $l \neq i$. We

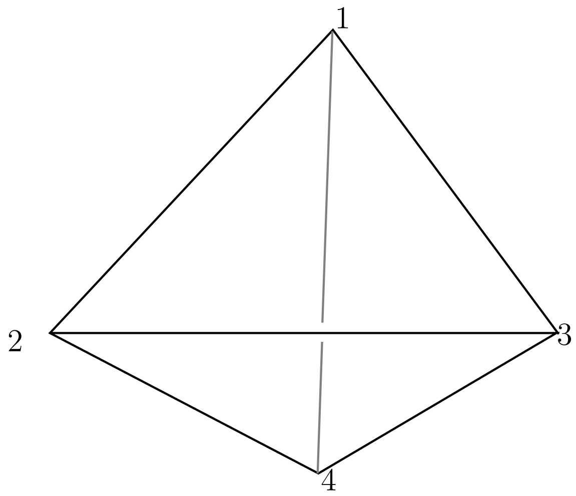

FIGURE 1. An ordered tetrahedron

define z_{ij} as the cross-ratio[1] of these four points:

$$z_{ij} := [\ker(f_i), (x_i x_j), (x_i x_k), (x_i x_l)].$$

We may rewrite this cross-ratio thanks to the following useful lemma.

2.7. Lemma. *We have* $z_{ij} = \frac{f_i(x_k)\det(x_i,x_j,x_l)}{f_i(x_l)\det(x_i,x_j,x_k)}$. *Here the determinant is w.r.t. the canonical basis on* V.

Proof. Consider the following figure:

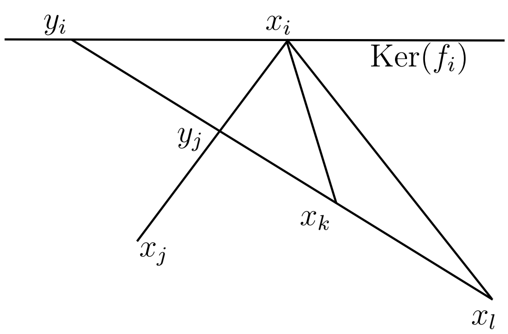

FIGURE 2. Cross-ratio

By duality, z_{ij} is the cross-ratio between the points y_i, y_j and x_k, x_l on the line $(x_k x_l)$. Now, f_i is a linear form vanishing at y_i and $\det(x_i, x_j, \cdot)$ is a linear form vanishing at y_j. Hence, on the line $(x_k x_l)$, the linear

[1]Note that we follow the usual convention (different from the one used by Fock and Goncharov) that the cross-ratio of four points x_1, x_2, x_3, x_4 on a line is the value at x_4 of a projective coordinate taking value ∞ at x_1, 0 at x_2, and 1 at x_3. So we employ the formula $\frac{(x_1-x_3)(x_2-x_4)}{(x_1-x_4)(x_2-x_3)}$ for the cross-ratio.

form $f_i(x)$ is proportional to $(\cdot - y_i)$ and $\det(x_i, x_j, \cdot)$ is proportional to $(\cdot - y_j)$. This proves the formula. □

2.8. Each face (ijk) inherits a canonical orientation as the boundary of the tetrahedron (1234). Hence to the face (ijk), we associate the 3-ratio of the corresponding cyclically oriented triple of flags:

$$z_{ijk} = \frac{f_i(x_j)f_j(x_k)f_k(x_i)}{f_i(x_k)f_j(x_i)f_k(x_j)}.$$

Observe that if the same face (ikj) (with opposite orientation) is common to a second tetrahedron then

$$z_{ikj} = \frac{1}{z_{ijk}}.$$

Figure 3 displays the coordinates.

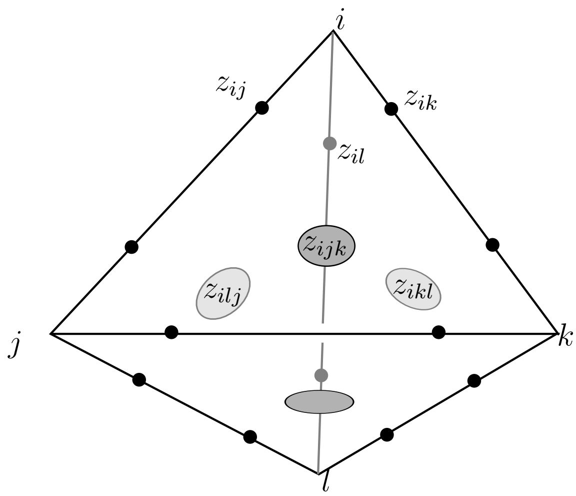

FIGURE 3. The z-coordinates for a tetrahedron

2.9. Of course there are relations between the whole set of coordinates. Fix an even permutation (i, j, k, l) of $(1, 2, 3, 4)$. First, for each face (ijk), the 3-ratio is the opposite of the product of all cross-ratios "leaving" this face:

$$z_{ijk} = -z_{il}z_{jl}z_{kl}. \tag{2.9.1}$$

Second, the three cross-ratio leaving a vertex are algebraically related:

$$\begin{aligned} z_{ik} &= \frac{1}{1 - z_{ij}} \\ z_{il} &= 1 - \frac{1}{z_{ij}} \end{aligned} \tag{2.9.2}$$

Relations 2.9.2 are directly deduced from the definition of the coordinates z_{ij}, while relation 2.9.1 is a direct consequence of lemma 2.7.

At this point, we choose four coordinates, one for each vertex: z_{12}, z_{21}, z_{34}, z_{43}. The next proposition shows that a tetrahedron is uniquely determined by these four numbers, up to the action of PGL(3). It also shows that the space of *cross-ratio structures* on a tetrahedron defined in [4] coincides with the space of generic tetrahedra as defined above.

2.10. Proposition. *A tetrahedron of flags is parametrized by the* 4*-tuple* $(z_{12}, z_{21}, z_{34}, z_{43})$ *of elements in* $k \setminus \{0, 1\}$.

Proof. Let e_1, e_2, e_3 be the canonical basis of V and (e_1^*, e_2^*, e_3^*) its dual basis. Up to the action of SL(3), an element $([x_i], [f_i])$ of $\mathcal{C}_3^r(\mathcal{F}l)$ is uniquely given, in these coordinates, by:

- $x_1 = (1, 0, 0)$, $f_1 = (0, z_1, -1)$,
- $x_2 = (0, 1, 0)$, $f_2 = (z_2, 0, -1)$,
- $x_3 = (0, 0, 1)$, $f_3 = (z_3, -1, 0)$ and
- $x_4 = (1, 1, 1)$, $f_4 = z_4(1, -1, 0) + (0, 1, -1)$.

Observe that $z_i \neq 0$ and $z_i \neq 1$ by the genericity condition. Now we compute, using lemma 2.7 for instance, that $z_{12} = \frac{1}{1-z_1}$, $z_{21} = 1 - z_2$, $z_{34} = z_3$, $z_{43} = 1 - z_4$, completing the proof. ☐

We note that one can then compute $\bar{d}_3 : C_3^r(\mathcal{F}l)_G \to C_2^r(\mathcal{F}l)_G$ on the generators of $C_3^r(\mathcal{F}l)_G$ to be

$$\bar{d}_3(z_{12}, z_{21}, z_{34}, z_{43}) = [z_{123}] - [z_{124}] + [z_{134}] - [z_{234}].$$

2.11. Coordinates for affine flags. We will also need coordinates for a tetrahedron of affine flags (the $\mathcal{A}$-coordinates in Fock and Goncharov [7]). Let $(x_i, f_i)_{1 \leq i \leq 4}$ be an element of $C_3(\mathcal{A}\mathcal{F}l)$. We also define a set of 12 coordinates on the edges of the thetrahedron (one for each oriented edge) and four coordinates associated to the faces:

We associate to the edge ij the number $a_{ij} = f_i(x_j)$ and to the face ijk (oriented as the boundary of the tetrahedron) the number $a_{ijk} = \det(x_i, x_j, x_k)$.

We remark that for a tetrahedron of affine flags, the z-coordinates are well-defined, and are ratios of the affine coordinates:

$$z_{ij} = \frac{a_{ik}a_{ijl}}{a_{il}a_{ijk}} \quad \text{and} \quad z_{ijk} = \frac{a_{ij}a_{jk}a_{ki}}{a_{ik}a_{ji}a_{kj}}. \tag{2.11.1}$$

3. Tetrahedra of flags and volume

In this section we define the *volume* of a tetrahedron of flags, generalizing and unifying the volume of hyperbolic tetrahedra (see section 3.7) and CR tetrahedra (see [4] and section 3.8). Via proposition 2.10, it coincides with the volume function on cross-ratio structures on a tetrahedron as defined in [4]. We then define the volume of a simplicial complex of flags tetrahedra. This volume is invariant under a change of triangulation of the simplicial complex (2-3 move) hence is naturally an element of the pre-Bloch group and the volume is defined on the third homology group of flag configurations (see also [6]). Eventually we get a map, still called volume, from the third (discrete) homology group of SL(3) to the Bloch group, through the natural projection from $H_3(\mathrm{SL}(3))$ to $H_3(\mathcal{F}l)$. We conclude the section with the proof that this last map actually coincides with the Suslin map from $H_3(\mathrm{SL}(3))$ to the Bloch group.

3.1. The pre-Bloch and Bloch groups, the dilogarithm. We define a volume for a tetrahedron of flags by constructing an element of the pre-Bloch group and then taking the dilogarithm map.

The *pre-Bloch group* $\mathcal{P}(k)$ is the quotient of the free abelian group $\mathbb{Z}[k \setminus \{0,1\}]$ by the subgroup generated by the 5-term relations

$$[x]-[y]+[\frac{y}{x}]-[\frac{1-x^{-1}}{1-y^{-1}}]+[\frac{1-x}{1-y}], \quad \forall\, x,y \in k\setminus\{0,1\}. \tag{3.1.1}$$

For a tetrahedron of flags T, let $z_{ij}=z_{ij}(T)$ and $z_{ijk}=z_{ijk}(T)$ be its coordinates.

3.2. To each tetrahedron define the element

$$\beta(T)=[z_{12}]+[z_{21}]+[z_{34}]+[z_{43}] \in \mathcal{P}(\mathbb{C})$$

and extend it – by linearity – to a function

$$\beta\ :\ \mathcal{C}_3(\mathcal{F}l(k)) \to \mathcal{P}(\mathbb{C}).$$

We emphasize here that $\beta(T)$ depends on the ordering of the vertices of each tetrahedron[2] T. The following proposition implies that β is well defined on $H_3(\mathcal{F}l)$.

3.3. Proposition. *We have:* $\beta(\overline{d}_4(\mathcal{C}_4(\mathcal{F}l)))=0$.

Proof. We have to show that $\mathrm{Im}(\overline{d}_4)$ is contained in the subgroup generated by the 5-term relations. This is proven by computation and is exactly the content of [4, Theorem 5.2]. □

3.4. We use wedge $\wedge_{\mathbb{Z}}$ for skew symmetric product on Abelian groups. Consider $k^\times \wedge_{\mathbb{Z}} k^\times$, where $k^\times$ is the multiplicative group of k. It is the abelian group generated by the set $x \wedge_{\mathbb{Z}} y$ factored by the relations

$$xy \wedge_{\mathbb{Z}} z = x \wedge_{\mathbb{Z}} z + y \wedge_{\mathbb{Z}} z \text{ and } x \wedge_{\mathbb{Z}} y = -y \wedge_{\mathbb{Z}} x.$$

In particular, $1 \wedge_{\mathbb{Z}} x = 0$ for any $x \in k^\times$, and

$$x^n \wedge_{\mathbb{Z}} y = n(x \wedge_{\mathbb{Z}} y) = x \wedge_{\mathbb{Z}} y^n.$$

[2]This assumption may be removed by averaging β over all orderings of the vertices. In any case if c is a chain in $C_3(\mathcal{F}l(k))$ representing a cycle in $\mathcal{C}_3(\mathcal{F}l(k))$ we can represent c by a closed 3-cycle K together with a numbering of the vertices of each tetrahedron of K (see section 5.5).

3.5. The *Bloch group* $\mathcal{B}(k)$ is the kernel of the homomorphism

$$\delta : \mathcal{P}(k) \to k^\times \wedge_{\mathbb{Z}} k^\times,$$

which is defined on generators of $\mathcal{P}(k)$ by $\delta([z]) = z \wedge (1-z)$.

The *Bloch-Wigner dilogarithm* function is

$$\begin{aligned} D(x) &= \arg(1-x)\log|x| - \mathrm{Im}\left(\int_0^x \log(1-t)\frac{dt}{t}\right), \\ &= \arg(1-x)\log|x| + \mathrm{Im}(\ln_2(x)). \end{aligned}$$

Here $\ln_2(x) = \int_0^x \log(1-t)\frac{dt}{t}$ is the dilogarithm function. The function D is well-defined and real analytic on $\mathbb{C} - \{0,1\}$ and extends to a continuous function on $\mathbb{C}P^1$ by defining $D(0) = D(1) = D(\infty) = 0$. It satisfies the 5-term relation and therefore, for k a subfield of $\mathbb{C}$, gives rise to a well-defined map:

$$D : \mathcal{P}(k) \to \mathbb{R},$$

given by linear extension as

$$D\left(\sum_{i=1}^k n_i[x_i]\right) = \sum_{i=1}^k n_i D(x_i).$$

3.6. We finally define the *volume map* on $\mathcal{C}_3(\mathcal{F}l)$ via the dilogarithm (the constant will be explained in the next section):

$$\mathrm{Vol} = \frac{1}{4} D \circ \beta \ : \ \mathcal{C}_3(\mathcal{F}l(k)) \to \mathbb{C}.$$

From Proposition 3.3, Vol is well defined on $H_3(\mathcal{F}l)$.

3.7. The hyperbolic case. We briefly explain here how the hyperbolic volume for ideal tetrahedra in the hyperbolic space $\mathbb{H}^3$ fits into the framework described above.

An ideal hyperbolic tetrahedron is given by 4 points on the boundary of $\mathbb{H}^3$, i.e. $\mathbb{P}_1(\mathbb{C})$. Up to the action of $\mathrm{SL}(2,\mathbb{C})$, these points are in homogeneous coordinates $[0,1]$, $[1,0]$, $[1,1]$ and $[1,t]$ – the complex number t being the cross-ratio of these four points. So its volume is $D(t)$ (see e.g. [16]).

Identifying $\mathbb{C}^3$ with the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, we have the adjoint action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}^3$ preserving the quatratic form given by the determinant, given in canonical coordinates by $xz - y^2$. The group $\mathrm{SL}(2,\mathbb{C})$ preserves the isotropic cone of this form. The projectivization of this cone is identified to $\mathbb{P}_1(\mathbb{C})$ via the Veronese map (in canonical coordinates):

$$\begin{array}{rccc} h_1 : & \mathbb{P}_1(\mathbb{C}) & \to & \mathbb{P}_2(\mathbb{C}) \\ & [x,y] & \mapsto & [x^2, xy, y^2] \end{array}$$

The first jet of that map gives a map h from $\mathbb{P}_1(\mathbb{C})$ to the variety of flags $\mathcal{F}l$. A convenient description of that map is obtained thanks to the identification between $\mathbb{C}^3$ and its dual given by the quadratic form. Denote $\langle , \rangle$ the bilinear form associated to the determinant. Then we have

$$\begin{array}{rcl} h \,:\, \mathbb{P}_1(\mathbb{C}) & \to & \mathcal{F}l(\mathbb{C}) \\ p & \mapsto & (h_1(p), \langle h_1(p), \cdot \rangle)\,. \end{array}$$

Let T be the tetrahedron $h([0,1])$, $h([1,0])$, $h([1,1])$ and $h([1,t])$. An easy computation gives its coordinates:

$$z_{12}(T) = t \qquad z_{21}(T) = t \qquad z_{34}(T) = t \qquad z_{43}(T) = t.$$

It implies that $\beta(T) = 4t$ and our function Vol coincide with the hyperbolic volume:

$$\mathrm{Vol}(T) = D(t).$$

Remark. Define an involution σ on the z-coordinates by:

$$\sigma(z_{ijk}) = \frac{1}{z_{ijk}}$$

on the faces and

$$\sigma(z_{ij}) = \frac{z_{ji}(1+z_{ilj})}{z_{ilj}(1+z_{ijk})} \text{ and } \sigma(z_{ji}) = \frac{z_{ij}(1+z_{ijk})}{z_{ijk}(1+z_{ilj})}$$

on edges. The set of fixed points of σ correspond exactly with the hyperbolic tetrahedra.

3.8. The CR case. CR geometry is modeled on the sphere $\mathbb{S}^3$ equipped with a natural $\mathrm{PU}(2,1)$ action. More precisely, consider the group $\mathrm{U}(2,1)$ preserving the Hermitian form $\langle z, w \rangle = w^* J z$ defined on $\mathbb{C}^3$ by the matrix

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and the following cones in $\mathbb{C}^3$;

$$V_0 = \left\{ z \in \mathbb{C}^3 - \{0\} \;\; : \;\; \langle z, z \rangle = 0 \;\right\},$$
$$V_- = \left\{ z \in \mathbb{C}^3 \;\; : \;\; \langle z, z \rangle < 0 \;\right\}.$$

Let $\pi : \mathbb{C}^3 \setminus \{0\} \to \mathbb{CP}^2$ be the canonical projection. Then $\mathbb{H}^2_{\mathbb{C}} = \pi(V_-)$ is the complex hyperbolic space and its boundary is

$$\partial \mathbb{H}^2_{\mathbb{C}} = \mathbb{S}^3 = \pi(V_0) = \{[x,y,z] \in \mathbb{CP}^2 \mid x\bar{z} + |y|^2 + z\bar{x} = 0 \;\}.$$

The group of biholomorphic transformations of $\mathbb{H}^2_{\mathbb{C}}$ is then $\mathrm{PU}(2,1)$, the projectivization of $\mathrm{U}(2,1)$. It acts on $\mathbb{S}^3$ by CR transformations.

An element $x \in \mathbb{S}^3$ gives rise to an element $([x],[f]) \in \mathcal{F}l(\mathbb{C})$ where $[f]$ corresponds to the unique complex line tangent to $\mathbb{S}^3$ at x. As in the hyperbolic case we may consider the inclusion map

$$h_1 \,:\, \mathbb{S}^3 \;\to\; \mathbb{P}_2(\mathbb{C})$$

and the first complex jet of that map gives a map

$$\begin{array}{rcl} h\,:\,\mathbb{S}^3 & \rightarrow & \mathcal{F}l(\mathbb{C}) \\ x & \mapsto & (h_1(x), \langle ., h_1(x)\rangle) \end{array}$$

A generic configuration of four points in $\mathbb{S}^3$ is given, up to $\mathrm{PU}(2,1)$, by the following four elements in homogeneous coordinates (we also give for each point x_i, the corresponding dual f_i to the complex line containing it and tangent to $\mathbb{S}^3$):

- $x_1 = (1,0,0)$, $f_1 = (0,0,1)$,
- $x_2 = (0,0,1)$, $f_2 = (1,0,0)$,
- $x_3 = (\frac{-1+it}{2}, 1, 1)$, $f_3 = (1, 1, \frac{-1-it}{2})$ and
- $x_4 = (\frac{|z|^2(-1+is)}{2}, z, 1)$, $f_4 = (1, \bar{z}, \frac{-|z|^2(1+is)}{2})$

with $z \neq 0, 1$ and $\overline{z}\frac{s+i}{t+i} \neq 1$. Observe that $\mathrm{PU}(2,1)$ acts doubly transitively on $\mathbb{S}^3$ and for a generic triple of points x_1, x_2, x_3 the triple ratio of the corresponding flags is given by $\frac{1-it}{1+it}$. One can easily compute the invariants of the tetrahedron:

$$z_{12} = z,\ z_{21} = \frac{\overline{z}(s+i)}{t+i},\ z_{34} = \frac{z[(t+i) - \overline{z}(s+i)]}{(z-1)(t-i)},\ z_{43} = \frac{\overline{z}(z-1)(s-i)}{(t+i) - \overline{z}(s+i)}.$$

The following proposition describes the space of generic configurations of four points in $\mathbb{S}^3$.

3.9. Proposition. *Configurations (up to translations by* $\mathrm{PU}(2,1)$*) of four generic points in* $\mathbb{S}^3$ *are parametrised by elements in* $\mathcal{C}_3^r(\mathcal{F}l)$ *with coordinates* z_{ij}*,* $1 \leq i \neq j \leq 4$ *satisfying the three complex equations*

$$z_{ij}z_{ji} = \overline{z_{kl}z_{lk}} \tag{3.9.1}$$

with the exclusion of solutions such that $z_{ij}z_{ji}z_{ik}z_{ki}z_{il}z_{li} = -1$ *and* $z_{ij}z_{ji} \in \mathbb{R}$.

As in [4] (up to multiplication by 4) the volume of a CR tetrahedron T_{CR} is $\mathrm{Vol}(T_{CR}) = \frac{1}{4}D \circ \beta(T_{CR})$.

3.10. Relations with the work of Suslin. We show here how our map β allows a new and more geometric way to interpret Suslin map $S : H_3(\mathrm{SL}(3)) \to \mathcal{P}$ (see [14]). First of all, recall that the natural projection $\pi : \mathrm{SL}(3) \to \mathcal{F}l = \mathrm{PGL}(3)/B$ gives a map $\pi_* : H_3(\mathrm{SL}(3)) \to H_3(\mathcal{F}l)$.

3.11. Theorem. *We have* $\beta \circ \pi_* = 4S$.

Proof. Let T be the subgroup of diagonal matrices (in the canonical basis) of SL(3). Recall that SL(2) is seen as a subgroup of SL(3) via the adjoint representation (as in section 3.7). We find in the work of Suslin the three following results:

1. $H_3(\mathrm{SL}(3)) = H_3(\mathrm{SL}(2)) + H_3(T)$ [14, p. 227]
2. S vanishes on $H_3(T)$ [14, p. 227]
3. S coincide with the cross-ratio on $H_3(\mathrm{SL}(2))$ [14, lemma 3.4].

So we just have to understand the map $\beta \circ \pi_*$ on T and SL(2). As T is a subgroup of B, the map $\beta \circ \pi_*$ vanishes on T. And we have seen in the section 3.7 that, on a hyperbolic tetrahedron, β coincide with 4 times the cross-ratio.

This proves the theorem. □

Remark. After writing this section we became aware of Zickert's paper [17]. In it (see §7.1) Zickert defines a generalization – denoted $\widehat{\lambda}$ – of Suslin's map. When specialized to our case his definition coincides with $\frac{1}{4}\beta \circ \pi^*$. We believe that the construction above sheds some light on the "naturality" of this map.

4. Decoration of a tetrahedron and the pre-Bloch group

In this section we let T be an ordered tetrahedron of flags and compute in two different ways $\delta(\beta(T))$. The first – and most natural – way uses a-coordinates associated to some lifting of T as a tetrahedron of affine flags. In that respect we mainly follow Fock and Goncharov. The second way directly deals with z-coordinates and follows the approach of Neumann and Zagier. Finally we explain how the two ways are related; we will see in the remaining of the paper how fruitful it is to mix them.

4.1. Affine decorations and the pre-Bloch group. We first let $(x_i, f_i)_{1\leq i\leq 4}$ be an element of $\mathcal{C}_3(\mathcal{AF}l)$ lifting T. This allows us to associate a-coordinates to T.

Let $J_T^2 = \mathbb{Z}^I$ be the 16-dimensional abstract free $\mathbb{Z}$-module where (see figure 4)

$$I = \{\text{vertices of the (red) arrows in the 2-triangulation of the faces of } T\}.$$

We denote the canonical basis $\{e_\alpha\}_{\alpha\in I}$ of J_T^2. It contains *oriented edges* e_{ij} (edges oriented from j to i) and *faces* e_{ijk}. Given α and β in I we set:

$$\varepsilon_{\alpha\beta} = \#\{\text{oriented (red) arrows from } \alpha \text{ to } \beta\} - \#\{\text{oriented (red) arrows from } \beta \text{ to } \alpha\}.$$

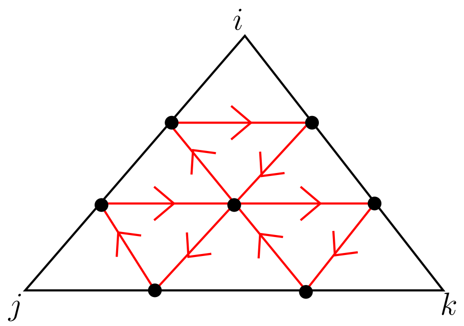

FIGURE 4. Combinatorics of W

4.2. The a-coordinates $\{a_\alpha\}_{\alpha\in I}$ of our tetrahedron of affine flags T now define an element $\sum_{\alpha\in I} a_\alpha e_\alpha$ of $k^\times \otimes_{\mathbb{Z}} J_T^2 \cong \mathrm{Hom}((J_T^2)^*, k^\times)$ where k is any field which contains all the a-coordinates.

Let V be a $\mathbb{Z}$-module equipped with a bilinear product

$$B : V \times V \to \mathbb{Z}.$$

We consider on the $k^\times$-module $V_k = k^\times \otimes_{\mathbb{Z}} V$ the bilinear product

$$\wedge_B : V_k \times V_k \to k^\times \wedge_{\mathbb{Z}} k^\times$$

defined on generators by

$$(z_1 \otimes v_1) \wedge (z_2 \otimes v_2) = B(v_1, v_2)(z_1 \wedge z_2).$$

In particular letting Ω^2 be the bilinear skew-symmetric form on J_T^2 given by[3]

$$\Omega^2(e_\alpha, e_\beta) = \varepsilon_{\alpha\beta},$$

we get:

$$a \wedge_{\Omega^2} a = \sum_{\alpha,\beta\in I} \varepsilon_{\alpha\beta} a_\alpha \wedge_{\mathbb{Z}} a_\beta.$$

4.3. Lemma. *We have:*

$$\delta(\beta(T)) = \frac{1}{2} a \wedge_{\Omega^2} a. \tag{4.3.1}$$

Proof. To each ordered face (ijk) of T we associate the element

$$W_{ijk} = a_{ijk} \wedge \frac{a_{ki}a_{jk}a_{ij}}{a_{ik}a_{kj}a_{ji}} + a_{ij}\wedge a_{ik} + a_{ki}\wedge a_{kj} + a_{jk}\wedge a_{ji} \in k^\times \wedge_{\mathbb{Z}} k^\times. \tag{4.3.2}$$

The proof in the CR case of [6, Lemma 4.9] obviously leads to[4]:

$$\delta(\beta(T)) = W_{143} + W_{234} + W_{132} + W_{124}.$$

Finally one easily sees that

$$W_{143} + W_{234} + W_{132} + W_{124} = \frac{1}{2} \sum_{\alpha,\beta\in I} \varepsilon_{\alpha\beta} a_\alpha \wedge_{\mathbb{Z}} a_\beta.$$

□

We let

$$W(T) = W_{143} + W_{234} + W_{132} + W_{124}.$$

Remark. 1. The element $W(T)$ coincides with the W invariant associated by Fock and Goncharov to the triangulation by a tetrahedron of a sphere with 4 punctures. (The orientation of the faces being induced by the orientation of the sphere.)

[3] Observe in particular that $\Omega^2(e_{ji}, e_{ijk}) = 1$ and so on, the logic being that the vector e_{ijk} is the outgoing vector on the face ijk and the vector e_{ji} (oriented from i to j) turns around it in the positive sense.

[4] Alternatively we may think of T as a geometric realization of a mutation between two triangulations of the quadrilateral (1324) and apply [7, Corollary 6.15].

2. Whereas T – being a tetrahedron of flags – only depends on the flag coordinates, each W associated to the faces depends on the *affine* flag coordinates.

In the next paragraph we make remark 2 more explicit by computing $\delta(\beta(T))$ using the z-coordinates.

4.4. The Neumann-Zagier symplectic space. In this section we analyse an extension of Neumann-Zagier symplectic space introduced by J. Genzmer [9] in the space of z-coordinates associated to the edges of a tetrahedron. We reinterpret her definitions in our context of flag tetrahedra. Recall that we have associated z-coordinates to a tetrahedron of flags T. These consists of 12 edge coordinates $\{z_{ij}\}$ and 4 face coordinates $\{z_{ijk}\}$ subject to the relations (2.9.1) and (2.9.2). Recall that relation (2.9.1) is $z_{ijk} = -z_{il}z_{jl}z_{kl}$ and note that (2.9.2) implies in particular that:

$$z_{ij}z_{ik}z_{il} = -1. \tag{4.4.1}$$

We linearize (2.9.1) and (4.4.1) in the following way: We let J_T be the $\mathbb{Z}$-module obtained as the quotient of $J_T^2 = \mathbb{Z}^I$ by the kernel of Ω^2. The latter is the subspace generated by elements of the form

$$\sum_{\alpha\in I} b_\alpha e_\alpha$$

for all $\{b_\alpha\} \in \mathbb{Z}^I$ such that $\sum_{\alpha\in I} b_\alpha \varepsilon_{\alpha\beta} = 0$ for every $\beta \in I$. Equivalently it is the subspace generated by $e_{ij} + e_{ik} + e_{il}$ and $e_{ijk} - (e_{il} + e_{jl} + e_{kl})$. We will rather use as generators the elements

$$v_i = e_{ij} + e_{ik} + e_{il} \text{ and } w_i = e_{ji} + e_{ki} + e_{li} + e_{ijk} + e_{ilj} + e_{ikl},$$

see Figure 5.

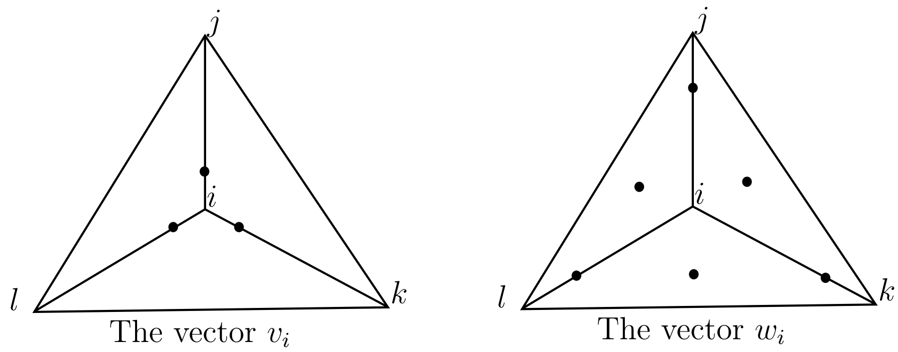

FIGURE 5. The vectors v_i and w_i in $\mathrm{Ker}(p)$

We let $J_T^* \subset (J_T^2)^*$ be the dual subspace which consists of the linear maps in $(J_T^2)^*$ which vanish on the kernel of Ω^2. Note that J_T (as well as J_T^*) is 8-dimensional.

4.5. The z-coordinates $\{z_\alpha\}_{\alpha\in I}$ of our tetrahedron of flags T now define an element

$$z = \sum_{\alpha\in I} z_\alpha e_\alpha^* \in \mathrm{Hom}(J_T^2, k^\times) \cong k^\times \otimes_{\mathbb{Z}} (J_T^2)^*$$

where k is any field which contains the z-coordinates. Because of (2.9.1) and (4.4.1) the image of the kernel of Ω^2 by z is the (torsion) subgroup $\{\pm 1\} \subset k^\times$ (that is easily checked on v_i and w_i). Denoting $V\left[\frac{1}{2}\right]$ the tensor product $V \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{2}\right]$ of a $\mathbb{Z}$-module V, we conclude that the element $z \in k^\times \otimes (J_T^2)^*\left[\frac{1}{2}\right]$ in fact belongs to $k^\times \otimes J_T^*\left[\frac{1}{2}\right]$.

4.6. The space J_T^* is 8-dimensional and we may associate to 8 oriented edges (two pointing at each vertex) of T a basis $\{f_{ij}\}$. Using this basis, the element $z \in k^\times \otimes_{\mathbb{Z}} J_T^*\left[\frac{1}{2}\right]$ is written $z = \sum z_{ij} f_{ij}$.

We then note that (up to eventually adding a root of -1 to k):

$$\begin{aligned}\delta(\beta(T)) &= z_{ij} \wedge_{\mathbb{Z}} (1 - z_{ij}) + z_{ji} \wedge_{\mathbb{Z}} (1 - z_{ji}) \\ &\qquad + z_{kl} \wedge_{\mathbb{Z}} (1 - z_{kl}) + z_{lk} \wedge_{\mathbb{Z}} (1 - z_{lk}) \\ &= \frac{1}{2} z \wedge_{\mathbb{Z}} Hz,\end{aligned} \tag{4.6.1}$$

where H is the linear map $J_T^* \to J_T^*$ which on generators of J_T^* is given by $H(f_{ij}) = f_{ik}$ and $H(f_{ik}) = -f_{ij}$. It yields a linear map $H : k^\times \otimes_{\mathbb{Z}} J_T^* \to k^\times \otimes_{\mathbb{Z}} J_T^*$. We note that in coordinates:

$$(Hz)_{f_{ij}} = \frac{1}{z_{ik}} \quad \text{and} \quad (Hz)_{f_{ik}} = z_{ij}.$$

4.7. The choice of the basis $\{f_{ij}\}$ of J_T^* and the choice of the map H are not canonical but they define a natural symplectic form

$$\Omega^*(\cdot,\cdot) = \langle H\cdot, \cdot\rangle \tag{4.7.1}$$

on J_T^* where $\langle , \rangle$ is the scalar product associated to the basis $\{f_{ij}\}$. Such a symplectic space was first considered by Neumann and Zagier (see [13, 11]) in the $\mathrm{SL}(2,\mathbb{C})$ context.

The following lemma now follows from (4.6.1) and (4.7.1).

4.8. Lemma. *We have:*

$$\delta(\beta(T)) = \frac{1}{2} z \wedge_{\Omega^*} z. \tag{4.8.1}$$

4.9. Relation between a and z-coordinates. Let

$$p : J_T^2 \to (J_T^2)^*$$

be the homomorphism $v \mapsto \Omega^2(v, \cdot)$. On the basis (e_α) and its dual (e_α^*), we can write

$$p(e_\alpha) = \sum_\beta \varepsilon_{\alpha\beta} e_\beta^*.$$

We define accordingly the dual map

$$p^* : \mathrm{Hom}((J_T^2)^*, k^\times) \to \mathrm{Hom}(J_T^2, k^\times).$$

Observe that if $a \in k^\times \otimes_{\mathbb{Z}} J_T^2$ and $z \in k^\times \otimes_{\mathbb{Z}} (J_T^2)^*$ are the elements associated to the a and z-coordinates of T then:

$$p^*(a) = z \quad \text{in } k^\times \otimes J_T^* \left[\frac{1}{2}\right].$$

Indeed,

$$p^*(a)(e_\alpha) = a(p(e_\alpha)) = a(\sum_\beta \epsilon_{\alpha\beta} e_\beta^*) = \prod_\beta a_\beta^{\varepsilon_{\alpha\beta}}.$$

In particular, we recuperate the formula

$$z_{ij} = \frac{a_{ik}a_{ijl}}{a_{il}a_{ijk}}.$$

Note however that in our conventions the coordinate a_{ijl} should be written a_{ilj}. There is therefore a sign missing here and $p^*(a) = z$ only holds modulo 2-torsion.

The image $p(J_T^2) \subset (J_T^2)^*$ coincides with J_T^* and one easily checks that $p^*(\Omega^*) = \Omega^2$. It then follows from the following lemma that

$$a \wedge_{\Omega^2} a = z \wedge_{\Omega^*} z \tag{4.9.1}$$

which explains the coincidence of lemma 4.3 and lemma 4.8.

4.10. Lemma. *If $\phi : V \to W$ is a homomorphism of $\mathbb{Z}$-modules equipped with bilinear forms B and b such that $\phi^*(b) = B$ then the induced map*

$$\phi : V_k \to W_k$$

satisfies

$$\phi^*(\wedge_b) = \wedge_B$$

Proof. This is a simple consequence of the definitions. □

4.11. Note that the form Ω^2 induces a – now non-degenerate – symplectic form Ω on J_T. This yields a canonical identification between J_T and J_T^*; the form Ω^* is the corresponding symplectic form. We may therefore as well work with (J_T, Ω) as with (J_T^*, Ω^*). The bilinear form Ω on J_T is characterized as the non-singular skew-symmetric form given by

$$\Omega(e_\alpha, e_\beta) = \varepsilon_{\alpha\beta}.$$

5. Decoration of a tetrahedra complex and its holonomy

In the previous sections we defined coordinates for a tetrahedron of flags and affine flags and defined its volume in $\mathcal{P}(\mathbb{C})$. We study here how one may decorate a complex of tetrahedra with these coordinates, compute the holonomy of its fondamental group. We also investigate the invariant β (in the pre-Bloch group) of the decorated complex. We eventually state the main theorem of the paper, theorem 5.14, which computes $\delta(\beta)$ in terms of the holonomy.

5.1. Quasi-simplicial complex and its decorations. Let us begin with the definition of a quasi-simplicial complex (see e.g. [11]): A *quasi-simplicial complex* K is a cell complex whose cells are simplices with injective simplicial attaching maps, but no requirement that closed simplices embed in $|K|$ – the underlying topological space. A *tetrahedra complex* is a quasi-simplicial complex of dimension 3.

5.2. From now on we let K be a tetrahedra complex. The *(open) star* of a vertex $v \in K^{(0)}$ is the union of all the open simplices that have v as a vertex. It is an open neighborhood of v and is the open cone on a simplicial complex L_v called the *link* of v.

A *quasi-simplicial* 3*-manifold* is a compact tetrahedra complex K such that $|K| - |K^{(0)}|$ is a 3-manifold (with boundary). By an orientation of K we mean an orientation of this manifold. A 3-*cycle* is a closed quasi-simplicial 3-manifold.

5.3. A quasi-simplicial 3-manifold is topologically a manifold except perhaps for the finitely many singular points $v \in |K^{(0)}|$ where the local structure is that of a cone on $|L_v|$ – a compact connected surface (with boundary). We will soon require that for each vertex $v \in K^{(0)}$, $|L_v|$ is homeomorphic to either a sphere, a torus or an annulus. Let $K_s^{(0)}$, $K_t^{(0)}$ and $K_a^{(0)}$ the corresponding subsets of vertices. We note that $|K| - |K_t^{(0)} \cup K_a^{(0)}|$ is an (open) 3-manifold with boundary that retracts onto a compact 3-manifold with boundary M. Note that ∂M is the disjoint union $T_1 \cup \ldots \cup T_\tau \cup S_1 \cup \ldots \cup S_\sigma$ where each T_i is a torus and each S_i a surface of genus $g_i \geq 2$. Moreover: each T_i corresponds to a vertex in $K_t^{(0)}$ and each S_i contains at least one simple closed essential curve each corresponding to a vertex in $K_a^{(0)}$, see figure 6.

Given such a compact oriented 3-manifold with boundary M. We call a quasi-simplicial 3-manifold as above a *triangulation* of M.

A *decoration* of a tetrahedra complex is an incarnation of this complex in our spaces of flags or affine flags:

5.4. A *parabolic decoration* of the tetrahedra complex is the data of a flag for each vertex (equivalently a map from the 0-skeleton of the complex to $\mathcal{F}l$) such that, for each tetrahedron of the complex, the

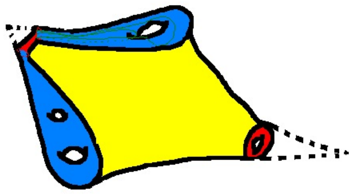

FIGURE 6. The retraction of a quasi-simplicial 3-manifold onto a compact 3-manifold with boundary

corresponding tetrahedron of flags is in generic position. Similarly, a *unipotent decoration* is the data of an affine flag for each vertex with the genericity condition.

Let us make two comments on these definitions. First, any parabolic decoration – together with an ordering of the vertices of each 3-simplex – equip each tetrahedron with a set of coordinates as defined in section 2.6. Second, an unipotent decoration induces a parabolic decoration via the canonical projection $\mathcal{AF}l \to \mathcal{F}l$, so we get these coordinates, as well as a set of affine coordinates (see section 2.11).

5.5. Neumann [12, §4] has proven that any element of $H_3(\mathrm{PGL}_3(\mathbb{C}))$ can be represented by an oriented 3-cycle K together with an ordering of the vertices of each 3-simplex of K so that these orderings agree on common faces, and a decoration of K.

In otherwords: Any class $\alpha \in H_3(\mathrm{PGL}_3(\mathbb{C}))$ can be represented as $f_*[K]$ where K is a quasi-simplicial complex such that $|K| - |K^{(0)}|$ is an oriented 3-manifold, $[K] \in H_3(|K|)$ is its fundamental class and $f : |K| \to \mathrm{BPGL}_3(\mathbb{C})$ is some map.

This motivates the study of decorated 3-cycles. From now on we fix K a decorated oriented quasi-simplicial 3-manifold together with an ordering of the vertices of each 3-simplex of K. Let N be the number of tetrahedra of K and denote by T_ν, $\nu = 1, \dots, N$, these tetrahedra. We let $z_{ij}(T_\nu)$ be the corresponding z-coordinates. We now describe the consistency relation on this coordinate in order to be able to glue back together the decorated tetrahedron

5.6. Consistency relations. (cf. [4]) Let F be an internal face (2-dim cell) of K and T, T' be the tetrahedra attached to F. In order to fix notations, suppose that the vertex of T are $1, 2, 3, 4$ and that the face F is 123. Let $4'$ be the remaining vertex of T'. The face F inherits two 3-ratio from the decoration: first $z_{123}(T)$ as a face of T and second

$z_{132}(T')$ as a face of T'. But considering F to be attached to T or T' only changes its orientation, not the flags at its vertex. So these two 3-ratios are inverse. Hence we get the:

(Face relation) Let T and T' be two tetrahedra of K with a common face (ijk) (oriented as a boundary of T), then $z_{ijk}(T) = \frac{1}{z_{ikj}(T')}$.

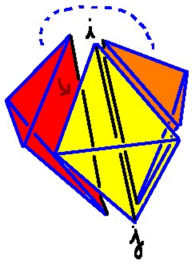

FIGURE 7. tetrahedra sharing a common edge

We should add another compatibility condition to ensure that the edges are not singularities: we are going to compute the holonomy of a path in a decorated complex and we want it to be invariant by crossing the edges. One way to state the condition is the following one: let $T_1, \ldots, T_\nu$ be a sequence of tetrahedra sharing a common edge ij and such that ij is an inner edge of the subcomplex composed by the T_μ's (they are making looping around the edge, see figure 7). Then we ask:

(Edge condition) $z_{ij}(T_1) \cdots z_{ij}(T_\nu) = z_{ji}(T_1) \cdots z_{ji}(T_\nu) = 1$

5.7. Holonomy of a decoration. Recall from §2.5 that, once we have a configuration of 3 generic flags $([x_i], [f_i])_{1\leq i\leq 3} \in \mathcal{C}_2^r(\mathcal{F}l)$ with triple ratio X, one defined a projective coordinate system of $\mathbb{P}(\mathbb{C}^3)$ as the one where the point x_1 has coordinates $[1:0:0]^t$, the point x_2 has coordinates $[0:0:1]^t$, the point x_3 has coordinates $[1:-1:1]^t$ and the intersection of $\mathrm{Ker} f_1$ and $\mathrm{Ker} f_2$ has coordinates $[0:1:0]^t$. The line $\mathrm{Ker} f_3$ then has coordinates $[X : X+1 : 1]$.

Given an oriented face we therefore get 3 projective basis associated to the triples (123), (231) and (312). The cyclic permutation of the flags induces the coordinate change given by the matrix

$$T(X) = \begin{pmatrix} X & X+1 & 1 \\ -X & -X & 0 \\ X & 0 & 0 \end{pmatrix}.$$

Namely: if a point p has coordinates $[u : v : w]^t$ in the basis associated to the triple (123) it has coordinates $T(X)[u : v : w]^t$ in the basis associated to (231).

5.8. Lemma. *If we have a tetrahedron of flags $(ijkl)$ with its z-coordinates, then the coordinate system related to the triple (ijk) is obtained from the coordinate system related to the triple (ijl) by the coordinate change given by the matrix*

$$E(z_{ij}, z_{ji}) = \begin{pmatrix} z_{ji}^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z_{ij} \end{pmatrix}.$$

Beware that the orientation of (ijl) is not the one given by the tetrahedron.

Proof. The matrix we are looking for fixes the flags $([x_1], [f_1])$ and $([x_2], [f_2])$ corresponding to the vertex i and j. In particular it should be diagonal. Finally it should send $[x_4]$ to $[x_3]$. But in the coordinate system associated to the triple (ijk) the point x_4 in the flag $([x_4], [f_4])$ corresponding to the vertex l has coordinates:

$$x_4 = [z_{ji} : -1 : z_{ij}^{-1}]^t.$$

This proves the lemma. ☐

5.9. From this we can explicitly compute the holonomy of a path in the complex. For that let us put three points in each face near the vertices denoting by (ijk) the point in the face ijk near i. As we have said before, each of these points corresponds to a projective basis of $\mathbb{C}^3$. Each path can be deformed so that it decomposes in two types of steps (see figure 8):

(1) a path inside an oriented face ijk from (ijk) to (jki),
(2) a path through a tetrahedron $ijkl$ from (ijk) to (ijl) (i.e. turning left around the edge ij oriented from j to i).

Now the holonomy of the path is the coordinate change matrix so that: in case 1, you have to left multiply by the matrix $T(z_{ijk})$ and in case 2 by the matrix $E(z_{ij}, z_{ji})$.

5.10. In particular the holonomy of the path turning left around an edge, i.e. the path $(ijk) \to (ijl)$, is given by

$$L_{ij} = E(z_{ij}, z_{ji}) = \begin{pmatrix} z_{ji}^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z_{ij} \end{pmatrix}. \tag{5.10.1}$$

As an example which we will use latter on, one may also compute the holonomy of the path turning right around an edge, i.e. the path

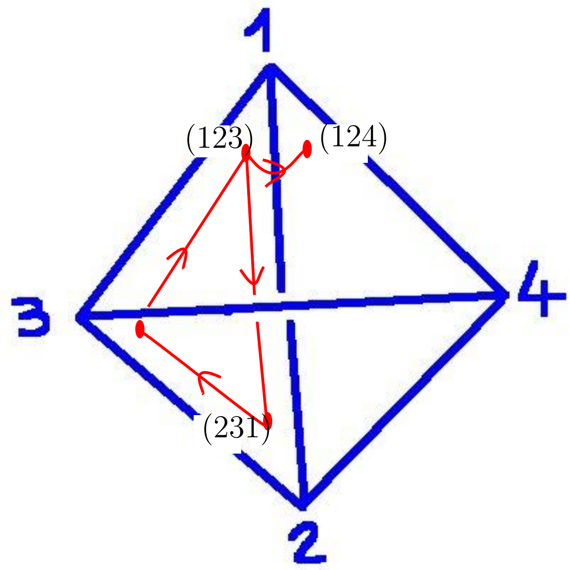

FIGURE 8. Two elementary steps for computing holonomy

$(ilj) \to (ikj)$. We consider the sequence of coordinate changes (see figure 9 for the path going from (231) to (241)):

$$(ilj) \to (lji) \to (jil) \to (jik) \to (ikj).$$

The first two operations are cyclic permutations both given by the matrix $T(z_{ilj})$. It follows from lemma 5.8 that the third is given by the matrix $E(z_{ji}, z_{ij})$. Finally the last operation is again a cyclic permutation given by the matrix $T(z_{ikj})$. The coordinate change from the basis (ilj) to (ikj) is therefore given by

$$T(z_{ikj})E(z_{ji}, z_{ij})T(z_{ilj})^2 = \begin{pmatrix} z_{ji}z_{ilj} & \star & \star \\ & z_{ikj} & \star \\ & & \frac{z_{ikj}}{z_{ij}} \end{pmatrix}$$

Using $z_{ikj} = \frac{1}{z_{ijk}}$, we get that the holonomy matrix, in $\mathrm{PGL}(3,\mathbb{C})$, of the path turning right around an edge ij is

$$R_{ij} = \begin{pmatrix} z_{ji}z_{ilj}z_{ijk} & \star & \star \\ & 1 & \star \\ & & \frac{1}{z_{ij}} \end{pmatrix}. \tag{5.10.2}$$

Remark. Beware that $L_{ij}R_{ij}$ is not the identity in $\mathrm{PGL}(3,\mathbb{C})$. This is due to the choices of orientations of the faces which prevents $L_{ij}R_{ij}$ to be a matrix of coordinate change. When computing the holonomy of a path we therefore have to avoid backtracking.

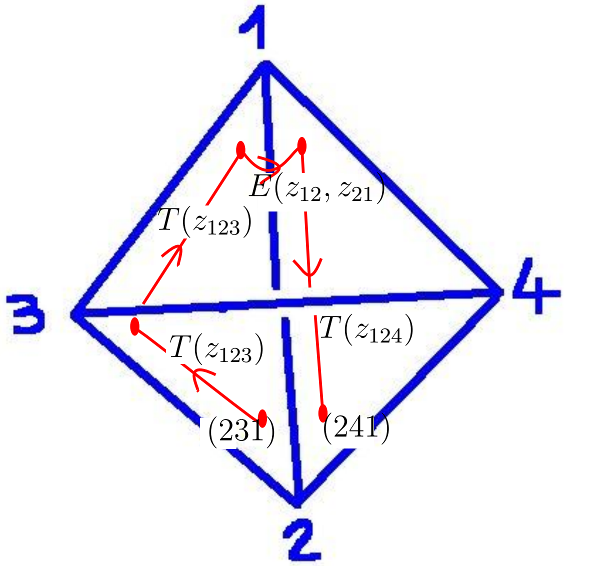

FIGURE 9. Turning right

5.11. Coordinates for the boundary of the complex. The boundary Σ of the complex K is a triangulated punctured surface. As in section 4 and in [7] we associate to Σ the set I_Σ of the vertices of the (red) arrows of the triangulation of Σ obtained using figure 4. As in the preceeding section we set $J^2_\Sigma = \mathbb{Z}^{I_\Sigma}$ and consider the skew-symmetric form Ω^2_Σ on J^2_Σ introduced by Fock and Goncharov in [7]. Here again we let $J^*_\Sigma \subset (J^2_\Sigma)^*$ be the image of J^2_Σ by the linear map $v \mapsto \Omega^2_\Sigma(v, \cdot)$.

5.12. The decoration of K yields a decoration of the punctures of Σ by flags, as in [7] and hence a point in J^*_Σ. Here is a more descriptive point of view, using the holonomy of the decoration of K: it provides Σ with coordinates associated to each $\alpha \in I_\Sigma$. To each face we associate the face z-coordinate of the corresponding tetrahedra of K. To each oriented edge ij of the triangulation of Σ we associate the last eigenvalue of the holonomy of the path joining the two adjacent faces by turning left around ij in K. It is equal to the product $z_{ij}(T_1) \cdots z_{ij}(T_\nu)$ where $T_1, \dots, T_\nu$ is the sequence of tetrahedra sharing ij as a common edge.

We denote by z_Σ the above defined element of $k^\times \otimes_{\mathbb{Z}} J^*_\Sigma \left[\frac{1}{2}\right]$.

Note that when K has a unipotent decoration, then the punctures are decorated by affine flags. We immediately get an element $a_\Sigma \in k^\times \otimes_{\mathbb{Z}} J^2_\Sigma$ which projects onto z_Σ in $k^\times \otimes_{\mathbb{Z}} J^*_\Sigma \left[\frac{1}{2}\right]$. Here again we have:

$$a_\Sigma \wedge_{\Omega^2_\Sigma} a_\Sigma = z_\Sigma \wedge_{\Omega^*_\Sigma} z_\Sigma.$$

The first expression is the W-element $W(\Sigma)$ associated to the decorated Σ by Fock and Goncharov.

5.13. Decoration and the pre-Bloch group. Let k be a field containing all the z-coordinates of the tetrahedra T_ν, $\nu = 1, \dots, N$. To any of these (ordered) tetrahedra we have associated an element $\beta(T_\nu) \in \mathcal{P}(k)$. Set:

$$\beta(K) = \sum_\nu \beta(T_\nu) \in \mathcal{P}(k).$$

From now on we assume that for each vertex $v \in K^{(0)}$, $|L_v|$ is homeomorphic to either a torus or an annulus. We fix symplectic bases (a_s, b_s) for each of the tori components and we fix c_r (resp. d_r) a generator of each homology group $H_1(L_r)$ (resp. $H_1(L_r, \partial L_r)$) where the L_r's are the annuli boundary components. We furthermore assume that the algebraic intersection number $\iota(c_r, d_r) = 1$.

Each one of these homology elements may be represented as a path as in section 5.7 which remains close to the associated vertex. So we may compute its holonomy using only matrices L_{ij} and R_{ij} : we will get an upper triangular matrix. More conceptually, the path is looping around a vertex decorated by a flag, so must preserve the flag. So it may be conjugated to an upper traingular matrix. Recall also that the diagonal part of a triangular matrix is invariant under conjugation by an upper-triangular matrix.

The following theorem computes $\delta(\beta(K))$ in terms of the *holonomy elements* A_s, B_s, C_r, D_r and A_s^*, B_s^*, C_r^*, D_r^* such that the holonomy matrices associated to a_s, b_s, c_r, d_r have the following form in a basis adapted to the flag decorating the link (see also §7.4 for a more explicit description):

$$\begin{pmatrix} \frac{1}{A_s^*} & * & * \\ 0 & 1 & * \\ 0 & 0 & A_s \end{pmatrix}.$$

5.14. Theorem. *The invariant $\delta(\beta(K))$ only depends on the boundary coordinates z_Σ, A_s, B_s, C_r, D_r and A_s^*, B_s^*, C_r^*, D_r^*. Moreover:*

1. *If the decoration of K is unipotent then $2\delta(\beta(K)) = z_\Sigma \wedge_{\Omega_\Sigma^*} z_\Sigma$.*
2. *If K is closed, i.e. $\Sigma = \emptyset$, and each link is a torus, we have the following formula for $3\delta(\beta(K))$:*

$$\sum_s \left(2A_s \wedge_{\mathbb{Z}} B_s + 2A_s^* \wedge_{\mathbb{Z}} B_s^* + A_s^* \wedge_{\mathbb{Z}} B_s + A_s \wedge_{\mathbb{Z}} B_s^*\right).$$

Theorem 5.14 generalizes several results known in the $\mathrm{SL}(2,\mathbb{C})$-case, see Neumann [11] – when K is closed – and Kabaya [10] – when all the connected components of Σ are spheres with 3 vertices. A related formula – still in the $\mathrm{SL}(2,\mathbb{C})$-case – is obtained by Bonahon [2, 3]. One may extract from our proof a formula for the general case. Though it should be related to the Weil-Petersson form on ∂M we are not able yet to explicit this relation.

Remark. Thanks to theorem 5.14, the fact that β lies inside the Bloch group is a boundary condition (the only non-vanishing part is $\frac{3}{2} z_\Sigma \wedge_{\Omega^*_\Sigma} z_\Sigma$). As a consequence, if the boundary is empty, it will automatically belong to the Bloch group. Using the work of Suslin, it allows to construct geometrically any class in $K_3^{\mathrm{ind}}(k)$, empowering a remark of Fock and Goncharov, see [8, Proposition 6.16].

6. Some linear algebra and the unipotent case

The goal of this section is to prove theorem 5.14 when K has a unipotent decoration. Along the way, we lay down the first basis for the homological proof in the general case.

6.1. First let (J^i, Ω^i) $(i = \emptyset, 2)$ denote the orthogonal sum of the spaces $(J^i_{T_\nu}, \Omega^i)$. We denote by e^μ_α the e_α-element in $J^i_{T_\mu}$.

A decoration provides us with an element

$$z \in \mathrm{Hom}(J, k^\times)\left[\frac{1}{2}\right] \simeq k^\times \otimes_{\mathbb{Z}} J^* \left[\frac{1}{2}\right] = k^\times \otimes_{\mathbb{Z}} \mathrm{Im}(p^*)\left[\frac{1}{2}\right]$$

which satisfies the face and edge conditions.[5] We first translate these two consistency relations into linear algebra.

Let C_1^{or} be the free $\mathbb{Z}$-module generated by the oriented internal[6] 1-simplices of K and C_2 the free $\mathbb{Z}$-module generated by the internal 2-faces of K. Introduce the map

$$F : C_1^{\mathrm{or}} + C_2 \to J^2$$

defined by, for $\bar{e}_{ij}$ an internal oriented edge of K,

$$F(\bar{e}_{ij}) = e^1_{ij} + \ldots + e^\nu_{ij}$$

where $T_1, \ldots, T_\nu$ is the sequence of tetrahedra sharing the edge $\bar{e}_{ij}$ such that $\bar{e}_{ij}$ is an inner edge of the subcomplex composed by the T_μ's and each e^μ_{ij} gets identified with the *oriented* edge $\bar{e}_{ij}$ in K (recall figure 7). And for a 2-face $\bar{e}_{ijk}$,

$$F(\bar{e}_{ijk}) = e^\mu_{ijk} + e^\nu_{ikj},$$

where μ and ν index the two 3-simplices having the common face $\bar{e}_{ijk}$. An element $z \in \mathrm{Hom}(J^2, k^\times)$ satisfies the face and edge conditions if and only if it vanishes on $\mathrm{Im}(F)$.

---

[5]Note that z moreover satisfies the non-linear equations

$$z_{ik}(T_\nu) = \frac{1}{1 - z_{ij}(T_\nu)}.$$

[6]Recall that our complex may have boundary.

Let $(J^2_{\text{int}})^*$ be the subspace of $(J^2)^*$ generated by internal edges and faces of K.

The dual map $F^* : (J^2)^* \to C_1^{\text{or}} + C_2$ (here we identify $C_1^{\text{or}} + C_2$ with its dual by using the canonical basis) is the "projection map":

$$(e^{\mu}_{\alpha})^* \mapsto \bar{e}_{\alpha}$$

when $(e^{\mu}_{\alpha})^* \in (J^2_{\text{int}})^*$ and maps $(e^{\mu}_{\alpha})^*$ to 0 if $(e^{\mu}_{\alpha})^* \notin (J^2_{\text{int}})^*$.

From the definitions we get the following:

6.2. Lemma. *An element $z \in k^{\times} \otimes_{\mathbb{Z}} (J^2)^*$ satisfies the face and edge conditions if and only if*

$$z \in k^{\times} \otimes_{\mathbb{Z}} \operatorname{Ker}(F^*).$$

A decorated tetrahedra complex thus provides us with and element $z \in k^{\times} \otimes (J^* \cap \operatorname{Ker}(F^*)) \left[\frac{1}{2}\right]$ and $\delta(\beta(K)) = \frac{1}{2} z \wedge_{\Omega^*} z$.

6.3. In this section we assume that K is equipped with a *unipotent* decoration. The boundary surface Σ is then a union of ideally *triangulated* closed oriented[7] surfaces with punctures decorated by affine flags in the sense of Fock and Goncharov [7]: the triangles are decorated by *affine* flags coordinates in such a way that the edge coordinates on the common edge of two triangles coincide. Each triangle being oriented we may define the W-invariant:

$$W(\Sigma) = \sum_{\Delta} W_{\Delta}$$

where W_{Δ} is defined by (4.3.2).[8]

Recall from §5.12 that the unipotent decoration of Σ provides us with an element $a_{\Sigma} \in k^{\times} \otimes_{\mathbb{Z}} J^2_{\Sigma}$ which projects onto $z_{\Sigma} \in k^{\times} \otimes_{\mathbb{Z}} J^*_{\Sigma} \left[\frac{1}{2}\right]$. We have:[9]

$$W(\Sigma) = \frac{1}{2} a_{\Sigma} \wedge_{\Omega^2_{\Sigma}} a_{\Sigma} = \frac{1}{2} z_{\Sigma} \wedge_{\Omega^*_{\Sigma}} z_{\Sigma}.$$

We have already done the computations leading to the proof of the theorem 5.14 in the unipotent case:

6.4. Proposition. *In the unipotent case we have:*

$$\delta(\beta(K)) = W(\Sigma).$$

Proof. The proof is the same as that of [6, Theorem 4.13]: we compute $\sum \beta(T_{\nu})$ for the tetrahedra complex using the a-coordinates as in §4.1. This gives a sum of W-invariants associated to the faces of the T_{ν}'s. The terms corresponding to a common face between two tetrahedra

[7]The orientation being induced by that of K.

[8]Note that in the case of $K = T$ the boundary of T is a sphere with 4 punctures and the definition of $W(T)$ in section 4 matches this one.

[9]Note in particular that $W(\Sigma)$ only depends on the flag z-coordinates, see also [8, Lem. 6.6]. Moreover, in case $K = T$, we recover lemma 4.3.

appear with opposite sign. The sum of the remaining terms is precisely $W(\Sigma)$. □

6.5. A unipotent decoration corresponds to a point $z \in k^\times \otimes (\mathrm{Im}(p \circ F)) \left[\frac{1}{2}\right]$. In §5.12 we therefore have defined a map

$$k^\times \otimes (\mathrm{Im}(p \circ F)) \left[\frac{1}{2}\right] \to k^\times \otimes J_\Sigma^* \left[\frac{1}{2}\right].$$

The following proposition states that this map respects the 2-forms Ω^* and Ω_Σ^*.

6.6. Proposition. *In the unipotent case, Ω^* is the pullback of Ω_Σ^*.*

Proof. We have seen that on each tetrahedron $p^*(\Omega^*(T)) = \Omega^2(T)$.

Since $\mathrm{Im}(p \circ F)$ is the image by p of the subspace $\mathrm{Im}(F)$ of J^2, each face f of T is an oriented triangle with a-coordinates, so we define a 2-form $\Omega^2(f, T)$ by the usual formula. If the face f is internal between T and T', we have $\Omega^2(f, T) = -\Omega^2(f, T')$ as the only difference is the orientation of the face (and hence of its red triangulation, see figure 4).

Moreover $p^*(\Omega^*)$ is the sum of the $\Omega^2(T)$. Hence it reduces to the sum on external faces of $\Omega^2(f, T)$, that is exactly $\Omega_\Sigma^2 = p^*(\Omega^*)$. □

Our goal is now to extend this result beyond the unipotent case; to this end we develop a theory analogous to the one of Neumann-Zagier but in the $\mathrm{PGL}(3, \mathbb{C})$-case. We first treat in details the case where K is closed.

7. Neumann-Zagier bilinear relations for $\mathrm{PGL}(3, \mathbb{C})$

A decorated tetrahedra complex provides us with and element $z \in k^\times \otimes (J^* \cap \mathrm{Ker}(F^*)) \left[\frac{1}{2}\right]$ and $\delta(\beta(K)) = \frac{1}{2} z \wedge_{\Omega^*} z$. Our final goal is to compute this last expression. But here we first describe the right set up to state the generalization of proposition 6.6 to general – non-unipotent – decorations. This leads to a more precise version of theorem 5.14, see corollary 7.11. We first deal with the case where K is a (closed) 3-cycle. We will later explain how to modify the definitions and proofs to deal with the general case.

7.1. Coordinates on the boundary. Let K be a quasi-simplicial triangulation of M. Assume that K is closed so that $\Sigma = \emptyset$ and each $|L_v|$ is a torus. We first define coordinates for ∂M and a symplectic structure on these coordinates.

Each torus boundary surface S in the link of a vertex is triangulated by the traces of the tetrahedra; from this we build the CW-complex $\mathcal{D}$ whose edges consist of the inner edges of the first barycentric subdivision, see figure 10. We denote by $\mathcal{D}'$ the dual cell division Let $C_1(\mathcal{D}) = C_1(\mathcal{D}, \mathbb{Z})$ and $C_1(\mathcal{D}') = C_1(\mathcal{D}', \mathbb{Z})$ be the corresponding chain groups. Given two chains $c \in C_1(\mathcal{D})$ and $c' \in C_1(\mathcal{D}')$ we denote by $\iota(c, c')$ the (integer) intersection number of c and c'. This defines a bilinear form $\iota : C_1(\mathcal{D}) \times C_1(\mathcal{D}') \to \mathbb{Z}$ which induces the usual intersection form on $H_1(S)$. In that way $C_1(\mathcal{D}')$ is canonically isomorphic to the dual of $C_1(\mathcal{D})$.

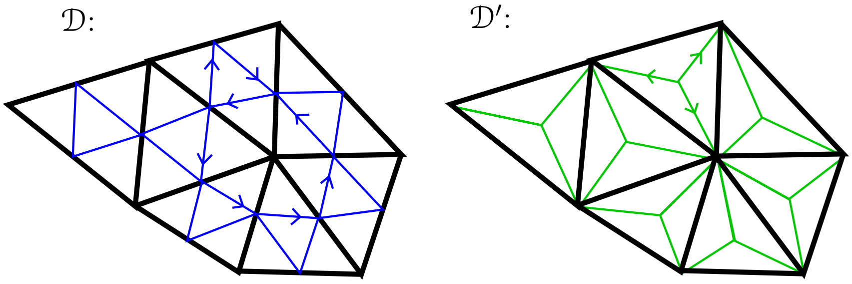

FIGURE 10. The two cell decompositions of the link

7.2. Goldman-Weil-Petersson form for tori. Here we equip

$$C_1(\mathcal{D}, \mathbb{R}^2) = C_1(\mathcal{D}) \otimes \mathbb{R}^2$$

with the bilinear form ω defined by coupling the intersection form ι with the scalar product on $\mathbb{R}^2$ seen as the space of roots of $\mathfrak{sl}(3)$ with its Killing form. We describe more precisely an integral version of this.

From now on we identify $\mathbb{R}^2$ with the subspace $V = \{(x_1, x_2, x_3)^t \in \mathbb{R}^3 \ : \ x_1 + x_2 + x_3 = 0\}$ via

$$\begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}-1\\0\\1\end{pmatrix} \text{ and } \begin{pmatrix}0\\1\end{pmatrix} \mapsto \begin{pmatrix}0\\-1\\1\end{pmatrix}.$$

We let $L \subset V$ be the standard lattice in V where all three coordinates are in $\mathbb{Z}$. We identify it with $\mathbb{Z}^2$ using the above basis of V. The restriction of the usual euclidean product of $\mathbb{R}^3$ gives a product, denoted $[,]$, on V (the "Killing form")[10]. In other words, we have:

$$\left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right] = \left[\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right] = 2 \text{ and } \left[\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right] = 1.$$

Identifying V with V^* using the scalar product $[,]$, the dual lattice $L^* \subset V^*$ becomes a lattice L' in V; an element $y \in V$ belongs to L' if and only if $[x, y] \in \mathbb{Z}$ for every $x \in L$.

We let $C_1(\mathcal{D}, L)$ and define $\omega = \iota \otimes [\cdot, \cdot] : C_1(\mathcal{D}, L) \times C_1(\mathcal{D}', L') \to \mathbb{Z}$ by the formula

$$\omega\,(c \otimes l, c' \otimes l') = \iota(c, c')\,[l, l'].$$

This induces a (symplectic) bilinear form on $H_1(S, \mathbb{R}^2)$ which we still denote by ω. Note that ω identifies $C_1(\mathcal{D}', L')$ with the dual of $C_1(\mathcal{D}, L)$.

[10]In terms of roots of $\mathfrak{sl}(3)$, the choosen basis is, in usual notations, $e_3 - e_1$, $e_3 - e_2$.

Remark. The canonical coupling $C_1(\mathcal{D}, L) \times C^1(\mathcal{D}, L^*) \to \mathbb{Z}$ identifies $C_1(\mathcal{D}, L)^*$ with $C^1(\mathcal{D}, L^*)$. This last space is naturally equipped with the "Goldman-Weil-Petersson" form wp, dual to ω. Let $\langle , \rangle$ be the natural scalar product on V^* dual to $[,]$: letting $d : V \to V^*$ be the map defined by $d(v) = [v, \cdot]$ we have $\langle d(v), d(v') \rangle = [v, v']$. In coordinates $d : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$d \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ 2y + x \end{pmatrix}.$$

Idenfying V^* with $\mathbb{R}^2$ using the dual basis we have:

$$\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle = \langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle = \frac{2}{3} \text{ and } \langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle = -\frac{1}{3}.$$

On $H^1(S, \mathbb{R}^2)$ the bilinear form wp induces a symplectic form – the usual Goldman-Weil-Petersson symplectic form – formally defined as the coupling of the cup-product and the scalar product $\langle , \rangle$.

7.3. To any decoration $z \in k^\times \otimes (J^* \cap \mathrm{Ker}(F^*)) \left[\frac{1}{2}\right]$ we now explain how to associate an element

$$R(z) \in \mathrm{Hom}(H_1(S, L), k^\times) \left[\frac{1}{2}\right].$$

We may represent any class in $H_1(S, L)$ by an element $c \otimes \begin{pmatrix} n \\ m \end{pmatrix}$ in $C_1(\mathcal{D}, L)$ where c is a closed path in S seen as the link of the corresponding vertex in the complex K. Using the decoration z we may compute the holonomy of the loop c, as explained in §5.7. This vertex being equipped with a flag stabilized by this holonomy, we may write it as an upper triangular matrix. Let $(\frac{1}{C^*}, 1, C)$ be the diagonal part. The application which maps $c \otimes \begin{pmatrix} n \\ m \end{pmatrix}$ to $C^{n+m}(C^*)^n$ is the announced element $R(z)$ of $k^\times \otimes H^1(S, L^*) \left[\frac{1}{2}\right]$.

7.4. Linearization for a torus. In the preceeding paragraph we have constructed a map

$$R : k^\times \otimes J^* \cap \mathrm{Ker}(F^*) \left[\frac{1}{2}\right] \to \mathrm{Hom}(H_1(S, L), k^\times) \left[\frac{1}{2}\right].$$

As we have done before for consistency relations we now linearize this map.

Let $h : C_1(\mathcal{D}, L) \to J^2$ be the linear map defined on the elements $e \otimes \begin{pmatrix} n \\ m \end{pmatrix}$ of $C_1(\mathcal{D}, L)$ by

$$h \left(e \otimes \begin{pmatrix} n \\ m \end{pmatrix} \right) = 2(m + n) e^\mu_{ij} + 2n e^\mu_{ji} + n(e^\mu_{ijk} + e^\mu_{ilj}).$$

Here we see the edge e as turning left around the edge (ij) in the tetrahedron $T_\mu = (ijkl)$, see figure 11.

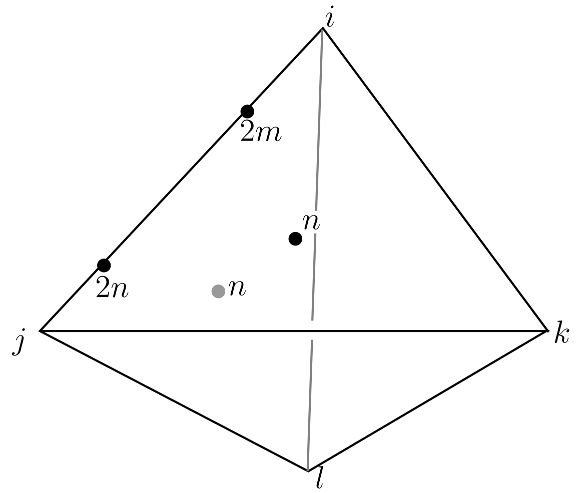

FIGURE 11. The map h

7.5. Lemma. *Let $z \in k^\times \otimes (J^* \cap \mathrm{Ker}(F^*)) \left[\frac{1}{2}\right]$. Seeing z as an element of $\mathrm{Hom}(J^2, k^\times)\left[\frac{1}{2}\right]$, we have:*

$$z \circ h = R(z)^2.$$

Proof. Let c be an element in $H_1(S)$. Recall that the torus is triangulated by the trace of the tetrahedra. To each triangle corresponds a tetrahedron T_μ and a vertex i of this tetrahedron. Now each vertex of the triangle corresponds to an edge ij of the tetrahedron T_μ oriented from the vertex j to i. Hence each edge of $\mathcal{D}$ may be canonically denoted by c^μ_{ij}: it is the edge in the link of i which turns left around the edge ij of the tetrahedron T_μ. We represent c as a cycle $\bar{c} = \sum \pm c^\mu_{ij}$. The cycle $\bar{c}$ turns left around some edges, denoted by e^μ_{ij}, and right around other edges, denoted by $e^{\mu'}_{ij}$. In other terms, we have $\bar{c} = \sum_\mu c^\mu_{ij} - \sum_{\mu'} c^{\mu'}_{ij}$. Then, using the matrices L^μ_{ij} (5.10.1) and $R^{\mu'}_{ij}$ (5.10.2), we see that the diagonal part of the holonomy of c is given by:

$$C = \frac{\prod z^\mu_{ij}}{\prod z^{\mu'}_{ij}} \tag{7.5.1}$$

$$\text{and } C^* = \frac{\prod z^\mu_{ji}}{\prod z^{\mu'}_{ji} z^{\mu'}_{ijk} z^{\mu'}_{ilj}}.$$

Let us simplify a bit the formula for C^*. Recall the face relation: if T and T' share the same face ijk, we have $z_{ijk}(T) z_{ikj}(T') = 1$. Hence if our path c was turning right before a face F and continues after crossing F, the corresponding face coordinate simplifies in the product

$\prod z^{\mu'}_{ijk} z^{\mu'}_{ilj}$. Let $\mathcal{F}$ be the set of faces (with multiplicity) at which α changes direction. For F in $\mathcal{F}$, let T be the tetrahedron containing F in which α turns right. We consider F oriented as a face of T and denote z_F its 3-ratio. We then have

$$C^* = \frac{\prod z^{\mu}_{ji}}{\prod z^{\mu'}_{ji} \prod_{\mathcal{F}} z_F}. \tag{7.5.2}$$

Now $h\left(c \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = 2\sum e^{\mu}_{ij} - 2\sum e^{\mu'}_{ij}$, as turning right is the opposite to turning left. It proves (with equation 7.5.1) that

$$z \circ h\left(c \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \left(\frac{\prod z^{\mu}_{ij}}{\prod z^{\mu'}_{ij}}\right)^2 = C^2.$$

We have to do a bit more rewriting to check it for $c \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Indeed, we have:

$$h\left(c \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \sum_{\mu}(2e^{\mu}_{ij} + 2e^{\mu}_{ji} + e^{\mu}_{ijk} + e^{\mu}_{ilj}) - \sum_{\mu'}(2e^{\mu'}_{ij} + 2e^{\mu'}_{ji} + e^{\mu'}_{ijk} + e^{\mu'}_{ilj}),$$

so that:

$$z \circ h\left(c \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \left(\frac{\prod z^{\mu}_{ij}}{\prod z^{\mu'}_{ij}}\right)^2 \left(\frac{\prod z^{\mu}_{ji}}{\prod z^{\mu'}_{ji}}\right)^2 \frac{\prod z^{\mu}_{ijk} z^{\mu}_{ilj}}{\prod z^{\mu'}_{ijk} z^{\mu'}_{ilj}}.$$

For the same reason as before the "internal faces" simplify in the product $\prod z^{\mu}_{ijk} z^{\mu}_{ilj}$. Moreover, for $F \in \mathcal{F}$, it appears with the opposite orientation: indeed the orientation given to F is the one given by the tetrahedron in which α turns *left*. Hence the coordinate that shows up is $\frac{1}{z_F}$. So the last formula rewrites:

$$z \circ h\left(c \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \left(\frac{\prod z^{\mu}_{ij}}{\prod z^{\mu'}_{ij}}\right)^2 \left(\frac{\prod z^{\mu}_{ji}}{\prod z^{\mu'}_{ji}}\right)^2 \left(\frac{1}{\prod_{\mathcal{F}} z_F}\right)^2 = C^2 (C^*)^2,$$

which proves the lemma. □

Let $h^* : (J^2)^* \to C_1(\mathcal{D}, L)^*$ be the map dual to h. Note that for any $e \in J^2$ and $c \in J_T$ we have

$$(h^* \circ p(e))(c) = p(e)(h(c)) = \Omega^2(e, h(c)). \tag{7.5.3}$$

Now composing p with h^* and identifying $C_1(\mathcal{D}, L)^*$ with $C_1(\mathcal{D}', L')$ using ω we get a map

$$g : J^2 \to C_1(\mathcal{D}', L')$$

and it follows from (7.5.3) that for any $e \in J^2$ and $c \in J_T$ we have

$$\omega(c, g(e)) = \Omega^2(e, h(c)). \tag{7.5.4}$$

In the following we let $J_{\partial M} = C_1(\partial M, L)$ and $C_1(\partial M', L')$ be the orthogonal sum of the $C_1(\mathcal{D}, L)$'s and $C_1(\mathcal{D}', L')$'s for each torus link S. We abusively denote by $h : J_{\partial M} \to J^2$ and $g : J^2 \to C_1(\partial M', L')$ the maps defined above on each T.

7.6. Homology of the complexes. First consider the composition of maps:

$$C_1^{\rm or} + C_2 \xrightarrow{F} J^2 \xrightarrow{p} (J^2)^* \xrightarrow{F^*} C_1^{\rm or} + C_2.$$

By inspection one may check that $F^* \circ p \circ F = 0$. Here is a geometric way to figure this after tensorization by $k^\times \otimes \mathbb{Z}\left[\frac{1}{2}\right]$: first note that if $z = p^*(a)$ then a can be thought as a set of affine coordinates lifting of z. Now a belongs to the image of F exactly when these a-coordinates agree on elements of J^2 corresponding to common oriented edges (resp. common faces) of K. In such a case the decoration of K has a unipotent decoration lifting z. Finally the map F^* computes the last eigenvalue of the holonomy matrix of paths going through and back a face (face relations) and of paths going around edges (edges relations). In case of a unipotent decoration these eigenvalues are trivial. This shows that $F^* \circ p \circ F = 0$.

In particular, letting $G : J \to C_1^{\rm or} + C_2$ be the map induced by $F^* \circ p$ and $F' : C_1^{\rm or} + C_2 \to J$ be the map F followed by the canonical projection from J^2 to J, we get a complex:

$$C_1^{\rm or} + C_2 \xrightarrow{F'} J \xrightarrow{G} C_1^{\rm or} + C_2. \tag{7.6.1}$$

Similarly, letting $G^* = p \circ F$ and $(F')^*$ be the restriction of F^* to $\mathrm{Im}(p) = J^*$ we get the dual complex:

$$C_1^{\rm or} + C_2 \xrightarrow{G^*} J^* \xrightarrow{(F')^*} C_1^{\rm or} + C_2. \tag{7.6.2}$$

We define the homology groups of these two complexes:

$$\mathcal{H}(J) = \mathrm{Ker}(G)/\mathrm{Im}(F') = \mathrm{Ker}(F^* \circ p)/(\mathrm{Im}(F) + \mathrm{Ker}(p))$$

and

$$\mathcal{H}(J^*) = \mathrm{Ker}((F')^*)/\mathrm{Im}(G^*) = (\mathrm{Ker}(F^*) \cap \mathrm{Im}(p))/\mathrm{Im}(p \circ F).$$

We note that:

$$\mathrm{Ker}(F') = \mathrm{Im}(G)^{\perp_\Omega} \text{ and } \mathrm{Ker}(G^*) = \mathrm{Im}((F')^*)^{\perp_{\Omega^*}}.$$

The symplectic forms Ω and Ω^* thus induce skew-symmetric bilinear forms on $\mathcal{H}(J)$ and $\mathcal{H}(J^*)$. These spaces are obviously dual spaces and the bilinear forms match through duality.

A decoration of K provides us with an element $z \in k^\times \otimes \mathrm{Ker}((F')^*)\left[\frac{1}{2}\right]$. We already have dealt with the subspace $k^\times \otimes \mathrm{Im}(p \circ F)\left[\frac{1}{2}\right]$ which corresponds to the unipotent decorations: in that case $\delta(\beta(K)) = 0$. We thus conclude that $\delta(\beta(K))$ only depends on the image of z in $k^\times \otimes_{\mathbb{Z}} \mathcal{H}(J^*)\left[\frac{1}{2}\right]$. We will describe this last space in terms of the homology of ∂M.

Let $Z_1(\mathcal{D}, L)$ and $B_1(\mathcal{D}, L)$ be the subspaces of cycles and boundaries in $C_1(\mathcal{D}, L)$. The following lemma is easily checked by inspection.

7.7. Lemma. *We have:*

$$h(Z_1(\mathcal{D}, L)) \subset \mathrm{Ker}(F^* \circ p)$$

and

$$h(B_1(\mathcal{D}, L)) \subset \mathrm{Ker}(p) + \mathrm{Im}(F).$$

In particular h induces a map $\bar{h} : H_1(\mathcal{D}, L) \to \mathcal{H}(J)$ in homology. By duality, the map g induces a map $\bar{g} : \mathcal{H}(J) \to H_1(\partial M, \mathcal{D}', L')$ as follows from:

7.8. Lemma. *We have:*

$$g(\mathrm{Ker}(F^* \circ p)) \subset Z_1(\mathcal{D}', L'),$$

and

$$g(\mathrm{Ker}(p) + \mathrm{Im}(F)) \subset B_1(\mathcal{D}', L').$$

Proof. First of all, $Z_1(\mathcal{D}', L')$ is the orthogonal of $B_1(\mathcal{D}, L)$ for the coupling ω. Moreover, by definition of g, if $e \in \mathrm{Ker}(F^* \circ p)$, we have:

$$\begin{aligned} g(e) \in Z_1(\mathcal{D}', L') &\Leftrightarrow \omega(B_1(\mathcal{D}, L), g(e)) = 0 \\ &\Leftrightarrow \Omega^2(h(B_1(\mathcal{D}, L)), e) = 0. \end{aligned}$$

The last condition is given by the previous lemma. The second point is similar. □

Note that $H_1(\mathcal{D}, L)$ and $H_1(\mathcal{D}', L')$ are canonically isomorphic so that we identified them (to $H_1(\partial M, L)$) in the following.

7.9. Theorem. (1) *The map $\bar{g} \circ \bar{h} : H_1(\partial M, L) \to H_1(\partial M, L)$ is multiplication by* 4.
(2) *Given $e \in \mathcal{H}(J)$ and $c \in H_1(\partial M, L)$, we have*

$$\omega(c, \bar{g}(e)) = \Omega(e, \bar{h}(c)).$$

As a corollary, one understands the homology of the various complexes.

7.10. Corollary. *The map $\bar{h}$ induces an isomorphism from $H_1(\partial M, L)\left[\frac{1}{2}\right]$ to $\mathcal{H}(J)\left[\frac{1}{2}\right]$. Moreover we have $\bar{h}^*\Omega = -4\omega$.*

7.11. Corollary. *The form Ω^* on $k^\times \otimes J^* \cap \mathrm{Ker}(F^*)\left[\frac{1}{2}\right]$ is the pullback of* wp *on $H^1(\partial M, L^*)$ by the map R.*

Theorem 5.14 will follow from corollary 7.11 and lemma 4.10 (see section 8.3 for an explicit computation). Corollary 7.11 is indeed the analog of proposition 6.6 in the closed case. We postpone the proof of theorem 7.9 until the next section and, in the remaining part of this section, deduce corollaries 7.10 and 7.11 from it.

7.12. Proof of corollary 7.11. We first compute the dimension of the spaces $\mathcal{H}(J)$ and $\mathcal{H}(J^*)$. Recall that l is the number of vertices in K.

7.13. Lemma. *The dimension of* $\mathcal{H}(J)$ *and* $\mathcal{H}(J^*)$ *is* $4l$

Proof. By the rank formula we have

$$\dim J^2 = \dim \mathrm{Ker}(F^* \circ p) + \dim \mathrm{Im}(F^* \circ p)$$

and by definition we have

$$\dim \mathrm{Ker}(F^* \circ p) = \dim(\mathrm{Ker}(p) + \mathrm{Im}(F)) + \dim \mathcal{H}(J).$$

We obviously have:

$$\dim(\mathrm{Ker}(p)+\mathrm{Im}(F)) = \dim \mathrm{Ker}(p) + \dim \mathrm{Im}(F) - \dim(\mathrm{Ker}(p) \cap \mathrm{Im}(F)),$$

$$\dim \mathrm{Im}(F^* \circ p) = \dim \mathrm{Im}(p) - \dim(\mathrm{Im}(p) \cap \mathrm{Ker}(F^*)),$$

and

$$\dim(\mathrm{Im}(p) \cap \mathrm{Ker}(F^*)) = \dim(J^2)^* - \dim(\mathrm{Im}(F) + \mathrm{Ker}(p)).$$

The map F is injective and therefore F^* is surjective. We conclude that

$$\dim \mathrm{Im}(F) = \dim \mathrm{Im}(F^*) = \dim C_1^{\mathrm{or}} + \dim C_2.$$

But $\dim J^2 = 16N$, $\dim \mathrm{Ker}(p) = 8N$, $\dim C_2 = 2N$ and, since the Euler caracteristic of M is 0, $\dim C_1^{\mathrm{or}} = 2N$. We are therefore reduced to prove that $\dim(\mathrm{Ker}(p) \cap \mathrm{Im}(F)) = 2l$. Restricted to a single tetrahedron T_μ, the kernel of p is generated by the elements $v_i^\mu = e_{ij}^\mu + e_{ik}^\mu + e_{il}^\mu$ and $w_i^\mu = e_{ji}^\mu + e_{ki}^\mu + e_{li}^\mu + e_{ijk}^\mu + e_{ilj}^\mu + e_{ikl}^\mu$ in $J^2(T_\mu)$ for i a vertex of T_μ (see section 4.4).

In $\mathrm{Im}(F)$, all the coordinates of e_{ij}^μ that projects on the same edge $\bar{e}_{ij}$ must be equal, as does the two coordinates of e_{ijk}^μ and $e_{ikj}^{\mu'}$ projecting on the same face. Hence, $\mathrm{Im}(F) \cap \mathrm{Ker}(p)$ is generated by the vectors $F(v_i)$ and $F(w_i)$ where

$$v_i = \sum_{\bar{e}_{ij} \text{oriented edge from } i} \bar{e}_{ij} \text{ and}$$

$$w_i = \sum_{\bar{e}_{ji} \text{ oriented edge toward } i} \bar{e}_{ji} + \sum_{\bar{e}_{ijk} \text{ a face containing } si} \bar{e}_{ijk}.$$

One verifies easily that these vectors are free, proving the lemma. □

Since it follows from theorem 7.9 (1) that $\bar{h}$ has an inverse after tensorization by $\mathbb{Z}\left[\frac{1}{2}\right]$ we conclude from lemma 7.13 that $\mathcal{H}(J)\left[\frac{1}{2}\right]$ and $H_1(\partial M, L)\left[\frac{1}{2}\right]$ are isomorphic. Now 7.9 (2) implies that $\bar{h}$ and $\bar{g}$ are adjoint maps w.r.t. the forms ω on $H_1(\partial M, L)\left[\frac{1}{2}\right]$ and Ω on $\mathcal{H}(J)\left[\frac{1}{2}\right]$. The corollary follows.

The second corollary is merely a dual statement: recall from 7.4 that the map R^2 is induced by the map $h^* : J^* \to C^1(\mathcal{D}, L^*)$ dual to h. Now the map $c' \mapsto \omega(\cdot, c')$ induces a symplectic isomorphism between $(H_1(\partial M, L'), \omega)$ and $(H^1(\partial M, L^*), \mathrm{wp})$. It therefore follows from corollary 7.10 that the symplectic form Ω^* on $\mathcal{H}(J^*)$ is four times the pullback of wp by the map $\mathcal{H}(J^*) \to H^1(\partial M, L^*)$ induced by h^*. Remembering that h^* induces the *square* of R the statement of corollary 7.11 follows.

8. Homologies and symplectic forms

In this section we first prove theorem 7.9 (in the closed case). We then explain how to deduce theorem 5.14 from it and its corollary 7.11.

8.1. Proof of theorem 7.9. We first compute $g \circ h : C_1(\mathcal{D}, L) \to C_1(\mathcal{D}', L')$ using (7.5.4). We work in a fix tetraedron and therefore forget about the μ's. We denote by c_{ij} the edge of $\mathcal{D}$ corresponding to a (left) turn around the edge e_{ij} and we denote by c'_{ij} its dual edge in $\mathcal{D}'$, see figure 10. The following computations are straightforward:

$$\Omega\left(h\left(c_{ij}\otimes\begin{pmatrix}n\\m\end{pmatrix}\right), h\left(c_{ik}\otimes\begin{pmatrix}n'\\m'\end{pmatrix}\right)\right) = 2\left[\begin{pmatrix}n\\m\end{pmatrix}, \begin{pmatrix}n'\\m'\end{pmatrix}\right],$$
$$\Omega\left(h\left(c_{ij}\otimes\begin{pmatrix}n\\m\end{pmatrix}\right), h\left(c_{jk}\otimes\begin{pmatrix}n'\\m'\end{pmatrix}\right)\right) = -2\left[\begin{pmatrix}n-2m/3\\m/3\end{pmatrix}, \begin{pmatrix}n'\\m'\end{pmatrix}\right],$$
$$\Omega\left(h\left(c_{ij}\otimes\begin{pmatrix}n\\m\end{pmatrix}\right), h\left(c_{ji}\otimes\begin{pmatrix}n'\\m'\end{pmatrix}\right)\right) = 0,$$
$$\Omega\left(h\left(c_{ij}\otimes\begin{pmatrix}n\\m\end{pmatrix}\right), h\left(c_{ki}\otimes\begin{pmatrix}n'\\m'\end{pmatrix}\right)\right) = 2\left[\begin{pmatrix}n-2m/3\\m/3\end{pmatrix}, \begin{pmatrix}n'\\m'\end{pmatrix}\right],$$

and so on... Since it follows from (7.5.4) that

$$\omega\left(c\otimes\begin{pmatrix}n'\\m'\end{pmatrix}, g\circ h\left(c_{ij}\otimes\begin{pmatrix}n\\m\end{pmatrix}\right)\right)$$
$$= \Omega\left(h\left(c_{ij}\otimes\begin{pmatrix}n\\m\end{pmatrix}\right), h\left(c\otimes\begin{pmatrix}n'\\m'\end{pmatrix}\right)\right)$$

we conclude that the element $g\circ h\left(c_{ij}\otimes\begin{pmatrix}n\\m\end{pmatrix}\right)$ in $C_1(\mathcal{D}', L)$ is:

$$g\circ h\left(c_{ij}\otimes\begin{pmatrix}n\\m\end{pmatrix}\right)$$
$$= 2(c'_{ik}-c'_{il})\otimes\begin{pmatrix}n\\m\end{pmatrix} + 2(c'_{ki}-c'_{kj}+c'_{jl}-c'_{jk}+c'_{lj}-c'_{li})\otimes\begin{pmatrix}n-2m/3\\m/3\end{pmatrix}.$$

Consider now a cycle $c = \sum c^{\mu}_{ij}$. We compute:

$$g\circ h\left(c\otimes\begin{pmatrix}n\\m\end{pmatrix}\right) = \left(2\sum c'_{ik}-c'_{il}\right)\otimes\begin{pmatrix}n\\m\end{pmatrix}$$
$$+\left(2\sum c'_{ki}-c'_{kj}+c'_{jl}-c'_{jk}+c'_{lj}-c'_{li}\right)\otimes\begin{pmatrix}n-2m/3\\m/3\end{pmatrix}$$

Interestingly, we are now reduced to a problem in the homology of ∂M and the lattice L does not play any role here. Indeed, the first assertion in theorem 7.9 follows from the following lemma. The second assertion of theorem 7.9 then follows from (7.5.4).

8.2. Lemma.
- *The path $\sum c'_{ik} - c'_{il}$ is homologous to $2c$ in $H^1(\partial M)$,*
- *The path $\sum c'_{ki} - c'_{kj} + c'_{jl} - c'_{jk} + c'_{lj} - c'_{li}$ vanishes in $H^1(\partial M)$.*

This lemma is already proven by Neumann [11, Lemma 4.3]. The proof is a careful inspection using figures 12 and 13. The first point is quite easy: the path $\sum c'_{ik} - c'_{il}$ is the boundary of a regular neighborhood of c. The second part is the "far from the cusp" contribution in Neumann's paper. We draw on figure 13 four tetrahedra sharing an edge (the edges are displayed in dotted lines). The blue path is the path c in the upper link. The collection of green paths are the relative $\sum c'_{ki} - c'_{kj} + c'_{jl} - c'_{jk} + c'_{lj} - c'_{li}$ in the other links. It consists in a collection of boundaries.

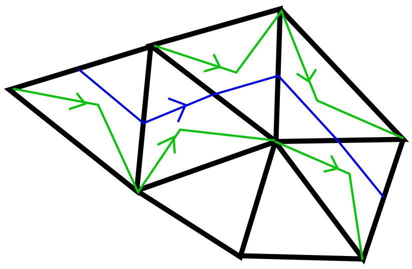

FIGURE 12. What happens inside the cusp: c in blue and $g \circ h(c)$ in green.

8.3. Proof of theorem 5.14 in the closed case. The theorem 5.14 is now a corollary. Indeed, we have from lemma 4.10 and corollary 7.11, if $z \in \mathcal{H}(J^*)$:

$$\begin{aligned} 3\delta(\beta(z)) &= \frac{3}{2} z \wedge_{\Omega^*} z \\ &= \frac{3}{2} R(z) \wedge_{\mathrm{wp}} R(z). \end{aligned}$$

It remains to compute the last quantity. Recall from the previous section the definition of $R(z)$: if a loop c represents a class in homology, let $(\frac{1}{C^*}, 1, C)$ be the diagonal part of its holonomy. Then $R(z)$ applied to $c \otimes \begin{pmatrix} n \\ m \end{pmatrix}$ equals $C^{n+m}(C^*)^n$. In other terms, denoting $[a_s]$ and $[b_s]$ the classes dual to a_s and b_s, we have (see §7.2):

$$R(z) = [a_s] \otimes \begin{pmatrix} A_s A_s^* \\ A_s \end{pmatrix} + [b_s] \otimes \begin{pmatrix} B_s B_s^* \\ B_s \end{pmatrix}.$$

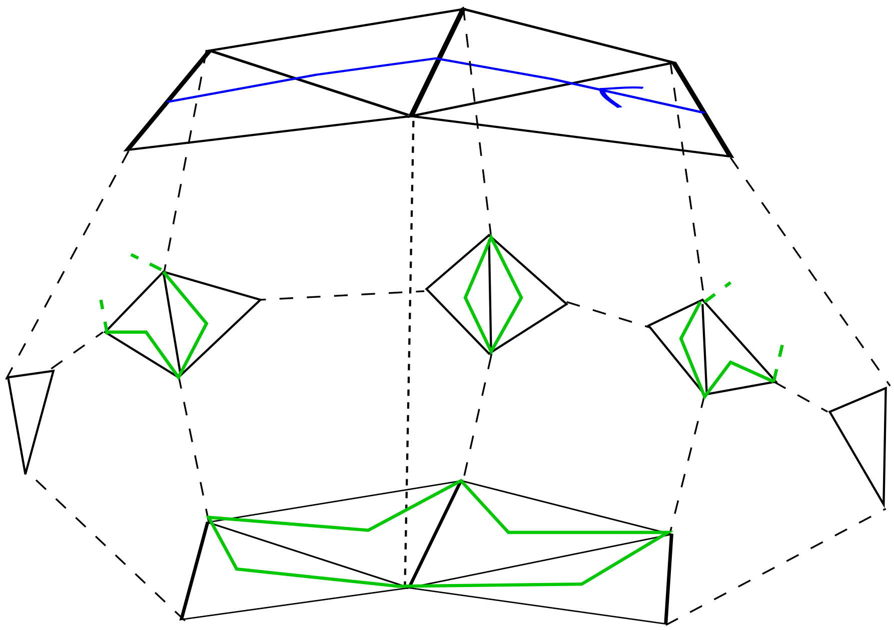

Figure 13. What happens far from the cusp

Recall from §7.2 that the form wp is the coupling of the cup product and the scalar product $\langle ; \rangle$ on $\mathbb{Z}^2$. Hence we conclude by:

$$\begin{aligned} 3\delta(\beta(z)) &= \sum_s 3\langle (A_sA_s^*, A_s), (B_sB_s^*, B_s)\rangle \\ &= \sum_s 2(A_sA_s^*) \wedge (B_sB_s^*) + 2A_s \wedge B_s - (A_sA_s^*) \wedge B_s - A_s \wedge (B_sB_s^*) \\ &= \sum_s 2A_s \wedge B_s + 2A_s^* \wedge B_s^* + A_s^* \wedge B_s + A_s \wedge B_s^*. \end{aligned}$$

9. Extension to the general case

We consider now the case of a complex K with boundary and explain how the preceeding proof of theorem 5.14 shall be adapted to deal with it. Recall that the boundary of $K - K^{(0)}$ decomposes as the union of a triangulated surface Σ and the links. The latter are further decomposed as tori links S_s and annuli links L_r. We proceed as in the closed case and indicate the modifications to be done. For simplicity we suppose that $k = \mathbb{C}$.

9.1. We denote by $C_1^{\text{or}} + C_2$ the $\mathbb{Z}$-module generated by *internal* (oriented) edges and faces. A parabolic decoration of K gives a parabolic decoration of Σ, i.e. an element $z_\Sigma \in k^\times \otimes_{\mathbb{Z}} (J_\Sigma^2)^* \left[\frac{1}{2}\right]$, whose interpretation is that one may glue the decorated surface Σ to the decorated complex fulfilling the consistency relations. More precisely, if e_α is a basis vector of J^2_Σ, one defines the e^*_α component of z_Σ by:

$$z^\Sigma_\alpha \prod_\nu z^\nu_\alpha = 1,$$

where the product is over all the e^ν_α identified with e_α. As usual we will rather consider the corresponding linear map:

$$h_\Sigma : J^2_\Sigma \to J^2; e_\alpha \mapsto -\sum_\nu e^\nu_\alpha$$

as well as the dual map $h^*_\Sigma : (J^2)^* \to (J^2_\Sigma)^*$. Note that if e^ν_α in J^2 corresponds to an *internal* edge or face then $h^*((e^\nu_\alpha)^*) = 0$ whereas if it correspond to a boundary element $e_\alpha \in J^2_\Sigma$ we have: $h^*((e^\nu_\alpha)^*) = -e^*_\alpha$. In particular one easily check that the following diagram is commutative:

(9.1.1)
$$\begin{array}{ccc} J^2 & \xrightarrow{p} & (J^2)^* \\ h\uparrow & & \downarrow h^* \\ J^2_\Sigma & \xrightarrow{p_\Sigma} & (J^2_\Sigma)^* \end{array}$$

Recall that $J_\Sigma = J^2_\Sigma/\mathrm{Ker}(p_\Sigma)$.

The cell decomposition $\mathcal{D}$ is now defined for every cusp, each of which being either a torus or an annulus. In the latter case we may consider cycles relative to the boundary. We denote by $Z^{\mathrm{rel}}_1(\mathcal{D}, L)$, resp. $Z^{\mathrm{rel}}_1(\mathcal{D}', L')$, the subspace of relative cycles in $C_1(\mathcal{D}, L)$, resp. $C_1(\mathcal{D}', L')$. It is the orthogonal of $B_1(\mathcal{D}', L')$, resp. $B_1(\mathcal{D}, L)$, w.r.t. to the form ω defined as above, see §7.2.

9.2. We now set

$$J^2_{\partial M} = J^2_\Sigma \oplus C_1(\mathcal{D}, L), \quad (J^2_{\partial M})' = J^2_\Sigma \oplus C_1(\mathcal{D}', L')$$

and let

$$\Omega^2_{\partial M} : J^2_{\partial M} \times (J^2_{\partial M})' \to \mathbb{Z}$$

be the bilinear coupling obtained as the orthogonal sum of Ω^2_Σ and ω. As above it corresponds to these data the map $p_{\partial M} : J^2_{\partial M} \to ((J^2_{\partial M})')^*$, $p_{\partial M}(c) = \Omega^2_{\partial M}(c, \cdot)$, as well as the spaces

$$J_{\partial M} = J^2_{\partial M}/\mathrm{Ker}(p_{\partial M}) = J_\Sigma \oplus C_1(\mathcal{D}, L)$$

and

$$(J'_{\partial M})^* = \mathrm{Im}(p_{\partial M}) = J^*_\Sigma \oplus C_1(\mathcal{D}', L').$$

The bilinear coupling induces a canonical perfect coupling

$$\Omega_{\partial M} : J_{\partial M} \times J'_{\partial M} \to \mathbb{Z}$$

which identifies $J^*_{\partial M}$ with $J'_{\partial M}$.

9.3. As in the closed case (see §7.4) the linearization of the holonomy yields an extension of h_Σ to a map $h : J^2_{\partial M} \to J^2$. We then have the following diagram:

$$\begin{array}{ccccccc} C_1^{\rm or} + C_2 & \xrightarrow{F} & J^2 & \xrightarrow{p} & (J^2)^* & \xrightarrow{F^*} & C_1^{\rm or}+C_2 \\ & & h\big\uparrow & & h^*\big\downarrow & & \\ & & J^2_{\partial M} & & (J^2_{\partial M})^* & & \end{array}$$

Now it follows from (9.1.1) that the image of $h^* \circ p$ is contained in $J^*_{\partial M}$. Identifying it with $J'_{\partial M}$ using $\Omega_{\partial M}$ we get a map $g : J^2 \to J'_{\partial M}$. As in the closed case, for any $c \in J^2_{\partial M}$ and $e \in J^2$, we have:[11]

$$\Omega_{\partial M}(c, g(e)) = \Omega^2(e, h(c)). \tag{9.3.1}$$

We moreover have the following inclusions:
- $h(J^2_\Sigma \oplus Z_1^{\rm rel}(\mathcal{D}, L)) \subset \mathrm{Ker}(F^* \circ p)$,
- $h(J^2_\Sigma \oplus B_1(\mathcal{D}, L)) \subset \mathrm{Im}(F) + \mathrm{Ker}(p)$.

Denoting

$$\mathcal{H}_{\partial M} = (J^2_\Sigma \oplus Z_1^{\rm rel}(\mathcal{D}, L))/h^{-1}(\mathrm{Im}(F) + \mathrm{Ker}(p))$$

and

$$\mathcal{H}'_{\partial M} = (J_\Sigma \oplus Z_1^{\rm rel}(\mathcal{D}', L'))/g(\mathrm{Im}(F) + \mathrm{Ker}(p)),$$

we conclude that the maps h and g induce maps

$$\bar{h} : \mathcal{H}_{\partial M} \to \mathcal{H}(J) \quad \text{and} \quad \bar{g} : \mathcal{H}(J) \to \mathcal{H}'_{\partial M}.$$

It furthermore follows from (9.3.1) that $\Omega_{\partial M}$ induces a bilinear coupling

$$\bar{\Omega}_{\partial M} : \mathcal{H}_{\partial M} \times \mathcal{H}'_{\partial M} \to \mathbb{Z}.$$

9.4. Lemma. *The bilinear coupling $\bar{\Omega}_{\partial M}$ is non-degenerate.*

Proof. Denote by $\partial M \setminus \Sigma$ the union of the links (tori and annuli). The quotient J_Σ of J^2_Σ naturally identifies with the quotient of $\mathrm{Im}(F) + h(J^2_\Sigma)$ by $\mathrm{Im}(F) + \mathrm{Ker}(p)$. Note that the former identifies with the image of the $\mathbb{Z}$-module generated by *all* (oriented) edges and faces of K into J^2. We then have two short exact sequences

$$0 \to J_\Sigma \to \mathcal{H}_{\partial M} \to H_1^{\rm rel}(\partial M \setminus \Sigma, L) \to 0$$

and

$$0 \to H_1(\partial M \setminus \Sigma, L') \to \mathcal{H}'_{\partial M} \to J_\Sigma \to 0.$$

These are in duality w.r.t. $\Omega_{\partial M}$. Moreover this duality yields Ω_Σ on the product $J_\Sigma \times J_\Sigma$ and the intersection form, coupled with $[,]$, on $H_1^{\rm rel}(\partial M \setminus \Sigma, L) \times H_1(\partial M \setminus \Sigma, L)$. Since both are non-degenerate this proves the lemma. □

[11] Here we abusively use the same notation for c and its image in $J_{\partial M}$.

It now follows from (9.3.1) that $\Omega_{\partial M}(\cdot, g\circ h(\cdot)) = \Omega^2(h(\cdot), h(\cdot))$. And computations similar to §8.1 show that the right-hand side has a trivial kernel on $\mathcal{H}_{\partial M}$. The coupling $\bar{\Omega}_{\partial M}$ being non-degenerate we conclude that $\bar{h}$ is injective. As in the closed case, we may furthermore compute the dimension of $\mathcal{H}(J)$. Let ν_t be the number of tori and ν_a be the number of annuli. Then, computing the Euler characteristic of the double of K along Σ, the proof of lemma 7.13 yields the following:

9.5. Lemma. *The dimension of* $\mathcal{H}(J)$ *is* $4\nu_t + 2\nu_a + \dim(J_\Sigma)$.

This is easily seen to be the same as both the dimensions of $\mathcal{H}_{\partial M}$ and $\mathcal{H}'_{\partial M}$, see the proof of lemma 9.4. Over $\mathbb{C}$ the maps $\bar{h}$ and $\bar{g}$ are therefore invertible and we conclude that the form Ω on J induces a form $\bar{\Omega}$ on $\mathcal{H}(J)$ such that

$$\bar{\Omega}_{\partial M}(c, \bar{g}(e)) = \bar{\Omega}(e, \bar{h}(c)).$$

In particular $\bar{\Omega}$ is determined by $\Omega_{\partial M}$ and the invariant $\delta(\beta(K))$ only depends on the boundary coordinates. This concludes the proof of theorem 5.14.

10. Examples

In this section we describe the complement of the figure eight knot obtained by gluing two tetrahedra. Let z_{ij} and w_{ij} be the coordinates associated to the edge ij of each of them.

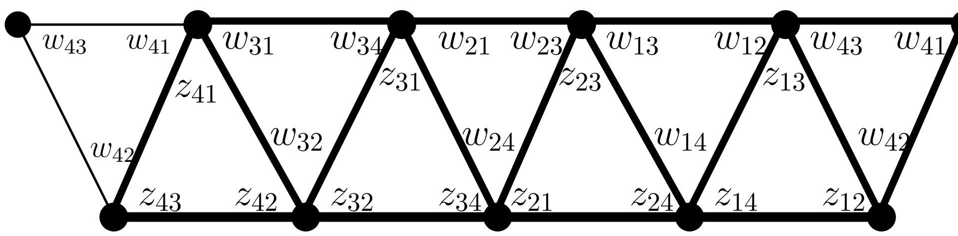

Figure 14. The link at the boundary for the figure eight knot

The edge equations are:

- $z_{12}w_{12}z_{13}w_{43}z_{43}w_{42} = 1$
- $z_{21}w_{21}z_{31}w_{34}z_{34}w_{24} = 1$
- $z_{42}w_{32}z_{32}w_{31}z_{41}w_{41} = 1$
- $z_{24}w_{23}z_{23}w_{13}z_{14}w_{14} = 1$

The face equations are:

- $z_{13}z_{43}z_{23}w_{14}w_{34}w_{24} = 1$
- $z_{14}z_{24}z_{34}w_{21}w_{41}w_{31} = 1$
- $z_{12}z_{42}z_{32}w_{13}w_{43}w_{23} = 1$
- $z_{21}z_{31}z_{41}w_{12}w_{32}w_{42} = 1$

And the holonomies are:

- $A = z_{41}\frac{1}{w_{32}}z_{31}\frac{1}{w_{24}}z_{23}\frac{1}{w_{14}}z_{13}\frac{1}{w_{41}}$
- $B = z_{43}\frac{1}{w_{41}}$
- $A^* = \frac{1}{z_{14}}\frac{w_{14}w_{41}}{w_{32}}\frac{1}{z_{13}}\frac{w_{13}w_{31}}{w_{24}}\frac{1}{z_{32}}\frac{w_{23}w_{32}}{w_{14}}\frac{1}{z_{31}}\frac{w_{31}w_{13}}{w_{42}}$
- $B^* = \frac{1}{z_{34}}\frac{w_{23}w_{32}}{w_{41}}$

If $A = B = A^* = B^* = 1$ the solutions of the equations correspond to unipotent structures. The complete hyperbolic structure on the complement of the figure eight knot determines a solution of the above equations. In fact, in that case, if $\omega = \frac{1+i\sqrt{3}}{2}$ then

$$z_{12} = z_{21} = z_{34} = z_{43} = w_{12} = w_{21} = w_{34} = w_{43} = \omega$$

is a solution the equations as obtained in [15].

The spherical CR structures with unipotent boundary holonomy were obtained in [5] as the following solutions (up to conjugation):

$$z_{12} = \bar{z}_{21} = z_{34} = \bar{z}_{43} = w_{12} = \bar{w}_{21} = w_{34} = \bar{w}_{43} = \omega,$$

$$z_{12} = \frac{5-i\sqrt{7}}{4}, z_{21} = \frac{3-i\sqrt{7}}{8}, z_{34} = \frac{5+i\sqrt{7}}{4}, z_{43} = \frac{3+i\sqrt{7}}{8}$$

$$w_{12} = \frac{3-i\sqrt{7}}{8}, w_{21} = \frac{5-i\sqrt{7}}{4}, w_{34} = \frac{3+i\sqrt{7}}{8}, w_{43} = \frac{5+i\sqrt{7}}{4}$$

and

$$z_{12} = \frac{-1+i\sqrt{7}}{4}, z_{21} = \frac{3-i\sqrt{7}}{2}, z_{34} = \frac{-1-i\sqrt{7}}{4}, z_{43} = \frac{3+i\sqrt{7}}{2}$$

$$w_{12} = \frac{3+i\sqrt{7}}{2}, w_{21} = \frac{-1-i\sqrt{7}}{4}w_{34} = \frac{3-i\sqrt{7}}{2}, w_{43} = \frac{-1+i\sqrt{7}}{4}$$

The first solution above corresponds to a discrete representation of the fundamental group of the complement of the figure eight knot in $\mathrm{PU}(2,1)$ with faithful boundary holonomy. Moreover, its action on complex hyperbolic space has limit set the full boundary sphere. The other solutions have cyclic boundary holonomy.

We will call these solutions standard structures on the complement of the figure eight knot. Recently, P.-V. Koseleff proved that they are the only solutions to the equations:

10.1. Proposition. *The only unipotent flag* $\mathrm{SL}(3,\mathbb{C})$*-structures on the complement of the figure eight knot are the standard structures.*

11. Applications

11.1. Volumes of decorated tetrahedra complex. A decorated closed tetrahedra complex K provides us with an element $z \in \mathbb{C}^\times \otimes_{\mathbb{Z}} J^*\left[\frac{1}{2}\right]$ which satisfies the face and edge conditions as well as the non-linear equations

$$z_{ik}(T_\nu) = \frac{1}{1 - z_{ij}(T_\nu)}.$$

Let $X = \mathbb{C}^\times \otimes_{\mathbb{Z}} J^*\left[\frac{1}{2}\right]$; this is a complex variety.

Following §3.6 we define the *volume* of K as:

$$\mathrm{Vol}(K) = \frac{1}{4} D(\beta(K)). \tag{11.1.1}$$

This defines a real analytic function on X:

$$\mathrm{Vol} : X \to \mathbb{C}.$$

Let $\mathcal{F}(X)^\times$ be the group of invertible real analytic functions on X and $\Omega^1(X)$ the space of real analytic 1-form on X. The holonomy elements A_s , A_s^* and B_s, B_s^* of theorem 5.14 define elements in $\mathcal{F}(X)^\times$. Now there is a map $\mathrm{Im}(d\log \wedge_{\mathbb{Z}} \log) : \mathcal{F}(X)^\times \wedge_{\mathbb{Z}} \mathcal{F}(X)^\times \to \Omega^1(X)$ defined by:

$$\mathrm{Im}(d\log \wedge_{\mathbb{Z}} \log)(f \wedge_{\mathbb{Z}} g) = \mathrm{Im}\left(\log|g| \cdot d(\log f) - \log|f| \cdot d(\log g)\right).$$

Following Neumann-Zagier [13] we want to compute the variation of $\mathrm{Vol}(K)$ as we vary $z \in X$. Equivalently we compute $d\mathrm{Vol} \in \Omega^1(X)$ in the following:

11.2. Proposition. *We have:*

$$d\mathrm{Vol} = \frac{1}{12} \sum_s \mathrm{Im}(d\log \wedge_{\mathbb{Z}} \log)(2A_s \wedge_{\mathbb{Z}} B_s + 2A_s^* \wedge_{\mathbb{Z}} B_s^* + A_s^* \wedge_{\mathbb{Z}} B_s + A_s \wedge_{\mathbb{Z}} B_s^*).$$

Proof. The derivatives of $D(z)$ are elementary functions:

$$\frac{\partial D}{\partial z} = \frac{i}{2}\left(\frac{\log|1-z|}{z} + \frac{\log|z|}{1-z}\right), \quad \frac{\partial D}{\partial \overline{z}} = -\frac{i}{2}\left(\frac{\log|1-z|}{\overline{z}} + \frac{\log|z|}{1-\overline{z}}\right). \tag{11.2.1}$$

Assume that the parameter $z \in \mathbb{C}^*$ is varying in dependence on a single variable t. Then:

$$\begin{aligned}\frac{d}{dt} D(z_t) &= \frac{i}{2}\left[\left(\frac{\log|1-z|}{z} + \frac{\log|z|}{1-z}\right)\frac{dz}{dt} - \left(\frac{\log|1-z|}{\overline{z}} + \frac{\log|z|}{1-\overline{z}}\right)\frac{d\overline{z}}{dt}\right] \\ &= \mathrm{Im}\left(\left(\frac{d}{dt}\log(z)\right)\log|1-z| - \left(\frac{d}{dt}\log(1-z)\right)\log|z|\right).\end{aligned}$$

In otherwords: $dD = \mathrm{Im}(d\log \wedge_{\mathbb{Z}} \log)(z \wedge_{\mathbb{Z}} (1-z))$. And proposition 11.2 follows from theorem 5.14 and (11.1.1). □

Remark. Proposition 11.2 implies in particular that the variation of the volume only depends on the contribution of the boundary. Specializing to the hyperbolic case we recover the result of Neumann-Zagier [13], see also Bonahon [3, Theorem 3].

11.3. Weil-Petersson forms. Let k be an arbitrary field. The Milnor group $K_2(k)$ is the cokernel of $\delta : \mathcal{P}(k) \to k^\times \wedge_{\mathbb{Z}} k^\times$.

Let $X_\Sigma = \mathbb{C}^\times \otimes_{\mathbb{Z}} J_\Sigma^* \left[\frac{1}{2}\right]$; it is a complex manifold. As above we may consider the field $\mathcal{F}(X_\Sigma)^\times$; we let $\Omega^2_{\text{hol}}(X_\Sigma)$ denote the space of holomorphic 2-forms on X_Σ. The element z_Σ defines an element in $\mathcal{F}(X_\Sigma)^\times$. We still denote the projection of $z_\Sigma \wedge_{\Omega_\Sigma^*} z_\Sigma$ into $K_2(\mathcal{F}(X_\Sigma)^\times)$.

Now, since $d\log \wedge_{\mathbb{Z}} d\log((1-f) \wedge_{\mathbb{Z}} f) = 0$, there is a group homomorphism:

$$d\log \wedge_{\mathbb{Z}} d\log : K_2(\mathcal{F}(X)_\Sigma^\times) \to \Omega^2(X_\Sigma), \quad f \wedge_{\mathbb{Z}} g \mapsto d\log(f) \wedge_{\mathbb{Z}} d\log(g).$$

In the hyperbolic case and when the decoration is unipotent, Fock and Goncharov [7] prove that

$$\frac{1}{2} d\log z_\Sigma \wedge_{\Omega_\Sigma^*} d\log z_\Sigma = d\log \wedge_{\mathbb{Z}} d\log(W(\Sigma))$$

is the Weil-Petersson form. Although expected, the analogous statement in the SL(3)-case seems to be open. In any case theorem 5.14 implies that this form vanishes, equivalently the "Weil-Petersson forms" corresponding to the different components of Σ add up to zero.

References

[1] Nicolas Bergeron, Elisha Falbel, and Antonin Guilloux. Local rigidity for $\mathrm{SL}(3,\mathbb{C})$ representations of 3-manifolds groups. 2012.

[2] Francis Bonahon. Shearing hyperbolic surfaces, bending pleated surfaces and Thurston's symplectic form. *Ann. Fac. Sci. Toulouse Math. (6)*, 5(2):233–297, 1996.

[3] Francis Bonahon. A Schläfli-type formula for convex cores of hyperbolic 3-manifolds. *J. Differential Geom.*, 50(1):25–58, 1998.

[4] Elisha Falbel. A volume function for CR tetrahedra. preprint http://arxiv.org/abs/0907.2305.

[5] Elisha Falbel. A spherical CR structure on the complement of the figure eight knot with discrete holonomy. *Journal of Differential Geometry*, 79:69–110, 2008.

[6] Elisha Falbel and Qingxue Wang. A combinatorial invariant for spherical CR structures. preprint http://arxiv.org/abs/1007.5228.

[7] Vladimir Fock and Alexander Goncharov. Moduli spaces of local systems and higher Teichmüller theory. *Publ. Math. Inst. Hautes Études Sci.*, (103):1–211, 2006.

[8] Vladimir V. Fock and Alexander B. Goncharov. Cluster ensembles, quantization and the dilogarithm. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(6):865–930, 2009.

[9] Juliette Genzmer. Sur les triangulations des structures CR-sphériques. Thesis, Paris 6 June 2010.

[10] Yuichi Kabaya. Pre-Bloch invariants of 3-manifolds with boundary. *Topology Appl.*, 154(14):2656–2671, 2007.

[11] Walter D. Neumann. Combinatorics of triangulations and the Chern-Simons invariant for hyperbolic 3-manifolds. In *Topology '90 (Columbus, OH, 1990)*, volume 1 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 243–271. de Gruyter, Berlin, 1992.

[12] Walter D. Neumann. Extended Bloch group and the Cheeger-Chern-Simons class. *Geom. Topol.*, 8:413–474 (electronic), 2004.

[13] Walter D. Neumann and Don Zagier. Volumes of hyperbolic three-manifolds. *Topology*, 24(3):307–332, 1985.

[14] A. A. Suslin. Homology of GL_n, characteristic classes and Milnor K-theory. *Trudy Mat. Inst. Steklov.*, 165:188–204, 1984. Algebraic geometry and its applications.

[15] W. Thurston. The geometry and topology of 3-manifolds. 1979. Lecture Notes.

[16] Don Zagier. The dilogarithm function. In *Frontiers in number theory, physics, and geometry. II*, pages 3–65. Springer, Berlin, 2007.

[17] Christian K. Zickert. The extended Bloch group and algebraic K-theory. preprint http://arxiv.org/abs/0910.4005v3.

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