REPRESENTATIONS OF FUNDAMENTAL GROUPS OF 3-MANIFOLDS INTO PGL$(3,\mathbb{C})$: EXACT COMPUTATIONS IN LOW COMPLEXITY.

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Abstract. In this paper we are interested in computing representations of the fundamental group of a 3-manifold into PSL$(3,\mathbb{C})$ (in particular in PSL$(2,\mathbb{C})$, PSL$(3,\mathbb{R})$ and PU$(2,1)$). The representations are obtained by gluing decorated tetrahedra of flags as in [11, 1]. We list complete computations (giving 0-dimensional or 1-dimensional solution sets) for the first complete hyperbolic non-compact manifolds with finite volume which are obtained gluing less than three tetrahedra with a description of the computer methods used to find them.

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1. Introduction

In this paper we are interested in obtaining representations of the fundamental group of a 3-manifold into PGL$(3,\mathbb{C})$. They will be defined via certain topological triangulations with additional geometric data carried by the 0-skeleton of the triangulation.

The most important example being hyperbolic geometry we will mainly consider 3-manifolds which carry a complete hyperbolic structure. That is, complete riemannian manifolds with constant negative curvature equal to -1.

In order to simplify the description we first treat open manifolds which are the interior of manifolds with boundary. If the hyperbolic manifold is complete and of finite volume one shows that its ideal boundary is a union of tori. In that case we consider ideal triangulations such that the 0-skeleton is in the ideal boundary. That is, identifying each boundary to a

This work was supported in part by the ANR through the project "Structures Géométriques et Triangulations".

point, we consider a triangulation of the quotient space such that the 0-skeleton coincides with the set of boundary points.

In hyperbolic space the ideal tetrahedra are described by a configuration of four points in the boundary of hyperbolic space, that is, $\mathbb{CP}^1$. The cross-ratio is sufficient to parameterize ideal tetrahedra up to the isometry group. Ideal triangulations were used by Thurston to obtain examples of hyperbolic manifolds (see [20]). A complete hyperbolic structure is obtained on the manifold once a system of equations on the cross-ratio coordinates of those ideal tetrahedra are solved. A very successful computational tool, Snappea, was developed by Jeff Weeks (see its very readable description in [21]) which solves numerically the system of equations.

Other geometric structures in 3-manifolds are associated to subgroups of $\mathrm{PGL}(3, \mathbb{C})$, namely, flag structures which are associated to $\mathrm{PGL}(3, \mathbb{R})$ and spherical Cauchy-Riemann structures (CR structures) which are associated to the group $\mathrm{PU}(2, 1)$. These geometric structures are not well understood, in particular one does not know which 3-manifolds can carry one of them.

The first step in trying to find a structure is to obtain a representation of the fundamental group in one of these groups (see [11] for the case $\mathrm{PU}(2, 1)$). We will use the raw data of triangulation of 3-manifolds as obtained in Snappea. Then we proceed as in [1]; to a triangulation we associate a decoration given by a choice of flags at each vertex of the tetrahedra. We set equations imposing that gluing of the tetrahedra is compatible with the decoration of flags. The solution of the equations would give immediately a representation of the fundamental group of the 3-manifold into $\mathrm{PGL}(3, \mathbb{C})$ (called the holonomy representation). Imposing that the holonomy restricted to the boundary tori be unipotent is a natural condition which is analogous to the condition of completeness in the hyperbolic case.

A generalization of the gluing equations for higher dimensions was described in [13]. A simplified set of equations describing representations into $\mathrm{SL}(n, \mathbb{C})$ is also described in [14]. They are based on decorations of tetrahedra by affine flags (see also the a-coordinates in [1]) but they miss many solutions related to $\mathrm{PGL}(n, \mathbb{C})$ representations obtained via projective flags. On the other hand it is not clear if the computations are actually simpler in the affine flag case as shown by limitations in the computations in [15].

In this paper we deal with methods to solve the equations and obtain a list of solutions for manifolds with low complexity. In particular we obtain all solutions for ideal triangulations of complete cusped hyperbolic manifolds with less than four tetrahedra.

In the first sections, in order to make the paper self-contained, we review results in [1] (see also [11] for the $\mathrm{PU}(2, 1)$ case). That contains the parametrization of decorated tetrahedra (configurations of four flags in $\mathbb{CP}^2$), the description of decorated ideal triangulations, the compatibility equations which will lead us to a system of equations and the computation of the holonomy representation. The special case we treat in this paper has unipotent boundary holonomy.

In section 6 we describe the methods used to solve the system of equations.

In the remaining sections we describe several important examples which illustrate the methods and the results. The complete census up to three tetrahedra is shown in Table 1 (more details can be obtained in our webpage .../SGT).

A very important observation which came out from the examples is that solutions of the gluing equations with unipotent boundary holonomy might not be zero dimensional even in low complexity. Computations for representations in $\mathrm{PSL}(2,\mathbb{C})$ show that, for cusped hyperbolic manifolds obtained with four tetrahedra, only two have a one parameter family component (there are no examples with less than four tetrahedra). On the other hand, we don't have an efficient method to decide when the solution will have a positive dimensional component. The one dimensional components in the examples were not systematically studied in this paper. We did not search for $\mathrm{PU}(2,1)$ or $\mathrm{PSL}(3,\mathbb{R})$ solutions in tall hese components. We verified, though, that the one dimensional components (in all examples) have at most a finite number of $\mathrm{PSL}(2,\mathbb{C})$ solutions. Moreover, they are all in $\mathrm{PSL}(2,\mathbb{R})$ so the volumes presented in the table concern indeed all $\mathrm{PSL}(2,\mathbb{C})$ representations arising from the given triangulation. On the other hand, $\mathrm{PU}(2,1)$ solutions might appear in families (of real dimension one or two) inside one dimensional components (see the worked example m003 in [2] where there is a whole complex one dimensional component consisting of CR solutions). Other examples are harder to analyse and we plan to study them in a future paper.

In the case of the figure eight knot the non-hyperbolic representations in $\mathrm{PGL}(3,\mathbb{C})$ were obtained in [11]. They happen to be discrete representations in $\mathrm{PU}(2,1)$ and two of them give rise to spherical structures on the complement of the figure eight knot (see [7, 8]). In the case of the Whitehead link, R. Schwartz ([19]) was able to analyse one representation (obtained in a completely independent way as a subgroup of $\mathrm{PU}(2,1)$ generated by reflections) and showed that the complement of the link has a spherical CR structure. Other representations of the fundamental group of the complement of the Whitehead link obtained similarly using reflection groups were analysed in [17]. In their case the representations form a one parameter family parameterized by the boundary holonomy. We obtain here independently the unipotent representation which they prove to be the holonomy of a CR structure on the complement of the Whitehead link.

It turns out that for the examples of CR structures obtained until now, the boundary holonomy of the $\mathrm{PU}(2,1)$ representations are abelian of rank one. This fact made us try to chase representations into $\mathrm{PU}(2,1)$ with rank one boundary holonomy in the hope that they will correspond to CR structures. For the ideal triangulation of the Whitehead link complement with four tetrahedra there is only one unipotent representation with rank one boundary holonomy (it is the same one obtained independently in [17]).

Above all, the computations encourage us to study CR structures. There is a great proportion of $\mathrm{PU}(2,1)$ representations obtained among all $\mathrm{PGL}(3,\mathbb{C})$ representations and some of them might correspond to holonomies of geometric structures. On the other hand it is a challenging problem to decide if a 3-manifold has a geometric structure. It is not clear when the representations obtained here are discrete (for general discrete subgroups of $\mathrm{PGL}(3,\mathbb{C})$ see [5]) . On the other hand, a criterium for rigidity of representations is given in [2] and a discussion of generic discreteness of the boundary holonomy is given in [16]. In this paper we don't address these questions.

Another interesting observation from Table 1 is the fact that the hyperbolic volume is the maximum among the volumes of all unipotent $\mathrm{PGL}(3,\mathbb{C})$ representations. This was shown recently using methods of bounded cohomology in [4]. On the other hand, $\mathrm{PU}(2,1)$ representations have null volume by cohomological reasons (see [9]).

We thank M. Thistlethwaite for introducing us to Snappea and making available his list of cusped 3-manifolds with particularly simple generators for the boundary fundamental group. We used his list for our computations. We also thank N. Bergeron, M. Deraux, A. Guilloux, M. Thistlethwaite and S. Tillmann for all the stimulating discussions.

2. Three geometric structures on three manifolds

Geometric structures on three manifolds have been extensively studied. The usual setting is an action of a Lie group G on a homogeneous 3-manifold X. An (X, G) structure on a 3-manifold M being a family of charts $\phi_i : U_i \to X$ from open subsets forming a cover of M with transition functions g_{ij} (given by $\phi_j = g_{ji} \circ \phi_i$) in the Lie group G.

Hyperbolic structures, that is, $(\mathbb{H}^3_{\mathbb{R}}, \mathrm{PSL}(2,\mathbb{C}))$ structures (where $\mathbb{H}^3_{\mathbb{R}}$ is hyperbolic 3-space) were shown by Thurston to be very important and his theory made possible, as a far reaching consequence of his ideas, to solve Poincaré's conjecture. The boundary of the three dimensional hyperbolic space can be identified to $\mathbb{CP}^1$. By embedding $\mathbb{CP}^1$ into $\mathbb{CP}^2$ as a conic we observe that a point in $\mathbb{CP}^1$ defines a point and a line containing it in $\mathbb{CP}^2$. Namely the point obtained by the embedding and the tangent line to the embedding passing through that point. This justifies considering configuration of flags associated to triangulations (recall that a pair consisting of a point and a projective line containing the point is called a flag). We will associate to each vertex of a tetrahedron a flag in the following section.

CR geometry is modelled on the sphere $\mathbb{S}^3$ equipped with a natural $\mathrm{PU}(2,1)$ action. More precisely, consider the group $\mathrm{U}(2,1)$ preserving the Hermitian form $\langle z, w\rangle = w^* J z$ defined on $\mathbb{C}^3$ by the matrix

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and the following cones in $\mathbb{C}^3$;

$$V_0 = \left\{ z \in \mathbb{C}^3 - \{0\} \quad : \quad \langle z, z\rangle = 0 \ \right\},$$

$$V_- = \left\{ z \in \mathbb{C}^3 \quad : \quad \langle z, z\rangle < 0 \ \right\}.$$

Let $\pi : \mathbb{C}^3 \setminus \{0\} \to \mathbb{CP}^2$ be the canonical projection. Then $\mathbb{H}^2_{\mathbb{C}} = \pi(V_-)$ is the complex hyperbolic space and its boundary is

$$\partial \mathbb{H}^2_{\mathbb{C}} = \mathbb{S}^3 = \pi(V_0) = \{[x, y, z] \in \mathbb{CP}^2 \mid x\bar{z} + |y|^2 + z\bar{x} = 0 \ \}.$$

The group of biholomorphic transformations of $\mathbb{H}^2_{\mathbb{C}}$ is then $\mathrm{PU}(2,1)$, the projectivization of $\mathrm{U}(2,1)$. It acts on $\mathbb{S}^3$ by CR transformations. An element $x \in \mathbb{S}^3$ gives rise to a flag in $\mathbb{CP}^2$ where the line corresponds to the unique complex line tangent to $\mathbb{S}^3$ at x.

A third real three dimensional geometry is the geometry of real flags in $\mathbb{R}^3$. That is the geometry of the space of all couples $[p, l]$ where $p \in \mathbb{RP}^2$ and l is a projective line containing p. The space of flags is identified to the quotient

$$\mathrm{SL}(3,\mathbb{R})/B$$

where B is the Borel group of all upper triangular matrices.

In the next section we describe the space of flags in $\mathbb{CP}^2$ which will be the common framework to describe the three geometries based on $\mathrm{PSL}(2,\mathbb{C}), \mathrm{PU}(2,1)$ and $\mathrm{PGL}(3,\mathbb{R})$.

3. Flag tetrahedra

In this section we recall the parametrization of configurations of four flags in the projective space $\mathbb{CP}^2$ (more details can be seen in [1]). Let $V = \mathbb{C}^3$. A flag in V is usually seen as a line and a plane, the line belonging to the plane. Using the dual vector space V^* and the projective spaces $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$, define the spaces of *flags* $\mathcal{F}l$ by the following:

$$\mathcal{F}l \quad = \quad \{([x],[f]) \in \mathbb{P}(V) \times \mathbb{P}(V^*) \mid f(x) = 0\}.$$

The natural action of $\mathrm{SL}(3,\mathbb{C})$ on $\mathcal{F}l$ makes us identify the space of flags with the homogeneous space $\mathrm{SL}(3,\mathbb{C})/B$, where B is the Borel subgroup of upper-triangular matrices in $\mathrm{SL}(3,\mathbb{C})$. A *generic configuration of flags* $([x_i],[f_i])$, $1 \leq i \leq n+1$ is given by $n+1$ points $[x_i]$ in general position and $n+1$ lines f_i in $\mathbb{P}(V)$ such that $f_j(x_i) \neq 0$ if $i \neq j$. A configuration of ordered points in $\mathbb{P}(V)$ is said to be in *general position* when they are all distinct and no three points are contained in the same line.

Up to the action of $\mathrm{SL}(3)$, a generic configuration of three flags $([x_i],[f_i])_{1\leq i\leq 3}$ has only one invariant given by the triple ratio

$$X = \frac{f_1(x_2)f_2(x_3)f_3(x_1)}{f_1(x_3)f_2(x_1)f_3(x_2)} \in \mathbb{C}^\times$$

3.1. **Coordinates for a tetrahedron of flags.**

We recall the parametrization used in [1]. We refer to Figure 3.1 which displays the coordinates.

Let $([x_i],[f_i])_{1\leq i\leq 4}$ be a generic tetrahedron. We define a set of 12 coordinates on the edges of the tetrahedron (1 for each oriented edge). To define the coordinate z_{ij} associated to the edge ij, we first define k and l such that the permutation $(1,2,3,4) \mapsto (i,j,k,l)$ is even. The pencil of (projective) lines through the point x_i is a projective line $\mathbb{P}_1(\mathbb{C})$. We have four points in this projective line: the line $\ker(f_i)$ and the three lines through x_i and one of the x_l for $l \neq i$. We define z_{ij} as the cross-ratio of four flags by

$$z_{ij} := [\ker(f_i), (x_i x_j), (x_i x_k), (x_i x_l)].$$

Note that we follow the usual convention that the cross-ratio of four points x_1, x_2, x_3, x_4 on a line is the value at x_4 of a projective coordinate taking value ∞ at x_1, 0 at x_2, and 1 at x_3. So we employ the formula

$$[x_1, x_2, x_3, x_4] = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_4)(x_2 - x_3)}$$

for the cross-ratio.

At each face (ijk) (oriented as the boundary of the tetrahedron (1234)), the 3-ratio is the opposite of the product of all cross-ratios "leaving" this face:

$$z_{ijk} = \frac{f_i(x_j)f_j(x_k)f_k(x_i)}{f_i(x_k)f_j(x_i)f_k(x_j)} = -z_{il}z_{jl}z_{kl}.$$

Observe that if the same face (ikj) (with opposite orientation) is common to a second tetrahedron T' then

$$z_{ikj}(T) = \frac{1}{z_{ijk}(T')}.$$

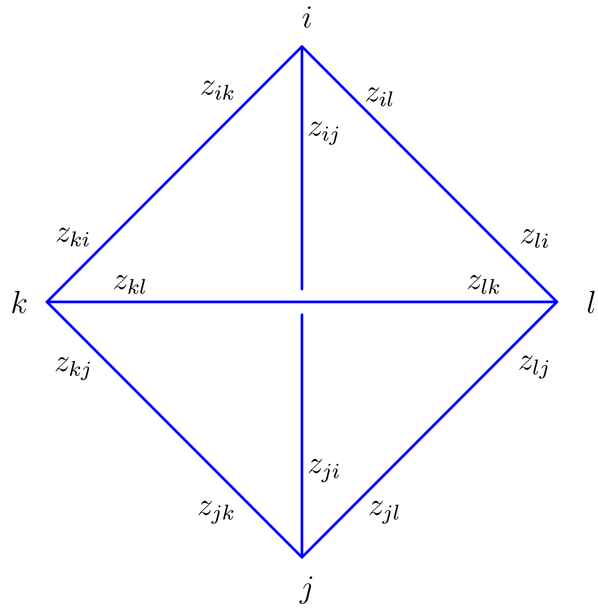

FIGURE 1. The cross-ratio coordinates (or z-coordinates).

Of course there are relations between the whole set of coordinates, namely, the three cross-ratio leaving a vertex are algebraically related:

$$z_{ik} = \frac{1}{1 - z_{ij}}, \qquad z_{il} = 1 - \frac{1}{z_{ij}}. \tag{3.1.1}$$

Observe that the relations follow a cyclic order around each vertex which is defined by the orientation of the tetrahedron. We also have

$$z_{ij}\, z_{ik}\, z_{il} = -1. \tag{3.1.2}$$

The next proposition shows that a tetrahedron is uniquely determined, up to the action of SL(3), by four numbers. One can pick a variable at each vertex. In fact, the space of flags is a complex manifold $(\mathrm{PSL}(3, \mathbb{C})/B)$ and, therefore, configurations of flags have a natural complex structure.

3.2. **Proposition.** [1] *The space of generic tetrahedra is biholomorphic to* $(\mathbb{C} \setminus \{0, 1\})^4$.

One can use one cross-ratio coordinate at each vertex, for instance $(z_{12}, z_{21}, z_{34}, z_{43})$ or $(z_{13}, z_{21}, z_{32}, z_{43})$.

3.3. **Hyperbolic ideal tetrahedra.** An ideal hyperbolic tetrahedron is given by 4 points on the boundary of $\mathbb{H}^3_{\mathbb{R}}$, i.e. $\mathbb{P}_1(\mathbb{C})$. Up to the action of $\mathrm{PSL}(2, \mathbb{C})$, these points are in homogeneous coordinates $[0,1]$, $[1,0]$, $[1,1]$ and $[1,z]$ – the complex number z being the cross-ratio of these four points.

We may embed $\mathbb{P}_1(\mathbb{C})$ into $\mathbb{P}_2(\mathbb{C})$ via the Veronese map, given in homogeneous coordinates by

$$[x,y] \quad \mapsto \quad [x^2, xy, y^2]$$

and therefore this map and its derivative defines a map from $\mathbb{P}_1(\mathbb{C})$ to the variety of flags $\mathcal{F}l$.

Let T be the tetrahedron $h([0,1])$, $h([1,0])$, $h([1,1])$ and $h([1,z])$. Its image in the variety of flags given by the above map has coordinates (see [1])

$$z_{12}(T) = z_{21}(T) = z_{34}(T) = z_{43}(T) = z.$$

Conversely, given parameters satisfying the equations above, they define a unique hyperbolic tetrahedron.

3.4. **The CR case.** Recall that an element $x \in \mathbb{S}^3$ gives rise to an element $([x],[f]) \in \mathcal{F}l(\mathbb{C})$ where $[f]$ corresponds to the unique complex line tangent to $\mathbb{S}^3$ at x. The following proposition describes the space of generic configurations of four points in $\mathbb{S}^3$.

3.5. **Proposition** ([11, 1])**.** *Generic configurations (up to translations by* $\mathrm{PU}(2,1)$*) of four points in* $\mathbb{S}^3$ *not contained in an* $\mathbb{R}$*-circle are parametrized by generic configurations of four flags with coordinates* z_{ij}*,* $1 \leq i \neq j \leq 4$ *satisfying the three complex equations*

$$z_{ij} z_{ji} = \overline{z_{kl}}\,\overline{z_{lk}} \tag{3.5.1}$$

not all of them being real and such that $z_{ji} z_{ki} z_{li} \neq 1$ *for each face with vertices* $\{j,k,l\}$*.*

The conditions (3.5.1) together with relation (3.1.2) imply that the triple ratio of each face satisfies $|z_{ijk}| = 1$. For the tetrahedron to be CR we need to verify the condition $z_{ijk} = -z_{ji} z_{ki} z_{li} \neq -1$ for each face with vertices $\{j,k,l\}$ (the triple ratio is never -1 for a CR generic triple of flags).

Conditions $z_{ij} z_{ji} = \overline{z_{kl}}\,\overline{z_{lk}} \in \mathbb{R}$ might describe other configurations of four flags which are not CR. From $z_{ijk} z_{ilj} = \dfrac{z_{lk} z_{kl}}{z_{ij} z_{ji}}$, we deduce that in this case $z_{ijk} z_{ilj} = 1$ and therefore $z_{ijk} = z_{ilj} = z_{ikl} = z_{jkl} = \pm 1$. If the cross-ratio invariants are all real, we obtain not necessarily CR configurations (when one of the triple ratios is -1), but including the ones contained in an $\mathbb{R}$-circle which coincide with the real hyperbolic ones with real cross ratio. That is,

$$z_{12}(T) = z_{21}(T) = z_{34}(T) = z_{43}(T) = x \in \mathbb{R} - \{0,1\}.$$

3.6. **The SL$(3,\mathbb{R})$ case.** Clearly, a configuration contained in the space of real flags is characterized by having all its invariants z_{ij} real. Observe that the three cases intersect precisely for configurations corresponding to degenerate real hyperbolic configurations.

4. Ideal triangulations by flag tetrahedra

We use the definition of an ideal triangulation of a 3-manifold as it is used usually in computations with Snappea. It is a union of 3-simplices $K = \bigcup_\nu T_\nu$ with face identifications (which are simplicial maps). We let $K^{(0)}$ be the union of vertices in K. The manifold $K - K^{(0)}$ is, topologically, the interior of a compact manifold when the vertices $K^{(0)}$ are deleted. In fact, $K - K^{(0)}$ is an (open) 3-manifold that retracts onto a compact 3-manifold with boundary M (the boundary being the link of $K^{(0)}$).

Given such a compact oriented 3-manifold M with boundary, we call a triangulation K as above an *ideal triangulation* of M. A *parabolic decoration* of an ideal triangulation is the data of a flag for each vertex (equivalently a map from the 0-skeleton of the complex to $\mathcal{F}l$). A parabolic decoration together with an ordering of the vertices of each 3-simplex equip each tetrahedron with a set of coordinates as defined in section 3.

From now on we fix K a decorated oriented ideal triangulation of a 3-manifold together with an ordering of the vertices of each 3-simplex of K. Denote by T_ν, $\nu = 1, \ldots, N$, the tetrahedra of K and $z_{ij}(T_\nu)$ the corresponding z-coordinates (or cross-ratio coordinates). We impose the following compatibility conditions which will imply the existence of a well defined representation of the fundamental group of the manifold M into $\mathrm{PGL}(3, \mathbb{C})$.

4.1. **Consistency relations.** (cf. [10, 1])

(Face equations) Let T and T' be two tetrahedra of K with a common face (ijk) (oriented as a boundary of T), then $z_{ijk}(T) z_{ikj}(T') = 1$.

For a fixed edge $e \in K$ let $T_{\nu_1}, \ldots, T_{\nu_{n_e}}$ be the n_e tetrahedra in K which contain an edge which projects unto $e \in K$ (counted with multiplicity). For each tetrahedron in K as above we consider its edge ij corresponding to e.

(Edge equations) $z_{ij}(T_{\nu_1}) \cdots z_{ij}(T_{\nu_{n_e}}) = z_{ji}(T_{\nu_1}) \cdots z_{ji}(T_{\nu_{n_e}}) = 1.$

4.2. **Volume.**

We recall the definition of volume of a tetrahedron of flags in [1]. The *Bloch-Wigner dilogarithm* function is

$$\begin{aligned} D(x) &= \arg(1-x)\log|x| - \mathrm{Im}\left(\int_0^x \log(1-t)\tfrac{dt}{t}\right), \\ &= \arg(1-x)\log|x| + \mathrm{Im}(\ln_2(x)). \end{aligned}$$

Here $\ln_2(x) = -\int_0^x \log(1-t)\frac{dt}{t}$ is the dilogarithm function. The function D is well-defined and real analytic on $\mathbb{C} - \{0, 1\}$ and extends to a continuous function on $\mathbb{C}P^1$ by defining $D(0) = D(1) = D(\infty) = 0$.

Given a tetrahedron of flags with coordinates $(z_1(T), z_2(T), z_3(T), z_4(T))$, we define its volume as

$$\mathrm{Vol}(T) = \tfrac{1}{4}(D(z_1) + D(z_2) + D(z_3) + D(z_4)).$$

The volume of a triangulation is the sum of the volumes of its tetrahedra. If the tetrahedron is hyperbolic this definition coincides with the volume of volume in hyperbolic geometry. If the tetrahedron is real its volume is zero. Although CR tetrahedra have generically a non zero volume, if a triangulation is such that all tetrahedra are CR then the total volume is zero ([9]).

5. Holonomy of a decoration

In this section we recall how to compute the holonomy of a decoration described in [1]. A decoration of a triangulation of a manifold M gives rise to a representation (called holonomy representation)

$$\rho : \pi_1(M, p_0) \to \mathrm{PGL}(3, \mathbb{C}).$$

The base point is not important if we study representations up to conjugation. In fact, each solution of the consistency equations gives a conjugacy class of representations but a special choice of base point is needed in explicit computations.

In order to compute holonomies we first fix a base point which in fact will be a face (with ordered vertices) of one of the tetrahedra. We then follow paths representing the generators of the fundamental group of the manifold which we decompose in arcs contained in each tetrahedron and which are transverse to their sides or totally contained in one side. For each arc we write the contribution to the holonomy as is explained in the following.

It is useful to associate to a face three points located near each vertex of the face. The face (ijk) is labelled then by the point which is in the link of the vertex i and is in the segment joining the edges (ij) and (ik). With that convention, given a tetrahedron $(ijkl)$, the arc going from face (ijk) to face (ijl) can be represented by a "left turn" arc in the link of the vertex i. We also consider arcs contained in one face which correspond to a permutation of the face. For example, the permutation $(ijk) \to (jki)$ is represented by the segment in the face (ijk) joining the point near i to the point near j.

To follow a path along these arcs we associate to each face, that is, to a configuration of 3 generic flags $([x_i], [f_i])_{1 \le i \le 3}$ with triple ratio X, a projective coordinate system of $\mathbb{CP}^2$: take the one where the point $x_1 = [1:0:0]^t$, $f_1 = [0:0:1]$, $x_2 = [0:0:1]^t$, $f_2 = [1:0:0]$, the point x_3 has coordinates $[1:-1:1]^t$ and $f_3 = [X : X+1 : 1]$.

The cyclic permutation of the flags $(ijk) \to (jki)$ with triple ratio X induces the coordinate change given by the matrix

$$T(X) = \begin{pmatrix} X & X+1 & 1 \\ -X & -X & 0 \\ X & 0 & 0 \end{pmatrix}.$$

It is also useful to consider the transposition $(ijk) \to (jik)$ which is given by the matrix

$$I = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Remark then that

$$T(\tfrac{1}{X}) = I \circ T^{-1}(X) \circ I.$$

If we have a tetrahedron of flags $(ijkl)$ with its z-coordinates, then the basis related to the triple (ijl) is obtained in the coordinate system related to the triple (ijk) by the coordinate change given by the matrix

$$(ijk) \to (ijl) = \begin{pmatrix} \dfrac{1}{z_{ji}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z_{ij} \end{pmatrix}.$$

5.1. **Representation of the fundamental group.** We obtain the change of coordinate matrix of a path in the triangulation by decomposing it in the two elementary steps described in the previous section and multiplying the change of coordinate matrices for each step from left to right. Finally, to obtain the representation of the fundamental group we take the inverse of the coordinate change matrix for each path representative of an element of the fundamental group.

One has to be careful if we want to obtain a representation in $\mathrm{PU}(2,1)$. In general we need to conjugate the above representation by an element so that the group preserves a chosen hermitian form.

Let $(i_0j_0k_0)$ (an ordered triple of points in a face of tetrahedron T_0) be the base point and $Z_0 = z_{i_0j_0k_0}(T_0)$ be its 3-ratio. We consider the transformation $A = \begin{pmatrix} -\frac{1+Z_0}{Z_0} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ such that $A\cdot \begin{pmatrix} \frac{1}{1+Z_0} \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. The hermitian form J is preserved if we conjugate the group elements by A, or equivalently the group preserves the hermitian form $J_0 = {}^t\overline{A}^{-1} J A^{-1}$ with eigenvalues $1, 1/|1+Z_0|, -1/|1+Z_0|$. We will show computations in a specific example in a future section.

5.2. **Boundary holonomy.** If we follow a path in a boundary torus of a triangulated manifold we will obtain, applying the procedure above, a coordinate change. Choosing a base point in each torus we obtain then a representation of the fundamental group of the boundary in $\mathrm{PGL}(3,\mathbb{C})$. Each torus is associated to a vertex and therefore the representation of its fundamental group fixes a flag. Choosing the flag properly up to conjugation, we can arrange it so that the boundary representation is upper triangular. Given two generators of a torus group represented by paths a and b one can compute easily the eigenvalues of the holonomy along them.

We suppose a path is contained in the link of the vertex and is transverse to the edges of the triangulation induced by the tetrahedra. Consider a path segment at the link of vertex i which is turning left around the vertex j, that is $(ijk) \to (ijl)$. Then, as before, the holonomy change of coordinate matrix is

$$\begin{pmatrix} \frac{1}{z_{ji}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z_{ij} \end{pmatrix}.$$

Consider the path segment at the link of vertex i which is turning right around the vertex j, that is

$$(ilj) \to (ikj).$$

Then the holonomy change of coordinate matrix is

$$\begin{pmatrix} \frac{z_{kl}z_{lk}}{z_{ij}} & -\frac{z_{kl}}{z_{ij}z_{lj}} - \frac{1}{z_{ki}} & -\frac{z_{il}}{z_{ki}} \\ 0 & 1 & z_{il} \\ 0 & 0 & \frac{1}{z_{ij}} \end{pmatrix}.$$

Both matrices being upper diagonal, to find the eigenvalues we need only to multiply the diagonal terms. The eigenvalues of the boundary holonomy are the diagonal elements obtained as above. We will note A^* and A the two eigenvalues (the first and the third with the normalization as above so that the second eigenvalue be one) of the path a and B^* and B the two eigenvalues of the path b.

5.3. **Boundary unipotent holonomy equations.** The boundary holonomy gives a representation of the fundamental group of the boundary tori into $\mathrm{PGL}(3,\mathbb{C})$. If t is the number of boundary tori, we note

$$\rho_b : (\mathbb{Z} \oplus \mathbb{Z})^t \to \mathrm{PGL}(3,\mathbb{C}).$$

For knots we can define canonical generators of the boundary torus called meridian and longitude. But, for simplicity, we will refer to generators of boundary tori for other manifolds as well as meridians and longitudes, denoted m and l. The abelian group generated by the image of ρ_b is called the peripheral group. It turns out that an important information on the representations is the rank of the peripheral group. For instance, examples of uniformizable CR manifolds were obtained precisely when the rank of the peripheral group is one (see [19, 7, 17]).

We say the representation is *unipotent* if the boundary holonomy is unipotent. That is, if the boundary holonomy representation can be given by matrices (up to scalar multiplication) of the form

$$\begin{pmatrix} 1 & \star & \star \\ 0 & 1 & \star \\ 0 & 0 & 1 \end{pmatrix}.$$

That is, all eigenvalues are equal.

In the following we will find unipotent representations associated to parabolic decorations. That means that for each boundary torus we compute the eigenvalues A^*, A, B^*, B corresponding to generators a and b at each torus and impose the equations

$$A^* = A = B^* = B = 1.$$

Remark that each eigenvalue is a rational function of the cross-ratio coordinates.

6. Computational steps

As mentioned in the introduction, we will use ideal triangulations of low complexity hyperbolic manifolds given to us by M. Thistlethwaite. From that list we generate the compatibility and holonomy equations:

Our goal is to obtain the solutions of the system in the cross-ratio coordinates z_{ij} and then the holonomy representations. Such a computation is decomposed into three main steps:

1. Solving the constraints on gluing decorated tetrahedra, which reduces to study a constructible set defined implicitly by
   (a) The consistency relations which define Edge and Face equations (4.1),
   (b) The holonomy equations (see section 5),
   (c) The cross-ratio equations (see 3.1),
   (d) The restrictions on the coordinate values, precisely $z_{i,j}$ must be different from 0 and 1.

(2) Discriminating hyperbolic, real flags and CR solutions (see 3.5.1)
(3) Computing the resulting volumes (4.2)
(4) Computing representations of the fundamental group (see section 5).

Each system depends on the $12n$ variables $\mathcal{Z} = \{z_{ij}(T)\}$, where T denotes a tetrahedron and ij one of its oriented edges.

— L_e (resp. L_f) denotes the set of the $2n$ edge (resp. face) equations;
— L_h denotes the set of the polynomials defining unipotent holonomy equations (their number is $4t$, where t is the number of tori in the boundary).
— L_c denotes the polynomials defining the cross-ratio relations, say $z_{ik}(T)(1-z_{ij}(T))-1$ and $z_{il}(T)z_{ij}(T) - z_{ij}(T) + 1$;
— $P_0 = \prod_{z\in\mathcal{Z}} z(1-z)$ is the polynomial defining the forbidden values for the $z_{ij}(T)$.

We have to compute an exhaustive and exact description of the constructible set

$$\mathcal{S} = \{z \in \mathcal{Z} \mid P(z) = 0, P \in L_e \cup L_f \cup L_h \cup L_c,\ P_0(z) \neq 0\}.$$

Some preliminary remarks are essential in order to understand the contents of the present section.

A straightforward approach that would consist in first solving the large algebraic system given by the consistency equations, the holonomy equations and the cross-ratio relation would certainly fail because of the number of variables ($12n$) and the degree. Moreover it may happen that this system is not 0-dimensional.

We chose to perform exclusively exact computation, using elimination techniques. We will first simplify the system using specific adapted pre-processing, then we will use general methods essentially based on Gröbner basis computations (see [6]). If the system we obtain is 0-dimensional, we use the Rational Univariate Representation ([18]).

6.1. **The pre-processing.** Using the cross-ratio relations at each vertex of a tetrahedron one can reduce the number of variables to $4n$. We pick a variable at each vertex and write the equations in terms of these variables. For example, such a substitution reduces the study of the 4_1 knot complement to the solutions of a zero-dimensional system depending on 8 variables (instead of 24).

After these simplifications, some new simple affine relations appear in form $Az_{ij} + B$, where A divides some power of P_0. A is then a product of z or $(1-z)$ for some $z \in \mathcal{Z} - \{z_{ij}\}$. The coordinate z_{ij} may be replaced everywhere by A/B. We then obtain a simplified system.

This simplification pre-processing reduces the study of the 4_1 knot complement to a simplified system of two equations in two variables. The other coordinates of the solutions can then be recovered from simple relations.

Note that the degree of the equations in the final simplified system depends strongly on the way these simplifications are performed. Blind simplifications may lead to consider systems of equations with very large degrees. The general methods for solving such systems formally have polynomial complexity in the Bézout's bound. Unappropriate choices in the simplifications will generate huge systems that are impossible to solve with *state of the art* algorithms.

We have implemented some tricky choice functions that minimize the degree of the final simplified system. There is no interest in describing them in detail here, but this straightforward step must be carefully implemented if one wants to succeed in the determination of the solutions.

6.2. **Solving the reduced system using Gröbner Basis.** We now have to solve a system of equations and (simple) inequalities. Three cases may occur:

- — The set of equations defines a 0-dimensional variety (finite set of points);
- — The set of equations defines a variety of positive dimension but the full constructible set is 0-dimensional;
- — The constructible set has positive dimension.

The first case happened only for the `m004` variety (4_1 complement). The last case occurs for example for the `m003` variety (4_1-sister) and also for the `m006`. It will be described in a separate section. The second case occurs for example for the `m007` and the `m015` (5_2 complement). The procedure we follow is divided into the following steps:

(1) **Compute a Gröbner basis** of the reduced system of equations.
(2) **Saturate the ideal** by the polynomials defining the inequalities
(3) **Compute the dimension** of this ideal

These computations are based on Gröbner Basis computations.

Even if the basic principles used for computing a Gröbner basis are simple (see [6] for a general description), their effective computation is known to be hard. The first algorithm due to Buchberger has been a lot improved and we use, in practice, the algorithm F_4 by J.-C. Faugère (see [12]), implemented in recent versions of MAPLE, and known to be currently the fastest variant.

A Gröbner basis of a polynomial ideal $\mathcal{I}$ is a set of generators of $\mathcal{I}$, such that there is a natural way of reducing canonically a polynomial $P \,(\mathrm{mod}\, \mathcal{I})$. A Gröbner basis is uniquely defined for a given admissible ordering on the monomials and equipped with a canonical *NormalForm* that gives a unique expression of the reduction of any polynomial modulo the considered ideal. Some orderings, call *elimination orderings* allow to eliminate variables. Given $I \subset \mathbb{Q}[Y_1, .., Y_k][X_1, .., X_n]$, the use of an elimination ordering for which $Y_i < X_j; \forall i, j$ allows to deduce straightforwardly a Gröbner basis of $I \cap \mathbb{Q}[Y_1, .., Y_k]$ (see [6] Chapter 3).

Saturating an ideal $I \subset \mathbb{Q}[X_1, .., X_n]$ by a polynomial f just remains to compute a Gröbner basis of $I + \langle Tf - 1 \rangle \cap \mathbb{Q}[X_1, .., X_n]$, where T is a new variable independent from $X_1, .., X_n$.

Once a Gröbner Basis of an ideal $\mathcal{I}$ is known, one can compute the associated Hilbert polynomial and then deduce the (Hilbert) dimension and (Hilbert) degree of $\mathcal{I}$ (see [6] Chapter 9 - Section 3).

6.3. **Rational Univariate Representation.** When the system we obtain is zero-dimensional, the so called Rational Univariate Representation (see [18]) defines a bijection between the solutions and the roots of some univariate polynomial, preserving the multiplicities.

This full parametrization is computed in two steps: first pre-process the initial system, then compute a Rational Univariate Representation of the reduced zero-dimensional system. If this systems depends on m variables $z_1, \ldots, z_m$, the Rational Univariate Representation consists in a polynomial $f \in \mathbb{Q}[Y]$ and a set of $m+1$ polynomials such that the solutions are $\{z_i = \dfrac{f_{z_i}(Y)}{f_0(Y)},\ i = 1,\dots,m \mid f(Y) = 0\}$.

Factorization of f leads to consider that the solutions are parameterized by a finite number of algebraic numbers γ, defined by their minimal polynomials f and each coordinate z is a polynomial $f_z(\gamma)$.

Then, we substitute the variables that have been extracted during the pre-processing by their expression in $\mathbb{Q}[Y]/(f)$ using modular computation. We obtain rational parametrization of the full system. Each of these parametrization may be described as

$$\text{rur} := \{z = f_z(Y),\ z \in \mathcal{Z} \mid f(Y) = 0\}.$$

Furthermore, $\gamma \in \mathbb{R}$ if an only if $z(\gamma) = f_z(\gamma) \in \mathbb{R}$ for all $z \in \mathcal{Z}$.

6.4. **Discriminating the solutions.** Thanks to the previous computations, the solutions of the initial system are expressed by means of rational parameterizations (here f is a prime polynomial)

$$\text{rur} := \{z = f_z(Y),\ z \in \mathcal{Z} \mid f(Y) = 0\}.$$

Real solutions are in correspondence with the real zeroes of f.

The hyperbolic solutions are extracted from the global solutions by identifying some coordinates. We just have to test if

$$f_{z_{12}(T)} \equiv f_{z_{21}(T)} \equiv f_{z_{34}(T)} \equiv f_{z_{43}(T)} \pmod f.$$

For the CR solutions, one has to consider the $3n$ relations $z_{ij}(T)z_{ji}(T) = \overline{z_{kl}}(T)\,\overline{z_{lk}}(T)$. Let $\gamma = x + iy$ be one of the root f. We obtain a new 0-dimensional system in the two variables x and y consisting of the 2 polynomial equations: $\operatorname{Re} f(\gamma) = 0,\ \operatorname{Im} f(\gamma) = 0$ and $2 \times 3n$ equations $\operatorname{Re}\left[f_{z_{ij}(\gamma)} f_{z_{ji}(\gamma)}\right] = \operatorname{Re}\left[f_{z_{kl}(\gamma)}\, f_{z_{lk}(\gamma)}\right], \operatorname{Im}\left[f_{z_{ij}(\gamma)} f_{z_{ji}(\gamma)}\right] = -\operatorname{Im}\left[f_{z_{kl}(\gamma)}\, f_{z_{lk}(\gamma)}\right]$. This last system may be solved by computing a Rational Univariate Representation. There exists an integer λ such that the algebraic number $Z = x + \lambda y$ describes the solutions:

$$\text{rur}_{\text{CR}} := \left\{x = \frac{g_x(Z)}{g_1(Z)}, y = \frac{g_y(Z)}{g_1(Z)} \mid g(Z) = 0\right\}.$$

Real solutions of the systems are in a one-to-one correspondance with the real roots of g.

6.5. **Computing the rank of the boundary holonomy representation.** We compute first the meridian and the longitude as upper triangular matrices, say

$$g_M = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \quad g_L = \begin{pmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{pmatrix}.$$

We easily obtain that $g_M^n = \begin{pmatrix} 1 & n\,a & n\,c + \frac{1}{2}n(n-1)ab \\ 0 & 1 & n\,b \\ 0 & 0 & 1 \end{pmatrix}$. We thus deduce that $g_M^n = g_L^{n'}$ iff $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{2c - ab}{2c' - a'b'} = \dfrac{n'}{n} \in \mathbb{Q}$. This can be decided by computing the Hermite form of $\begin{pmatrix} a & b & 2c - ab \\ a' & b' & 2c' - a'b' \end{pmatrix}$ in $\mathbb{Q}[Y]/(f(Y))$ where f is the minimal irreducible polynomial parameterizing the solutions.

6.6. **Certified numerical approximations.** For simplicity, and in order to get a human readable output, we often express the results by means of numerical approximations instead of formal (large) expressions. We precise that all these approximations are certified (in practice, all the digits are correct but the last one). We make use of multi-precision floating point numbers coupled with interval arithmetic or well known results on error control for the evaluation of univariate polynomials that allows to accurately evaluate the Rational Univariate Representations at the roots of a given univariate polynomial. A fully treated example in the case of systems with two variables is described in [3].

7. An example : the figure-eight knot

The figure eight knot is given as a gluing of two tetrahedra as in Figure 2. We refer to it to set the equations of compatibility.

Let z_{ij} and w_{ij} be the coordinates associated to the edge ij of each of the tetrahedra.

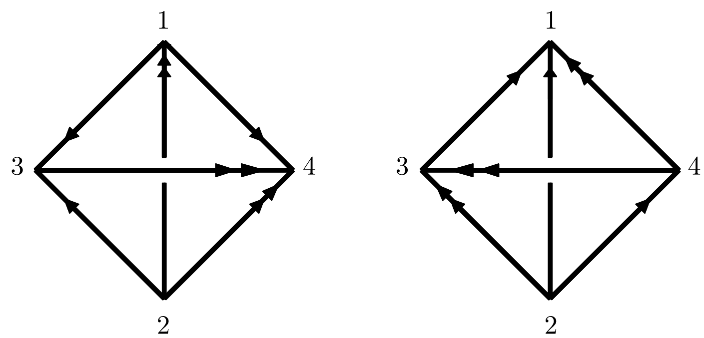

Figure 2. Triangulation of `m004`

We obtain the edge and face equations:

$$(L_e)\left\{\begin{array}{lcl} z_{23}w_{24}z_{13}w_{31}z_{14}w_{21} & = & 1 \\ z_{32}w_{42}z_{31}w_{13}z_{41}w_{12} & = & 1 \\ z_{24}w_{23}z_{21}w_{43}z_{34}w_{41} & = & 1 \\ z_{42}w_{32}z_{12}w_{34}z_{43}w_{14} & = & 1. \end{array}\right. \qquad (L_f)\left\{\begin{array}{lcl} z_{21}z_{31}z_{41}w_{21}w_{31}w_{41} & = & 1 \\ z_{12}z_{32}z_{42}w_{13}w_{23}w_{43} & = & 1 \\ z_{13}z_{23}z_{43}w_{12}w_{32}w_{42} & = & 1 \\ z_{14}z_{24}z_{34}w_{14}w_{24}w_{34} & = & 1. \end{array}\right.$$

7.1. **Boundary holonomy representation.** The meridian and the longitude are computed from the Snappea triangulation. We follow the algorithm to compute the holonomy along the meridian and the longitude along appropriate paths:

$(421)^1 \to (423)^1 \to (324)^0 \to (314)^0$

$(421)^1 \to (431)^1 \to (214)^0 \to (234)^0 \to (243)^1 \to (241)^1 \to (134)^0 \to (132)^0 \to$
$(312)^1 \to (342)^1 \to (432)^0 \to (412)^0 \to (134)^1 \to (132)^1 \to (312)^0 \to (314)^0$

The corresponding matrices are:

$$g_M = \begin{bmatrix} \dfrac{z_{12}z_{21}}{z_{34}w_{24}} & -\dfrac{z_{12}z_{13}+z_{34}z_{24}}{z_{34}z_{24}z_{13}} & -\dfrac{z_{32}w_{42}}{z_{13}} \\ 0 & 1 & z_{32}w_{42} \\ 0 & 0 & \dfrac{w_{42}}{z_{34}} \end{bmatrix} g_L = \begin{bmatrix} \dfrac{z_{13}z_{31}w_{14}w_{23}}{w_{31}z_{42}w_{42}z_{24}} & * & * \\ 0 & 1 & * \\ 0 & 0 & \dfrac{z_{31}w_{13}z_{13}w_{24}}{z_{42}w_{32}z_{24}w_{41}} \end{bmatrix}$$

We therefore deduce the unipotent holonomy equations: $A = A^* = B = B^* = 1$ where

$$A = \frac{z_{12}z_{21}}{z_{34}w_{24}},\ A^* = \frac{w_{42}}{z_{34}},\ B = \frac{z_{13}z_{31}w_{14}w_{23}}{w_{31}z_{42}w_{42}z_{24}},\ B^* = \frac{z_{31}w_{13}z_{13}w_{24}}{z_{42}w_{32}z_{24}w_{41}}.$$

7.2. **Solutions.** Here the pre-processing provides a reduced system of 7 polynomial equations in the variables $\{z_{14}, z_{43}\}$. All of the 22 other variables belong to $\mathbb{Q}(z_{14}, z_{43})$. This reduced system is 0-dimensional and one can easily compute Rational Univariate Representations of its zeroes.

One obtain four sets given by their minimal polynomials.

- ► $f_1 = Y^2 - Y + 1$.
  The complete hyperbolic structure on the complement of the figure-eight knot is obtained from two conjugate solutions. In fact, in that case, if $\omega^{\pm} = \frac{1 \pm i\sqrt{3}}{2}$ is one root of f_1, then
  $$z_{12} = z_{21} = z_{34} = z_{43} = w_{12} = w_{21} = w_{34} = w_{43} = \omega^{\pm}$$
  is a solution of the equations as obtained in [20]. Its volume is $2.029883212\cdots$.
- ► $f_2 = Y^2 - Y + 1$.
  $$z_{12} = z_{34} = w_{34} = w_{43} = \overline{z}_{21} = \overline{z}_{43} = \overline{w}_{12} = \overline{w}_{21} = \omega^{\pm}.$$
  This solution corresponds to a discrete representation of the fundamental group of the complement of knot in $\mathrm{PU}(2,1)$ with faithful boundary holonomy. Moreover, its action on complex hyperbolic space has limit set the full boundary sphere ([11]).
- ► $f_3 = Y^2 + Y + 2$. Let $\gamma^{\pm} = -\frac{1}{2} \pm i\frac{1}{2}\sqrt{7}$ be a root of f_3. The coordinates are
  $$z_{21} = w_{21} = \overline{z}_{43} = \overline{w}_{12} = \tfrac{5-i\sqrt{7}}{4},\ z_{12} = w_{2,1} = \tfrac{3-i\sqrt{7}}{8}, w_{34} = \overline{w}_{43} = -\tfrac{1+i\sqrt{7}}{2}.$$
- ► $f_4 = 2Y^2 + Y + 1$. $1/\gamma^{\pm} = -\frac{1}{4} \pm i\frac{1}{4}\sqrt{7}$ is a root of f_4. We obtain
  $$z_{12} = \overline{z}_{34} = \tfrac{3-i\sqrt{7}}{2},\ w_{43} = \overline{w}_{34} = \tfrac{5+i\sqrt{7}}{8},\ z_{21} = w_{21} = \overline{z}_{43} = \overline{w}_{12} = \tfrac{-1+i\sqrt{7}}{4}.$$

All solutions corresponding to f_2, f_3, f_4 were obtained in [11] and f_3, f_4 correspond to spherical CR structures with unipotent boundary holonomy of rank one ([7]).

7.3. **Holonomy representation.** In order to obtain the representation of the fundamental group of the complement of the figure eight knot we identify the face $(324)^0$ of the tetrahedron 0 to the face $(423)^1$ of the tetrahedron 1. The generators are explicitly written from the remaining three face identifications following paths which pass through the chosen face $(324)^0 = (423)^1$ inside the tetrahedra and transport the projective basis defined by each oriented face.

Here are the paths generating the identifications on the polyhedron:

(1) $(134)^0 \to (341)^0 \to (342)^0 \to (423)^0 \to (234)^0 = (243)^1 \to (241)^1$
(2) $(214)^0 \to (142)^0 \to (421)^0 \to (423)^0 \to (234)^0 = (243)^1 \to (432)^1 \to (431)^1$
(3) $(123)^0 \to (231)^0 \to (234)^0 = (243)^1 \to (432)^1 \to (324)^1 \to (321)^1$

In order to obtain paths generating the fundamental group with a base point at face $(234)^0$, the paths have to be conjugated, respectively, by

(1) $(234)^0 \to (342)^0 \to (341)^0 \to (413)^0 \to (134)^0$
(2) $(234)^0 \to (342)^0 \to (423)^0 \to (421)^0 \to (214)^0$
(3) $(234)^0 \to (231)^0 \to (312)^0 \to (123)^0$

The coordinate transformations corresponding to each path are obtained by multiplying the coordinate change matrices. The generators of the group representation are obtained as inverse matrices of the coordinate change matrices.

Applying the computations above to the solution f_2 we obtain the generators

$$g_1 = \begin{bmatrix} 1 & 0 & 0 \\ -w^{\pm} & 1 & 0 \\ -w^{\mp} & -2w^{\mp} & 1 \end{bmatrix}, \; g_2 = \begin{bmatrix} 3w^{\mp} & 6w^{\mp} & 1 \\ 3w^{\pm} & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; g_3 = \begin{bmatrix} 1 & 2w^{\mp} & -w^{\mp} \\ 0 & 1 & w^{\mp} \\ 0 & 0 & 1 \end{bmatrix}.$$

The group $\langle g_1, g_2, g_3 \rangle$ preserves the hermitian form $\begin{bmatrix} 0 & 0 & -w^{\pm} \\ 0 & 1 & 0 \\ -w^{\mp} & 0 & 0 \end{bmatrix}$.

For the solution f_3 we obtain $g_1 = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{\gamma^{\pm}} & 1 & 0 \\ -\frac{\gamma^{\pm}}{2} & \frac{2}{\gamma^{\pm}-1} & 1 \end{bmatrix}$, $g_2 = \begin{bmatrix} \frac{8}{\gamma^{\pm}+3} & \frac{3}{\gamma^{\pm}+2} & -\frac{4}{3\gamma^{\pm}+2} \\ -\frac{3}{\gamma^{\pm}} & 1 & 0 \\ \frac{2}{\gamma^{\pm}+2} & 0 & 0 \end{bmatrix}$,

$g_3 = \begin{bmatrix} 1 & \frac{2}{\gamma^{\pm}+2} & -\frac{1}{\gamma^{\pm}} \\ 0 & 1 & \gamma^{\pm} \\ 0 & 0 & 1 \end{bmatrix}$. The group $\langle g_1, g_2, g_3 \rangle$ preserves the hermitian form $\begin{bmatrix} 0 & 0 & \gamma^{\mp} \\ 0 & 1 & 0 \\ \gamma^{\pm} & 0 & 0 \end{bmatrix}$.

Finally, for the solution f_4 we obtain $g_1 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{\gamma^{\pm}-2} & 1 & 0 \\ -\frac{2}{\gamma^{\pm}} & \frac{1}{2}\gamma^{\pm}-1 & 1 \end{bmatrix}$, $g_2 = \begin{bmatrix} -2 & -\frac{1}{2}\gamma^{\pm}-3 & -1 \\ \frac{8}{\gamma^{\pm}+6} & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$,

$g_3 = \begin{bmatrix} 1 & \frac{4}{\gamma^{\pm}+3} & -\gamma^{\pm} \\ 0 & 1 & -\frac{2}{\gamma^{\pm}-2} \\ 0 & 0 & 1 \end{bmatrix}$. The group $\langle g_1, g_2, g_3 \rangle$ preserves the hermitian form $\begin{bmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix}$.

7.4. **Boundary holonomy.** For all solutions in $\mathrm{PU}(2,1)$ we always obtain $g_M \cdot g_1 = 1$. The longitude is respectively

$$g_L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \pm\sqrt{3} & 0 & 1 \end{bmatrix}, \; g_L = {g_M}^{-3}, \; g_L = {g_M}^{3}.$$

8. Description of the solutions

As explained in the last section we solve the system consisting of compatibility equations and the unipotent holonomy conditions plus the inequalities which prevent the z-coordinates to be 0 or 1. The solutions will, in turn, give rise to representations in $\mathrm{PSL}(3,\mathbb{C})$.

We have computed an exhaustive list of solutions for the eleven first 3-manifolds and for the Whitehead link complement. Their main properties are summarized in the Table 1. For these examples the dimension of solutions is at most 1 and there are always isolated solutions besides the hyperbolic one. Five of them are 0-dimensional: `m004`, `m007`, `m009`, `m015` and the Whitehead link complement. All others are 1-dimensional: `m003`, `m006`, `m010`, `m011`, `m016`, `m017`, `m019`. We further analyse the solutions to decide if the corresponding representation is in $\mathrm{PSL}_3(\mathbb{R})$, $\mathrm{PSL}_2(\mathbb{C})$, $\mathrm{PSL}_2(\mathbb{R})$ or $\mathrm{PU}(2,1)$ and compute their volume.

In Table 1, we indicate the *name* of the variety, the number *1-D* of 1-dimensional prime components, the *degrees* of the prime 0-dimensional components. The isolated solutions studied are the ones not contained in the one-dimensional components. We indicate the total number of solutions such that the corresponding representations are in $\mathrm{PSL}_3(\mathbb{C})$, $\mathrm{PSL}_3(\mathbb{R})$, $\mathrm{PSL}_2(\mathbb{C})$, $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PU}(2,1)$. Observe that $\mathrm{PSL}_3(\mathbb{R}) \cap \mathrm{PSL}_2(\mathbb{C}) = \mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_3(\mathbb{R}) \cap \mathrm{PU}(2,1) = \mathrm{PSL}_2(\mathbb{R})$ so $\mathrm{PSL}_2(\mathbb{R})$ solutions are being also redundantly counted in $\mathrm{PSL}_3(\mathbb{R})$, $\mathrm{PSL}_2(\mathbb{C})$ and $\mathrm{PU}(2,1)$. We also indicate the different positive volumes we obtain, writing in boldface the volumes of representations in $\mathrm{PSL}(2,\mathbb{C})$. Keep in mind though that real solutions as well as solutions giving rise to $\mathrm{PU}(2,1)$ representations have zero volume.

For the 0-dimensional case, we briefly describe the solutions and in some cases the group representation, when the boundary holonomy has rank 1. More details can be obtained in the website, .../SGT including matrix representations of the fundamental groups. In the following description, for the sake of simplicity, we will say that a solution is CR (short for Cauchy-Riemann), when the corresponding representation is conjugated to a representation in $\mathrm{PU}(2,1)$. For each variety we singled out a $\mathrm{PU}(2,1)$ representation with boundary holonomy of rank one. We think that they are natural candidates for being holonomies of a spherical CR structure on the variety.

The one dimensional components in the examples are not completely described. We don't enumerate the $\mathrm{PSL}(2,\mathbb{C}), \mathrm{PU}(2,1)$ or $\mathrm{PSL}(3,\mathbb{C})$ solutions in these components. But an important observation is that in all examples up to 3 tetrahedra the one dimensional components contain at most a finite number of $\mathrm{PSL}(2,\mathbb{C})$ solutions which turn out to be real so the volumes presented in the table contain indeed all volumes of $\mathrm{PSL}(2,\mathbb{C})$ representations arising from the given triangulation. In fact, the $\mathrm{PSL}(2,\mathbb{C})$ solutions contained in one dimensional $\mathrm{PSL}(3,\mathbb{C})$ components occur only in the following manifolds: m003 (2 solutions), m006 (2 solutions), m010 (1 solution) and m017 (3 solutions).

On the other hand, $\mathrm{PU}(2,1)$ solutions might appear in families inside complex one dimensional components (see the worked example m003 in [2]). In that example there is a whole complex one dimensional component consisting of CR solutions. But most of our examples have only isolated $\mathrm{PU}(2,1)$ solutions, namely for m004, m007, m009, m011, m015, m016, m019 and the Whitehead link. The detailed study of $\mathrm{PU}(2,1)$ solutions in the one dimensional components will be addressed elsewhere.

		Zero-dimensional prime components						
		Number(s) of Solutions						
Name	1-D	Ext. Degrees	$PGL_3(\mathbb{C})$	$PSL_3(\mathbb{R})$	$PSL_2(\mathbb{C})$	$PSL_2(\mathbb{R})$	$PU(2,1)$	*Volumes*
m003	2	$2, 2, 8, 8$	20	0	2	0	2	0.648847 **2.029883**
m004	0	$2, 2, 2, 2$	8	0	2	0	6	**2.029883**
m006	2	$6, 6, 12, 28$	43	1	3	1	15	0.707031 0.719829 0.971648 1.284485 **2.568971**
m007	0	$3, 6, 8, 8, 8$	33	1	3	1	15	0.707031 0.822744 1.336688 **2.568971**
m009	0	$2, 4, 4, 4, 6, 8$	28	2	2	0	8	0.507471 0.791583 1.417971 **2.666745**
m010	2	$4, 6, 6, 12, 12$	38	0	2	0	4	0.251617 0.791583 0.809805 0.982389 1.323430 **2.666745**
m011	1	$3, 4, 16, 64$	87	5	7	3	21	0.226838 0.251809 0.328272 0.397457 0.452710 0.643302 0.685598 0.700395 0.724553 0.770297 0.879768 **0.942707** 0.988006 1.099133 1.184650 1.846570 **2.781834**
m015	0	$3, 4, 4, 6, 6$	23	3	3	1	11	0.794323 1.583167 **2.828122**
m016	1	$3, 3, 10, 50$	66	4	6	4	24	0.296355 0.403707 0.710033 0.753403 0.773505 0.796590 0.886451 1.135560 1.422985 1.505989 **2.828122**
m017	3	$3, 4, 6, 6, 44$	63	1	3	1	21	0.527032 0.794323 0.801984 0.828705 1.252969 1.588647 **2.828122**
m019	1	$4, 4, 22, 84$	114	6	8	4	24	0.027351 0.062112 0.323395 0.332856 0.347159 0.411244 0.467624 0.524801 0.544151 0.599455 0.638404 0.738805 0.758111 0.798098 0.851139 0.916588 1.101800 1.130263 1.190919 **1.263709** 1.340255 2.111776 **2.944106**
Wh. link	0	$2, 2, 4, 4, 10, 10$	32	0	2	0	14	1.132196 1.683102 **3.663862**

TABLE 1. Description of the solutions

8.1. **The variety `m007`.** There are three simplices with parameters u_{ij}, v_{ij} and w_{ij}, $1 \leq i, j \leq 4$.

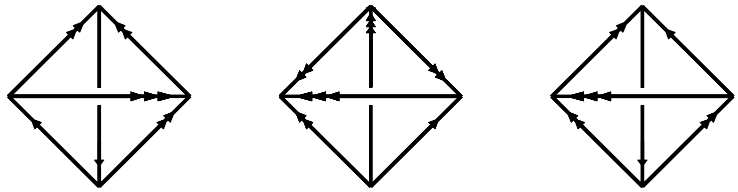

FIGURE 3. Triangulation of m007

- ► $f_1 = Y^3 - 2Y^2 - 1$.
  The solutions giving rise to $\mathrm{PSL}_2(\mathbb{C})$ representations are

$$\begin{aligned} u_{12} = u_{21} = u_{34} = u_{43} &= \gamma, \\ v_{12} = v_{21} = v_{34} = v_{43} &= 2\gamma - \gamma^2 = -1/\gamma, \\ w_{12} = w_{21} = w_{34} = w_{43} &= \gamma. \end{aligned}$$

  where γ is a root of $f_1 = Y^3 - 2Y^2 - 1$. The volume of the hyperbolic solution is $\mathrm{vol} = 2.5689706009\cdots$.
- ► $f_2 = Y^6 - Y^5 + Y^4 - 2Y^3 + Y^2 + 1$.
  2 solutions are CR. The four others define representations with volume $.70703052208\cdots$.
- ► $f_3 = Y^8 - 2Y^7 + 2Y^6 - 6Y^5 + 7Y^4 - 2Y^3 + 3Y^2 - Y + 1$.
  4 solutions are CR. The volume of the other representations is $0.82274406556\cdots$
- ► $f_4 = 3Y^8 - 6Y^7 + 7Y^6 - 4Y^5 + 4Y^4 + 4Y^3 + 4Y^2 + 1$.
  4 solutions are CR. Their volume is $.82274406556\cdots$
- ► $f_5 = 16Y^8 - 20Y^7 + 23Y^6 - 27Y^5 + 10Y^4 - 5Y^3 + 9Y^2 + 2Y + 4$.
  4 solutions are CR. The four others have volume $1.3366875264\cdots$

In conclusion, there are 33 solutions, 15 being CR.

8.2. **The variety m009.** There are three simplices with parameters u_{ij}, v_{ij} and w_{ij}, $1 \leq i, j \leq 4$.

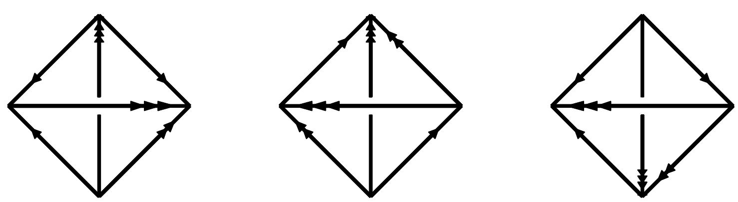

FIGURE 4. Triangulation of m009

We obtain

- ► $f_1 = Y^2 - Y + 2$.
  The $\mathrm{PSL}(2,\mathbb{C})$ solutions are given by

$$\begin{array}{rcl} u_{12} = u_{21} = u_{34} = u_{43} & = & \gamma, \\ v_{12} = v_{21} = v_{34} = v_{43} & = & \frac{1}{4}(\gamma + 1) \\ w_{12} = w_{21} = w_{34} = w_{43} & = & \gamma. \end{array}$$

  where $\gamma = \frac{1}{2}(1 \pm i\sqrt{7})$ is a root of f_1. The volume of the hyperbolic solution is $\mathrm{vol} = 2.6667447834\cdots$.
- ► $f_2 = f_3 = Y^4 + Y^3 - Y^2 - Y + 1$.
  They correspond to 8 solutions with the same volume $0.50747080320\cdots$.
- ► $f_4 = Y^4 + 2Y^3 - Y^2 - 2Y - 4 = (Y^2 + Y - 1 - \sqrt{5})(Y^2 + Y - 1 + \sqrt{5})$.
  There are two real solutions $\gamma^{\pm} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{5 + 4\sqrt{5}}$ and two conjugate CR solutions corresponding to $\gamma^{\pm} = -\frac{1}{2} \pm \frac{1}{2} i\sqrt{-5 + 4\sqrt{5}}$. We have

$$u_{12} = w_{34} = \frac{\gamma^{\pm} + 3}{\gamma^{\pm} + 1},\ u_{21} = w_{43} = \gamma^{\pm},\ u_{34} = w_{12} = \frac{\gamma^{\pm} - 2}{\gamma^{\pm}},$$
$$u_{43} = w_{21} = -1 - \gamma^{\pm},\ v_{12} = v_{34} = \frac{1}{\gamma^{\pm} + 3},\ v_{21} = v_{43} = \frac{1}{2 - \gamma^{\pm}}.$$

  The two real solutions give rise to representations in $\mathrm{PSL}(3\mathbb{R}) \backslash \mathrm{PSL}(2,\mathbb{R})$. Moreover, for all of these solutions we find that the meridian g_M and the longitude g_L satisfy $g_M g_L^2 = 1$. We find

$$g_M = \begin{bmatrix} 1 & -2\,\dfrac{\gamma^{\pm} + 2}{\gamma^{\pm}\,(\gamma^{\pm} + 1)} & -2\dfrac{\gamma^{\pm} + 2}{\gamma^{\pm}\,(\gamma^{\pm} + 3)} \\ 0 & 1 & \dfrac{2}{3 + \gamma^{\pm}} \\ 0 & 0 & 1 \end{bmatrix}.$$

- ► $f_5 = Y^6 - Y^4 + 2Y^3 + Y^2 - Y + 1$.
  There are two CR solutions and 4 solutions with the same volume $0.79158333031\cdots$.
- ► $f_6 = 8Y^8 - 16Y^7 + 22Y^6 - 25Y^5 + 16Y^4 - 6Y^3 + Y^2 + 3Y + 1$.
  There are 4 CR solutions and 4 solutions with the same volume $1.4179708859\cdots$.

In conclusion we found 28 solutions, 8 being CR. There are 3 different volumes apart from the hyperbolic one.

8.3. **The 5_2 knot complement.** There are three simplices with parameters u_{ij}, v_{ij} and w_{ij}, $1 \leq i, j \leq 4$. We find 23 solutions in 5 sets of Galois conjugate solutions.

- ► $f_1 = Y^3 - Y + 1$.
  The solutions correspond to the two conjugate hyperbolic solutions and a real one.

$$\begin{array}{rcl} u_{12} = u_{21} = u_{34} = u_{43} & = & \gamma, \\ v_{12} = v_{21} = v_{34} = v_{43} & = & \gamma, \\ w_{12} = w_{21} = w_{34} = w_{43} & = & \gamma. \end{array}$$

  where γ is a root of f_1. The volume is $2.828122\cdots$.
- ► $f_2 = Y^4 - Y^2 + 1$.
  We obtain 4 solutions (the 4 12th-roots of unity) that are all CR.

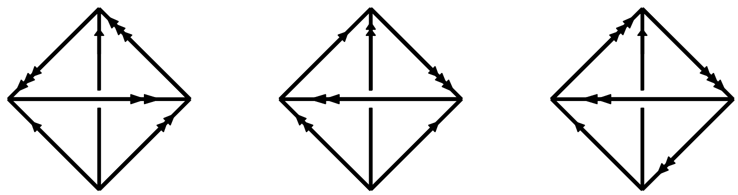

FIGURE 5. Triangulation of m015

- ► $f_3 = 4Y^4 - 4Y^3 - Y^2 + Y - 1$.
  We obtain 2 real solutions (giving representations in $\mathrm{PSL}(3,\mathbb{R}) \setminus \mathrm{PSL}(2,\mathbb{R})$) that are the roots of

$$4Y^2 - 2Y - (1 - \sqrt{5}),$$

and 2 complex conjugate solutions that are the roots of

$$4Y^2 - 2Y - (1 + \sqrt{5}).$$

These two last solutions are CR. We obtain, for these four solutions:

$$\begin{aligned} u_{21} = v_{21} = v_{34} = w_{21} &= \frac{1}{1-\gamma} \\ v_{12} = u_{43} = v_{43} = w_{43} &= \frac{2}{2\gamma+1}, \\ u_{12} = w_{12} = -\gamma(\gamma-1),\ u_{34} = w_{34} &= \tfrac{1}{4} - \gamma^2. \end{aligned}$$

Note that in this case the meridian and the longitude are equal:

$$g_M = g_L = \begin{bmatrix} 1 & \dfrac{2}{(2\gamma-1)^2} & \dfrac{1}{\gamma(2\gamma-1)^2} \\ 0 & 1 & \dfrac{1}{\gamma} \\ 0 & 0 & 1 \end{bmatrix}.$$

The holonomy group, in this case, is generated by

$$g_1 = g_M^{-1},\ g_2 = \begin{bmatrix} 1 & \dfrac{2}{(2\gamma-1)^2} & \dfrac{1}{\gamma(2\gamma-1)^2} \\ \dfrac{2}{1-2\gamma} & \dfrac{1-4\gamma}{1-4\gamma^2} & \gamma - 1 \\ \dfrac{1}{2\gamma^2(1-2\gamma)} & \dfrac{\gamma-1}{2\gamma} & \gamma \end{bmatrix},$$

$$g_3 = \begin{bmatrix} 1 & 0 & 0 \\ -\dfrac{1}{2\gamma-1} & 1 & 0 \\ \dfrac{1}{2\gamma^2(1-2\gamma)} & -\dfrac{1}{2\gamma^2} & 1 \end{bmatrix}, \; g_4 = \begin{bmatrix} \dfrac{1+4\gamma}{2\gamma} & \dfrac{1}{2\gamma(2\gamma-1)} & \dfrac{2\gamma}{1-2\gamma} \\ \dfrac{1}{2\gamma} & \dfrac{2\gamma-1}{2\gamma} & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

where γ is one of the four roots of f_3.

- ▶ $f_4 = Y^6 - Y^5 - 3Y^3 + 2Y^2 + Y + 1$.
  Only two of these roots correspond to CR structures. The four others have same volume: $1.583167\cdots$.
- ▶ $f_5 = 17Y^6 - 5Y^5 + 33Y^4 - 11Y^3 + 26Y^2 - 4Y + 8$.
  Only two of these roots correspond to CR structures. The others define representations of volume $0.794323\cdots$.

In conclusion we obtain 23 solutions and 11 of them are CR.

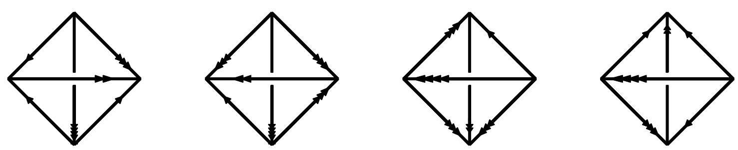

FIGURE 6. Triangulation of Whitehead link complement

8.4. **The Whitehead link.** Here we have four tetrahedra with parameters u_{ij}, v_{ij}, w_{ij} and x_{ij} with $1 \le i, j \le 4$ glued as in Figure 6.

There are 6 different groups of solutions.

- ▶ $f_1 = Y^2 + 1$. The conjugate hyperbolic solutions are given by

$$u_{12} = u_{21} = u_{34} = u_{43} = v_{12} = v_{21} = v_{34} = v_{43} = \pm i,$$

$$w_{12} = w_{21} = w_{34} = w_{43} = w_{12} = w_{21} = w_{34} = w_{43} = \pm i.$$

- ▶ $f_2 = Y^2 + Y + 4$.
  We obtain

$$\begin{aligned} u_{12} = v_{34} = w_{12} = x_{34} &= \tfrac{3}{8} - \tfrac{1}{8}\gamma, \\ u_{21} = v_{43} = w_{21} = x_{43} &= \gamma, \\ u_{34} = v_{12} = w_{34} = x_{12} &= \overline{\gamma}, \\ u_{43} = v_{21} = w_{43} = x_{21} &= \tfrac{3}{8} - \tfrac{1}{8}\overline{\gamma}. \end{aligned}$$

where $\gamma = -\frac{1}{2} \pm \frac{1}{2} i\sqrt{15}$.

The holonomies of the meridian and longitude on the first torus (it corresponds to the vertex 1 of the first tetrahedron) are both equal to

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

while the holonomies of the meridian and longitude of the second torus are equal to

$$\begin{bmatrix} \frac{1}{4}+\frac{1}{4}\,i\sqrt{15} & \frac{3}{4}+\frac{3}{4}\,i\sqrt{15} & \frac{3}{2}+\frac{1}{2}\,i\sqrt{15} \\ -\frac{3}{8}-\frac{1}{8}\,i\sqrt{15} & -2-\frac{1}{2}\,i\sqrt{15} & -\frac{21}{8}-\frac{3}{8}\,i\sqrt{15} \\ \frac{3}{2} & \frac{21}{4}+\frac{1}{4}\,i\sqrt{15} & \frac{19}{4}+\frac{1}{4}\,i\sqrt{15} \end{bmatrix}.$$

The representation corresponding to this solution was also obtained in [17] where it is shown that it is the holonomy of a spherical CR structure on the complement of the Whitehead link.

- ▶ $f_3 = f_4 = Y^4 + Y^3 + 2Y^2 - Y + 1$. All of these 8 solutions are CR.
- ▶ $f_5 = f_6 = Y^{10} - 4Y^8 + Y^7 + 7Y^6 - 4Y^5 - 8Y^4 + 4Y^3 + 5Y^2 + 1$. 4 of these 20 solutions are CR-spherical.

Among these 32 solutions, 14 are CR.

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