A simple proof of the Riemann Mapping Theorem for Domains with Spherical Boundary

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1 Introduction

A hypersurface of a complex manifold is spherical if it is locally equivalent to $S^{2n-1} \subset \mathbf{C}^n$ as a CR manifold (see section 4 for definitions). We prove the following theorem

Theorem 1.1 *Let Ω be a simply connected domain of a Stein manifold of dimension ≥ 3. Suppose the boundary of Ω is connected and spherical. Then Ω is biholomorphic to the complex ball.*

The theorem is announced in [CJ] in the case of domains of $\mathbf{C}^n$ and all $n \geq 2$. We give a simple proof for the case $n \geq 3$ (see section 4). But we don't know what happens in dimension 2 (see the remarks at the end of the paper). The hypothesis about connectedness of the boundary is superfluous if we assume that Ω is a Stein domain.

As the proof is based essentially on the topology of Stein manifolds and Hartogs theorem we also give a topological flavored proof of Hartogs theorem, proving it for Riemann domains over Stein manifolds. The fundamental result we use is that the envelope of holomorphy of a Riemann domain has a strictly plurisubharmonic exhaustion function. This result follows from Oka's theorem and this plurisubharmonic function is used as a Morse function to obtain topological properties as in Andreotti-Frankel's theorem[AF].

The proof of theorem 1 is independent of section 3 and the reader is invited to read section 4 independently on the previous section.

The author would like to thank Professors J.-M. Trépreux and L. Siebenman for enlightning discussions.

2 Envelopes of holomorphy and Stein manifolds

Definition 2.1 *A Riemann domain over a complex manifold S is a pair (M,π) where M is a complex manifold and $\pi : M \to S$ is a local biholomorphism.*

Definition 2.2 *The envelope of holomorphy of (M,π) is another Riemann domain $(E(M),\underline{\pi})$ together with a map $p : M \to E(M)$ such that any holomorphic function on M is the pull-back of a holomorphic function on $E(M)$ and such that*

1) $\underline{\pi} \circ p = \pi$

2) it is maximal, in the sense that if $(E(M)', \underline{\pi}')$ is another such pair, then there exists a map $e : E(M)' \to E(M)$ satisfying $\underline{\pi}' = \underline{\pi} \circ e$.

It can be proved that the envelope of a Riemann domain exists [C]. The fundamental result we will use is the following theorem by Doquier-Grauert [DG], which generalizes the work of Oka on Riemann domains over $\mathbf{C}^n$.

Theorem 2.1 ([DG]) *The envelope of holomorphy of a Riemann domain over a Stein manifold is a Stein manifold.*

Stein manifolds are generalizations of open Riemann surfaces to higher dimensions, in the sense that they have plenty of holomorphic functions. For the sake of completeness we give the basic definitions and Andreotti-Frankel-Milnor theorem on the homotopical type of a Stein manifold. It is based on the existence of a strictly plurisubharmonic exhaustion function.

Definition 2.3 *A complex manifold S is a Stein manifold if there exists a proper imbedding $i: S \to \mathbf{C}^n$, for some n.*

Theorem 2.2 ([AF,M1]) *If S is a Stein manifold of complex dimension n , then S is homotopic to a CW-complex of dimension n.*

See [G], [El] for more information on the topology of Stein manifolds.

In the next section it will be important to consider information on what occurs at neighborhoods of boundaries of Stein manifolds, we then give some background on the topology of the boundaries.

Let X be a topological space. Suppose X is a relatively compact subset of a larger space Y. We say that an open set V $\subset$ X is a neighborhood of ∂X if it is the intersection of a neighborhood of ∂X in Y and X. A distinguished neighborhood of ∂X is defined to be such a neighborhood which doesn‘t have relatively compact components in X. If X is not a subspace of a larger space we still can define a distinguished neighborhood of the boundary of X as an open set V such that V = X - K, where K is a compact in X and there are no relatively compact components of X - K. We say then, that V is a neighborhood of ∂X, as we will consider only distinguished neighborhoods.

Observe that this definition doesn't imply that the neighborhood is connected, as the cylinder $S^1\times$(0,1) shows. As any open Riemann surface is a Stein space, there aren't restrictions on the possible boundaries of 1-dimensional Stein spaces. The situation changes completely in higher dimensions. In fact we have the following theorem, where we identify, by abuse of language, ∂S with a neighborhood of ∂S.

Theorem 2.3 (see [M1]) *If S is a Stein manifold of complex dimension ≥ 2, then $\pi_1(S) = \pi_1(\partial S)$.*

3 Hartogs Theorem

To pave the way towards Hartogs theorem, we start with the following theorem which guarantees the Jordan-Brouwer separation property in Stein manifolds.

Theorem 3.1 *Let $H \subset S$ be a compact connected orientable hypersurface imbedded in a Stein manifold S of complex dimension ≥ 2. Then S - H has two connected components, one of them relatively compact.*

Proof. Observe first that there exists a tubular neighborhood of H in S. That is an open set $T \subset S$, such that $H \subset T$, and such that T - H has two connected components. There are at most two connected components of S - H, because if A is a connected component, we must have that $A \cap T \neq \emptyset$. Suppose, by contradiction, that S - H has only one component. By cutting along H and pasting along H another copy of S, we obtain a connected complex manifold which has a strictly plurisubharmonic exhaustion function (Observe that by Grauert's theorem [G], this implies that this manifold is a Stein manifold). Now, this is a contradiction with the last theorem, as in complex dimensions greater than or equal to two, ∂S is connected. For a diferent proof, see [D], corollary 3.6, Chap. VIII. □

Corollary 3.1 *Let S be a Stein manifold of complex dimension ≥ 2 and $\Omega \subset S$ be a domain such that $S - \Omega$ doesn't have relatively compact components in S. Then there exists a connected neighborhood of $\partial\Omega$.*

Theorem 3.2 (Hartogs) *Let $\Omega \subset S$ be a domain in a Stein manifold of dimension ≥ 2 and $f : \Omega - K \to \mathbf{C}$ a holomorphic function, where $K \subset \Omega$ is compact and $\Omega - K$ connected. Then f admits an extension $F : \Omega \to \mathbf{C}$.*

Proof. Without loss of generality we might suppose that $S - \Omega$ doesn't have compact components. Let E be the envelope of holomorphy of the domain $\Omega - K$. By corollary 3.1 $\partial\Omega$ has a connected neighborhood, and therefore, there exists a compact connected hypersurface H, such that K is contained in the relatively compact component S_H of $S - H$. We claim that the relatively compact component E_H of $E - p(H)$ is biholomorphic to S_H, and this finishes the proof. Observe that the image of E_H by the projection is an open set, but the image of $E_H \cup p(H)$ is compact. The only possibility is that the image of E_H by the projection is contained in S_H. Consider now the map $p : E_H \cup p(H) \to S_H \cup H$. It is a local homeomorphism and $E_H \cup p(H)$ is compact, therefore it is a covering map. As, when restricted to $p(H)$ this map is a homeomorphism, this completes our proof. □

We will prove now a generalization of Hartogs theorem to covers of domains in Stein manifolds.

Theorem 3.3 *Let (M, π) be a Riemann domain which is a finite cover of $\Omega - K$, where $\Omega \subset S$ is domain in a Stein manifold of dimension ≥ 2 and $K \subset \Omega$ is compact with $\Omega - K$ connected. Then $\underline{\pi}$ restricted to $p(M) \subset E(M)$ is a covering of $\Omega - K$ which extends to a cover of Ω.*

Proof. It is easy to see that $\underline{\pi}$ restricted to $p(M) \subset E(M)$ is a covering of $\Omega - K$. We will prove that $\underline{\pi}$ extends to a covering of Ω. We supoose as in the proof of Hartogs theorem that $S - \Omega$ doesn't have compact components. Let $H \subset \Omega$ be a hypersurface such that $\Omega - H$ has two connected components and such that K is contained in the relatively compact one. Observe that $\underline{H} = \underline{\pi}^{-1}(H)$ is a compact hypersurface because $\underline{\pi}$ is proper. Let $\underline{H^i}$ be one connected component of $\underline{H}$, then $E(M) - \underline{H^i}$ has two connected components, using theorem 3.1 as in the proof of Hartogs theorem above, we get that the relatively compact one is projected onto the relatively compact component of $\Omega - H$, therefore it is a covering of $\Omega - H$. Repeating this argument for each component $\underline{H^i}$ we get the theorem. Observe that if we choose a hypersurface H near enouh to $\partial\Omega$, than $\underline{H}$ has only one connected component. □

Corollary 3.2 *Let (M, π) be a Riemann domain which is a finite cover of $\Omega - K$, where $\Omega \subset S$ is a domain in a Stein manifold of dimension ≥ 2 and $K \subset \Omega$ is compact with $\Omega - K$ connected. Suppose Ω is simply connected, then $E(M) = E(\pi(M))$.*

An important situation occurs when $\Omega - K$ is a tubular neighborhood of a compact hypersurface. A theorem by Lewy [L] guarantees that we can work with CR-structures, rather than tubular neighborhoods of hypersurfaces (see section 4).

Example

Consider the Reinhardt domain

$$\Omega = \{ \ (z_1, z_2) \in \mathbf{C}^2 \mid (|z_1| - 1)^2 + (|z_2| - 1)^2 < 1/2 \ \}$$

Observe that it is homeomorphic to $S^1 \times S^1 \times D^1$, whereas $\partial\Omega$ is homeomorphic to $S^1 \times S^1 \times S^1$. It is easy to see that it is a Stein manifold with a strictly pseudoconvex boundary. We describe two completely different possible 2-coverings of $\partial\Omega$.

i) If we consider a finite cover of the last S^1 we obtain a CR-manifold whose CR-functions are pull-backs of CR-functions on $\partial\Omega$, that is, the envelope of holomorphy of the cover is equal to Ω. In particular, the finite cover cannot be the boundary of a domain.

ii) Consider now a double cover of the first S^1, we can give it explicitly as

$$M = \{ \ (z_1, z_2, w) \in \mathbf{C}^3 \mid w^2 = z_1 \text{ and } \ (|z_1| - 1)^2 + (|z_2| - 1)^2 = 1/2 \ \}$$

In this case we see clearly that M is embedded into $\mathbf{C}^3$. M extends to a cover E(M) of Ω given by

$$M = \{ \ (z_1, z_2, w) \in \mathbf{C}^3 \mid w^2 = z_1 \text{ and } \ (|z_1| - 1)^2 + (|z_2| - 1)^2 \leq 1/2 \ \}$$

4 Proof of theorem 1.1

Let M be a smooth manifold equipped with a contact distribution D and suppose that a complex structure J is defined on D, that is, J is a smooth endomorphism $D \to D$ such that $J^2 = -I$ at each point of M. Let $D^{1,0} \oplus D^{0,1}$ be the decomposition of the complexification of D in $\pm$i-eigenbundles of J. we say that the triple (M, D, J) is a strictly pseudoconvex CR manifold if $D^{1,0}$ is involutive. Natural examples arise considering convex hypersurfaces in $\mathbf{C}^n$, perhaps the most natural is the sphere $S^{2n-1} \subset \mathbf{C}^n$. A function $F : M \to N$, where M is a CR manifold and N is a CR manifold with distribution D', or a complex manifold, is called a CR function if $dF \circ J_M = J_N \circ dF$ (if N is a CR manifold we impose that $dF(D) = D'$). A *spherical* CR manifold is a manifold M which is locally CR-equivalent to $S^{2n-1} \subset \mathbf{C}^n$, that is, for each $x \in M$ there exists a neighborhood U of that point and a diffeomorphism $F : U \to V$ for $V \subset S^{2n-1}$ such that $dF \circ J_M = J_{S^{2n-1}} \circ dF$.

let M be a strictly pseudoconvex CR-manifold embedded in a complex manifold as connected component of the boundary of an open set D. We say that a CR-function F defined on M extends to D if there exists a continuous function defined on D$\cup$M which coincides with F on M and is holomorphic on D. By a result of H. Lewy, if M is a strictly pseudoconvex CR-manifold embedded in a complex manifold then CR-functions extend to one half of a tubular neighborhood of M. We will use the following form of Hartogs theorem:

Theorem 4.1 ([Hartogs-Lewy]) *let M be a strictly pseudoconvex closed CR-manifold embedded in a Stein manifold bounding a domain Ω. Then CR-functions extend to Ω.*

The following theorem is a consequence of a Liouville type theorem for spherical CR structures and hence of the existence of a development map (see [BS]).

Theorem 4.2 ([BS]) *Let M be a simply connected closed spherical CR-manifold. Then M is CR-equivalent to the sphere.*

Using Hartogs theorem one obtains easily the following

Corollary 4.1 ([BS]) *Let Ω be a domain with spherical simply connected boundary. Then it is biholomorphic to the ball.*

Prof of theorem 1 As the boundary of a simply connected domain is simply connected in dimensions greater then 2 (by theorem 2), we may apply the corollary above. This ends the proof.

Remarks

1. In $\mathbf{C}^2$ the theorem is true as proved in [CJ].
2. For a Stein manifold of dimension two the theorem above might still be true. But the above proof breaksdown as it is not true that a simply connected strictly pseudoconvex domain has simply connected boundary (see the example in section 3 and Andreotti-Rossi's example as described in [F]).
3. One can use the same proof to establish a Riemann mapping theorem for domains whose boundary is locally equivalent to other homogeneous pseudoconvex CR manifolds.

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