A Circle of Modular Groups in PU(2,1)

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Abstract

We prove that there exists a circle of discrete and faithful embeddings of the triangle group of type $(2, 3, \infty)$ in the automorphisms group of complex hyperbolic space. The proof is obtained by a construction of a fundamental domain using **C**-spheres.

1 Introduction

Let Γ be the triangular group of type (p, q, ∞), that is, the abstract group presented by

$$< \iota_0, \iota_1, \iota_2 : \iota_0^2 = 1, \iota_1^2 = 1, \iota_2^2 = 1, (\iota_0 \circ \iota_1)^p = 1, (\iota_0 \circ \iota_2)^q = 1 > .$$

By an *embedding* of Γ in $\widehat{\mathbf{PU}(2,1)}$ (containing the holomorphic and the anti-holomorphic transformations), the isometry group of the two dimensional complex ball (with the natural metric invariant under biholomorphisms), we will consider a homomorphism such that ι_i are mapped to anti-holomorphic generators and such that $\iota_1 \circ \iota_2$ is parabolic. In this paper we prove

Theorem 1.1 *There exists a circle of discrete faithful embeddings of the triangle group* $(2, 3, \infty)$ *in* $\widehat{\mathbf{PU}(2,1)}$. *Up to conjugation in* $\widehat{\mathbf{PU}(2,1)}$ *the family is reduced to a quotient by the dihedral group* $\mathbf{Z_2} \times \mathbf{Z_2}$. *Moreover the family contains embeddings fixing a complex geodesic and embeddings fixing a totally real totally geodesic plane.*

Embeddings of triangle groups of type (p, q, ∞) in the neighborhood of an embedding fixing a complex geodesic were analyzed previously in [FK2]. A family connecting embeddings preserving a real and a complex geodesic for the triangle modular group were also obtained in [GuP1] independently, but there, fundamental domains are constructed for a subgroup of the triangle group. We prove discretness of the embeddings by constructing explicitly fundamental domains using **C**-spheres (see [FZ, FK1]).

The technique used in this paper is also sufficient to prove the same result for triangle groups (p, q, ∞) for $p \leq q \leq 4$. In the case $3 \leq p$ we obtain in fact an open neighborhood of a circle in the set of embeddings in $\widehat{\mathbf{PU}(2,1)}$. On the other hand, for other triangle groups the technique proves the existence of deformations of embeddings fixing either a complex geodesic or a totally real one. We thank S. Markham and J. Parker for help with trigonometry.

2 Complex hyperbolic space and its boundary

In this section and the following we collect general results about complex hyperbolic space. As a reference we use [G2],[FK1] and [FK2].

By a Cayley transform, complex hyperbolic space

$$H^2_{\mathbf{C}} = \{ \ w \in \mathbf{C}^2 \ : \ |w_1|^2 + |w_2|^2 < 1 \ \}$$

is biholomorphic to

$$V = \{ \ z \in \mathbf{C}^2 \ : \ \mathrm{Im}\,(z_2) > |z_1|^2 \ \}.$$

It is well known that its boundary S^3 can be identified to the one point compactification of the Heisenberg group $\overline{\mathbf{H}}$. The group of all conformal transformations of $\overline{\mathbf{H}}$ is isomorphic to $\widehat{\mathbf{PU}(2,1)}$ (see [G2]).

There are two kinds of totally geodesic submanifolds of real dimension 2 in $H^2_{\mathbf{C}}$: *complex geodesics* (represented by $H^1_{\mathbf{C}} \subset H^2_{\mathbf{C}}$) and *totally real geodesic 2-planes* (represented by $H^2_{\mathbf{R}} \subset H^2_{\mathbf{C}}$). Each of these totally geodesic submanifold is a model of the hyperbolic plane.

We define $\mathbf{C}$-*circles* to be the intersections of S^3 with the boundaries of totally geodesic complex submanifolds $H^1_{\mathbf{C}}$ in $H^2_{\mathbf{C}}$. Analogously, we define $\mathbf{R}$-*circles* to be the intersections of S^3 with the boundaries of totally geodesic totally real submanifolds $H^2_{\mathbf{R}}$ in $H^2_{\mathbf{C}}$.

Definition 2.1 *An inversion in an* $\mathbf{R}$*-circle is a non-trivial element of* $\widehat{\mathbf{PU}(2,1)}$ *transformation which fixes it point-wise.*

There is, then, a one-to-one correspondence between inversions and $\mathbf{R}$-circles. For instance the transformation $\hat{m}(z,t) = (\overline{z}, -t)$ on the Heisenberg group is the inversion that fixes point-wise the $\mathbf{R}$-circle $\mathrm{Im}\,(z) = 0$.

2.1 R-circles in the Heisenberg model

We denote by I_0 the inversion which leaves point-wise fixed the standard $\mathbf{R}$-circle $\mathbf{R}_0$ (see [G2] and [FK1] for details)

$$r^2 + iz = -e^{-2i\theta}.$$

$\mathbf{R}_0$ is given in cylindrical coordinates by $r = \sqrt{-\cos(2\theta)}, z = \sin(2\theta)$.

Definition 2.2 *The contact plane at* $M = (r, \theta, z)$ *is :* $z - Z - 2r(\cos(\theta)Y - \sin(\theta)X) = 0$.

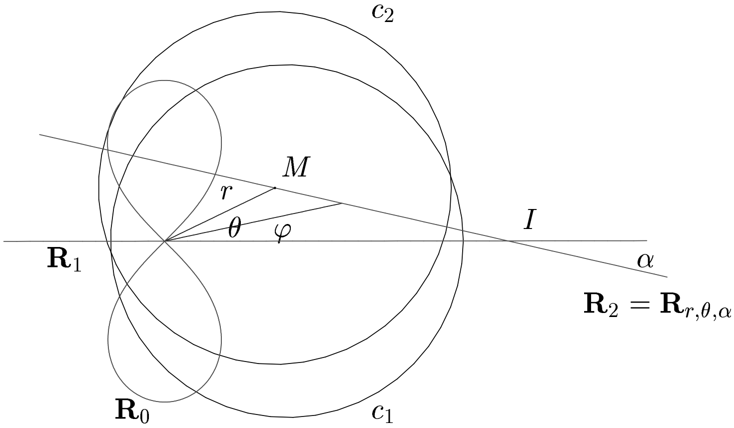
Any infinite $\mathbf{R}$-circle lies in the contact plane at each of its points.

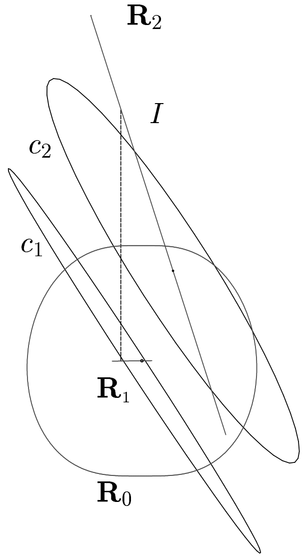
In the Heisenberg model, $\mathbf{C}$-circles are either vertical lines or else ellipses that belong to the contact plane of their centers and whose projections onto the z-plane are circles.

Lemma 2.3 (see [FK1]) *Invariant* $\mathbf{C}$*-circles under the inversion on* $\mathbf{R}_0$ *are centered in the surface of centers* $\{z = \sin(2\theta)\}$ *and have radius* $R^2 = r^2 + \cos(2\theta)$. *If* $r^2 + \cos(2\theta) \leq 0$ *we say that the* $\mathbf{C}$*-circle is imaginary.*

Definition 2.4 $\mathbf{R}_{r,\theta,\alpha}$ *is the infinite* $\mathbf{R}$*-circle passing through* $M = (r, \theta, \sin(2\theta))$ *whose projection on the* z*-plane is the line of slope* $\tan(\alpha)$.

Observe that $\mathbf{R}_{r,\theta,\alpha}$ is horizontal if and only if $\alpha = \theta \mod \pi$.





Left: Projection on the z-plane of the configuration of $\mathbf{R}_0$, $\mathbf{R}_1$ and $\mathbf{R}_2(\theta)$ for the modular group showing the invariant $\mathbf{C}$-circles c_1 and c_2. In the picture we fixed $\theta = \pi/7$. $\mathbf{R}_2(\theta)$ has polar coordinates ρ and φ.

Right: A space view of the configuration.

2.2 Configurations

Let ι_0, ι_1 and ι_2 be the standard generators of the triangle group $(2, 3, \infty)$. Consider an embedding $r : (2, 3, \infty) \to \mathbf{P}\widehat{\mathbf{U}(2,1)}$ such that $r(\iota_i) = I_i$ are inversions fixing three $\mathbf{R}$-circles $\mathbf{R}_i$ and such that $I_1 \circ I_2$ is parabolic.

Observe that $I_1 \circ I_0$ is an elliptic element of order 2 and $I_2 \circ I_0$ is an elliptic element of order 3. $I_1 \circ I_0$ is conjugate to an element in $U(2)$. Its eigenvalues may be ± 1. Similarly $I_2 \circ I_0$ could have eigenvalues $1, e^{2\pi/3}, e^{-2\pi/3}$. There are 6 possible families of embeddings.

In [FK2] The family of embeddings such that $I_1 \circ I_0$ have both eigenvalues ± 1 and such that $I_2 \circ I_0$ have both eigenvalues $e^{2\pi/3}, e^{-2\pi/3}$ are called of type $(\pi/2, -\pi/2, \pi/3, -\pi/3)$. It contains two distinguished discrete and faithful representations, namely one preserving a complex line and one preserving a real line. A parameterization of those embeddings is given by the following proposition.

Consider in the Heisenberg group $\mathbf{R}_0$ to be the standard $\mathbf{R}$-circle, $\mathbf{R}_1$ to be the infinite $\mathbf{R}$-circle given by the x-axis and $\mathbf{R}_2$ to be the family of infinite $\mathbf{R}$-circles $\mathbf{R}_2(\theta) = \mathbf{R}_{r,\theta,\alpha}$ where

$$\alpha = -\pi/2 + 3\theta, r^2 = -\frac{\cos(2\theta - \pi/3)\cos(2\theta + \pi/3)}{\cos(2\theta)} = -\frac{\cos(6\theta)}{4\cos^2(2\theta)}$$

for $\frac{\pi}{12} \leq \theta \leq \frac{\pi}{6}$. We denote by I_0 the inversion fixing $\mathbf{R}_0$, by I_1 the inversion fixing the $\mathbf{R}_1$ and by $I_2(\theta)$ the inversion on the $\mathbf{R}$-circle $\mathbf{R}_2(\theta)$.

Theorem 2.5 (FK2) *The representations r of the triangle group $(2, 3, \infty)$ with $r(\iota_0) = I_0$, $r(\iota_1) = I_1$ and $r(\iota_2) = I_2(\theta)$ form a connected component of the representation space up to conjugations in $\widehat{\mathbf{PU}(2,1)}$. Moreover if $\theta = \pi/12$ the representation fixes a complex line (a $\mathbf{C}$-fuchsian representation) and if $\theta = \pi/6$, the representation fixes a totally real totally geodesic plane (a $\mathbf{R}$-fuchsian representation).*

A circle of representations in $\widehat{\mathbf{PU}(2,1)}$ is obtained by conjugating the group by the inversions fixing the x-axis and the y-axis. The index two subgroup generated by the order two and order three elements will have, a fortiori, a circle of representations in $\mathbf{PU}(2,1)$. The space of representations modulo conjugation in $\mathbf{PU}(2,1)$ is obtained by the quotient of this circle by a $\mathbf{Z}_2$-action (a rotation by π).

3 Families of non-linked C-circles

Let C_1 and C_2 be two $\mathbf{C}$-circles and M_1, M_2 be their centers with parameters (r_1, θ_1, z_1) and (r_2, θ_2, z_2). Their radius are R_i. They are not linked if and only if

$$E = d^4 - 2(R_1^2 + R_2^2)d^2 + (R_1^2 - R_2^2)^2 + (h + 4S)^2 > 0,$$

where $h = z_2 - z_1$ and $S = 1/2(x_1 y_2 - x_2 y_1)$, $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ (see [FK1]).

3.1 Good axis in the surface of centers

Let $M(\theta)$ be lying on the surface of centers of $\mathbf{R}_0$. That is $M(\theta) = (r, \theta, \sin(2\theta))$ It defines an invariant $\mathbf{C}$-circle $C(\theta)$ for $\mathbf{R}_0$ with radius $R^2(\theta) = r^2(\theta) + \cos(2\theta)$.

Lemma 3.1 *Let $r_0 \leq 1$. The curve $M(\theta) = (r_0, \theta, \sin(2\theta))$ in the surface of centers defines a family of invariant non-linked $\mathbf{C}$-circles for $\mathbf{R}_0$, when $\theta \in [-\pi/2, \pi/2]$.*

Proof. Let $r_1 = r_2 = r_0$, we get

$$\begin{aligned}
1/4\,E &= -r_0^2(\cos(2\theta_1) + \cos(2\theta_2) - 2\cos(\theta_1+\theta_2)) + \sin(\theta_1 - \theta_2)^2 \\
&= (1 - \cos(\theta_1 - \theta_2))\left((\cos(\theta_1 - \theta_2) + r_0^2\cos(\theta_1+\theta_2)) + (r_0^2\cos(\theta_1+\theta_2) + 1)\right) \\
&= (1 - \cos(\theta_1 - \theta_2))\left(\cos(\theta_1)\cos(\theta_2)(1 + 2r_0^2) + \sin(\theta_1)\sin(\theta_2)(1 - 2r_0^2) + 1\right).
\end{aligned}$$

But $\cos(\theta_1)\cos(\theta_2)(1 + 2r_0^2) + \sin(\theta_1)\sin(\theta_2)(1 - 2r_0^2) + 1 \geq \min(2r_0^2, 2 - 2r_0^2) \geq 0.$ □

We call such a curve a good axis (cf. [FK1]).

3.2 Dilated C-circles

We construct a two parameter family of unlinked **C**-circles using dilations $l_\lambda : (z,t) \mapsto (\lambda z, \lambda^2 t)$ in the Heisenberg group.

Lemma 3.2 *Let $r(\theta), \theta \in I$ be a curve in the surface of centers such that corresponding **C**-circles are not linked (*good axis *in [FK1]). Let $C_\lambda(\theta) = l_\lambda(C(\theta))$ be the two-parameter family of dilated **C**-circles $C(\theta)$ by dilation l_λ. For any $\lambda_1, \lambda_2 > 0$ and $(\theta_1, \theta_2) \in I \times I$. If $r(\theta_1)r(\theta_2)\cos(\theta_1+\theta_2) \geq -1$, then $C_{\lambda_1}(\theta_1)$ and $C_{\lambda_2}(\theta_2)$ are not linked.*

Remark 3.3 *Observe that the surfaces $C_\lambda(\theta)$ (constant λ) are disjoint. Moreover, by the previous lemma, as $r(\theta) = r_0 \leq 1$ is a good axis, the conclusion of the lemma is true in that case.*

Proof. Let $C(\theta)$ be a **C**-circle of center $M(\theta)$ and radius $R(\theta)$, then $C_\lambda(\theta)$ has center $M_\lambda(\theta)$ and radius λR. Linking formula for $C_{\lambda_1}(\theta_1)$ and $C_{\lambda_2}(\theta_2)$ is

$$\begin{aligned} F &= \lambda_1^4 + \lambda_2^4 - 2\lambda_1^2\lambda_2^2(2r_2^2\cos(2\theta_1) + 2r_1^2\cos(2\theta_2) + \cos(2\theta_1 - 2\theta_2)) + 4\lambda_1\lambda_2 r_1 r_2(\lambda_1^2+\lambda_2^2)\cos(\theta_1+\theta_2) \\ &= \lambda_1^2\lambda_2^2 E + (\lambda_1-\lambda_2)^2\left(\lambda_1^2+\lambda_2^2+2\lambda_1\lambda_2(1+2r_1r_2\cos(\theta_1+\theta_2))\right) \geq 0 \end{aligned}$$

when $E \geq 0$ and $r_1 r_2 \cos(\theta_1+\theta_2) \geq -1$. □

3.3 Invariant C-circles for an infinite R-circle

Lemma 3.4 *Let $C(\varphi), \varphi_1 \leq \varphi \leq \varphi_2$ be a family of **C**-circle of radius $R(\varphi)$ centered at $M(\varphi)$ which projects to a line in the the xy-plane . Suppose that $\varphi \mapsto R(\varphi)$ is differentiable and let $d(\varphi)$ be the horizontal distance between $M(\varphi)$ and $M(\varphi_2)$. If $\left[\frac{\mathrm{d}R^2(\varphi)}{\mathrm{d}\varphi}\right]^2 > 4R(\varphi)^2\left[\frac{\mathrm{d}d(\varphi)}{\mathrm{d}\varphi}\right]^2$, then the family $C(\varphi)$ has no links.*

Proof. In this case $R(\varphi) - d(\varphi)$ is monotonic and circles are not linked because their projections are not. □

4 Construction of a fundamental domain

Theorem 4.1 *The embeddings of the triangle group defined by $< I_0, I_1, I_2(\theta) >$ are faithful and discrete for all $\pi/12 \leq \theta \leq \pi/6$.*

It follows from the following construction.

Theorem 4.2 *For each $\pi/12 \leq \theta \leq \pi/6$, there exits three surfaces S_0, S_1 and S_2 such that*

- *each S_i is foliated by invariant **C**-circles under I_i*
- *$c_1 = S_0 \cap S_1$ is an invariant **C**-circle under both I_0 and I_1.*

- *$c_2 = S_0 \cap S_2$ is an invariant $\mathbf{C}$-circle under both I_0 and $I_2(\theta)$.*

- *$S_1 \cap S_2$ is the point at infinity*

Remark 4.3 *Those surfaces are called $\mathbf{C}$-surfaces in [FZ, FK1]. An application of Poincaré's theorem for complex hyperbolic geometry [FZ, FK2] states then that the region bounded by the three surfaces in the Heisenberg group is a fundamental domain for the triangle group. In [FK2] we proved the theorem for θ in a neighborhood of $\pi/12$.*

Proof. We construct three surfaces S_0, S_1 and S_2 and proof that they satisfy the properties in the theorem.

We recall that the $\mathbf{R}$-circle $\mathbf{R}_2(\theta)$ has an intersection with the surface of centers of $\mathbf{R}_0$ for $r^2 = -\dfrac{\cos(6\theta)}{4\cos^2(2\theta)}$. This intersection point determines a $\mathbf{C}$-circle c_2 which should be $S_0 \cap S_2$.

On the other hand any point in the x-axis could be a center of an invariant circle under both I_0 and I_1. For instance; the equator. To construct S_0 we must choose one of those invariant $\mathbf{C}$-circles.

S_0

For each fixed $\pi/12 \leq \theta < \pi/6$, we choose the family of invariant $\mathbf{C}$-circles under I_0 to be the family $C_0(\varphi)$ defined by the arc in the surface of centers $\rho_0(\varphi) = \rho_0 = r(\theta)$ of constant radius. This choice also fixes the invariant $\mathbf{C}$-circle c_1 under both I_0 and I_1 to have center at $(r_2(\theta), 0, 0)$ in the Heisenberg group. By lemma 3.1 this arc defines the surface S_0.

S_1

The $\mathbf{C}$-circle $c_1 = C_0(0)$ will be dilated into the $\mathbf{C}$-circles $C_1(t) = l_t c_1$ (where l_t is Heisenberg dilation by t) whose centers belong to $\mathbf{R}_1$, and $1 \leq t \leq \frac{\rho(0)}{\rho_0}$. By lemma 3.2 the dilated circles are disjoint and unlinked. Their union does not intersect S_0, and will be part of the surface S_1. We complete the surface S_1 by a union of concentric $\mathbf{C}$-circles centered at the point $\frac{\rho(0)}{\rho_0}$ of the x-axis with increasing radii up to infinity. We have $c_1 = S_0 \cap S_1$.

S_2

For each fixed $\pi/12 \leq \theta \leq \pi/6$, the $\mathbf{R}$-circle $\mathbf{R}_2(\theta)$ is parametrized by an angle φ such that $\alpha < \varphi < \alpha + \pi$, where $\alpha = -\pi/2 + 3\theta$. We have $[\rho(\varphi)\cos(\varphi), \rho(\varphi)\sin(\varphi), Z(\varphi)]$ with

$$\rho(\varphi) = r(\theta)\sin(\alpha - \theta)/\sin(\alpha - \varphi) = \frac{\sqrt{-\cos(6\theta)}}{2\cos(3\theta - \varphi)}$$

and

$$Z(\varphi) = \sin(2\theta) + 2r(\theta)\rho(\varphi)\sin(\theta - \varphi) = \sin(2\theta) + \frac{\cos 6\theta \sin(\theta - \varphi)}{2\cos 2\theta \cos(3\theta - \varphi)} = \frac{\sin(3\theta + \varphi)}{2\cos(3\theta - \varphi)}$$

Each **C**-circle $C_0(\varphi)$, for $0 \le \varphi \le \theta$, will be dilated into a **C**-circle $C_2'(\varphi) = l_{\frac{\rho(\varphi)}{\rho_0}} C_0(\varphi)$ (where $l_{\frac{\rho(\varphi)}{\rho_0}}$ is Heisenberg dilation by $\frac{\rho(\varphi)}{\rho_0}$) whose center belongs to the curve

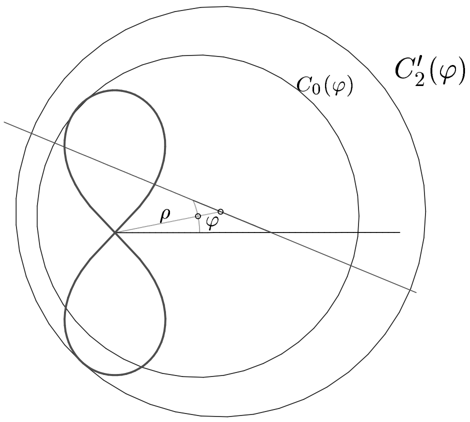
$$\mathbf{R}_2' = [\rho(\varphi)\cos(\varphi), \rho(\varphi)\sin(\varphi), \left(\frac{\rho(\varphi)}{\rho_0}\right)^2 \sin(2\varphi)].$$

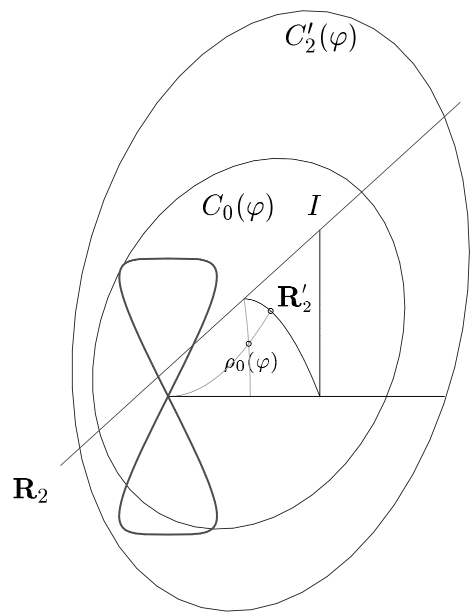
$\mathbf{R}_2'$ and $\mathbf{R}_2$ have same projection onto the xy-plane. But the height of $\mathbf{R}_2'$ is

$$Z'(\varphi) = \left(\frac{\rho(\varphi)}{\rho_0}\right)^2 \sin(2\varphi) = \frac{\cos^2 2\theta \sin 2\varphi}{\cos^2(3\theta - \varphi)}$$

We call S_2' the surface determined by those **C**-circles.

We translate (by a vertical translation on the Heisenberg group) each **C**-circle $C_2'(\varphi)$ in such a way that its center belongs to $\mathbf{R}_2$. We thus obtain a family S_2 of invariant **C**-circles $C_2(\varphi)$ for $\mathbf{R}_2$, for $0 \le \varphi \le \theta$. We complete the family by the union of concentric **C**-circles centered at the point of $\mathbf{R}_2$that projects over the x-axis.





Left: Projection on the z-plane of the **C**-circles of $C_0(\varphi)$ and $C_2'(\varphi)$; the second one is obtained from the first one by a dilation. In this configuration θ was fixed.

Right: A space view of the configuration. We see the curve $\mathbf{R}_2'(\varphi)$, which is a dilation of the constant radius curve $\rho_0(\varphi) = r(\theta)$. Shown in the picture the effect of the dilation at angle φ.

S_0, S_1 and S_2' have no linked C-circles

Any two of the **C**-circles of S_0, S_1 or S_2' are not linked because they are all obtained by dilation of **C**-circles of S_0 (by lemma 3.2).

S_2 has no linked C-circles

We will apply lemma 3.4 to the family of **C**-circles $C_2(\varphi)$. They all have centers in $\mathbf{R}_2$. Observe that $\mathbf{R}_2'$ and $\mathbf{R}_2$ project onto the same line in the xy-plane, so it is sufficient to show that the projections of the family $C_2'(\varphi)$ are disjoint. We have that $C_2'(\varphi) = l_{\frac{\rho(\varphi)}{\rho_0}} C_0(\varphi)$ has radii given by $R^2(\varphi) = \left(\frac{\rho(\varphi)}{\rho_0}\right)^2 (\rho_0^2 + \cos(2\varphi))$. Using lemma 3.4 we compute

$$\left[\frac{\mathrm{d}R^2(\varphi)}{\mathrm{d}\varphi}\right]^2 - 4R(\varphi)^2\left[\frac{\mathrm{d}d(\varphi)}{\mathrm{d}\varphi}\right]^2 = 4\frac{\sin(\alpha-\theta)^4}{\sin(\varphi-\alpha)^6}\left(\cos(\varphi+\alpha)^2 + \rho_0^2\cos(2\alpha) - \rho_0^4\sin(\varphi-\alpha)^2\right)$$

But

$$\begin{aligned}\cos(\varphi+\alpha)^2 + \rho_0^2\cos(2\alpha) - \rho_0^4\sin(\varphi-\alpha)^2 &\geq \cos(\varphi+\alpha)^2 + {\rho_0}^2\cos(2\alpha) - \sin(\varphi-\alpha)^2\\ &= \cos(2\alpha)\left({\rho_0}^2 + \cos(2\varphi)\right) \geq 0\end{aligned}$$

because $\cos(2\alpha) = \cos(6\theta - \pi) \geq 0$ for $\frac{\pi}{12} \leq \theta \leq \frac{\pi}{6}$.

$\mathbf{R}_2'$ is below $\mathbf{R}_2$

Let $Z'(\varphi)$ and $Z(\varphi)$ be the height of the centers of $C_2'(\varphi)$ and $C(\varphi)$. We compute

$$\begin{aligned}Z(\varphi) - Z'(\varphi) &= \frac{\sin(3\theta+\varphi)}{2\cos(3\theta-\varphi)} - \frac{\cos^2 2\theta \sin 2\varphi}{\cos^2(3\theta-\varphi)}\\ &= \frac{1}{4\cos^2(3\theta-\varphi)}\left(2\sin(3\theta+\varphi)\cos(3\theta-\varphi) - 4\cos^2 2\theta\sin 2\varphi\right)\\ &= \frac{1}{4\cos^2(3\theta-\varphi)}\left(\sin 6\theta + \sin 2\varphi - 4\cos^2 2\theta \sin 2\varphi\right)\\ &= \frac{1}{4\cos^2(3\theta-\varphi)}(\sin 6\theta - \frac{\sin 6\theta}{\sin 2\theta}\sin 2\varphi)\\ &= \frac{\sin 6\theta}{4\sin 2\theta\cos^2(3\theta-\varphi)}(\sin 2\theta - \sin 2\varphi)\end{aligned}$$

We thus deduce that $Z'(\varphi) \leq Z(\varphi), 0 \leq \varphi \leq \theta$.

S_0, S_1 and S_2 are disjoint

S_0, S_1 and S_2' are disjoint as seen previously. It remains to show that S_2 is disjoint from the other surfaces. As S_2 is above S_2', *a fortiori* S_2 does not intersect S_0 and S_1.

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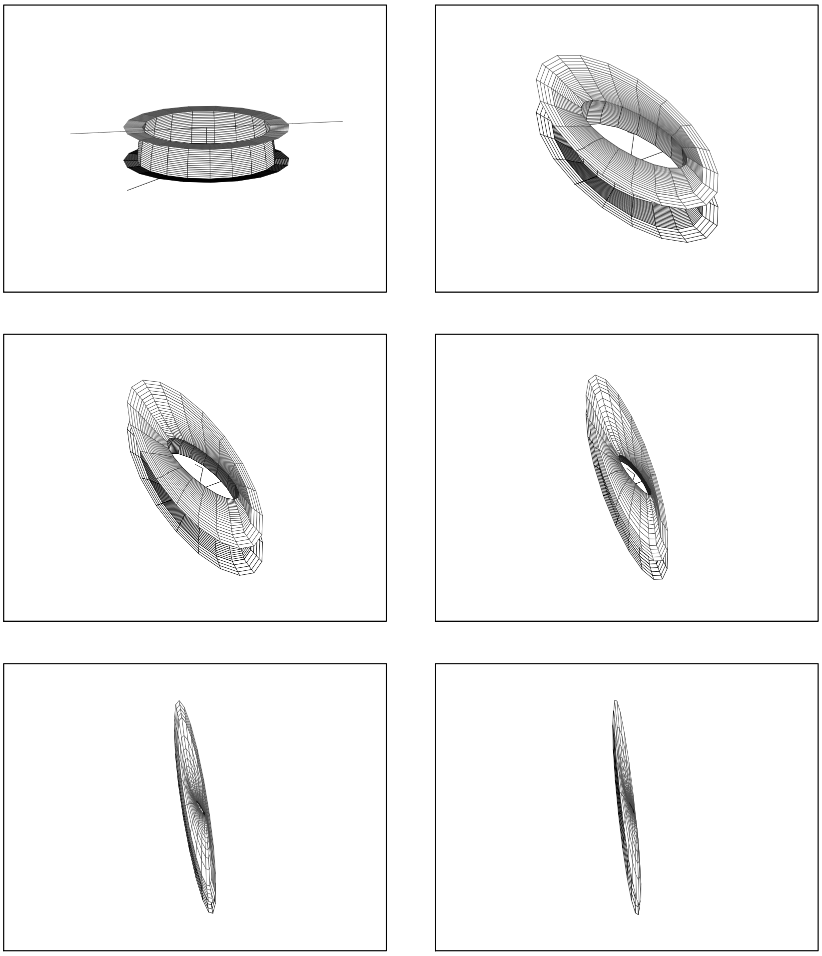

Six fundamental domains for θ between $\pi/12$ and $\pi/6$. Remark that scaling is not constant. The completion of the surfaces by concentric **C**-circles is partially drawn. The first upper left image corresponds to the embedding fixing a complex geodesic, while the last one fixes a totally real geodesic.

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