Diffeomorphisms preserving $\mathbb{R}$-circles in three dimensional CR manifolds

E. Falbel
Institut de Mathématiques
Analyse Complexe et Géométrie
4 place Jussieu - Paris France

J. M. Veloso
Faculdade de Matemática - ICEN
Universidade Federal do Pará
66059 - Belém- PA - Brazil

Abstract

$\mathbb{R}$-circles in general three dimensional CR manifolds (of contact type) are the analogues to traces of Lagrangian totally geodesic planes on S^3 viewed as the boundary of two dimensional complex hyperbolic space. They form a family of certain legendrian curves on the manifold. We prove that a diffeomorphism between three dimensional CR manifolds which preserve circles is either a CR diffeomorphism or conjugate CR diffeomorphism.

1 Introduction

Given a three manifold M equipped with a contact plane distribution D, we say M is a CR manifold if D is equipped with a complex operator $J : D \to D$ satisfying $J^2 = -Id$. Examples of CR manifolds arise naturally as real hypersurfaces in $\mathbb{C}^2$ or as boundaries of complex two-dimensional manifolds. The most important example appears as follows: the complex two-dimensional ball

$$H^2_{\mathbb{C}} = \left\{ \ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 < 1 \ \right\}$$

has boundary S^3 which is naturally equipped with a distribution $D = TS^3 \cap JTS^3$, where J denotes the standard complex structure of $\mathbb{C}^2$. The group of biholomorphisms of $H^2_{\mathbb{C}}$ is $PU(2,1)$ and it acts on S^3.

$\mathbb{C}$-circles (or chains) and $\mathbb{R}$-circles (or pseudocircles or circles for short) are analogues defined for general CR manifolds to curves in S^3 obtained as traces of complex lines and Lagrangian totally geodesic subspaces of $H^2_{\mathbb{C}}$ in its boundary. More precisely, complex lines through the origin are totally geodesic spaces and their intersection with S^3 are called chains. The group of biholomorphisms acts transitively on the space of chains. It turns out that a Lagrangian plane passing through the origin is a totally real totally geodesic surface of $H^2_{\mathbb{C}}$. Its intersection with the sphere is called an $\mathbb{R}$-circle. Again, $PU(2,1)$ acts transitively on the space of Lagrangian totally geodesic subspaces.

The generalization of these curves to real hypersurfaces of $\mathbb{C}^2$ was considered by E. Cartan [C], where chains and circles (what we call here $\mathbb{R}$-circles) are defined. Here we will restrict our attention to $\mathbb{R}$-circles. Cartan's definition was considered by Jacobowitz in [J], chapter 9, where these curves are called pseudocircles and are defined on an abstract three dimensional CR manifold. In this paper we make explicit the system of differential equations satisfied by $\mathbb{R}$-circles, and use these equations to prove our main theorem (Theorem 7.1) that a diffeomorphism between two CR manifolds which preserve circles is a CR map or an anti CR map. Our motivation was the analogue result for chains in [Ch].

We will first recall general results on CR structures, the group $SU(2,1)$ and the Cartan's connection on a principal bundle $Y \to M$ with group H (a subgroup of $SU(2,1)$ which is the isotropy of its action on S^3) which will be used in our main theorem. We also recall the definition of $\mathbb{R}$-circles (also called pseudocircles or circles for short) as projections of leaves of an integrable differential system (for more details we refer to [J]). We finally obtain the differential equations which circles satisfy in M (Theorem 6.3) and use them in the proof of our main theorem in the last section.

2 CR structures

For a general reference or this section see [CM, J]. We consider a three manifold M equipped with a contact plane distribution D. A 3-dimensional *CR-structure* (M, D, J) is the contact manifold M equipped with the complex operator $J : D \to D$ satisfying $J^2 = -Id$.

If (M, D, J) and $(\tilde{M}, \tilde{D}, \tilde{J})$ are 3-dimensional CR-structures, then a diffeomorphism $f : M \to \tilde{M}$ is a *CR-diffeomorphism* if $f_*(D) = \tilde{D}$ and $f_*J = \tilde{J}f_*$. If $f_*J = -\tilde{J}f_*$, we say that f is a *conjugate* CR-diffeomorphism.

A CR-structure induces an orientation on D and an orientation of the normal bundle TM/TD given by $X, JX, [X, JX]$ where X is a local section of D.

Fixing a local section X of D one can define a form θ such that $\theta(D) = 0$ and such that $\theta([X, JX]) = -2$.

Consider $D \otimes \mathbb{C} = D^{1,0} \oplus D^{0,1} \subset TM \otimes \mathbb{C}$. Taking $Z = \frac{1}{2}(X - iJX)$ and $\bar{Z} = \frac{1}{2}(X + iJX)$, we get $[Z, \bar{Z}] = \frac{i}{2}[X, JX]$. We define a form $\theta^1 \in D^{0,1\perp}$ such that $\theta^1(Z) = 1$. Then $d\theta(Z, \bar{Z}) = -\theta([Z, \bar{Z}]) = i$, so

$$d\theta = i\theta^1 \wedge \theta^{\bar{1}} \text{ modulo } \theta$$

where we define $\theta^{\bar{1}} = \overline{\theta^1}$. If θ'^1 is another form satisfying the equation we have

$$\theta'^1 = e^{i\alpha}\theta^1 \text{ modulo } \theta$$

for $\alpha \in \mathbb{R}$.

Let E to be the oriented line bundle of all forms θ as above. On E we define the tautological form ω. That is $\omega_\theta = \pi^*(\theta)$ where $\pi : E \to M$ is the natural projection.

We consider the tautological forms defined by the forms above over the line bundle E. That is, for each θ^1 as above, we let $\omega^1_\theta = \pi^*(\theta^1)$. At each point $\theta \in E$ we have the family

$$\omega' = \omega$$

$$\omega'^1 = e^{i\alpha}\omega^1 + v^1\omega$$

where we understand that the forms are defined over E. Those forms vanish on vertical vectors, that is, vectors in the kernel of the map $TE \to TM$. In order to define non-horizontal 1-forms we let θ be a section of E over M and introduce the coordinate $\lambda \in \mathbb{R}^+$ in E. By abuse of notation, let θ denote the tautological form on the section θ. Therefore the tautological form ω over E is

$$\omega_\lambda = \lambda\theta.$$

Differentiating this formula we obtain

$$d\omega = \omega \wedge \varphi + i\omega^1 \wedge \omega^{\bar{1}} \tag{1}$$

where $\varphi = -\frac{d\lambda}{\lambda}$ modulo $\omega, \omega^1, \omega^{\bar{1}}$, φ real.

Observe that $\frac{d\lambda}{\lambda}$ is a form intrinsically defined on E up to horizontal forms (the minus sign is just a matter of conventions). In fact, choosing a different section θ' with $\theta = \mu\theta'$ where μ is a function over M, we can write $\omega = \lambda\theta = \lambda\mu\theta'$ and obtain

$$\frac{d(\lambda\mu)}{\lambda\mu} = \frac{\mu d\lambda + \lambda d\mu}{\lambda\mu} = \frac{d\lambda}{\lambda} + \frac{d\mu}{\mu},$$

where $\frac{d\mu}{\mu}$ is a horizontal form.

If we write (1) as

$$\begin{aligned} d\omega &= \omega' \wedge \varphi' + i\omega'^1 \wedge \omega'^{\bar{1}} = \omega \wedge \varphi' + i(e^{i\alpha}\omega^1 + v^1\omega) \wedge (e^{-i\alpha}\omega^{\bar{1}} + v^{\bar{1}}\omega) \\ &= \omega \wedge (\varphi' - ie^{i\alpha}v^{\bar{1}}\omega^1 + ie^{-i\alpha}v^1\omega^{\bar{1}}) + i\omega^1 \wedge \omega^{\bar{1}} \end{aligned}$$

it follows that $\varphi' - ie^{i\alpha}v^{\bar{1}}\omega^1 + ie^{-i\alpha}v^1\omega^{\bar{1}} = \varphi$ modulo ω.

We obtain in this way a coframe bundle over E:

$$\omega' = \omega$$

$$\omega'^1 = e^{i\alpha}\omega^1 + v^1\omega$$

$$\varphi' = \varphi + ie^{i\alpha}v^{\bar{1}}\omega^1 - ie^{-i\alpha}v^1\omega^{\bar{1}} + s\omega$$

$v^1 \in \mathbb{C}$ and $s \in \mathbb{R}$ are arbitrary.

Definition 2.1 *We denote by Y the coframe bundle $Y \to E$ given by the set of 1-forms $\varphi, \omega^1, \omega^{\bar{1}}, \omega$. Two coframes are related by*

$$(\varphi', \omega'^1, \omega'^{\bar{1}}, \omega') = (\varphi, \omega^1, \omega^{\bar{1}}, \omega) \begin{pmatrix} 1 & 0 & 0 & 0 \\ ie^{i\alpha}v^{\bar{1}} & e^{i\alpha} & 0 & 0 \\ -ie^{-i\alpha}v^1 & 0 & e^{-i\alpha} & 0 \\ s & v^1 & v^{\bar{1}} & 1 \end{pmatrix}$$

where $s, \alpha \in \mathbb{R}$ and $v^1 \in \mathbb{C}$.

Theorem 2.2 ([C, CM]) *On Y we have unique globally defined forms $\omega, \omega^1, \omega_1^1, \varphi^1, \psi$ such that*

$$\begin{aligned}
d\omega &= \omega \wedge \varphi + i\omega^1 \wedge \omega^{\bar{1}} \\
d\omega^1 &= \tfrac{1}{2}\omega^1 \wedge \varphi + \omega^1 \wedge \omega_1^1 + \omega \wedge \varphi^1 \\
d\varphi &= i\omega^1 \wedge \varphi^{\bar{1}} - i\omega^{\bar{1}} \wedge \varphi^1 + \omega \wedge \psi \\
d\omega_1^1 &= \tfrac{3}{2}i\omega^1 \wedge \varphi^{\bar{1}} + \tfrac{3}{2}i\omega^{\bar{1}} \wedge \varphi^1 \\
d\varphi^1 &= \tfrac{1}{2}\varphi \wedge \varphi^1 - \omega_1^1 \wedge \varphi^1 + \tfrac{1}{2}\omega^1 \wedge \psi + Q_{\bar{1}}^1 \omega \wedge \omega^{\bar{1}} \\
d\psi &= 2i\varphi^1 \wedge \varphi^{\bar{1}} + \varphi \wedge \psi + \left(R_1\omega^1 + R_{\bar{1}}\omega^{\bar{1}}\right) \wedge \omega
\end{aligned}$$

with $\omega_1^1 + \overline{\omega_1^1} = 0$, ω, φ, ψ real and

$$dQ_{\bar{1}}^1 - 2Q_{\bar{1}}^1\varphi + 2Q_{\bar{1}}^1\omega_1^1 = S\omega - \frac{1}{2}R_{\bar{1}}\omega^1 + T\omega^{\bar{1}},$$

$$dR_1 - \frac{5}{2}R_1\varphi - R_1\omega_1^1 + 2i\overline{Q}_{\bar{1}}^1\varphi^1 = A\omega + B\omega^1 + C\omega^{\bar{1}}.$$

where A, B, C are functions on Y and C is real.

We can verify easily that structure equations in Cartan or Jacobowitz are the same as here, with the correspondance

$$\Omega = \omega,\ \ \Omega_1 = \omega^1,\ \ \Omega_2 = -\frac{1}{2}\varphi - \omega_1^1,\ \ \Omega_3 = -\varphi^1,\ \ \Omega_4 = \frac{1}{2}\psi$$

and

$$R = Q_{\bar{1}}^1,\ \ S = R_1.$$

3 SU(2, 1)

Define

$$SU(2,1) = \{\ g \in SL(3,\mathbb{C}) \mid \bar{g}^T Q g = Q\ \}$$

where the Hermitian form Q given by

$$Q = \begin{pmatrix} 0 & 0 & i/2 \\ 0 & 1 & 0 \\ -i/2 & 0 & 0 \end{pmatrix}.$$

The group $SU(2,1)$ acts on $\mathbb{C}^3$ on the left preserving the cone

$$\{\ z \in \mathbb{C}^3 \mid \bar{z}^T Q z = 0\ \}.$$

The projectivization of this cone is $S^3 \subset \mathbb{C}P^2$. $SU(2,1)$ has a finite center K which is a cyclic group of order 3 acting trivially on the sphere S^3. We define $PU(2,1) = SU(2,1)/K$.

The elements of the Lie algebra $\mathfrak{s}u(2,1)$ are represented by the matrices

$$\begin{pmatrix} u & -2i\bar{y} & w \\ x & a & y \\ z & 2i\bar{x} & -\bar{u} \end{pmatrix}$$

where $ia \in \mathbb{R}$, $z, w \in \mathbb{R}$, $x, y \in \mathbb{C}$, $u \in \mathbb{C}$ and $u - \bar{u} = -a$. Observe that the Lie algebra $\mathfrak{g} = \mathfrak{s}u(2,1)$ is graded:

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{1} \oplus \mathfrak{g}^{2}$$

where

$$\mathfrak{g}^{-2} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ z & 0 & 0 \end{pmatrix} \right\} \quad \mathfrak{g}^{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & 2i\bar{x} & 0 \end{pmatrix} \right\}$$

$$\mathfrak{g}^{0} = \left\{ \begin{pmatrix} u & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -\bar{u} \end{pmatrix} \right\} \quad \mathfrak{g}^{1} = \left\{ \begin{pmatrix} 0 & -2i\bar{y} & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \right\} \quad \mathfrak{g}^{2} = \left\{ \begin{pmatrix} 0 & 0 & w \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

We have

$$\mathfrak{g}^{0} = \mathbb{R} \oplus \mathfrak{u}(1),$$

where

$$\mathfrak{u}(1) = \left\{ \begin{pmatrix} -iq/2 & 0 & 0 \\ 0 & iq & 0 \\ 0 & 0 & -iq/2 \end{pmatrix} \right\}$$

with $q \in \mathbb{R}$ and

$$\mathbb{R} = \left\{ \begin{pmatrix} r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -r \end{pmatrix} \right\}$$

with $r \in \mathbb{R}$.

Define the subalgebra

$$\mathfrak{h} = \mathfrak{g}^{0} \oplus \mathfrak{g}^{1} \oplus \mathfrak{g}^{2}.$$

The isotropy of the action of $SU(2,1)/K$ on S^3 at the point $[1,0,0]^T$ is the group $H = CU(1) \ltimes N$ (whose Lie algebra is $\mathfrak{h}$), represented (up to K) by matrices of the form

$$\begin{pmatrix} a & -2i\bar{a}\bar{b} & a(s - ib\bar{b}) \\ 0 & \frac{\bar{a}}{a} & b \\ 0 & 0 & \bar{a}^{-1} \end{pmatrix} \tag{2}$$

where $s \in \mathbb{R}$, $b \in \mathbb{C}$. N is the Heisenberg group represented by matrices

$$\begin{pmatrix} 1 & -2i\bar{b} & s - ib\bar{b} \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

4 The Cartan connection

The bundle $Y \to M$ is a H-principal bundle and we can interpret Theorem 2.2 by introducing the concept of a Cartan connection. In fact, one can represent the structure equations of Theorem 2.2 as a matrix equation whose entries are differential forms. The forms are disposed in the Lie algebra $\mathfrak{s}u(2,1)$ as

$$\pi = \begin{pmatrix} -\frac{1}{2}\varphi - \frac{1}{3}\omega_1^1 & -i\varphi^{\bar{1}} & -\frac{1}{4}\psi \\ \omega^1 & \frac{2}{3}\omega_1^1 & \frac{1}{2}\varphi^1 \\ 2\omega & 2i\omega^{\bar{1}} & \frac{1}{2}\varphi - \frac{1}{3}\omega_1^1 \end{pmatrix}$$

It is a simple verification to show that

$$d\pi + \pi \wedge \pi = \Pi \tag{3}$$

where

$$\Pi = \begin{pmatrix} 0 & -i\Phi^{\bar{1}} & -\frac{1}{4}\Psi \\ 0 & 0 & \frac{1}{2}\Phi^1 \\ 0 & 0 & 0 \end{pmatrix}$$

Recall that $X^*(y) = \frac{d}{dt}_{t=0} ye^{tX}$ where e^{tX} is the one parameter group generated by the X.

Definition 4.1 *A Cartan connection on Y is a 1-form $\pi : TY \to su(2,1)$ satisfying:*
1. $\pi_p : T_pY \to su(2,1)$ is an isomorphism
2. If $X \in \mathfrak{h}$ and $X^* \in TY$ is the vertical vector field canonically associated to X then $\pi(X^*) = X$.
3. If $h \in H$ then $(R_h)^*\pi = Ad_{h^{-1}}\pi$

Theorem 4.2 ([C, CM]) *The form π is a Cartan connection.*

5 Formulae

In the following we will need the value of $B = Ad_{h^{-1}}\pi$, where $h \in H$ is as in (2). We get

$$B_{31} = 2(a\bar{a}\omega)$$

$$B_{21} = a^2\bar{a}^{-1}\omega^1 - 2a^2b\omega$$

$$B_{11} = -\frac{1}{3}\omega_1^1 - \frac{1}{2}\varphi + 2ia\bar{b}\omega^1 - 2a\bar{a}(s + ib\bar{b})\omega$$

$$B_{22} = \frac{2}{3}(\omega_1^1 - 3ia\bar{b}\omega^1 - 3i\bar{a}b\omega^{\bar{1}} + 6ia\bar{a}b\bar{b}\omega)$$

$$B_{23} = \frac{1}{2}\left(a\bar{a}^{-2}\varphi^1 - ba\bar{a}^{-1}\varphi + 2a\bar{a}^{-1}b\omega_1^1 + 2a^2\bar{a}^{-1}(s - ib\bar{b})\omega^1 - 4iab^2\omega^{\bar{1}} - 4a^2b(s - ib\bar{b})\omega\right)$$

$$B_{13} = -\frac{1}{4}\Big((a\bar{a})^{-1}\psi + 4ia^{-1}b\varphi^{\bar{1}} - 4i\bar{a}^{-1}\bar{b}\varphi^1 - 8ib\bar{b}\omega_1^1$$

$$-8ia\bar{b}(s-ib\bar{b})\omega^1+8i\bar{a}b(s+ib\bar{b})\omega^{\bar{1}}+4s\varphi+8a\bar{a}(s^2+(b\bar{b})^2)\omega\Big)$$

Observe that

$$B_{11}+\frac{1}{2}B_{22}=-\frac{1}{2}(\varphi-2ia\bar{b}\omega^1+2i\bar{a}b\omega^{\bar{1}}+4a\bar{a}s\omega).$$

Also, if h is a section (that is a function $h:M\to H$), then

$$h^{-1}dh=\left(\begin{array}{ccc} a^{-1}da & -2i(a^{-2}\bar{a}\bar{b}da+a^{-1}\bar{a}d\bar{b}) & ds+i\bar{b}db-ibd\bar{b}+a^{-1}(s-ib\bar{b})da+\bar{a}^{-1}(s+ib\bar{b})d\bar{a}\\ 0 & -a^{-1}da+\bar{a}^{-1}d\bar{a} & a\bar{a}^{-1}db+a\bar{a}^{-2}bd\bar{a}\\ 0 & 0 & -\bar{a}^{-1}d\bar{a}\end{array}\right) \tag{4}$$

If

$$\tilde{\pi}=R_h^*\pi=h^{-1}dh+Ad_{h^{-1}}\pi,$$

and writing

$$\tilde{\pi}=\left(\begin{array}{ccc} -\frac{1}{2}\tilde{\varphi}-\frac{1}{3}\tilde{\omega}_1^1 & -i\tilde{\varphi}^{\bar{1}} & -\frac{1}{4}\tilde{\psi}\\ \tilde{\omega}^1 & \frac{2}{3}\tilde{\omega}_1^1 & \frac{1}{2}\tilde{\varphi}^1\\ 2\tilde{\omega} & 2i\tilde{\omega}^{\bar{1}} & \frac{1}{2}\tilde{\varphi}-\frac{1}{3}\tilde{\omega}_1^1\end{array}\right)$$

we obtain from above

$$\tilde{\omega}=a\bar{a}\omega$$

$$\tilde{\omega}^1=a^2\bar{a}^{-1}\omega^1-2a^2b\omega$$

$$\tilde{\varphi}=\varphi-2ia\bar{b}\omega^1+2i\bar{a}b\omega^{\bar{1}}+4a\bar{a}s\omega+(-a^{-1}da-\bar{a}^{-1}d\bar{a})$$

$$\tilde{\omega}_1^1=\omega_1^1-3ia\bar{b}\omega^1-3i\bar{a}b\omega^{\bar{1}}+6ia\bar{a}b\bar{b}\omega+\frac{3}{2}(-a^{-1}da+\bar{a}^{-1}d\bar{a})$$

$$\tilde{\varphi}^1=a\bar{a}^{-2}\varphi^1-ba\bar{a}^{-1}\varphi+2a\bar{a}^{-1}b\omega_1^1+2a^2\bar{a}^{-1}(s-ib\bar{b})\omega^1-4iab^2\omega^{\bar{1}}-4a^2b(s-ib\bar{b})\omega+2(a\bar{a}^{-1}db+a\bar{a}^{-2}bd\bar{a})$$

$$\tilde{\psi}=(a\bar{a})^{-1}\psi+4ia^{-1}b\varphi^{\bar{1}}-4i\bar{a}^{-1}\bar{b}\varphi^1-8ib\bar{b}\omega_1^1-8ia\bar{b}(s-ib\bar{b})\omega^1+8i\bar{a}b(s+ib\bar{b})\omega^{\bar{1}}+4s\varphi+8a\bar{a}(s^2+(b\bar{b})^2)\omega$$
$$-4(ds+i\bar{b}db-ibd\bar{b}+a^{-1}(s-ib\bar{b})da+\bar{a}^{-1}(s+ib\bar{b})d\bar{a}).$$

We take a particular section h where

$$b=\frac{\bar{a}}{a},$$

and write $a=rv$ with $r>0$ and $|v|=1$. The section h depends on functions r, v and s. As $\frac{da}{a}=\frac{dr}{r}+\frac{dv}{v}$, our above formulas can be writen as

$$\tilde{\omega}=r^2\omega$$

$$\tilde{\omega}^1=rv^3\omega^1-2r^2\omega$$

$$\tilde{\varphi}=\varphi-2irv^3\omega^1+2ir\bar{v}^3\omega^{\bar{1}}+4r^2s\omega-2\frac{dr}{r}$$

$$\tilde{\omega}_1^1=\omega_1^1-3ir(v^3\omega^1+\bar{v}^3\omega^{\bar{1}})+6ir^2\omega-3\frac{dv}{v} \tag{5}$$

$$\tilde{\varphi}^1=\frac{1}{r}v^3\varphi^1+2\omega_1^1-\varphi+2r(s-i)v^3\omega^1-4ir\bar{v}^3\omega^{\bar{1}}-4r^2(s-i)\omega+2\frac{dr}{r}-6\frac{dv}{v}$$

$$\tilde{\psi}=\frac{1}{r^2}\psi-4i\frac{1}{r}v^3\varphi^1+4i\frac{1}{r}\bar{v}^3\varphi^{\bar{1}}-8i\omega_1^1+4s\varphi-8rv^3(1+is)\omega^1-8r\bar{v}^3(1-is)\omega^{\bar{1}}+8r^2(s^2+1)\omega$$
$$-4(ds+2s\frac{dr}{r}-6i\frac{dv}{v}).$$

6 $\mathbb{R}$-Circles

General references for this section is [C, J]. The definition (Definition 6.2) of circles by a differential system is due to Cartan, here we introduce a system of differential equations describing them (see Theorem 6.3). In S^3, $\mathbb{R}$-*circles* are the trace of Lagrangian planes of complex hyperbolic space in its boundary S^3. To define the analog of an $\mathbb{R}$-circle for a CR-structure we begin to impose that the curve is horizontal. In fact we will define the curve in the fiber bundle Y by means of a differential system. The projection of an integral curve will be an $\mathbb{R}$-circle.

Observe that a curve $\tilde{\gamma}$ in Y projects to a horizontal curve γ in M if and only if

$$\omega(\dot{\tilde{\gamma}}) = 0.$$

Indeed, as ω is a tautological form of Y over E,

$$\omega(\dot{\tilde{\gamma}}) = \lambda\theta(\dot{\gamma}).$$

where λ is positive and θ is a contact form.

Equation 1 can be written

$$d\omega = \omega \wedge \varphi + \frac{i}{2}(\omega^1 - \omega^{\bar{1}}) \wedge (\omega^1 + \omega^{\bar{1}}).$$

In order to obtain an integrable system we add

$$(\omega^1 - \omega^{\bar{1}}) = 0.$$

Observe that we could have added instead $(\omega^1 - \omega^{\bar{1}}) = 0$ and obtain an equivalent system. This differential system is still not integrable. It follows from the structure equations in Theorem 2.2 that

$$d(\omega^1 - \omega^{\bar{1}}) = \frac{1}{2}(\omega^1 - \omega^{\bar{1}}) \wedge \varphi + (\omega^1 + \omega^{\bar{1}}) \wedge \omega_1^1 + \omega \wedge (\varphi^1 - \varphi^{\bar{1}}),$$

so as $\omega^1 + \omega^{\bar{1}}$ is not null, we should add the equation

$$\omega_1^1 = 0.$$

We rewrite the equation of $d\omega_1^1$ in 2.2 as

$$d\omega_1^1 - \frac{3}{2}i\omega^1 \wedge (\varphi^{\bar{1}} + \varphi^1) - \frac{3}{2}i(\omega^1 - \omega^{\bar{1}}) \wedge \varphi^1 = 0 \qquad \text{mod } \omega.$$

In order to obtain an integrable system we add the equation

$$\varphi^1 + \varphi^{\bar{1}} = 0.$$

Finally, from the equation of $d\varphi^1$ in Theorem 2.2 we get

$$(\omega^1 + \omega^{\bar{1}}) \wedge \psi = 0 \qquad \text{mod } \omega,\ \omega^1 - \omega^{\bar{1}},\ \omega_1^1,\ \varphi^1 + \varphi^{\bar{1}}$$

so, as above, in order to obtain an integrable system, we impose

$$\psi = 0.$$

Observe that the structure equations shows that the system is integrable because $d\psi$ is already in the ideal generated by the previous forms. We have shown

Proposition 6.1 *The differential sistem on Y given by*

$$\omega = \omega^1 - \omega^{\bar{1}} = \omega_1^1 = \varphi^1 + \varphi^{\bar{1}} = \psi = 0 \tag{6}$$

is integrable

Definition 6.2 *The curves in the CR-manifold M that are projections of integral manifolds of the differential system 6 are called* $\mathbb{R}$-circles*.*

To find the equations of $\mathbb{R}$-circles directly in the CR manifold M we proceed as in [BS] pg 164 for the case of chains. Take a section $\sigma : M \to Y$. Suppose $\gamma : I \to M$ is a circle, and let's apply a transformation $R_{h(t)}$ on $\sigma\gamma(t)$ such that $R_{h(t)}\sigma\gamma(t)$ is inside an integral leaf of 6. Then

$$\tilde{\pi} = (R_{h(t)}\sigma\gamma(t))^*\pi = h^{-1}(t)h'(t) + Ad_{h^{-1}(t)}\gamma(t)^*(\sigma^*\pi).$$

Taking account of the formulas 5, it follows from $\tilde{\omega}^1 - \tilde{\omega}^{\bar{1}} = 0$ that $rv^3\omega^1 = r\bar{v}^3\omega^{\bar{1}}$, so

$$v^3\omega^1 = \bar{v}^3\omega^{\bar{1}}, \tag{7}$$

and therefore v is determined up to a sixth root of unity.

The following equation $\tilde{\omega}_1^1 = 0$ says that

$$0 = \omega_1^1 - 3ir(v^3\omega^1 + \bar{v}^3\omega^{\bar{1}}) - 3\frac{dv}{v} \tag{8}$$

which determines r.

The next equation is

$$0 = \tilde{\varphi}^1 + \tilde{\varphi}^{\bar{1}} = (\frac{1}{r}v^3\varphi^1 + 2\omega_1^1 - \varphi + 2r(s-i)v^3\omega^1 - 4ir\bar{v}^3\omega^{\bar{1}} + 2\frac{dr}{r} - 6\frac{dv}{v})$$
$$+(\frac{1}{r}\bar{v}^3\varphi^{\bar{1}} - 2\omega_1^1 - \varphi + 2r(s+i)\bar{v}^3\omega^{\bar{1}} + 4irv^3\omega^1 + 2\frac{dr}{r} + 6\frac{dv}{v})$$

so we obtain

$$\frac{1}{r}(v^3\varphi^1 + \bar{v}^3\varphi^{\bar{1}}) - 2\varphi + 2rs(v^3\omega^1 + \bar{v}^3\omega^{\bar{1}}) + 4\frac{dr}{r} = 0.$$

If we pose

$$z = r^2 s$$

we get

$$dr = -\frac{1}{4}(v^3\varphi^1 + \bar{v}^3\varphi^{\bar{1}}) + \frac{1}{2}r\varphi - \frac{1}{2}z(v^3\omega^1 + \bar{v}^3\omega^{\bar{1}}) \tag{9}$$

wich determines z, and so s.

Now the last equation of the integral system is $\tilde{\psi} = 0$, or

$$0 = \frac{1}{r^2}\psi - 4i\frac{1}{r}(v^3\varphi^1 - \bar{v}^3\varphi^{\bar{1}}) + 4s\varphi - 8r(v^3\omega^1 + \bar{v}^3\omega^{\bar{1}}) - 4(ds + 2s\frac{dr}{r} - 6i\frac{dv}{v}),$$

and from this equation we get

$$dz = \frac{1}{4}\psi - ir(v^3\varphi^1 - \bar{v}^3\varphi^{\bar{1}}) + z\varphi - 2r^3(v^3\omega^1 + \bar{v}^3\omega^{\bar{1}}) + 6ir^2\frac{dv}{v},$$

If we replace equation 8 we obtain

$$dz = \frac{1}{4}\psi - ir(v^3\varphi^1 - \bar{v}^3\varphi^{\bar{1}}) + z\varphi + 2ir^2\omega_1^1 + 4r^3(v^3\omega^1 + \bar{v}^3\omega^{\bar{1}}). \tag{10}$$

Consider

$$\gamma'(t) = cZ_1 + \bar{c}Z_{\bar{1}},$$

where Z_1, $Z_{\bar{1}}$, Z_0 are dual vectors to ω^1, $\omega^{\bar{1}}$, ω. It follows from 7 that $v^3c = \bar{v}\bar{c}^3$, so $v^3c = \rho$, with ρ real. Therefore $c = \rho\bar{v}^3$, and $(v^3\omega^1 + \bar{v}^3\omega^{\bar{1}})(\gamma') = 2\rho$. If we pose

$$u = \bar{v}^3,$$

we obtain

Theorem 6.3 *If $\gamma : I \to M$ is an $\mathbb{R}$-circle, and if we write $\gamma' = \rho(uZ_1 + \bar{u}Z_{\bar{1}})$, with $\rho \neq 0$ and $|u| = 1$, then γ satisfies the following system of differential equations:*

$$\begin{gathered}
\omega = 0 \\
\bar{u}\omega^1 - u\omega^{\bar{1}} = 0 \\
\frac{du}{u} + \omega_1^1 - 6ir\rho dt = 0 \\
dr + \frac{1}{4}(\bar{u}\varphi^1 + u\varphi^{\bar{1}}) - \frac{r}{2}\varphi + z\rho dt = 0 \\
dz - \frac{1}{4}\psi + ir(\bar{u}\varphi^1 - u\varphi^{\bar{1}}) - z\varphi - 2ir^2\omega_1^1 - 8r^3\rho dt = 0.
\end{gathered}$$

Observe that for each $p \in M$ one can determine a circle passing through p specifying as initial conditions u (with $|u| = 1$), $r > 0$ and $z \in \mathbb{R}$.

Example 6.4 (The quadric, see [J])

Consider the *quadric* Q in $\mathbb{C}^2$ defined as the null set of

$$\frac{1}{2i}(z_2 - \bar{z}_2) = z_1\bar{z}_1.$$

Take as

$$Z^1 = dz_1,$$
$$Z^{\bar{1}} = d\bar{z}_1$$

and

$$Z^0 = \frac{1}{2}\left(dx_2 - i\bar{z}_1dz_1 + iz_1d\bar{z}_1\right),$$

where $z_2 = x_2 + iy_2$. The dual fields are

$$Z_1 = \frac{\partial}{\partial z_1} + i\bar{z}_1\frac{\partial}{\partial x_2},$$

$$Z_{\bar{1}} = \frac{\partial}{\partial \bar{z}_1} - iz_1 \frac{\partial}{\partial x_2}$$

and

$$Z_0 = 2\frac{\partial}{\partial x_2}.$$

We have

$$dZ^0 = iZ^1 \wedge Z^{\bar{1}}$$

and

$$dZ^1 = 0.$$

Considering this section $\sigma : M \to Y$ defined by $Z^0, Z^1, Z^{\bar{1}}$, we obtain from Cartan equations that $\sigma^*\varphi = \sigma^*\omega_1^1 = \sigma^*\varphi^1 = \sigma^*\psi = 0$. In particular $Q_{\bar{1}}^1 = 0$. Replacing this values in equations of Theorem 6.3 we obtain the differential equations for the chains on the quadric,

$$\frac{u'}{u} - 6i\rho r = 0, \tag{11}$$

$$r' + z\rho = 0 \tag{12}$$

and

$$z' - 8\rho r^3 = 0. \tag{13}$$

One verifies that the curve

$$\gamma(t) = (x_1(t), y_1(t), x_2(t))$$

given by

$$x_1(t) = \frac{2t(1-t^2)}{1+6t^2+t^4}, \quad y_1(t) = -\frac{(1-t^2)(1+t^2)}{1+6t^2+t^4}, \quad x_2(t) = -\frac{4t(1+t^2)}{1+6t^2+t^4}$$

is an $\mathbb{R}$-circle corresponding to $z_1(0) = -i$, $x_2(0) = 0$, $u(0) = 1$, $r(0) = 1$ and $z(0) = 0$.

7 Circle preserving diffeomorphisms are CR or conjugate CR

Theorem 7.1 *Let M and $\tilde{M}$ be CR manifolds. If $f : M \to \tilde{M}$ is a (local) diffeomorphism such that for every $\mathbb{R}$-circle γ in M, $f \circ \gamma$ is an $\mathbb{R}$-circle in $\tilde{M}$, then f is a CR diffeomorphism or a conjugate CR diffeomorphism.*

Proof. Consider the structures Y and $\tilde{Y}$ on M and $\tilde{M}$ respectively, and sections $\sigma : M \to Y$ and $\tilde{\sigma} : \tilde{M} \to \tilde{Y}$. Taking the pull back by σ and $\tilde{\sigma}$ of connection forms on Y and $\tilde{Y}$ respectively, we obtain forms ω, ω^1, ω_1^1, φ^1, ψ on M, and forms $\tilde{\omega}$, $\tilde{\omega}^1$, $\tilde{\omega}_1^1$, $\tilde{\varphi}^1$, $\tilde{\psi}$ on $\tilde{M}$. Using these forms, we can write circle equations on M and $\tilde{M}$, as in theorem 6.3. We now consider forms $\tilde{\omega}$, $\tilde{\omega}^1$, $\tilde{\omega}_1^1$, $\tilde{\varphi}^1$, $\tilde{\psi}$ as forms on M (by a slight abuse of notions they denote the forms $f^*\tilde{\omega}$, $f^*\tilde{\omega}^1$, $f^*\tilde{\omega}_1^1$, $f^*\tilde{\varphi}^1$, $f^*\tilde{\psi}$).

As tangent vectors to circles generate D, then $\tilde{\omega} = 0$ on D, so we obtain $\tilde{\omega} = \lambda\omega$. That is, the diffeomorphism is a contactomorphism. We write

$$\tilde{\omega}^1 = \alpha\omega^1 + \beta\omega^{\bar{1}} + c\omega \tag{14}$$

and

$$\tilde{\omega}^{\bar{1}} = \bar{\beta}\omega^1 + \bar{\alpha}\omega^{\bar{1}} + \bar{c}\omega,$$

where α, β, c are functions on M. Observe that as $\tilde{\omega}^1 \wedge \tilde{\omega}^{\bar{1}}|_D = (\alpha\bar{\alpha} - \beta\bar{\beta})\omega^1 \wedge \omega^{\bar{1}}|_D$ is never null on D, we get

$$\alpha\bar{\alpha} - \beta\bar{\beta} \neq 0$$

everywhere on M. We have to show that $\beta = 0$ (for f to be a CR diffeomorphism) or $\alpha = 0$ (for f to be a conjugate CR diffeomorphism). The idea is to use the arbitrary initial conditions (so vary u, r and z) to obtain enough information on those functions.

The computations we are going to do are on a circle γ, so we can assume from theorem 6.3 that

$$\omega = \tilde{\omega} = \omega^1 - u^2\omega^{\bar{1}} = \tilde{\omega}^1 - \tilde{u}^2\tilde{\omega}^{\bar{1}} = 0.$$

Applying this in 14 we obtain

$$(\alpha u^2 + \beta)\omega^{\bar{1}} = \alpha\omega^1 + \beta\omega^{\bar{1}} = \tilde{\omega}^1 = \tilde{u}^2\tilde{\omega}^{\bar{1}} = \tilde{u}^2(\bar{\beta}\omega^1 + \bar{\alpha}\omega^{\bar{1}}) = \tilde{u}^2(\bar{\beta}u^2 + \bar{\alpha})\omega^{\bar{1}},$$

therefore

$$\tilde{u} = \left(\frac{\alpha u^2 + b}{\bar{\beta}u^2 + \bar{\alpha}}\right)^{\frac{1}{2}}. \tag{15}$$

Define $h : \mathbb{R} \times S^1 \to \mathbb{C}$ by

$$h(t, u) = \left(\frac{\alpha(\gamma(t))u^2 + \beta(\gamma(t))}{\bar{\beta}(\gamma(t))u^2 + \bar{\alpha}(\gamma(t))}\right)^{\frac{1}{2}} \tag{16}$$

for $t \in \mathbb{R}$ and $u \in S^1 \subset \mathbb{C}$. Applying again Theorem 6.3 we can write

$$d\tilde{u} = \tilde{u}(-\tilde{\omega}_1^1 + 6i\tilde{r}\tilde{\rho}dt) = h(-\tilde{\omega}_1^1 + 6i\tilde{\rho}\tilde{r}dt)$$

and taking the derivative of 16, we obtain

$$d\tilde{u} = \frac{\partial h}{\partial u}du + \frac{\partial h}{\partial t}dt = \frac{\partial h}{\partial u}u(-\omega_1^1 + 6ir\rho dt) + \frac{\partial h}{\partial t}dt.$$

From the equality of the right sides on the two lines above we obtain

$$\tilde{r} = Ar + B \tag{17}$$

where

$$A(t, u) = \frac{\rho u}{\tilde{\rho}h(t, u)}\frac{\partial h}{\partial u}(t, u) \tag{18}$$

and B is a function of t, u. Taking the derivative of 17 and replacing the circle equation

$$d\tilde{r} = -\frac{1}{4}(\bar{\tilde{u}}\tilde{\varphi}^1 + \tilde{u}\tilde{\varphi}^{\bar{1}}) + \frac{\tilde{r}}{2}\tilde{\varphi} - \tilde{z}\tilde{\rho}dt$$

we obtain

$$-\frac{1}{4}(\bar{\tilde{u}}\tilde{\varphi}^1 + \tilde{u}\tilde{\varphi}^{\bar{1}}) + \frac{1}{2}(Ar + B)\tilde{\varphi} - \tilde{z}\tilde{\rho}dt =$$

$$= A(-\frac{1}{4}(\bar{u}\varphi^1 + u\varphi^{\bar{1}}) + \frac{r}{2}\varphi - z\rho dt) + (\frac{\partial A}{\partial u}r + \frac{\partial B}{\partial u})(-u\omega_1^1 + 6iur\rho dt) + (\frac{\partial A}{\partial t}r + \frac{\partial B}{\partial t})dt,$$

and solving for $\tilde{z}$ we obtain

$$\tilde{z} = Cz + Dr^2 + Er + F, \tag{19}$$

where, in a short form,

$$C = \frac{A\rho}{\tilde{\rho}}, \tag{20}$$

$$D = -6i\frac{\rho u}{\tilde{\rho}}\frac{\partial A}{\partial u}. \tag{21}$$

Taking the derivative of 19 we obtain

$$d\tilde{z} = C\left(\frac{1}{4}\psi - ir(\bar{u}\varphi^1 - u\varphi^{\bar{1}}) + z\varphi + 2ir^2\omega_1^1 + 8r^3\rho dt\right)$$

$$+(2rD+E)\left(-\frac{1}{4}(\bar{u}\varphi^1 + u\varphi^{\bar{1}}) + \frac{r}{2}\varphi - z\rho dt\right) + (\frac{\partial C}{\partial u}z + \frac{\partial D}{\partial u}r^2 + \frac{\partial E}{\partial u}r + \frac{\partial F}{\partial u})(-u\omega_1^1 + 6iur\rho dt)$$

$$+(\frac{\partial C}{\partial t}z + \frac{\partial D}{\partial t}r^2 + \frac{\partial E}{\partial t}r + \frac{\partial F}{\partial t})dt.$$

By theorem 6.3 and 17, 19 we have

$$d\tilde{z} = \frac{1}{4}\tilde{\psi} - i(Ar+B)(\tilde{\bar{u}}\tilde{\varphi}^1 - \tilde{u}\tilde{\varphi}^{\bar{1}}) + (Cz + Dr^2 + Er + F)\tilde{\varphi} + 2i(Ar+B)^2\tilde{\omega}_1^1 + 8(Ar+B)^3\tilde{\rho}dt.$$

Replacing this in the above equation, we get a polynomial equation in z and r:

$$A_{30}r^3 + A_{11}rz + A_{20}r^2 + A_{10}r + A_{00} = 0 \tag{22}$$

where

$$A_{11} = (D - 3iu\frac{\partial C}{\partial u})\rho = -9iu\frac{\rho^2}{\tilde{\rho}}\frac{\partial A}{\partial u}. \tag{23}$$

As r and z can take arbitrary values in $\mathbb{R}$ we obtain that all coeficients of 22 are null. If we replace 18, 20 and 21 in $A_{11} = 0$ we obtain

$$18i\frac{\rho^3}{\tilde{\rho}^2}\frac{\left(u^2\alpha\bar{\beta} - \bar{u}^2\bar{\alpha}\beta\right)\left(\alpha\bar{\alpha} - \beta\bar{\beta}\right)}{\left(\alpha u + \beta\bar{u}\right)^2\left(\bar{\beta}u + \bar{\alpha}\bar{u}\right)^2} = 0.$$

As u is any complex number such that $|u| = 1$ and $\alpha\bar{\alpha} - \beta\bar{\beta} \neq 0$, we obtain $\alpha\bar{\beta} = 0$. It follows from 14 that if $\beta = 0$ then f is a CR diffeomorphism and if $\alpha = 0$ we get that f is a conjugate CR diffeomorphism.

□

References

[BDS] Burns, D., Diederich, K., Shnider, S.: Distinguished curves in pseudoconvex boundaries. Duke Math. J. 44 (1977), 407-431.

[BS] Burns, D., Shnider, S.: Real Hypersurfaces in complex manifolds. Proceedings of Symposia in Pure Mathematics, 30 (1977), 141-168.

[Ch] Cheng, J.-H.: Chain preserving diffeomorphisms and CR-equivalence, Proc. Amer. Math. Soc. 103 (1988),75-80.

[C] Cartan, E.: Sur la géométrie pseudo-conforme des hypersurfaces de deux variables complexes, I. Ann. Math. Pura Appl., (4) 11 (1932) 17-90 (or Ouevres II, 2, 1231-1304); II, Ann. Scuola Norm. Sup. Pisa, (2) 1 (1932) 333-354 (or Ouevres III, 2, 1217-1238).

[CM] Chern, S. S. , Moser, J.: Real Hypersurfaces in Complex Manifolds . Acta Math. 133 (1974) 219-271.

[J] Jacobowitz, H.: An Introduction to CR Structures, vol. 32,Mathematical Surveys and Monographs, American Mathematical Society, Providence,Rhode Island, 1990