A bilinear form associated to contact sub-conformal manifolds

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Abstract

We define a bilinear form associated to a sub-Riemannian contact manifold. It transforms by scalar multiples under sub-conformal transformations and with further hypothesis it is naturally defined on certain torus bundles over the contact manifold.

1 Introduction

A pseudohermitian structure on a contact manifold M gives rise to a Lorentz conformal metric on a circle bundle over M (see [D] for a survey). That conformal structure is the same for all pseudohermitian structures defining the same CR structure. This was the method given in [L] generalizing the construction of the Lorentz form associated to a real strictly pseudoconvex hypersurfaces in $\mathbf{C}^n$ in [F].

Another construction of the Lorentz structure for abstract CR manifolds is given in [BDS]. In fact, that was the first construction for an abstract CR manifold but its construction involves the construction of a Cartan connection ([C, CM, T]) on a fiber bundle canonically associated to a CR manifold.

A natural generalization of pseudohermitian geometry on a contact manifold is sub-Riemannian geometry which is a metric structure only defined on the contact distribution. The sub-conformal structure associated to the sub-Riemannian manifold has an associated fiber bundle but in general it is not a principal bundle so that the approach in [BDS] becomes impossible.

The goal of this paper is to construct a form which changes by a scalar factor under sub-conformal transformations following the construction in [L]. This construction is very general and holds even when there exists no natural circle bundle associated to the manifold. Instead, under appropriate hypothesis, there exists a torus bundle where the bilinear form is naturally defined (see Theorem 3.1) and eventually, under more restrictive conditions, is well defined on a circle bundle over the sub-Riemannian manifold (see Theorem 3.2). This generalizes the Fefferman's metric.

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2 sub-Riemannian and sub-conformal manifolds

In this section we define the geometrical structures we will use, namely the sub-Riemannian structures and sub-conformal structures. Let D be a contact distribution on a manifold M.

Definition 2.1

1. *(M, D, g) is a sub-Riemannian structure if g is a metric on D.*
2. *$(M, D, \tilde{g})$ is a sub-conformal structure if $\tilde{g}$ is a conformal class of sub-Riemannian metrics.*

Let $\pi : TM \to TM/D$ be the quotient map.

Definition 2.2 *The Levi form $\alpha : D \times D \to TM/D$ is the skew-symmetric form defined as $\alpha(X, Y) = -\pi([X, Y])$.*

Fixing a base v of TM/D defines the Levi form α_v as a real valued form. Let θ_v be the contact form of this distribution such that $\theta_v(\pi^{-1}v) = 1$, then the Levi form is given by

$$d\theta_v(X, Y) = \alpha_v(X, Y)$$

If we have a metric on D, define a skew-symmetric operator H_v on the distribution by

$$\alpha_v(X, Y) = g(H_v X, Y)$$

As α_v is non-degenerate, we can always choose v such that $det H_v = 1$ and this determines a unique v ignoring orientation effects. Observe that if we let tg be a new metric and choose v as above, then

$$\alpha_{\frac{1}{t}v}(X, Y) = tg(H_v X, Y)$$

so the definition of H_v does not depend on a metric inside a conformal class of metrics. Fixing a metric on D, denote by H this operator. Its normal form is given in the following lemma.

Lemma 2.1 *Let V be a 2n dimensional real vector space with a scalar product. If H is a nondegenerated skew-symmetric operator, then there exists an orthonormal basis of V such that the matrix of H is*

$$\Lambda = \begin{pmatrix} 0 & \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix} \\ \begin{pmatrix} -\lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & -\lambda_n \end{pmatrix} & 0 \end{pmatrix}$$

with $\lambda_i > 0$, $i = 1, \cdots, n$.

We also need the following simple lemma which characterizes the subgroup of $SO(2n)$ commuting with Λ.

Lemma 2.2 *Suppose that Λ is such that $\lambda_{d_1+\cdots+d_{k-1}+1} = \cdots = \lambda_{d_1+\cdots+d_k} = \nu_k$ for $1 \le k \le r$, with $d_1 + \cdots + d_r = n$, and $\nu_1 < \nu_2 < \cdots < \nu_r$, where the ν_k are real numbers, and that $A \in SO(2n)$ satisfies $A\Lambda A^T = \Lambda$. Then $A \in U(d_1) \times \cdots \times U(d_r)$.*

3 Sub-Riemannian geometry

Let (M, D, g) be a sub Riemannian manifold of dimension 2n+1, where D is a contact distribution on M, and g a positive quadratic form on D. We will further suppose that TM/D is oriented.

We consider the $SO(2n)$ bundle E of coframes (θ, θ^i) on M such that the θ^i restricted to D are orthonormal, θ is the positive contact form defined such that

$$d\theta = h_{ij}\theta^i \wedge \theta^j, \tag{1}$$

with $h_{ij} = -h_{ji}$ and $\det h_{ij} = 1$. Observe that a change in coframes given by $\overline{\theta}^i = a^i_j\theta^j$, $(a^i_j) \in SO(2n)$ implies that $\overline{h}_{ij} = a^i_k h_{kl} a^j_l$. On E we consider the tautological forms defined by θ, θ^i which we denote by the same letters. E is a principal fiber bundle with a right action by $SO(2n)$. We will consider the coframes as line vectors where $SO(2n)$ acts by matrix multiplication from the right.

Proposition 3.1 (see [W, FVV]) *There exist unique connection forms ω^i_j and torsion forms τ^i on E satisfying*

$$d\theta^i = \theta^j \wedge \omega^i_j + \theta \wedge \tau^i, \tag{2}$$

with $\omega^i_j = -\omega^j_i$ and

$$\sum \tau^i \wedge \theta^i = 0. \tag{3}$$

Proof. Let $\tilde{\omega}^i_j$ and $\tilde{\tau}^i$ be any forms satisfying the first equation. If ω^i_j and τ^i also satisfy the equation, then

$$\theta^j \wedge (\omega^i_j - \tilde{\omega}^i_j) + \theta \wedge (\tau^i - \tilde{\tau}^i) = 0$$

From Cartan's lemma we have

$$\omega^i_j - \tilde{\omega}^i_j = a^i_{jk}\theta^k + b^i_j\theta$$

$$\tau^i - \tilde{\tau}^i = b^i_k\theta^k$$

with $a^i_{jk} = a^i_{kj}$. We will choose a^i_{jk}, b^i_j such that the conditions in the theorem be satisfied for ω^i_j, τ^i. To verify condition 3 we must have

$$0 = \sum \tau^i \wedge \theta^i = \sum \tilde{\tau}^i \wedge \theta^i + \sum\sum b^i_k\theta^k \wedge \theta^i$$

If we write $\tilde{\tau}^i = \tilde{\tau}^i_k\theta^k$, then

$$\sum\sum(\tilde{\tau}^i_k + b^i_k)\theta^k \wedge \theta^i = 0$$

and using Cartan's lemma again $\tilde{\tau}^i_k + b^i_k = a^i_k$ with $a^i_k = a^k_i$. On the other hand if $\omega^i_j = -\omega^j_i$ is satisfied, and writing $\tilde{\omega}^i_j = \tilde{\omega}^i_{jk}\theta^k + \tilde{w}^i_j\theta$ we obtain

$$(\tilde{\omega}^i_{jk} + \tilde{\omega}^j_{ik} + a^i_{jk} + a^j_{ik})\theta^k + \ (\tilde{w}^i_j + \tilde{w}^j_i + b^i_j + b^j_i)\theta = 0$$

We get two equations

$$\tilde{w}^i_j + \tilde{w}^j_i + a^i_j + a^j_i - \tilde{\tau}^i_j - \tilde{\tau}^j_i = 0$$

$$\tilde{\omega}^i_{jk} + \tilde{\omega}^j_{ik} + a^i_{jk} + a^j_{ik} = 0$$

The first equation, recalling that a^i_j is symmetric, has solution $a^i_j = \frac{\tilde{\tau}^i_j+\tilde{\tau}^j_i}{2} - \frac{\tilde{w}^i_j+\tilde{w}^j_i}{2}$ therefore b^i_j is determined. The second equation can be solved using the permutation trick, as in Riemannian geometry. $\square$

Equation 3 is equivalent to $\tau^i = \tau^i_j\theta^j$, with $\tau^i_j = \tau^j_i$. If we differentiate 1, and apply 2 we get

$$(dh_{ij} - h_{kj}\omega^k_i + h_{ki}\omega^k_j)\theta^i \wedge \theta^j + 2h_{ij}\tau^i \wedge \theta^j \wedge \theta = 0.$$

A simple computation shows that there exist unique functions b_{ijk} and b_{ij} so that we can write

$$dh_{ij} - h_{kj}\omega^k_i + h_{ki}\omega^k_j = b_{ijk}\theta^k + b_{ij}\theta \tag{4}$$

and

$$2h_{ij}\tau^i \wedge \theta^j = -b_{ij}\theta^i \wedge \theta^j, \tag{5}$$

where $b_{ijk} = -b_{jik}, b_{ijk} + b_{jki} + b_{kij} = 0, b_{ij} = -b_{ji}$.

We will use the following notations:

$$\Omega = (\omega^i_j),$$

$$H = (h_{ij}) \text{ and } H^{-1} = (h^{ij}).$$

Definition 3.1 *On E we define,*

$$\varsigma = Tr(H^{-1}\Omega) = h^{ji}\omega^j_i.$$

3.1 Conformal change in the sub-Riemannian metric

In this section we will study the transformation in the connection forms when the metric on D undergoes a conformal change of the form

$$g' = e^{2f}g$$

where f is a real function on M. For this we need to compare the structure equations for the metrics g and g'. Let's first introduce some notation.

Define f_0 and f_i using the formula

$$df = f_0\theta + f_i\theta^i, \tag{6}$$

and write

$$f^i = h^{ij}f_j.$$

If we differentiate 6, we get

$$(df_0 - f_i\tau^i) \wedge \theta + (df_j - f_i\omega^i_j + f_0h_{ij}\theta^i) \wedge \theta^j = 0.$$

Applying Cartan's lemma we obtain

$$df_j - f_i\omega^i_j + f_0h_{ij}\theta^i = f_{jk}\theta^k + f_{j0}\theta,$$

$$df_0 - f_i\tau^i = f_{0j}\theta^j + f_{00}\theta,$$

with $f_{jk} = f_{kj}$ and $f_{0j} = f_{j0}$. It is a direct verification, applying 4, that

$$dh^{ij} = h^{jk}\omega_k^i - h^{ik}\omega_k^j + h^{ik}(b_{klm}\theta^m + b_{kl}\theta)h^{jl}.$$

Then

$$df^i + f^j\omega_j^i - f_0\theta^i = f_j^i\theta^j + f_0^i\theta, \tag{7}$$

where

$$f_j^i = h^{ik}(f_{kj} - b_{klj}f^l)$$

and $f_0^i = h^{ik}(f_{k0} - b_{kl}f^l)$.

The contact form associated to g' is

$$\theta' = e^{2f}\theta.$$

Verifying that

$$d\theta' = h_{ij}\theta'^i \wedge \theta'^j, \tag{8}$$

we obtain that the new coframes are given by

$$\theta'^i = e^f(\theta^i + f^i\theta) \tag{9}$$

with $h'_{ij} = h_{ij}$. Let E' be the bundle of coframes (θ', θ'^i) associated to the sub Riemannian manifold (M, D, g').

Proposition 3.2 *The application*

$$F : E \to E'$$

given by

$$F(\theta, \theta^i) = (\theta', \theta'^i) = (e^{2f}\theta, e^f(\theta^i + f^i\theta))$$

is a isomorphism of $SO(2n)$-bundles.

Proof. Let $(\theta, \overline{\theta}^i)$ be a new coframe of E, with $\overline{\theta}^i = a_j^i\theta^j$, $(a_j^i) \in SO(2n)$. From $d\theta = \overline{h}_{ij}\overline{\theta}^i \wedge \overline{\theta}^j$ we obtain $\overline{h}^{ij} = a_k^i h^{kl} a_l^j$. Then $F(\theta, \overline{\theta}^i) = (\theta', \overline{\theta}'^i)$, where $\overline{\theta}'^i = e^f(\overline{\theta}^i + \overline{f}^i\theta)$, $df = \overline{f}_j\overline{\theta}^j + \overline{f}_0\theta$, and $\overline{f}^i = \overline{h}^{ij}\overline{f}_j$. That implies $\overline{f}_i = a_j^i f_j$, $\overline{f}^i = a_k^i f^k$ and $\overline{\theta}'^i = e^f(a_j^i\theta^j + a_j^i f^j\theta) = a_j^i\theta'^j$. We proved that

$$F(\theta, a_j^i\theta^j) = (\theta', a_j^i\theta'^j),$$

what ends the proof. □

We consider from now on the tautological forms on E' using, by abuse of notation, the same letters θ'^j and θ'. As before there exist unique forms ω'^i_j and τ'^i such that

$$d\theta'^i = \theta'^j \wedge \omega'^i_j + \theta' \wedge \tau'^i, \tag{10}$$

$\omega'^i_j = -\omega'^j_i$, and $\sum \tau'^i \wedge \theta'^i = 0$.

In what follows, we will omit the function F when comparing the bundles E and E'. That is, we will write, by abuse of notation, $\alpha' = F^*(\alpha')$ for a form α' defined on E'.

Differentiating 9 and applying 2 and 10, we obtain

Proposition 3.3 *The connection and torsion forms of E and E' satisfy the following formulae*

$$\omega'^i_j = \omega^i_j + e^{-f}(f_j\delta^i_k - f_i\delta^j_k + f^i h_{jk} - f^j h_{ik} - f^k h_{ij})\theta'^k + e^{-2f}(f^j f_i - f^i f_j + \frac{1}{2}f^i_j - \frac{1}{2}f^j_i)\theta' \quad (11)$$

and

$$\tau'^i = e^{-2f}(\tau^i_j + f_0\delta^i_j + f^i f_j + f^j f_i - \frac{1}{2}f^j_i - \frac{1}{2}f^i_j)\theta'^j.$$

It follows from 11 the following

Proposition 3.4 $\varsigma = Tr(H^{-1}\Omega)$ *and* $\varsigma' = Tr(H^{-1}\Omega')$ *are related by*

$$\varsigma' = \varsigma + (2n+4)f^i\theta^i + ((2n+2)f^i f^i + h^{ij}f^i_j)\theta.$$

Applying 7 we get

$$d(f^i\theta^i) = f^j_i\theta^i \wedge \theta^j + \theta \wedge (f^i_0\theta^i + f^i\tau^i).$$

Then

$$d\varsigma' = d\varsigma + ((2n+4)f^j_i + ((2n+2)f^k f^k + h^{kl}f^k_l)h_{ij})\theta^i \wedge \theta^j \mod \theta. \quad (12)$$

Our goal is to construct a conformally invariant bilinear form on a certain bundle over M using ς, $d\varsigma$ and the tautological forms θ^i and θ. In order to do so we first need to reduce the structure group of E. Eventually we will impose that H is constant on the reduced bundle, but we will carry our computation in a more general setting with some regularity assumptions on H.

3.2 Reduction of the structure group

We suppose now that the canonical form Λ of $H = (h_{ij})$ as in lemma 2.2 has in every point of M r values ν_k, each one with multiplicity d_k. More explicitly, the ν_k are real functions on M.

Definition 3.2 *E_1 is the subset of points of E such that $H = \Lambda$.*

¿From lemma 2.2 we have that E_1 is a $U(d_1)\times\cdots\times U(d_r)$ subbundle of E. We denote a coframe in E_1 by θ^{i_k} where $1 \le k \le r$ and $d_1 + \cdots + d_{k-1} + 1 \le i_k \le d_1 + \cdots + d_k$ with $d_1 + \cdots + d_r = n$.

If $Y = Y_1 + \cdots + Y_r$ with $Y_k \in su(d_k)$, then $\Lambda^{-1}Y = 2\nu_1^{-1}J_1Y_1 + \cdots + 2\nu_r^{-1}J_rY_r$, where J_k is a linear operator such that $J^*_k\theta^{i_l} = -\delta^l_k\theta^{i_l+n}$ and $J^*_k\theta^{i_l+n} = \delta^l_k\theta^{i_l}$. Observe that $J^*_k\theta^i \neq 0$ only for $d_1 + \cdots + d_{k-1} + 1 \le i \le d_1 + \cdots + d_k$.

3.3 The torus bundle over M

The subgroup $SU(d_1)\times\cdots\times SU(d_r)$ is normal in $U(d_1)\times\cdots\times U(d_r)$. We define $\mathbf{T}$ as the quotient bundle of E_1 by $SU(d_1)\times\cdots\times SU(d_r)$. $\mathbf{T}$ is a $U(1)\times\cdots\times U(1)$ bundle, i.e. $\mathbf{T}$ is a r-torus principal bundle over M:

Proposition 3.5 $\mathbf{T} = E_1/SU(d_1)\times\cdots\times SU(d_r)$ *is a* $U(1)\times\cdots\times U(1)$*-principal bundle.*

Parts of the sub-Riemannian connection descend to forms defined on the torus bundle. We use the following criterion:

A differential form φ on E projects on $\mathbf{T}$ if and only if

1. $R^*_g\varphi = \varphi$ for every $g \in SU(d_1)\times\cdots\times SU(d_r)$,

2. $\varphi(X^*) = 0$ for every $X \in su(d_1) + \cdots + su(d_r)$.

Restricted to E_1 we have

$$\varsigma = Tr(\Lambda^{-1}\Omega) = \sum_k \sum_{i_k} \frac{2}{\nu_k} \omega^{i_k}_{i_k+n},$$

where $d_{k-1} + 1 \leq i_k \leq d_k$ and $1 \leq k \leq r$.

Also,

Definition 3.3 *On E_1 we define,*

$$\omega_k = 2 \sum_{i_k} \omega^{i_k}_{i_k+n}$$

where $d_{k-1} + 1 \leq i_k \leq d_k$.

Proposition 3.6 *The forms ς and ω_k project to $\mathbf{T} = E_1/SU(d_1) \times \cdots \times SU(d_r)$.*

PROOF. We prove first the result for the form ς. In order to verify $R^*_a\varsigma = \varsigma$ for $a \in U(d_1) \times \cdots \times U(d_r)$, it suffice to verify the formula on vertical vectors, because ς vanishes on horizontal vectors. Suppose $X \in u(d_1) + \cdots + u(d_r), a \in U(d_1) \times \cdots \times U(d_r)$ and let X^* be the vertical vector field on E_1 induced by X. From $R_{a*}X^* = (a^{-1}Xa)^*$ and observing that $a\Lambda = \Lambda a$, we have

$$R^*_a\varsigma(X^*) = \varsigma(R_{a*}X^*) = Tr(\Lambda^{-1}a^{-1}Xa) = Tr(\Lambda^{-1}X) = \varsigma(X^*).$$

To end the proof it is enough to verify that if $Y \in su(d_1) + \cdots + su(d_r)$

$$Tr\Lambda^{-1}Y = 0.$$

Recall that if $Y = Y_1 + \cdots + Y_r$ with $Y_k \in su(d_k)$, then $\Lambda^{-1}Y = 2\nu_1^{-1}J_1Y_1 + \cdots + 2\nu_r^{-1}J_rY_r$ and $Tr\Lambda^{-1}Y = 2\nu_k^{-1}Tr(J_kY_k) = 0$.

The same argument shows that the forms $\omega_k = 2\sum_{i_k} \omega^{i_k}_{i_k+n}$ project to $\mathbf{T}$. □

Proposition 3.7 *The projected forms $(\omega_1, \cdots, \omega_r)$ define a connection on $\mathbf{T}$ with values in $u(1) + \cdots + u(1)$.*

Proposition 3.8 *The form $d\varsigma$ descends to a form on M if and only if H is constant.*

PROOF. As $R^*_a\varsigma = \varsigma$ for every $a \in U(d_1) \times \cdots \times U(d_r)$, then $L_{X^*}\varsigma = 0$ for every $X \in u(d_1) + \cdots + u(d_r)$. We write $d(i(X^*)\varsigma) = d(\sum_k \frac{1}{\nu_k}\omega_k(X^*))$ and observe that $\omega_k(X^*)$ is constant for every X^*. Therefore $d(i(X^*)\varsigma) = 0$ if and only if ν_k are constant.

It follows from the formula $L_{X^*} = di(X^*) + i(X^*)d$ that $i(X^*)d\varsigma = 0$. We conclude that $d\varsigma$ is projectable on M. The same argument shows that the forms $d\omega_k$ can always be projected. □

In the rest of that section we suppose that H is constant. Given a two form

$$\varpi = V_{ij}\theta^i \wedge \theta^j + V_i\theta^i \wedge \theta$$

on M, define

$$Tr\varpi = h^{ij}V_{ji}.$$

We can now define on $\mathbf{T}$ the form

Definition 3.4

$$\sigma = \frac{1}{n+2}(\varsigma - \frac{1}{4(n+1)}Tr(d\varsigma)\theta).$$

We define now a bilinear form on $\mathbf{T}$ of type $(2n+1, 1, r-1)$, that is, with $2n+1$ positive eigenvalues, 1 negative eigenvalue and with a $(r-1)$ dimensional kernel.

Definition 3.5 *Let b be the bilinear symmetric form of type* $(2n+1, 1, r-1)$

$$b = \theta^i\theta^i + \theta\sigma$$

Observe that for any 2-form ϖ on M, $Tr'(\varpi) = Tr(\varpi)e^{2f}$. From 12 we obtain

Lemma 3.1

$$Tr'd\varsigma' = \big(Trd\varsigma + (4n+4)(f^i_j h^{ij} + nf^i f^i)\big)e^{-2f}$$

so

Proposition 3.9

$$\sigma' = \sigma - 2f^i\theta^i - f^i f^i\theta.$$

Putting together the formulas above we finally obtain the conformal invariance of the bilinear form.

Theorem 3.1 *Let (M, D, g) be a sub-Riemannian manifold such that H is constant on E_1. Then there exists a torus bundle $\mathbf{T}$ over M and the bilinear form*

$$b = \theta^i\theta^i + \theta\sigma$$

on $\mathbf{T}$ is a conformal invariant, i.e.,

$$b' = e^{2f}b,$$

if $g' = e^{2f}g$ on D.

Proof. From the definition we get

$$b' = e^{2f}(\theta^i + f^i\theta)(\theta^i + f^i\theta) + e^{2f}\theta(\sigma - 2f^i\theta^i - f^i f^i\theta) = e^{2f}b.$$

$\square$

Corollary 3.1 *When $H^2 = -Id$ on E_1 the torus bundle $\mathbf{T}$ is a $U(1)$ bundle and the bilinear form b is a conformally invariant Lorentz metric*

3.4 Circle bundles

In that section we suppose that H is constant on $\mathbf{T}$.

Consider now

$$g = \{X \in u(1) + \cdots + u(1) : \sigma(X^*) = 0\}.$$

As $\sigma(X^*) = \frac{1}{n+2}\sum_k \frac{\omega_k(X^*)}{\nu_k} = 0$, $G = \exp g$ is a closed subgroup of $U(1) \times \cdots \times U(1)$ if and only if there exists relatively prime positive integers $m_1, \cdots, m_r$ such that

$$m_1\nu_1 = m_2\nu_2 = \cdots = m_r\nu_r.$$

We may now define the $U(1)$ bundle $\mathbf{N} = \mathbf{T}/G$. We have proved

Theorem 3.2 *The bilinear form b descends to a Lorentz metric on $\mathbf{N}$ defined by*

$$L = \theta^i\theta^i + \theta\sigma$$

which is conformally invariant, that is, if $g' = e^{2f}g$ on D then $L' = e^{2f}L$ on $\mathbf{N}$.

References

[BDS] Burns, D. , Diederich, K. , Shnider, S. : Distinguished curves in pseudoconvex boundaries. Duke Math. J. 44 (1977), 407-431.

[C] Cartan, E. : Sur la géométrie pseudo-conforme des hypersurfaces de deux variables complexes, I. Ann. Math. Pura Appl., (4) 11 (1932) 17-90 (or Ouevres II, 2, 1231-1304); II, Ann. Scuola Norm. Sup. Pisa, (2) 1 (1932) 333-354 (or Ouevres III, 2, 1217-1238).

[CM] Chern, S. S. , Moser, J. : Real Hypersurfaces in Complex Manifolds . Acta Math. 133 (1974) 219-271.

[D] Dragomir, S. : Pseudohermitian Geometry. Bull. Math. Soc. Sc. Math. Roumanie 43(93), 3, (2000) 225-234.

[FV] Falbel, E., Veloso, J. M., : A parallelism for contact conformal sub-Riemannian geometry. Forum Mathematicum 10 (1998) 453-478.

[FVV] Falbel, E., Veloso, J. M., Verderesi, J. A. : Constant curvature models in sub-Riemannian geometry. Matematica Contemp. 4 (1993) 119-125.

[Fa] Farris, F. : An intrinsic construction of Fefferman's CR metric. Pac. J. Math. 123, 1, (1986) 33-45.

[F] Fefferman, C. : Monge-Ampre equations, the Bergman kernel, and the geometry of pseudo-convex domains. Ann. of Math. (2) 103 (1976) 395-416. Correction, 104 (1976) 393-394.

[Ku] Kuranishi, M. : CR geometry and Cartan geometry. Forum Math. 9 (1997), no. 2, 127–164.

[L] Lee, J. M. : The Fefferman metric and pseudoHermitian invariants. Trans. Am. Math. Soc. 296 (1986) 411-429.

[T] Tanaka, N. : On the equivalence problems associated with simple graded Lie algebras. Graded Lie algebras and geometrical structures. Hokkaido Math. J. v. 8 (1979) 23-84.

[W] Webster, S. : Pseudo-Hermitian structures on a real hypersurface. J. Diff. Geom. 13 (1978) 25-41.